# THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 1

CODEN: AMMYAE


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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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PRINTED IN THE UNITED STATES OF AMERICA

# THE AMERICAN MATHEMATICAL MONTHLY

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 83
1976

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND WASHINGTON, D.C.

# GROWTH FUNCTIONS ASSOCIATED WITH BIOLOGICAL DEVELOPMENT

GABOR T. HERMAN AND PAUL M. B. VITÁNYI

When an organism is growing under optimal conditions it may be assumed that its growth rate, and that of its parts, is governed by internal, inherited factors. The growth function of the organism is a function $f$ such that $f(t)$ is the number of cells in the organism at time $t$. In the last few years such growth functions have been actively studied by some researchers interested in mathematical models for biological development. We report on some of the results obtained.

**1. Introduction.** There have been many attempts to describe the process of biological development by mathematical models. In this article we shall only deal with mathematical models of development which are based on the approach first advocated by Lindenmayer [**10, 11, 12**]. Such models are commonly referred to as *L systems.*

The underlying idea of $L$ systems is that the natural basic unit for the discussion of biological development is the *cell.* For the sake of simplicity, let us concentrate on those organisms which consist of a simple linear array of cells, referred to as a *filament.* (For example, many algae are of this type. As we shall show, the mathematical theory is applicable to some more complicated organisms as well.) If we use different symbols to describe the different states a cell may be in, then a string of such symbols can be used to describe the whole filament. What happens to a cell at any given time depends on its own state and the states of its neighbors at that time. An L system contains a set of *production rules* which describe precisely how a cell changes depending on its own state and the state of its neighbors. This change may be a simple change of state, but it may also be a division of the cell into two or more cells or the cell might even disappear altogether, i.e., die. If a string of symbols (referred to as a *word*) describes the states of the cells in a filament, then by *simultaneous* application of the production rules to all the symbols in the word we obtain a new word which describes the next stage in the development of the filament. Repeating this process we can get the whole development history of the organism. Formal mathematical definitions of these concepts will be given in the next section.

As will be readily noticed, the approach taken to model development is by discretizing space and time. This is natural in the context of biological development: we discretize space in discrete cells and time in discrete time observations. The justification for assuming a finite set of states is that there are usually threshold values for parameters that determine the behavior of a cell. Thus, with respect to each of these parameters, it is sufficient to specify two conditions of the cell: "below threshold" and "above threshold," although the parameter itself may have infinitely many values. Even in those cases where such a simple minded scheme is insufficient, it is usually possible to approximate the infinite set of values by a sufficiently large finite set of values, without any serious detriment to the accuracy of the developmental model.

L systems have received a great deal of attention in recent years, both because they are biologically relevant and because they are a rich source of fascinating mathematical problems. We refer the interested reader to the book of Herman and Rozenberg [**7**] (for a shorther account see [**4**]), which also contains a detailed discussion, written by Lindenmayer, of the biological significance of L systems. For recent developments in the field see Rozenberg and Salomaa [**15**].

In this article we shall deal with only three types of L systems: D0L systems, D1L systems and D2L systems. In all three cases the D refers to the fact that the systems are *deterministic*: in any given situation there is only one production rule which is applicable to each cell of the filament. (Although non-deterministic L systems have also been studied, the restriction to the deterministic case is reasonable from the biological point of view, especially when we are interested in growth functions. We shall return to this point below.) The 2 in D2L systems refers to 2 *sided interaction,*

i.e., how a cell changes depends on the states of both of its neighbors and its own state. Similarly, the 1 in D1L systems refers to 1 *sided interaction*: how a cell changes depends only on its own state and that of its neighbor on one side, which for a particular D1L system is always either to the left or to the right. The 0 in D0L systems refers to 0 *sided interactions* (or no interactions): the change of a cell is determined solely by the state of the cell itself.

An L system of any of the three types under consideration will have three components. (i) A finite nonempty set of symbols, referred to as the *alphabet*, which contains a symbol for all the possible cellular states between which we wish to distinguish. (ii) A set of *production rules* which associates with every symbol (in the D0L case), with every pair of symbols (in the D1L case) or with every triple of symbols (in the D2L case), in the alphabet a unique string of symbols by which the symbol in question will have to be replaced. (iii) A word (string of symbols) over the alphabet, referred to as the *axiom*, which describes the organism at the beginning of the developmental process.

If we denote the axiom by $w_0$ then, by applying the production rules simultaneously to all symbols of $w_0$, we obtain a string $w_1$. We can repeat this process any number of times, obtaining the *developmental sequence* $w_0, w_1, w_2, w_3, \cdots$ associated with the L system under consideration. The *growth function* associated with this L system is a function from the natural numbers into the natural numbers, whose value for any number $t$ is the length of the word $w_t$, i.e., the number of cells in the organism at that time.

The study of the change in size and weight of a growing organism as a function of time constitutes a considerable part of the literature on developmental biology. Usually, genetically identical specimens of a specific organism are investigated in controlled environments and their changes of size and weight in time are described. The scientific presupposition is that identical genetic material and identical environment will result in identical growth rates, i.e., that the experiment is repeatable. This assumes a deterministic (causal) underlying structure, and makes a good case for the biological relevance of the study of growth functions of deterministic L systems, where we assume that the production rules reflect the simultaneous influence of the inherited genetic factors and a specific environment on the developmental behavior of the cells. Thus, when an organism is growing under optimal conditions it may be assumed that its growth rate, and that of its parts, is governed by internal, inherited factors. One of the easiest things to observe about a filamentous organism is the number of cells it has. Suppose, having observed the development of a particular organism, we generalize our observations by giving a function $f$, such that $f(t)$ is the number of cells in the organism after $t$ steps. The problem then arises to produce a developmental system whose growth function is $f$.

We must warn the reader that the growth models treated here apply only to organisms which consist of autonomous segments or compartments (living cells are naturally such compartments). By "autonomy" is meant primarily independence in the hereditary sense, in the sense that cells are known to carry their own genetic instructions and pass them on to their daughters, but also metabolic and functional independence. The existence of autonomous compartments enables us to describe such organisms in terms of automata arrays, from which the formalism of L-systems is easily derivable. But a consequence of this requirement is that these models cannot be directly applied to subcellular growth processes.

Another restriction of these models lies in their one-dimensional nature. This implies that at the present time they are only applicable to filamentous organisms, or to one-dimensional aspects of the growth of bulky organisms (such as length measurements). Multi-dimensional models similar to these (graph L-systems) have recently been introduced by various workers, their growth functions have been investigated (for example, by Professor K. Culik of the University of Waterloo, Canada) and were found to be simple extensions of the one-dimensional case. Thus the restriction to one-dimensional growth descriptions in the present paper can be viewed as a temporary and non-essential one.

The third point of warning we should mention is that these models are based on the assumptions

that the relative positions of cells (or compartments) cannot change during growth, neither can the neighborhoods of daughter cells be different from those of their mother cells. These assumptions are in agreement with development in plants but not necessarily with that in animals, where cells may slide past each other in the course of growth.

We conclude this section with a discussion of the biological motivation for one of the problems which will be considered in detail below. Clearly, any growth function which can be achieved by a D0L system can also be achieved by a D1L system, simply by giving production rules for the D1L system which for all practical purposes ignore the state of the neighbor. The question arises whether the converse is also true. We shall show that it is not: if a D0L system keeps growing at all, it must be growing "fast," as opposed to systems with interactions which are capable of "slow" but nevertheless unbounded growth. Thus interaction between cells provides organisms with the capability of controlling the rate of their growth in an orderly manner. When this interaction mechanism breaks down, tumors containing cells which do not interact with their neighbors may begin to grow at an exponential rate. For this reason, some early workers in the field of growth functions referred to such an exponential growth as "malignant."

**2. Definitions and problem statements.** In the following $\Sigma$ always denotes a finite nonempty set of *symbols*, $\Sigma^*$ denotes the set of all *words* over $\Sigma$, including the *empty word* $\Lambda$ (the word with no symbols in it). If $w$ is a word in $\Sigma^*$, then $\lg(w)$ denotes the *length* of $w$, i.e. the number of symbols in it. In particular $\lg(\Lambda) = 0$.

DEFINITION 1. A *DxL system*, $x \in \{0, 1, 2\}$, is a construct $G = \langle \Sigma, \delta, w \rangle$, where the *alphabet* $\Sigma$ is a finite nonempty set of symbols; the set of *production rules* $\delta$ is a mapping from $\bigcup_{i=1}^{x+1} \Sigma^i$ into $\Sigma^*$, i.e., for each ordered set $(a_1, \cdots, a_i)$ of $i$ elements of $\Sigma$, $1 \leqq i \leqq x+1$, there is one and only one $\alpha$ in $\Sigma^*$ such that $\delta(a_1, \cdots, a_i) = \alpha$; and the *axiom* $w$ is an element of $\Sigma^*$.

Given a DxL system $G = \langle \Sigma, \delta, w \rangle$, $\delta$ induces a mapping $\bar{\delta}$ from $\Sigma^*$ into $\Sigma^*$ defined as follows. $\bar{\delta}(\Lambda) = \Lambda$. For any word $a_1 a_2 \cdots a_n \in \Sigma^*$, $n \geqq 1$, $\bar{\delta}(a_1 a_2 \cdots a_n) = \alpha_1 \alpha_2 \cdots \alpha_n$ if and only if the following holds.

(i) If $x = 0$, then $\alpha_i = \delta(a_i)$ for all $i$, $1 \leqq i \leqq n$.
(ii) If $x = 1$, then $\alpha_i = \delta(a_{i-1}, a_i)$ for all $i$, $1 < i \leqq n$, and $\alpha_1 = \delta(a_1)$.
(iii) If $x = 2$, then $\alpha_i = \delta(a_{i-1}, a_i, a_{i+1})$ for all $i$, $1 < i < n$, $\alpha_1 = \delta(a_1)$ if $n = 1$, and $\alpha_1 = \delta(a_1, a_2)$ and $\alpha_n = \delta(a_{n-1}, a_n)$ if $n > 1$.

Hence, if $x \geqq 1$, then the end cells of a filament, sensing that they have no neighboring cells on one or both sides, follow special rules. We have defined D1L systems such that each cell is influenced by its left neighbor. The case where each cell of a D1L system is influenced by its right neighbor is entirely symmetric and yields exactly the same results with respect to growth functions.

Since for an element $a$ of $\Sigma$, $\bar{\delta}(a) = \delta(a)$, we shall from now on use the notation $\delta$ for the mapping $\bar{\delta}$ as well. Confusion is avoided by the format of the arguments.

For any natural number $i$, we define the $i$-fold composition $\delta^i$ of $\delta$ inductively by $\delta^0(v) = v$ and $\delta^{i+1}(v) = \delta(\delta^i(v))$, for each word $v \in \Sigma^*$.

*Example* 1. Let $G = \langle \{a, b, o, r\}, \delta, ar \rangle$ be a D1L system where $\delta$ consists of the following productions.

$\delta(o) = a$,
$\delta(x, o) = a$, for $x \in \{a, b, o, r\}$,
$\delta(a) = o$,
$\delta(o, a) = b$,
$\delta(o, b) = o$,
$\delta(o, r) = ar$,
$\delta(x, y) = y$ and $\delta(y) = y$, otherwise.

Thus,

$\delta^0(ar) = ar$, $\delta^4(ar) = abr$,
$\delta^1(ar) = or$, $\delta^5(ar) = obr$,
$\delta^2(ar) = aar$, $\delta^6(ar) = aor$,
$\delta^3(ar) = oar$, $\delta^7(ar) = oaar$, etc.

*Example* 2. Let $G = \langle\{0,1,2,3,4,5,6,7,8,9,(,)\}, \delta, 4\rangle$ be a D0L system, where $\delta(0) = 10$, $\delta(1) = 32$, $\delta(2) = 3(4)$, $\delta(3) = 3$, $\delta(4) = 56$, $\delta(5) = 37$, $\delta(6) = 58$, $\delta(7) = 3(9)$, $\delta(8) = 50$, $\delta(9) = 39$, $\delta(() = (, \delta()) = )$.

These rules were devised in [5] in an attempt to model without cellular interactions the developmental behavior of certain red algae. In order to see whether we were successful, we would have to follow the developmental sequence through quite a few steps. Because this organism grows very fast, this would be quite tedious to do by hand. We have therefore used a computer to work out the developmental sequence for us. The program CELIA (CEllular Linear Iterative Array simulator, described in [1, 2, 6]) was provided with the description of $G$ and produced for us the developmental sequence of $G$, the first 16 stages of which are reproduced below.

1 4

2 56

3 3758

4 33(9)3750

5 33(39)33(9)3710

6 33(339)33(39)33(9)3210

7 33(3339)33(339)33(39)33(4)3210

8 33(33339)33(3339)33(339)33(56)33(4)3210

9 33(333339)33(33339)33(3339)33(3758)33(56)33(4)3210

10 33(3333339)33(333339)33(33339)33(33(9)3750)33(3758)33(56)33(4)3210

11 33(33333339)33(3333339)33(333339)33(33(39)33(9)3710)33(33(9)3750)33(3758)33(56)33(4)3210

12 33(333333339)33(33333339)33(3333339)33(33(339)33(39)33(9)3210)33(33(39)33(9)3710)33(33(9)3750)33(3758)
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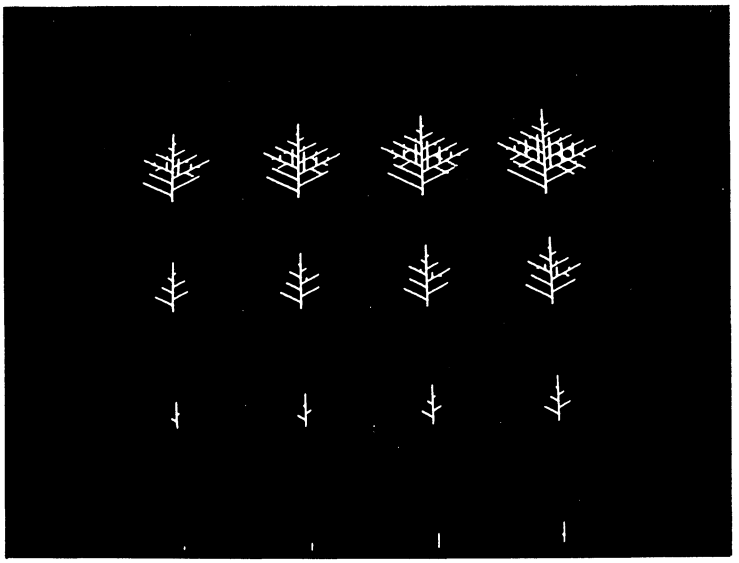
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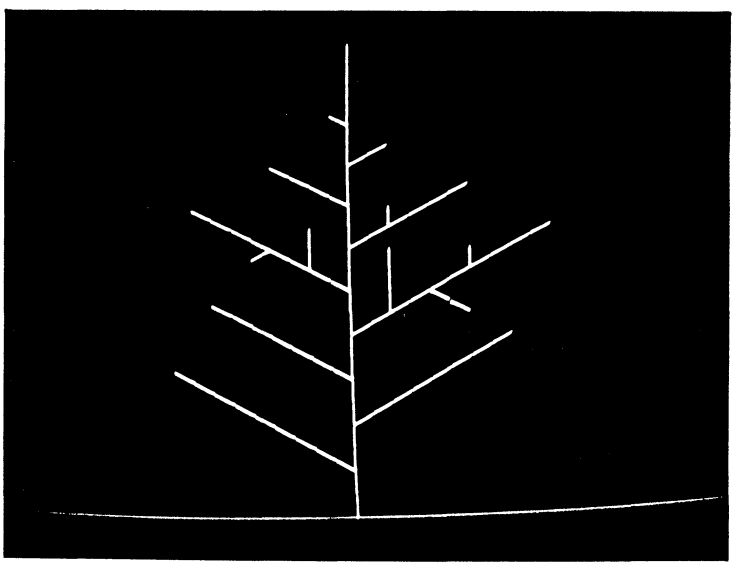
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16 33(3333333333339)33(333333333339)33(33333333339)33(33(3333339)33(333339)33(33339)33(33(9)3750)33(3758)
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This example is particularly interesting because it also demonstrates our earlier claim that the mathematical formalism developed can be used to investigate structures more complicated than simple linear arrays. Interpreting the left parenthesis as the beginning of a branch and right parenthesis as the end of a branch (thus parentheses within parentheses indicate branches on branches), the computer has displayed on the screen the following, which represents the first 16 stages of the development.



Since we wanted to have a look at the details of this development, we requested the computer to display stage 12 in some detail. This is shown below.



Such studies made us conclude that the development of branching patterns of certain red algae can be achieved without cellular interactions.

Examples 1 and 2 emphasize our comments at the end of the last section. In Example 1, a slow rate of growth is controlled by the 1-sided interaction, while in Example 2 the lack of interaction causes a fast rate of growth.

DEFINITION 2. If $G = \langle \Sigma, \delta, w \rangle$ is a DxL system, $x \in \{0, 1, 2\}$, then the function $f_G$ from the

nonnegative integers into the nonnegative integers defined by

$$f_G(t) = \lg(\delta^t(w))$$

for all $t$, is said to be the *growth function* of $G$.

*Example* 3. Let $G = \langle\{a, b\}, \{\delta(a) = b, \delta(b) = ab\}, a\rangle$ be a D0L system. Then,

$$f_G(0) = f_G(1) = 1,$$

and for all $t$ such that $t \geqq 0$,

$$f_G(t+2) = f_G(t+1) + f_G(t).$$

Thus, $f_G(t)$ is the $t$th element of the well-known Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \cdots$.

*Example* 4. Let $G = \langle\{a, b, c\}, \{\delta(a) = abc^2, \delta(b) = bc^2, \delta(c) = c\}, a\rangle$ be a D0L system. Then,

$$f_G(0) = \lg(a) = 1,$$

$$f_G(1) = \lg(abc^2) = 4,$$

$$f_G(2) = \lg(abc^2bc^4) = 9,$$

$$f_G(3) = \lg(abc^2bc^4bc^6) = 16.$$

In fact, for all $t > 0$,

$$f_G(t) = f_G(t-1) + 2t + 1.$$

By induction it follows that $f_G(t) = (t+1)^2$.

In investigating growth functions, one of the first questions we ask is what rates of growth are possible. That the rate of growth of a DxL system is at most exponential follows from the next lemma which is immediate from the definitions.

LEMMA 1. *For any DxL system* $G = \langle\Sigma, \delta, w\rangle$, $x \in \{0, 1, 2\}$,

$$f_G(t) \leqq \lg(w)m^t,$$

*where* $m$ *is the maximum length of a value* $\delta$ *may have. (I.e.,* $m = \max\{\lg(\alpha) \mid \alpha$ *is in the range of the set of production rules* $\delta\}$.)

The problems which have been investigated with respect to growth functions fall roughly into the following six categories.

(i) *Analysis problems.* Given a DxL system, describe its growth function in some fixed predetermined formalism.

(ii) *Synthesis problems.* Given a function $f$ in some fixed predetermined formalism and an $x \in \{0, 1, 2\}$, find a DxL system whose growth function is $f$. Related to this is the problem: which functions can be growth functions of DxL systems?

(iii) *Growth equivalence problems.* Given two DxL systems, decide whether or not they have the same growth function.

(iv) *Classification problems.* Given a DxL system decide what is its growth type. (E.g., is there a polynomial or even a constant which bounds its growth function. Growth types will be rigorously defined in Section 5.)

(v) *Structural problems.* What properties of production rules induce what type of growth.

(vi) *Hierarchy problems.* Is the set of growth functions of DxL systems a proper subset of the set of growth functions of D(x + 1)L systems and similar problems.

In the first five cases we would like to solve our problems effectively. That is, we would like to be able to write computer programs (*algorithms*) which, in the case of the analysis problem say,

provide us with an explicit description of the growth function whenever they are given the description of the DxL system. (We shall return in a more rigorous way to the concept of an algorithm in Section 4, where we shall show that for some of the tasks described above there is no algorithm which does the job.)

In this article we shall deal with only two questions in detail: the analysis problem for D0L systems and the nature of subpolynomial growth functions (these are unbounded functions which nevertheless grow slower than any unbounded polynomial). These are the topics of the next two sections. The section after them summarizes some other known results about growth functions.

**3. Analysis of growth functions of D0L systems.** The next definition and lemma provide the essence of much that we know about growth functions of D0L systems.

DEFINITION 3. For a D0L system $G = \langle \Sigma, \delta, w \rangle$, with alphabet $\Sigma = \{a_1, a_2, \cdots, a_n\}$, we define the following matrices. The *initial vector* $\pi_G$ is the $n$ dimensional row vector such that its $i$th component equals the number of occurrences of the letter $a_i$ in $w$, for $i = 1, 2, \cdots, n$. The *final vector* $\eta_G$ is the $n$ dimensional column vector with all its components equal to 1. The *growth matrix* $M_G$ is the $n \times n$ matrix whose $(i, j)$-th entry equals the number of occurrences of $a_j$ in $\delta(a_i)$.

These matrices are introduced because from the point of view of growth of D0L systems the order of the letters in $w$ and in the values of $\delta$ is immaterial. In fact, the following result has an easy inductive proof.

LEMMA 2. *If $G$ is a D0L system and $t$ is a natural number, then*

$$f_G(t) = \pi_G M_G^t \eta_G.$$

*Example* 5. Consider the D0L system $G$ of Example 4. We have

$$\pi_G = (1,0,0), \qquad M_G = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \eta_G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Therefore

$$M_G^2 = \begin{pmatrix} 1 & 2 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_G^3 = \begin{pmatrix} 1 & 3 & 12 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} = 3M_G^2 - 3M_G + M_G^0,$$

where $M_G^0$ is the $3 \times 3$ identity matrix. From this and Lemma 2 it follows that

$$f_G(t+3) = 3f_G(t+2) - 3f_G(t+1) + f_G(t),$$

which can be used to prove that $f_G(t) = (t+1)^2$ for all $t$.

We now proceed to give an explicit formula for the growth functions of D0L systems, by using known facts concerning homogeneous linear difference equations with constant coefficients.

THEOREM 1. *For any D0L system $G = \langle \Sigma, \delta, w \rangle$ the form of $f_G$ is*

$$f_G(t) = \sum_{i=1}^{k} p_i(t)\, c_i^t,$$

*where the $c_i$'s are the distinct characteristic values of $M_G$, and $p_i$ is an $r_i$-th degree polynomial in $t$, where $r_i + 1$ is the multiplicity of the characteristic value $c_i$ of $M_G$, $1 \leqq i \leqq k$. (Therefore $\sum_{i=1}^{k} (r_i + 1) = \# \Sigma$.) The coefficients of the polynomials are determined by the first $\# \Sigma$ values of $f_G$.*

*Proof.* Let $q(x) = \sum_{i=0}^{n} a_i x^i$ be the polynomial $\det(Ix - M_G)$, i.e., the characteristic polynomial of

$M_G$. By the Cayley-Hamilton Theorem, $q(M_G) = 0$ (where 0 denotes the zero matrix). By Lemma 2,

$$\sum_{i=0}^{n} a_i f_G(t+i) = \sum_{i=0}^{n} a_i \pi_G M_G^{t+i} \eta_G$$

$$= \pi_G M_G^t \left( \sum_{i=0}^{n} a_i M_G^i \right) \eta_G$$

$$= 0.$$

Hence the growth function $f_G$ satisfies a homogeneous difference equation of order $n = \#\Sigma$ with coefficients identical to those of the characteristic polynomial of $M_G$. It is well known that such difference equations have a solution of the form

$$f_G(t) = \sum_{i=1}^{k} p_i(t) c_i^t,$$

where the $c_i$'s are the distinct roots of $q(x) = 0$ (i.e., the distinct characteristic values of $M_G$), and the $p_i$'s are polynomials in $t$ of degree $r_i$, where $r_i + 1$ is the multiplicity of $c_i$, $1 \leqq i \leqq k$. The coefficients of the polynomials $p_i$ are determined by $f_G(s), \cdots, f_G(n-1)$, where $s$ is the multiplicity of the zero root ($s = 0$ if zero is not a root). Hence we see that the growth function of a D0L system is a generalized exponential polynomial which has positive integer values for positive integer arguments.

*Example* 6. Let $G = \langle \{a, b, c\}, \{\delta(a) = a^2, \delta(b) = a^5 b, \delta(c) = b^3 c\}, a^m b^n c^p \rangle$ be a D0L system. The characteristic equation $x^3 - 4x^2 + 5x - 2 = 0$ of the growth matrix

$$M_G = \begin{pmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

has roots $x_1 = x_2 = 1$ and $x_3 = 2$. (Note that $M_G$ is independent of the axiom.) Since the axiom has $m$ occurrences of $a$, $n$ occurrences of $b$ and $p$ occurrences of $c$, we obtain as the growth function of $G$:

$$f_G(t) = a_1 + a_2 t + a_3 2^t,$$

where

$$f_G(0) = a_1 + a_3 = m + n + p,$$

$$f_G(1) = a_1 + a_2 + 2a_3 = 2m + 6n + 4p,$$

$$f_G(2) = a_1 + 2a_2 + 4a_3 = 4m + 16n + 22p.$$

Consequently,

$$f_G(t) = (m + 5n + 15p)2^t - 12pt - 4n - 14p.$$

This shows immediately, that $G$ has an exponentially increasing growth function for all axioms not equal to $\Lambda$.

*Example* 7. In Example 5 we produced the homogeneous difference equation

$$f_G(t+3) - 3f_G(t+2) + 3f_G(t+1) - f_G(t) = 0.$$

The characteristic equation of this difference equation (and of the matrix it derives from) is

$$x^3 - 3x^2 + 3x - 1 = 0.$$

The roots are $x_1 = x_2 = x_3 = 1$. Hence, solutions to this equation are all of the form

$$f_G(t) = at^2 + bt + c.$$

Using $f_G(0) = 1$, $f_G(1) = 4$ and $f_G(2) = 9$, we obtain $a = 1$, $b = 2$ and $c = 1$, proving again that

$$f_G(t) = (t+1)^2.$$

An alternative approach for solving the analysis problem, which is also of use for the growth equivalence problem and the synthesis problem, is an application of the theory of generating functions.

DEFINITION 4. With any function $f$ from the nonnegative integers into the nonnegative integers we associate its *generating function* $F(x)$ which is defined to be the formal infinite power series $\sum_{t=0}^{\infty} f(t)x^t$. We also say that $F(x)$ *generates* $f$.

The reason for such a definition is that very often the function $F(x)$ can be represented in a simple way. For example, if $f(t) = 2^t$ then $F(x) = 1/(1-2x) = 1 + 2x + 4x^2 + 8x^3 + \cdots$.

The following lemmas are well known and easily proven mathematical facts; $(p(x)/q(x)$ denotes the quotient, $p(x)q(x)$ the product of the polynomials $p$ and $q)$.

LEMMA 3.

(i) *If $p(x)$ and $q(x)$ are two polynomials with integer coefficients such that $q(0) = 1$, then $p(x)/q(x)$ uniquely determines an infinite power series with integer coefficients, i.e. $p(x)/q(x) = \sum_{t=0}^{\infty} f(t)x^t$, where $f(t)$ is an integer for all $t$. Thus $p(x)/q(x)$ generates the function $f$. Furthermore, given $p(x)$ and $q(x)$, $f(t)$ is effectively computable for every nonnegative integer $t$.*

(ii) *Let $p(x)$, $q(x)$, $p'(x)$ and $q'(x)$ be polynomials with integer coefficients such that $q(0) = q'(0) = 1$, and let $f$ and $f'$ be functions generated by $p(x)/q(x)$ and $p'(x)/q'(x)$, respectively. Then $f(t) = f'(t)$ for all $t$ if and only if $p(x)q'(x) = p'(x)q(x)$, for all $x$. Thus it is effectively decidable whether or not $p(x)/q(x)$ and $p'(x)/q'(x)$ generate the same function.*

LEMMA 4. *Let $n$ be any positive integer and let $A$ be an $n \times n$ matrix whose entries are polynomials in $x$ with integer coefficients. Let $q(x) = \det(A)$. If there exists a value of $x$ such that $q(x) \neq 0$, then $A$ is invertible, i.e. there exists an $n \times n$ matrix $A^{-1}$ such that $AA^{-1} = A^{-1}A = I$, where $I$ denotes the $n \times n$ identity matrix. Furthermore, given $A$, $A^{-1}$ can be effectively obtained, and each entry of $A^{-1}$ will be of the form $p_{i,j}(x)/q(x)$, where $p_{i,j}(x)$ is a polynomial with integer coefficients.*

These lemmas lead us to the following theorem.

THEOREM 2. *There is an algorithm which, for any D0L system $G$, effectively computes two polynomials $p(x)$ and $q(x)$ with integer coefficients where $q(0) = 1$, such that $p(x)/q(x)$ generates the growth function $f_G$ of $G$.*

*Proof.* Let $G$ be the given D0L system and let $\pi_G$, $M_G$ and $\eta_G$ be as usual. Suppose the alphabet of $G$ contains $n$ elements. Let $M_G x$ be the $n \times n$ matrix obtained by multiplying each entry of $M_G$ by the variable $x$. Let $I$ denote the $n \times n$ identity matrix. Then $I - M_G x$ is an $n \times n$ matrix whose entries are polynomials with integer coefficients. Let $q(x) = \det(I - M_G x)$. Since $q(0) = 1$ we see that $I - M_G x$ is an invertible matrix. According to Lemma 4, we can effectively produce an $n \times n$ matrix $(I - M_G x)^{-1}$ whose entries are all of the form $p_{i,j}(x)/q(x)$, where $p_{i,j}(x)$ and $q(x)$ are polynomials with integer coefficients. Clearly, $\pi_G(I - M_G x)^{-1}\eta_G$ is of the form $p(x)/q(x)$ where $p(x)$ is a polynomial with integer coefficients and can be effectively computed. All we need to complete the proof of the theorem is to show that $p(x)/q(x)$ generates the growth function $f_G$ of $G$.

For $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, let $f_{i,j}$ be the function generated by $p_{i,j}(x)/q(x)$, i.e., $p_{i,j}(x)/q(x) = \sum_{t=0}^{\infty} f_{i,j}(t)x^t$. (That such an $f_{i,j}$ exists and is unique follows from Lemma 3(i).) For $t \geqq 0$, let $F_t$ be the $n \times n$ matrix whose typical entry is $f_{i,j}(t)$. Then we have that

$$I = (I - M_G x)(I - M_G x)^{-1}$$

$$= (I - M_G x)\left(\sum_{t=0}^{\infty} F_t x^t\right)$$

$$= \sum_{t=0}^{\infty} (F_t x^t) - \sum_{t=0}^{\infty} (M_G F_t x^{t+1})$$

$$= \sum_{t=0}^{\infty} (F_t x^t) - \sum_{t=1}^{\infty} (M_G F_{t-1} x^t).$$

Identifying coefficients of powers of $x$ we get that $F_0 = I$, and, for $t \geqq 1$, $F_t = M_G F_{t-1}$. From this it follows that, for $t \geqq 0$, $F_t = M_G^t$. Hence,

$$\begin{aligned} p(x)/q(x) &= \pi_G (I - M_G x)^{-1} \eta_G \\ &= \pi_G \left( \sum_{t=0}^{\infty} F_t x^t \right) \eta_G \\ &= \pi_G \left( \sum_{t=0}^{\infty} M_G^t x^t \right) \eta_G \\ &= \sum_{t=0}^{\infty} (\pi_G M_G^t \eta_G) x^t \\ &= \sum_{t=0}^{\infty} f_G(t) x^t. \end{aligned}$$

Thus, $p(x)/q(x)$ is the generating function of $f_G$.

This theorem can certainly be considered as a solution to the analysis problem for D0L systems, since given a D0L system the algorithm provides us with a description of its growth function in the form of a rational generating function.

*Example* 8. Consider the D0L system $G = \langle \{a, b, c\}, \delta, a \rangle$, where $\delta(a) = abc^2$, $\delta(b) = bc^2$, $\delta(c) = c$, of Examples 4, 5, and 7. As in Example 5,

$$\pi_G = (1, 0, 0), \qquad M_G = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \eta_G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Using Cramer's rule we see that

$$\pi_G (I - M_G x)^{-1} \eta_G = \frac{\det \begin{pmatrix} 1 & -x & -2x \\ 1 & 1-x & -2x \\ 1 & 0 & 1-x \end{pmatrix}}{\det \begin{pmatrix} 1-x & -x & -2x \\ 0 & 1-x & -2x \\ 0 & 0 & 1-x \end{pmatrix}}$$

$$= \frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \cdots.$$

**4. Subpolynomial growth functions.** In this section we investigate some of the properties of the class of growth functions which are unbounded and yet grow slower than any unbounded polynomial.

DEFINITION 5. A function is said to be *unbounded* if $\lim_{t \to \infty} f(t) = \infty$.

DEFINITION 6. The growth type of a DxL system $G = \langle \Sigma, \delta, w \rangle (x \in \{0, 1, 2\})$ is said to be *subpolynomial* if and only if $f_G$ is unbounded and for every unbounded polynomial $p$ it is the case that $\lim_{t \to \infty} (f_G(t)/p(t)) = 0$.

*Example* 9. The growth type of the D1L system $G$ of Example 1 is subpolynomial. It is in fact easy to show that $f_G$ is unbounded and that, for all positive integers $k$, there exists an integer $t$ such that

$$f_G(t+1) = \cdots = f_G(t+2^k),$$

and so, for any unbounded polynomial $p$, $\lim_{t\to\infty}(f_G(t)/p(t)) = 0$.

The next theorem makes precise our claim that unbounded growth functions of D0L systems must grow "fast."

THEOREM 3. *If $G = \langle \Sigma, \delta, w\rangle$ is a D0L system and $m$ is an integer such that $f_G(m) = f_G(m+1) = \cdots = f_G(m+n)$, where $n$ is the number of symbols in $\Sigma$, then for all $t \geqq 0$, $f_G(m+t) = f_G(m)$.*

*Proof.* Let $q(x) = \Sigma_{i=0}^{n} a_i x^i$ be the characteristic polynomial of $M_G$ ($n = \#\Sigma$ and $a_n = 1$). We prove that $f_G(m+t) = f_G(m)$ for all $t \geqq 0$ by induction. The cases $0 \leqq t \leqq n$ follow from the condition in the theorem. Suppose now that the result is valid for $0 \leqq t \leqq s$, where $s \geqq n$. We have shown in the proof of Theorem 1 that for all $k \geqq 0$, $\Sigma_{i=0}^{n} a_i f_G(k+i) = 0$. Letting $k = m+s-n+1$, we get (using the induction hypothesis) that

$$\begin{aligned} f_G(m+s+1) &= -\sum_{i=0}^{n-1} a_i f_G(m+s-n+1+i) \\ &= -\sum_{i=0}^{n-1} a_i f_G(m+s-n+i) \\ &= f_G(m+s). \end{aligned}$$

This leads directly to the following hierarchy result.

THEOREM 4. *The set of growth functions of D0L systems is a proper subset of the set of growth functions of D1L systems.*

*Proof.* That the set of growth functions of D0L systems is a subset of the set of growth functions of D1L systems follows by definition. That it is a proper subset follows from Theorem 3 and Example 9.

The essence of the proof above is that unbounded growth functions of D0L systems must grow with at least a certain speed, and we have found an unbounded growth function of a D1L system which grows slower than this. It is interesting to note that the same type of argument cannot be repeated to show the existence of a growth function of a D2L system which is not also a growth function of a D1L system. This is because the longest period for which an unbounded growth function of a D2L system $G = \langle \Sigma, \delta, w\rangle$ can retain the value $k$ is clearly $n^k$ where $n = \#\Sigma$. It is not too difficult to prove that for any integer $n > 1$ there is a D1L system $G = \langle \Sigma, \delta, w\rangle$ with $\#\Sigma = n+2$, such that $f_G$ is unbounded and $f_G$ retains the value $k$ for at least $n^k$ consecutive arguments. (Example 1 serves this purpose in case $n = 2$.) It is at present an open problem whether or not there exists a D2L system which has a growth function which is not also the growth function of a D1L system.

We complete this section with an example of a task for the execution of which there is no algorithm.

THEOREM 5. *There is no algorithm which for any given D1L system $G$ decides whether or not the growth type of $G$ is subpolynomial.*

*Proof.* The proof of this theorem makes use of the theory of Turing machines. A Turing machine is a logical device consisting of a finite control with an attached read-write head travelling about on an infinitely expandable tape divided into squares. Each square contains one of a finite set of symbols, and according to the current state of its finite control and the symbol in the scanned tape

square, the Turing machine prints a new symbol in the square under scan, moves one square to the left or to the right and enters a new state. If the Turing machine enters a special state, then it is said to halt. Consider the following task: give an algorithm which for any Turing machine decides whether or not that Turing machine halts if it is started on a blank tape. This is referred to as the "blank tape halting problem."

It is standard in the theory of computation to identify the intuitive concept of an "algorithm" with the mathematically precise concept of a "Turing machine" (Church's thesis). It is also well known (see, e.g., Minsky [13]) that there is no Turing machine which "solves" the blank tape halting problem. However, it can be shown (for details see Vitányi [19]) that if there was an algorithm which decides for any given D1L system $G$ whether or not the growth type of $G$ is subpolynomial, then it could be used to construct an algorithm which solves the blank tape halting problem. Since the latter does not exist, the former cannot exist either.

**5. Summary of other results on growth functions.**

(i) ANALYSIS PROBLEMS. In section 3 we have discussed and solved the analysis problem for D0L systems. There we have used two different formalisms: sums of exponential functions with polynomial coefficients and rational generating functions. We have described procedures for obtaining the growth function in either of these formalisms for an arbitrary D0L system.

Theorem 5 and similar results have interesting consequences regarding the analysis problem for D1L and D2L systems. It is true that we may be able to find a suitable mathematical expression for the growth function of any given D2L system by *ad hoc* methods. However, if we fix a formalism in which we want to express the growth function in a way which clearly indicates the growth type of the function, then there is no algorithm which, for an arbitrary D1L system $G$, gives an explicit expression for $f_G$ in the predetermined formalism.

(ii) SYNTHESIS PROBLEMS. A major result in the direction of synthesis of growth functions for D0L systems is the following. It can be shown (Sziłard [17]) that any positive, nondecreasing, ultimately polynomial function is the growth function of a D0L system. The proof provides an algorithm which for any such function (described in some predetermined way) produces the required D0L system.

The method uses many results on the nature of polynomial functions. On the way to proving the main theorem Szilard showed, for example, that if the generating functions $F(x)$ and $F'(x)$ generate growth functions of D0L systems, then so do $F(x)+F'(x)$, $1+xF(x)$ and $F(x)/(1-x)$. His proofs were effective, given the D0L systems whose growth functions are generated by $F(x)$ and $F'(x)$, he showed how one can obtain the D0L systems whose growth functions are generated by $F(x)+F'(x)$, $1+xF(x)$ and $F(x)/(1-x)$. Thus, if we know how to obtain D0L systems whose growth functions are generated by certain basic generating functions, results such as this provide us with the ability to construct D0L systems whose growth functions are generated by more and more complicated generating functions put together from the basic ones by the operations described above.

Such method can certainly be used to synthesize a function like $(t+1)^2$, i.e., to produce a D0L system whose growth function is $(t+1)^2$. In fact much more complicated growth functions can also be synthesized.

Presently it is an open problem whether or not there exists an algorithm such that given a function $f$, either by an exponential polynomial as in Theorem 1 or by its generating function as in Theorem 2, the algorithm decides whether or not $f$ is a D0L growth function. On the other hand, there are algorithms which, given a D0L growth function $f$ in either of these formalisms, will produce a D0L system whose growth function is $f$ (see, e.g., Paz and Salomaa [14]).

When we come to L systems with interaction, the situation is again much worse. There are no general algorithms for synthesizing growth functions of D1L and D2L systems, but some partial results have been obtained. For example, Vitányi [19] proved that for each rational number

$0 < r < 1$, we can effectively find a D2L system whose growth function is of the order of magnitude of $t^r$.

(iii) GROWTH EQUIVALENCE PROBLEMS. It is a consequence of Theorem 2 and Lemma 3(ii) that there is an algorithm which, for any two D0L systems, decides whether or not they have the same growth function.

Exactly the opposite is the case for D1L and D2L systems. Vitányi [19] proved that there does not exist an algorithm which decides the growth equivalence of two arbitrary D1L systems. However, such an algorithm exists for the rather restricted case of those D1L systems whose growth functions are bounded or which have an alphabet of one letter only.

(iv) CLASSIFICATION PROBLEMS. In view of Lemma 1, the following gives an exhaustive classification of growth types.

The growth of a DxL system $G = \langle \Sigma, \delta, w \rangle$ $(x \in \{0, 1, 2\})$ is said to be

(i) *exponential* (*type* 3) if and only if there exists a real number $x > 1$, such that $\lim_{t\to\infty} f_G(t)/x^t > 0$;

(ii) *subexponential* (*type* $2\frac{1}{2}$) if and only if the growth is not exponential and there does not exist a polynomial $p$ such that $f_G(t) \leqq p(t)$ for all $t$;

(iii) *polynomial* (*type* 2) if and only if $f_G$ is unbounded and there exist polynomials $p$ and $q$ such that $p(t) \leqq f_G(t) \leqq q(t)$ for all $t$;

(iv) *subpolynomial* (*type* $1\frac{1}{2}$) if and only if $f_G$ is unbounded and for each unbounded polynomial $p$ $\lim_{t\to\infty} f_G(t)/p(t) = 0$;

(v) *limited* (*type* 1) if and only if there exists an integer $m$ such that $0 < f_G(t) < m$ for all $t$;

(vi) *terminating* (*type* 0) if and only if there exists an integer $t_0$ such that $f_G(t) = 0$ for all $t \geqq t_0$.

It is known [18] that growth types $2\frac{1}{2}$ and $1\frac{1}{2}$ cannot occur in the D0L case (cf. Theorem 3). There are D0L systems with growth types 3, 2, 1, 0. The system of Example 4 is clearly polynomial (type 2), while the system of Example 6 is exponential (type 3) for a nonempty axiom. Theorem 1 can also be used to find the growth type of a given D0L system: we work out an expression of the form $\sum_{i=1}^{k} p_i(t) c_i^t$ as described in Theorem 1, and read off what the growth type is from this expression.

There exist D1L systems with growth type $1\frac{1}{2}$, the system in Example 1 is such. Karhumäki [9] has given an example of a D2L system of growth type $2\frac{1}{2}$, and by a result in [19] this implies that there is also a D1L system of growth type $2\frac{1}{2}$.

Earlier on we have pointed out that we need interaction in order to make an unbounded growth rate "slow." More formally this can now be expressed by saying that in the D0L case if a growth function is not limited it must be at least polynomial; however, in the D1L case it may be subpolynomial. Nevertheless, even in the D2L case, a growth function which is not limited must grow with at least an approximately logarithmic rate. Vitányi [19] proved that if $G = \langle \Sigma, \delta, w \rangle$ is a D2L system such that $r = \#\Sigma > 1$ and $f_G$ is unbounded, then

$$\lim_{t\to\infty} \left[ \left( \sum_{i=0}^{t} f_G(t) \right) \Big/ \left( \sum_{i=0}^{t} \lfloor \log_r ((r-1)i + r) \rfloor \right) \right] \geqq 1.$$

As far as growth classification problems for L systems with interactions are concerned, the results are all negative. Vitányi [19] proved that if $x \in \{1, 2\}$ and $i \in \{0, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3\}$, then there is *no* algorithm which decides for an arbitrary DxL system whether or not its growth type is $i$ (see Theorem 5).

(v) STRUCTURAL PROBLEMS. The view we have taken until now is global in its approach, i.e., we have not yet considered what properties of the production rules cause the different types of growth. We now give an example of a typical structural result.

Let $G = \langle \Sigma, \delta, w \rangle$ be a D0L system, and let $a$ be a symbol in $\Sigma$. Then $a$ is said to be ***expanding*** if and only if there exists a $t \geqq 0$ and $\alpha_1, \alpha_2, \alpha_3$ in $\Sigma^*$, such that $\delta^t(a) = \alpha_1 a \alpha_2 a \alpha_3$, and $a$ is said to be

*accessible* if and only if there exists a $t \geqq 0$ and $\alpha_1$ and $\alpha_2$ in $\Sigma^*$, such that $\delta^t(w) = \alpha_1 a \alpha_2$. Salomaa [16] proved that a D0L system $G = \langle \Sigma, \delta, w \rangle$ is exponential (type 3) if and only if there exists a symbol $a$ in $\Sigma$ which is both accessible and expanding.

Vitányi [18] proved similar results on the inter-relationship between the nature of production rules and the type of growth of D0L systems for the cases of type 0, 1 and 2. He has proved for example that it is possible to give four D0L systems which differ from each other only in their axioms, and which nevertheless are of type 0, 1, 2 and 3, respectively. In fact, if a D0L system $G = \langle \Sigma, \delta, w \rangle$ is of growth type 2, then there is a substring $v$ of $\delta^{\#_\Sigma}(w)$ such that $\langle \Sigma, \delta, v \rangle$ is of growth type 1.

Thus our knowledge of structural problems for D0L systems is rather exhaustive. As opposed to this, little work has been done on structural problems for D1L and D2L systems. In view of the results on the classification problems for such systems, an exhaustive set of solutions to the structural problems is impossible.

(vi) HIERARCHY PROBLEMS. Theorem 4 is a typical hierarchy result. As we have mentioned, the problem whether or not there exists a D2L growth function $f$ which is not the growth function of a D1L system is still unknown.

A simple hierarchy result is the following [19]. If $G$ is a D2L system with a one letter alphabet, then $f_G$ is the growth function of a D0L system. However, there are D1L systems with two letter alphabets whose growth functions are not D0L growth functions.

Finally, let us point out that in Definition 1 we have allowed a symbol to map into the empty string. It is natural to consider the so-called *propagating* L systems, in which we do not allow production rules which have $\Lambda$ in their range. Although such restrictions can drastically limit the possibilities of a system, much, but by no means all of the theory described above, remains valid even under the propagating restriction.

For example, the impossibilities of algorithmic solutions to the analysis, growth equivalence and classification problems carry over to the propagating DxL systems ($x \in \{1, 2\}$), except for the growth equivalence problem for propagating D1L systems of subpolynomial growth type, which is still open. There are propagating D1L systems using an alphabet of two letters whose growth functions are not D0L growth functions. For each propagating DxL system $G, x \in \{0, 1, 2\}$,

$$\lim_{t \to \infty} (f_G(t) / \lfloor \log_r((r-1)t + r) \rfloor) \geqq 1,$$

where $r$ is the cardinality of the alphabet of $G$, a result which does not hold without the propagating restriction.

There exist propagating D2L systems whose growth function is of the order of magnitude $t^r$ for $0 < r < 1$. It is conjectured [19] that there is no such propagating D1L system. If this conjecture is true, it follows that the set of growth functions of propagating D1L systems is a proper subset of the set of growth functions of propagating D2L systems.

The study of growth functions as described in this paper is a very active field and many interesting results have been recently obtained. For a sample of such results the reader should consult the article of Arto Salomaa on *Growth functions of Lindenmayer systems: some new approaches*, which is to appear in *Automata, languages and development* (A. Lindenmayer and G. Rozenberg, eds.) North Holland, 1976.

**6. Historical notes.** The first paper in the field of growth functions was by Szilard [17] who treated the analysis and synthesis problems for D0L systems with the generating function approach. In Paz and Salomaa [14] growth functions of D0L systems are investigated from the point of view of integral sequential word functions and algorithms are obtained for the solution of the analysis, synthesis and growth equivalence problems. The difference equation method appears in Doucet [3], Paz and Salomaa [14] and Salomaa [16] which latter paper contains the closed form expression for

D0L growth functions and a classification of growth types of D0L systems together with a result on the structure of D0L growth. The present classification of growth types, as well as most of the results on the classification and structural problems, appears in Vitányi [**18**]. The first example of a D1L system with subpolynomial growth, and Theorems 3 and 4 are due to Herman (see [**8**] or [**14**]). Karhumäki [**9**] provided an example of a (propagating) D2L system of growth type $2\frac{1}{2}$. The remainder of the results on growth functions of L systems with interaction appearing in this article are due to Vitányi [**19**], which contains further results as well as some interesting conjectures. Further relevant references can be found in [**7**].

**7. Acknowledgements.** The inspiration, as well as some of the material, for this article was provided by an excellent survey lecture on this field by Professor Arto Salomaa, delivered at McMaster University in the spring of 1973. Earlier versions of this paper have been read and criticized by a number of people, and the authors would like to acknowledge the help of G. Albertelli, K. P. Lee, H. K. Liu, W. H. Liu, G. Rozenberg, J. van Leeuwen, A. Walker, and, especially, Professor Aristid Lindenmayer. This work is supported by NSF grant GJ 998, NATO grant 574, The Netherlands Organization for the Advancement of Pure Research (Z.W.O.) and the Mathematical Center under # IW 27/74.

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# MEASURING THE ASSOCIATION OF POINT PROCESSES: A CASE HISTORY

DAVID R. BRILLINGER

**1. Introduction.** Modern applied statistics typically involves elements of computation, probability theory, statistical theory and collaboration with specialists in the subject matter of some substantive field. In this article I shall describe part of a continuing experience of collaboration with two neurophysiologists from U.C.L.A., H. L. Bryant Jr. and J. P. Segundo. In formal terms, the problem considered is one of measuring the degree of association of points of two different sorts distributed along a straight line in an irregular manner. In real terms, the problem is one of investigating the behavior of a simple nerve cell network in a sea slug (*Aplysia californica*). The paper discusses a summary measure of association that has proved useful in assessing whether two nerve cells are behaving in a related manner or are behaving independently. The experiments by means of which the data were collected are described in Bryant, Ruiz Marcos and Segundo [4], as are the results of preliminary statistical analyses. The paper [4] is representative of the extent to which quantification is now occuring in the life sciences.




FIG. 1. A typical record of the changing voltage level of a nerve cell.

**2. Some neurophysiology.** The nerve cell (or neuron) is the basic unit of the animal involved in the transmission of information. Described schematically, it consists of a central cell body (or soma), branches (called dendrites) carrying impulses to the body and a long outgrowth (the axon) conducting impulses from the body. One way information is transmitted through the dendrites and axon is through changes in electrical activity. Figure 1 is an example of the changing voltage recorded when a microelectrode is inserted into a nerve cell. The record is seen to be made up of pulses of large amplitude compared to their duration. Because of its appearance, such a record is often called a spike train.

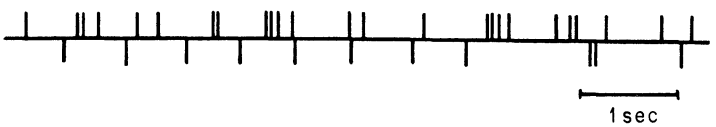



FIG. 2. A record of the times of spikes of two simultaneously firing nerve cells.

The junction whereby one neuron may influence another is called the synapse. When a pulse reaches the terminal point of an axon it provokes the release of a transmitter substance which alters the permeability of the dendrite of the next cell to certain ions. The resulting flow of ions generates a small electric current which moves down the dendrite to the soma. If the junction is excitory, the spike activity of the second cell is increased, if inhibitory it is decreased. Figure 2 is an example of the times of spikes for two nearby cells, the times for one cell corresponding to spikes above the line and for the other corresponding to spikes below. In practise, given two neurons, it may not be known whether either is influencing the other and it may be of interest to determine if there is some influence or association. Doing this by eye from records such as those of Figure 2 can be very difficult. Researchers have therefore been led to compute summary values from the records (see Griffith and Horn [9] for example) and this is the concern of the present paper.

The data discussed is recorded simultaneously on cells L3 and L10 of the sea hare. This

particular animal and these particular cells were used because the cells may be identified in different specimens and consequently experiments may be repeated. The experimental methods are described in detail in [4]. Further information concerning neurons and synapses may be found in Eccles [7].

**3. Some probability theory.** Commonly in his work on applied problems, a statistician brings the apparatus of probability theory into use. This involves his asking the experimentalists and himself whether or not it is reasonable to talk about random outcomes and probabilities of events connected with outcomes. The statistician seeks to bring probability theory into a problem because it provides a precise means of defining parameters and models and it allows him to interpret and assess various manipulations of experimental data. Not all problems of data analysis require the introduction of probability theory, but many seem to benefit from its appearance—among the latter are problems concerning nerve cell spike trains.

The branch of probability theory concerned with entities like irregular spike trains is that of stochastic point processes. A stochastic point process is a random, non-negative, integer-valued measure. If $I$ is an interval of the real line and $\omega$ is a random element, then the values of this measure may be denoted by $N(I, \omega)$, with $N(I, \omega)$ denoting the number of points in the interval $I$ for the realization corresponding to $\omega$. Here the atoms of the measure $N(I, \omega)$ correspond to the times of spikes of a particular spike train. Repeating the experiment would most likely yield a different set of spike times and consequently a different measure $N(I, \omega')$. In this sense $N$ is a random measure. (We remark that in many problems one can suppress the dependence of $N$ on $\omega$, however, it is an essential element of the approach.) Point processes were considered recently in the MONTHLY by Chung [5] and are discussed in Cox and Lewis [6] and in a volume [11] edited by Lewis, for example.

For nerve cell trains, it is appropriate to assume that the point process is without multiple points; that is, the spike times are isolated, separated by positive distances. Because the spikes proceed from no inherent origin, it also seems appropriate to assume that the point process is stationary in time in the sense that the probability distribution of the random vector

$$\{N(I_1, \omega), \cdots, N(I_k, \omega)\}$$

is the same as that of the shifted vector

$$\{N(I_1 + t, \omega), \cdots, N(I_k + t, \omega)\}$$

for all $t$ and $k = 1, 2, \cdots$, where $I + t$ denotes the interval $(a + t, b + t)$ if $I = (a, b)$.

Important parameters of a stationary point process $N$ include the **mean intensity,** $p_N$, and the **second-order product density,** $p_{NN}(u)$, given by

$$p_N = \lim_{h \downarrow 0} \text{Prob}\{\text{point in the interval } (t, t + h)\}/h \tag{1}$$

and

$$p_{NN}(u) = \lim_{h,h' \downarrow 0} \text{Prob}\{\text{point in } (t + u - h, t + u + h) \text{ and point in } (t - h', t + h')\}/(4hh') \tag{2}$$

$-\infty < u < \infty$, respectively when these limits exist.

In fact we shall be concerned with two different types of points, say $M$ points and $N$ points, with $M(I, \omega)$ referring to the number of $M$ points in the interval $I$ and $N(I, \omega)$ the number of $N$ points in the interval $I$. We denote the mean intensity of $M$ points by $p_M$ and the second-order product density of $M$ points by $p_{MM}(u)$. We also define a **cross-product density,** $p_{MN}(u)$, by

$$p_{MN}(u) = \lim_{h,h' \downarrow 0} \text{Prob}\{M \text{ point in } (t + u - h, t + u + h) \text{ and } N \text{ point in } (t - h', t + h')\}/(4hh'). \tag{3}$$

The parameters in (1), (2), (3) do not depend on $t$ because the process is stationary.

The first thing that one tends to notice when examining a spike train is whether there are a lot of spikes or only a few. The mean intensity of the process gives information in this connection. Expression (1) implies that the probability of there being an $N$ point in a small interval of length $h$ is approximately $p_N h$. The next thing that one tends to notice is the relative positioning of pairs of spikes of a single train or from one train to another. Expressions (2) and (3) give information in this connection. From expression (3), for example, we have

$$\text{Prob}\{M \text{ point in } (t+u-h, t+u+h) \text{ and } N \text{ point in } (t-h', t+h')\} \sim p_{MN}(u) 4hh' \tag{4}$$

for $h$, $h'$ non-negative and small. Using the definition of conditional probability and (1), this implies that

$$\text{Prob}\{M \text{ point in } (t+u-h, t+u+h) \text{ given an } N \text{ point at } t\} \sim 2h\, p_{MN}(u)/p_N. \tag{5}$$

In the case that the $M$ points are distributed independently of the $N$ points, the probability referred to in expression (5) is just $\text{Prob}\{M \text{ point in } (t+u-h,\ t+u+h)\}$ and so

$$p_{MN}(u)/p_N = p_M \quad \text{or} \quad p_{MN}(u) = p_M p_N \tag{6}$$

for all $u$. This last suggests that the function $p_{MN}(u)$, and related functions such as

$$\frac{p_{MN}(u)}{p_M p_N} \quad \text{or} \quad \sqrt{\frac{p_{MN}(u)}{p_M p_N}} \tag{7}$$

might prove useful measures of the degree of association of points of the $M$ process with points of the $N$ process. They are identically 1.00 in the case of independence.

We remark that, since we have assumed the points of the processes to be isolated, we can replace the probabilities of expressions (1)–(3) by expected values, for example we could write for (3)

$$p_{MN}(u) = \lim_{h,h' \downarrow 0} E\{N(t+u-h, t+u+h)\, N(t-h', t+h')\}/(4hh').$$

**4. Some statistical theory.** The preceding section described a mathematical idealization that could be of use in examining the degree of relationship of two given spike trains. The idealization suggested the definition of parameters $p_M$, $p_N$, $p_{MN}(u)$ based on the probabilities of certain events. In order to make concrete use of these parameters we need to have some idea of their values for the spike trains at hand.

Statistical theory has long been concerned with the problem of estimating the probability of an event given experimental results. In elementary situations one estimates the probability of an event $A$ by $n_A/n$, where $n_A$ denotes the number of times the event $A$ occurred out of $n$ times when it might have occurred. Let us use this approach to construct estimates of $p_M, p_N, p_{MN}(u)$.

Suppose that spike trains $M$ and $N$ are observed throughout the time interval $(0, T)$. Let $s_1 < s_2 < \cdots < s_{M(T)}$ be the observed times of $M$ spikes and $t_1 < t_2 < \cdots < t_{N(T)}$ be the observed times of $N$ spikes where we have observed $M(T)$ $M$ spikes and $N(T)$ $N$ spikes in all. Let $h$ be small and imagine the interval $(0, T)$ divided into $T/h$ intervals of length $h$. The number of times the event "$M$ spike in small interval of length $h$" occurred is $M(T)$. It might have occurred $T/h$ times. This suggests estimating $p_M h$ by $M(T)/(T/h)$ and so estimating $p_M$ by

$$\hat{p}_M = M(T)/T. \tag{8}$$

Likewise we could estimate $p_N$ by $\hat{p}_N = N(T)/T$.

Next we consider the estimation of $p_{MN}(u)$. For small $h$, let

(9) $n(u,h)$ = the number of $s_i$ such that $t_j + u - h < s_i < t_j + u + h$ for some $j$,

then the probability of expression (5) may be estimated by $n(u,h)/N(T)$. This suggests the estimation of $p_{MN}(u)$ by

$$\hat{p}_{MN}(u) = \frac{n(u,h)\hat{p}_N}{N(T)2h} = \frac{n(u,h)}{2hT}. \tag{10}$$

This estimate may be used in turn to construct estimates of the functions of expression (7) if so desired. We must not forget, however, that we have not measured $p_{MN}(u)$ exactly. Rather we have constructed an expression that should be near it, especially in the case that $T$ is large. We must also remember that were we to repeat the experiment, almost certainly, we would obtain a different value for $\hat{p}_{MN}(u)$. This last variation is called sampling fluctuation. Figure 3 is a graph of $\hat{p}_{MN}(u)/(\hat{p}_M\hat{p}_N)$ for the nerve cell data described in Section 2. Here $N(T) = 1232$, $M(T) = 816$, $\hat{p}_N = 1.77$ events/sec. and $\hat{p}_M = 1.17$ events/sec.. The Figure suggests that the probability of occurrence of an $M$ event is depressed for about 5 seconds after the occurrence of an $N$ event. However, before we can come to a belief that the outputs of the two nerve cells are in fact related, we must first have some confidence that the deviation of the function from the value 1.00 is not due simply to sampling fluctuations.

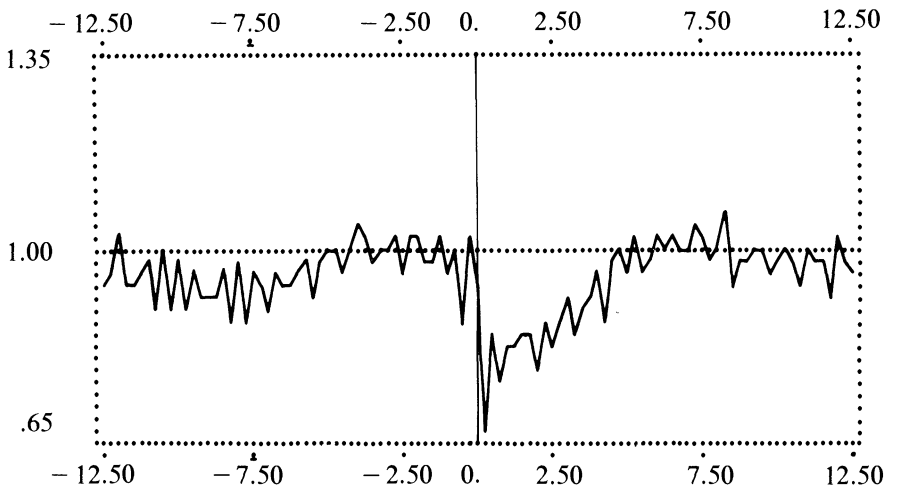



FIG. 3. An example of $\hat{p}_{MN}(u)/(\hat{p}_M\hat{p}_N)$ for cells L3 and L10 of *Aplysia californica*. The horizontal axis gives $u$ in seconds.

The estimate (10) was proposed in Griffith and Horn [9]. Its direct computation involves the comparison of $M(T) \cdot N(T)$ values. In many of the experiments referred to $M(T)$ and $N(T)$ are both about 1000, so fairly clearly a high speed computer must be used in its computation. Hugh Bryant, Jr. has noted that if the spike times are recorded by increasing time, with a simple indicator to say whether a spike was an $M$ or an $N$, then there exists a direct algorithm for computing (9) with one pass through the data. Suppose the data is denoted $(u_j, \alpha_j)$, $j = 1,2,\cdots$ where $u_1 < u_2 < u_3 \cdots$ and $u_j$ is an $s_k$ if $\alpha_j = 0$, $u_j$ is a $t_k$ if $\alpha_j = 1$. The algorithm is the following: (1) initialize $n(lh,h)$ to 0 for $l = 0, \pm 1, \pm 3, \cdots$, (2) for $j = 1,2,\cdots$ and $k = j+1, j+2, \cdots$ if $\alpha_j = 1$, $\alpha_k = 0$ compute $l = [(u_k - u_j)/(2h)]$ and set $n(lh,h) = n(lh,h) + 1$ or if $\alpha_j = 0$, $\alpha_k = 1$ compute $l = -[(u_k - u_j)/(2h) + 1/2]$ and set $n(lh,h) = n(lh,h) + 1$. (Here $[x]$ means the integral part of the number $x$.)

**5. Some more probability theory.** Figure 3 is a graph of an estimate of $p_{MN}(u)/(p_Mp_N)$ rather than the function itself. Part of the irregular nature of the figure is undoubtedly due to sampling fluctuations. Before we can come to a reasonable decision that the two spike trains are related, with

a spike of the $N$ train associated with an apparent depression in the rate of $M$ spikes for example, we must assess the magnitude of sampling fluctuations. The key random variate appearing in $\hat{p}_{MN}(u)/(\hat{p}_M\hat{p}_N)$ is $n(u,h)$ given by expression (9). Let us attempt to approximate the distribution of this variate for large $T$.

$n(u,h)$ is a counting variate. As $h$ is small it is counting rare events. Now in many situations, counts of rare events are approximately Poisson (see for example Feller [8] p. 282). Volkonski and Rozanov [12] demonstrate the related result that if $N^T(I,\omega)$, $T=1,2,\cdots$ is a sequence of point processes with mean intensities $p_N^T \to 0$ as $T\to\infty$, then under a further regularity condition, the sequence of processes with rescaled time, $N^T(I/p_N^T,\omega)$, $T=1,2,\cdots$ tends to a Poisson process. Perhaps $n(u,h)$ here is approximately Poisson with mean $2hTp_{MN}(u)$.

Supposing $h=L/T$, with $L$ constant, we define a sequence of processes $N^T(I,\omega)$ by saying that $N^T(\cdot,\omega)$ has a spike at $t$, if $N(\cdot,\omega)$ has a spike at $t$ and if $M(\cdot,\omega)$ has a spike in the interval $(t+u-L/T, t+u+L/T)$. The mean intensity of the process $N^T(\cdot,\omega)$ is $\sim p_{MN}(u)\,2L/T\to 0$. In Brillinger [2] it is shown that the result of Volkonski and Rozanov [12] may be applied to conclude that for large $T$ the process $N^T(IT,\omega)$ is approximately Poisson and in particular $n(u,T)\sim N^T((0,T),\omega)$ is approximately Poisson with mean $2hTp_{MN}(u)$.

**6. Some statistical inference.** A common procedure that a statistician employs to communicate an interval of plausible values for an unknown parameter, in the light of data collected, is a confidence interval. For example, a 95 per cent confidence interval has the formal interpretation that 95 is the long run percentage of such intervals that actually contained the true parameter value. At this point we could use Table 40 of Biometrika Tables, based on the Poisson distribution, to construct a confidence interval for $p_{MN}(u)$ and by division through by $\hat{p}_M\hat{p}_N$ a confidence interval for $p_{MN}(u)/(p_Mp_N)$.

A less troublesome way in which to proceed is to take advantage of the fact that if $P$ is a Poisson variate with mean $\mu$, then $\sqrt{P}$ is approximately a normal variate with mean $\sqrt{\mu}$ and standard deviation 1/2 (see pp. 88–96 in Kendall and Stuart [10]). The application of this square root transformation has two advantages; tables of the normal distribution are widely available for constructing confidence intervals, and the approximate standard deviation does not depend on the unknown parameter. All of this suggests a consideration of the estimate $\sqrt{\hat{p}_{MN}(u)/(\hat{p}_M\hat{p}_N)}$ and of the approximation of its distribution by a normal with mean $\sqrt{p_{MN}(u)/(p_Mp_N)}$ and standard deviation $1/(2\sqrt{2hTp_Mp_N})$. Approximate 95 per cent confidence intervals are constructed by adding and subtracting $1.96/(2\sqrt{2h\hat{p}_M\hat{p}_N})$ along the curve.

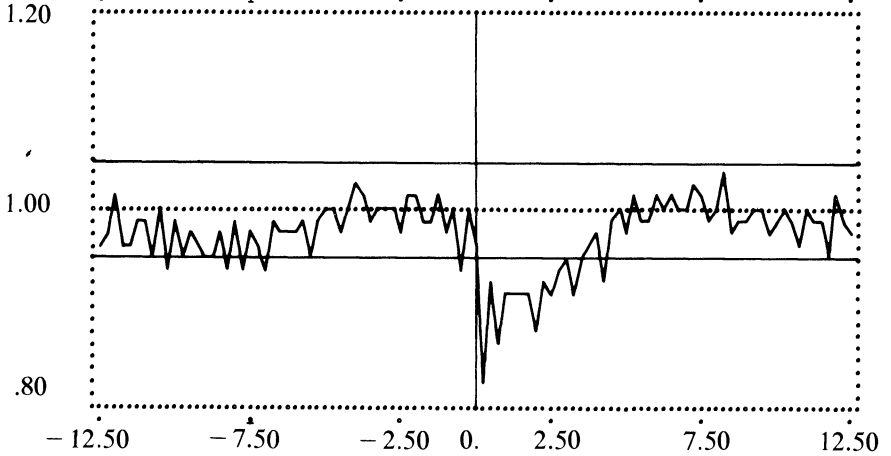



FIG. 4. $\sqrt{\hat{p}_{MN}(u)/(\hat{p}_M\hat{p}_N)}$ and 95 per cent confidence limits about the level 1.00.

In the case of independence of the two processes $p_{MN}(u)/(p_M p_N) = 1$ and this hypothesis may be easily checked into by plotting horizontal lines at the levels $1.00 \pm 1.96/(2\sqrt{2hT\hat{p}_M\hat{p}_N})$. This has been done in Figure 4, for the data of Figure 3. This new Figure is strongly suggestive of the association of a reduced rate of $M$ spikes for a period after the occurrence of an $N$ spike.

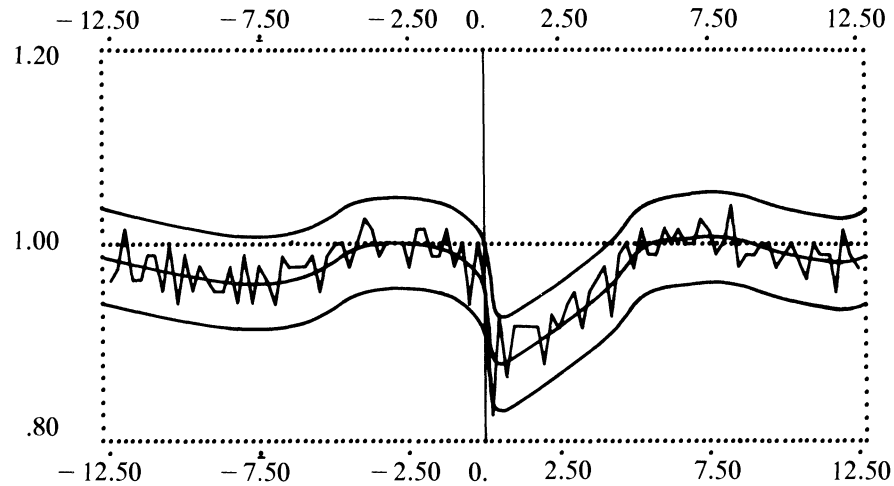



FIG. 5. $\sqrt{\hat{p}_{MN}(u)/(\hat{p}_M\hat{p}_N)}$ and 95 per cent confidence limits about a smoothed version of these values.

We might have chosen to indicate sampling fluctuations in the manner of Figure 5 where we have plotted $\pm 1.96/(2\sqrt{2hT\hat{p}_M\hat{p}_N})$ limits around a heavily smoothed version of the estimate. The points where the upper line is below 1.00 might be assessed significant. Figure 5 is also suggestive of a direction of causation for the two cells, namely the $N$ spikes seem to be associated only with later $M$ spikes (i.e., at positive $u$). This last is consistent with the neurophysiologists' understanding of the relation of the two particular cells for which this data was collected.

**7. Final remarks.** In this article I have sought to describe some of the stages involved in a modern applied statistics problem. These include: (i) the experimenter collects interesting data, (ii) the experimenter recognizes relevent scientific parameters to estimate, (iii) the experimenter consults a statistician as to whether or not his estimates are significant, (iv) the statistician suggests means of assessing sampling fluctuations and possibly suggests transformations in order that the data be more simply described and (v) the experimenter and statistician collaborate to determine and fit a statistical model and to design future experiments to confirm that model. The distinction between these stages is not always apparent nor is it clear whose ideas are whose. The experimenter must learn a fair amount of statistical methodology and the statistician must learn a fair amount of the experimenter's subject matter before real progress can be made.

Previous to my involvement in this work and the carrying out of the research leading to [3], another applied statistician, Peter Lewis, was involved. He suggested, [4], some clever and widely applicable alternate procedures for assessing the significance of the estimate (10). These were: (i) look at the variations of $n(u, h)$ for $|u|$ large enough that the interactions of the two cells are independent, (ii) substitute for the $N$ train, a spike train of $N(T)$ independent spikes and examine the variation of $n(u, h)$ here, (iii) split the observed trains into $J$ (say 20) pairs of shorter records and examine the variation of $n(u, h)$ when computed for the shorter records. The easiest of these procedures to carry out is (i). When done, it does lead to an estimate of the variation of the same order of magnitude as that of this paper.

Reference [3] is a joint paper carrying the analysis of the data described in this paper considerably further forward. I should like to thank Hugh Bryant, Jr. and Jose Segundo for the great pleasure I have derived from working with them on this problem.

Research supported in part by NSF Grant GP-31411.

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## ADDENDUM TO "STRONG DERIVATIVES AND INVERSE MAPPINGS"

(This MONTHLY, 81(1974) 969–981)

ALBERT NIJENHUIS

**8. Further comments.** An excellent discussion of strong derivatives and other properties of functions appears in Gleason's lectures [8], which also make reference to [7].

A treatment of the subject of the first four sections of the present paper, ostensibly finite dimensional but general in spirit, appears in [9]. Their use of the "homeomorphism theorems" 5.1.6–7 is very clarifying.

A correction: formula (4.7) should be

$$\varphi(y, z) = x_0 + \varphi^*(\lambda^{-1}(y - y_0), z). \tag{4.7}$$

**Additional References**

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## QUERIES

EDITED BY A. C. ZITRONENBAUM

*This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses of teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.*

**27. S. C. Kopf.** I would be interested in sharing experiences of other college mathematics professors who have team taught integrated courses in the physical and engineering sciences and calculus.

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## MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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### ON A PROBLEM OF HIRSCHHORN

PAUL ERDÖS AND MIKLOS SIMONOVITS

**Introduction.** Hirschhorn gave the following problem [1]: Let $q_1 > 1$ be given and

$$q_{n+1} = q_n + \prod_{i \leqq n} \left(1 - \frac{1}{q_i}\right)^{-1} \tag{1}$$

Is it true that $q_n = (1 + o(1))\, n \log n$?

The background of this problem is that, if $p_n$ denotes the $n$th prime, then the well-known sieve method gives that the number of integers between $a$ and $b$ which are not divisible by any of $p_1, \cdots, p_n$, is approximately

$$(b - a) \prod_{i \leqq n} \left(1 - \frac{1}{p_i}\right).$$

The interval $(p_n, p_{n+1}]$, contains exactly one prime, i.e., exactly one integer not divisible by any $p_i$ $(i \leqq n)$. This suggests that

$$p_{n+1} - p_n = \prod_{i \leqq n} \left(1 - \frac{1}{p_i}\right)^{-1}.$$

We know in this special case by the prime number theorem that $q_n = (1 + o(1))\, n \log n$. This shows why we are interested in this particular sequence. (Of course, the argument is only a heuristic one, the sieve method cannot be applied to short intervals and from our point of view $(p_n, p_{n+1}]$ is too short.)

We shall prove that if $q_n$ is defined by (1), then

$$(q_{n+1} - q_n)/\log n \to 1 \tag{2}$$

and, consequently,

$$q_n = (1 + o(1))n \log n, \tag{3}$$

so that the conjecture of Hirschhorn is proved.

REMARK. We can prove (2) and (3) in two different ways, and the way we shall prove them is the shorter, more formal one. But just because of this we first give a short heuristic argument why (1) implies (2).

Let

$$r_n = \prod_{i \leqq n} \left(1 - \frac{1}{q_i}\right)^{-1}, \qquad r_0 = 1. \tag{4}$$

Then

$$q_{n+1} = q_n + r_n \tag{5}$$

and

$$r_{n+1} = r_n\left(1 - \frac{1}{q_{n+1}}\right)^{-1} = r_n + \frac{r_n}{q_{n+1}} + O\left(\frac{r_n}{q_{n+1}^2}\right). \tag{6}$$

Now, $r_n$ and $q_n$ form a "self-regulating" system in the following sense: if $q_n$ were essentially larger than $n \log n$ for $n \in (a, b)$ and the interval $(a, b)$ were long enough, then, by (4) $r_n$ should become much smaller than $\log n$ and, by (5), after a while $q_n$ would become smaller than $n \log n$. Similarly, if $q_n$ were essentially smaller than $n \log n$ for a long period, then $r_n$ would become larger than $\log n$ and, consequently, $q_n$ also would become larger than $n \log n$. Of course, this argument does not exclude the possibility that $r_n$ and $q_n$ are oscillating about $\log n$ and $n \log n$ respectively, but, because of (6) the "inertia" of $r_n$ is too great, more exactly, $r_n$ changes only very slowly and does not "feel" minor changes in $q_n$. Thus the system $\{r_n, q_n\}$ is unable to oscillate.

*The exact proof.* We introduce two new sequences: $s_n = nr_n$ and $d_n = s_n - q_n$. Since

$$\frac{1}{q_n} - \frac{1}{q_{n+1}} = \frac{r_n}{q_n q_{n+1}}$$

and $q_n$ is monotone increasing, (6) can be replaced by the more convenient

$$r_{n+1} = r_n + \frac{r_n}{q_n} + O\left(\frac{r_n}{q_n^2}\right) = r_n + \frac{s_n - q_n}{nq_n} + \frac{1}{n} + O\left(\frac{r_n}{q_n^2}\right). \tag{7}$$

Further by (7)

$$\begin{aligned} d_{n+1} - d_n &= (s_{n+1} - s_n) - (q_{n+1} - q_n) = n(r_{n+1} - r_n) + r_{n+1} - r_n \\ &= (n+1)(r_{n+1} - r_n) = \left(1 + \frac{1}{n}\right)\frac{s_n}{q_n} + O\left(\frac{s_n}{q_n^2}\right). \end{aligned} \tag{8}$$

(A) First we need that $q_n/n$ tends to infinity. A trivial induction gives that $q_n > n$ and it is also trivial that $r_n$ is monotone increasing. From the left side of (7) we get

$$r_{n+1} - r_n = (1 + o(1))\frac{r_n}{q_n} = (1 + o(1))\frac{s_n}{nq_n}. \tag{9}$$

Further,

$$q_n = q_1 + \sum_{i=1}^{n-1} r_i < q_1 + (n-1)\, r_n < s_n + q_1.$$

Thus

$$r_{n+1} - r_n \geqq (1 + o(1))\frac{1}{n}.$$

This implies that $r_n \geqq \log n - o(\log n)$ and, consequently,

$$q_n \geqq n \log n - o(n \log n). \tag{10}$$

(B) By (10) $s_n/q_n < 2s_n/n \log n = o(r_n)$, i.e., $s_n/q_n = o(q_{n+1} - q_n)$, and it follows from (8) that

$$s_{n+1} - s_n = (1 + o(1))(q_{n+1} - q_n). \tag{11}$$

Since $s_n$ and $q_n$ tend to infinity, (11) yields

$$s_n/q_n \to 1. \tag{12}$$

Now, applying (12) to (9) we obtain

$$r_{n+1} - r_n = (1 + o(1))/n. \tag{13}$$

Thus

$$r_n = (1 + o(1)) \log n \quad \text{and} \quad q_n = (1 + o(1))\, n \log n \tag{14}$$

which proves (2) and (3).

(C) (14) can be improved in the following way: By (12) and (8)

$$d_{n+1} - d_n = 1 + o(1), \tag{15}$$

and hence

$$d_n = n + o(n). \tag{16}$$

But (16) and (7) give

$$r_{n+1} - r_n = \frac{1}{n} + (1 + o(1))\,\frac{1}{n \log n} + O\left(\frac{1}{n^2}\right).$$

Thus

$$r_n = \log n + (1 + o(1)) \log \log n \tag{17}$$

and

$$q_n = n \log n + (1 + o(1)) n \log \log n. \tag{18}$$

Here we used $q_n = \Sigma_{i=1}^{n-1} r_i + q_1$ and

$$\sum_{i=1}^{n} \log i = n \log n - n + O(\log n), \tag{19}$$

$$\sum_{i=2}^{n} \log \log i = n \log \log n + O(n/\log n). \tag{20}$$

(D) Iterating the method of (C) we can improve (17) and (18) in the following way: From (8)

$$d_{n+1} - d_n = (1 + 1/n)\left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) = 1 + O\left(\frac{\log \log n}{\log n}\right).$$

Therefore

$$d_n = n + O\left(\frac{n \log\log n}{\log n}\right).$$

Substituting in (7) we obtain that

$$r_{n+1} - r_n = \frac{1}{n} + \left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)\frac{1}{q_n} = \frac{1}{n} + \left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)^2 \frac{1}{n \log n}$$

$$= \frac{1}{n} + \frac{1}{n \log n} + O\left(\frac{\log\log n}{n \log^2 n}\right).$$

Thus $r_n = \log n + \log\log n + O(1)$ and hence $q_n = n \log n + n \log\log n + O(n)$.

By the same method but with more labor one could probably obtain further terms in the asymptotic expansion of $q_n$.

*Added in proof.* After writing this paper we learned that R. C. Vaughan, Imperial College, London, obtained stronger results.

**Reference**

1. M. D. Hirschhorn, How unexpected is the prime number theorem, this MONTHLY, 80 (1973) 675–677.

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## INEQUALITIES AND IDENTITIES FOR SUMS AND INTEGRALS

M. S. KLAMKIN AND D. J. NEWMAN

**I. Introduction.** We will show that the well-known elementary identity

$$\sum_{j=1}^{n} j^3 = \left\{\sum_{j=1}^{n} j\right\}^2$$

is a special case (equality) of the inequality

$$\sum_{j=1}^{n} A_j^3 \leqq \left\{\sum_{j=1}^{n} A_j\right\}^2, \tag{1}$$

where $\{A_j\}$ is a non-decreasing sequence with $A_0 = 0$ and $A_j - A_{j-1} \leqq 1$.

It is somewhat suprising that the proof of (1) turns out to be more troublesome than the analogous integral inequality

$$\int_a^b f(x)^3\,dx \leqq \left\{\int_a^b f(x)\,dx\right\}^2, \tag{2}$$

where $f$ is $C^1$, $f(a) = 0$ and $1 \geqq f'(x) \geqq 0$ in $[a, b]$. (The proof for (2) was set as a problem in the 1973 William Lowell Putnam Mathematics Competition by the second author.) Consequently, we shall first present a proof of a generalization of (2) and give a corresponding probabilistic interpretation. Then in the last section, we shall establish (1), a generalization and also give probabilistic interpretations.

**II. Integral inequalities and identities.** We first obtain a generalization of (2) under the same conditions on $f(x)$ with the exception that $M = \max f'(x)$ in $[a, b]$. We have successively that

$$f'(x)f(x)^n \leqq Mf(x)^n, \qquad n \geqq 0,$$

$$f(x)^{n+1} \leqq M(n+1)\int_a^x f(t)^n\,dt,$$

$$f(x)^{r(n+1)+n} \leqq M^r(n+1)^r f(x)^n\left\{\int_a^x f(t)^n\,dt\right\}^r, \qquad r > 0.$$

Integrating again with respect to $x$ and letting $m = r(n+1)+n$, we obtain

$$M^n\left\{(m+1)\int_a^b f(x)^m\,dx\right\}^{n+1} \leqq M^m\left\{(n+1)\int_a^b f(x)^n\,dx\right\}^{m+1}, \tag{3}$$

where $m > n$ and with equality **iff** $f(x) = Mx$. Inequality (2) corresponds to the special case here of $m = 3$, $n = 1$ and $M = 1$.

If we now let $[a, b] = [0, 1]$, $M = 1$ and $f(x) = \int_{1-x}^1 g(x)\,dx$, then

$$\int_0^1 f(x)^2\,dx = \int_0^1 dx\int_{1-x}^1 g(r)dr\int_{1-x}^1 g(s)ds = \int_0^1 g(r)dr\int_0^1 g(s)ds\int_{\substack{1\geqq x\geqq 1-r,\\ 1\geqq x\geqq 1-s.}} dx$$

$$= \int_0^1 g(r)dr\int_0^1 g(s)ds\int_{1-\min(r,s)}^1 dx = \int_0^1\int_0^1 \min(r, s)g(r)g(s)drds.$$

Similarly,

$$\int_0^1 f(x)^n\,dx = \int_0^1\cdots\int_0^1 \min(x_1, x_2, \cdots, x_n)g(x_1)g(x_2)\cdots g(x_n)dx_1\,dx_2\cdots dx_n, \tag{4}$$

$$\int_0^1 f(x)^n\,dx = \int_0^1\cdots\int_0^1 \{1 - \max(x_1, x_2, \cdots, x_n)\}h(x_1)h(x_2)\cdots h(x_n)dx_1\,dx_2\cdots dx_n, \tag{5}$$

where $f(x) = \int_0^x h(x)dx$. Here, $1 \geqq g(x)$, $h(x) \geqq 0$.

If $f(1) = 1$, then $g(x)$ and $h(x)$ are probability density functions over the interval [0, 1] and (4) and (5) are related to the known probability distribution functions for the min and max of a number $(n)$ of identically distributed random variables **[1]**. Corresponding to (3), we now have that

$$\{(m+1)\,E\,[\min(x_1, x_2, \cdots, x_m)]\}^{n+1} \leqq \{(n+1)\,E\,[\min(x_1, x_2, \cdots, x_n]\}^{m+1}, \tag{6}$$

where $E$ denotes "expectation" with respect to $g(x)$. There is equality only for the uniform distribution $g(x) = 1$.

With respect to the equality case, it is known more generally that

$$\int_0^1\cdots\int_0^1 M_r(x_1, x_2, \cdots, x_n)dx_1\,dx_2\cdots dx_n = \frac{r}{n+1},$$

where $M_r$ denotes the $r$th largest of $x_1, x_2, \cdots, x_n$ (starting with the smallest). Recently **[2]**, it was shown by solving difference equations that if

$$A(r, n, k) = \int_0^1\cdots\int_0^1 M_r^k(x_1, x_2, \cdots, x_n)dx_1\,dx_2\cdots dx_n,$$

and

$$A(1, n, k) = 1\Big/\binom{n+k}{n}, \qquad A(n, n, k) = n/(n+k).$$

One can also obtain these latter results by a slight extension of (4) and (5). However, the following

symmetry derivation for general $A(r,n,k)$ is simpler. By symmetry, $A(r,n,k) = n! \times$ the same integral over the region $1 \geqq x_n \geqq x_{n-1} \geqq \cdots \geqq x_2 \geqq x_1 \geqq 0$. Since $M_r$ is now $x_r$, we simply perform the integrations to obtain

$$A(r,n,k) = \frac{n!\Gamma(r+k)}{\Gamma(r)\Gamma(n+k+1)}. \tag{7}$$

**III. Summation inequalities and identities.** We now give a proof of (1) by double induction. Then we shall indicate an alternate proof by a transformation analogous to the one we used in the previous section.

Assuming that (1) is valid for $n = k$, we then have

$$A_{k+1}^3 + \sum_1^k A_j^3 \leqq A_{k+1}^3 + \left\{\sum_1^k A_j\right\}^2. \tag{8}$$

(1) will also be valid for $n = k+1$ if we can show that the right hand side of (8) is $\leqq \{\Sigma_1^{k+1} A_j\}^2$ or equivalently that

$$2\sum_1^k A_j \geqq A_{k+1}^2 - A_{k+1}. \tag{9}$$

To prove (9) by induction, it suffices to show that

$$A_{k+1}^2 + A_{k+1} \geqq A_{k+2}^2 - A_{k+2}$$

which follows since $1 \geqq A_{k+2} - A_{k+1}$. Since (1) and (9) are valid for $k = 1$, they are also valid for all non-negative integers $k$. There is equality in (1) and (9) **iff** $A_i = 0$ $(i \leqq n)$, $A_{i+n} = i$ $(i > 0)$ where $n$ is an arbitrary non-negative integer.

For a sketch of an alternate proof, we first transform (1) by letting

$$A_r = \sum_{i=n+1-r}^{n} c_i$$

so that $0 \leqq c_i \leqq 1$. Then

$$\begin{aligned}\sum_{r=1}^{n} A_r^3 &= \sum_{r=1}^{n} \sum_{i,j,k>n-r} c_i c_j c_k = \sum_{i,j,k=1}^{n} c_i c_j c_k \sum_{\substack{r>n-i\\ r>n-j\\ r>n-k}} 1 \\ &= \sum_{i,j,k=1}^{n} \min(i,j,k) c_i c_j c_k.\end{aligned} \tag{10}$$

Similarly, $\Sigma_{r=1}^n A_r = \Sigma_{i=1}^n i c_i$. Thus (1) takes the alternate form

$$\sum_{i,j,k}^{n} \min(i,j,k)\, c_i c_j c_k \leqq \left\{\sum_{i=1}^{n} i c_i\right\}^2 \tag{11}$$

for any numbers $c_1, c_2, \cdots, c_n$ all lying in [0, 1].

We can assume without loss of generality that the $c_i$'s are non-decreasing with respect to $i$. If we now take every $c_i c_j c_k$ term on the left hand side of (11), permute the indices to be non-decreasing and then replace every resulting term $c_x c_y c_z$ by $(c_x c_z + 2 c_y c_z)/3$, we shall find that we obtain exactly the right hand side. Since $c_x \leqq c_y \leqq c_z$, the replacement process does not decrease any term and the inequality follows immediately. The verification that the replacement process always does transform the left hand side of (11) into the right hand side is elementary but tedious.

If $A_n = \Sigma c_n = 1$, the $c_i$'s denote a discrete probability density function and then (11) can be rewritten in the form

$$E[\min(i,j,k)] \leqq \{E(i)\}^2 \tag{12}$$

with equality **iff** $c_1 = 1$.

The identity (10) can be generalized to

$$\sum_{r=1}^{n} \{(n-r+1)^k - (n-r)^k\} A_r^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} \min^k(i_1, i_2, \cdots, i_m)\, c_{i_1} c_{i_2} \cdots c_{i_m}. \tag{13}$$

For $A_r = r$, (13) reduces to

$$\sum_{r=1}^{n} \{(n-r+1)^k - (n-r)^k\} r^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} \min^k(i_1, i_2, \cdots, i_m). \tag{14}$$

The right hand side of (14) cannot be evaluated as easily as (7) by using a symmetry argument since one also has to take care of the troublesome cases of equality for various groups of indices.

A corresponding form to (13) in terms of "max" can be obtained by letting $B_r = \sum_{i=1}^{r} b_i$. Then, $1 \geqq b_i \geqq 0$ and

$$\sum_{r=1}^{n} \{r^k - (r+1)^k (1-\delta_{r,n})\} B_r^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} \max^k(i_1, i_2, \cdots, i_m) \cdot b_{i_1} b_{i_2} \cdots b_{i_m}, \tag{15}$$

where $\delta_{r,n} = 0$ or 1 according to whether or not $r \neq n$ or $r = n$. For $B_r = r$, (15) reduces to

$$(n+1)^k n^m + \sum_{r=1}^{n} \{r^k - (r+1)^k\} r^m = \sum_{i_1,i_2,\cdots,i_m=1}^{n} \max^k(i_1, i_2, \cdots, i_m). \tag{16}$$

Since the discrete version (1) of (3) for $m = 3$, $n = 1$ was shown to be valid, it is a natural question to ask if

$$\left\{(r+1)\sum_{i=1}^{n} A_i^r\right\}^{s+1} \leqq \left\{(s+1)\sum_{i=1}^{n} A_i^s\right\}^{r+1} \tag{17}$$

for all $r > s > 0$ and the $A_i$'s satisfy the same conditions as for (1). Although an inductive procedure appears to be difficult in general, we can establish (17) for the one parameter family $r = 2p-1$, $s = p-1$, i.e.,

$$2\sum_{i=1}^{n} A_i^{2p-1} \leqq p\left\{\sum_{i=1}^{n} A_i^{p-1}\right\}^2. \tag{18}$$

Proceeding by double induction as in the proof of (1), we end up having to prove that

$$p(A_n^{p-1} + A_{n+1}^{p-1}) \geqq 2(A_{n+1}^p - A_n^p).$$

The latter inequality follows immediately from the stronger result:

LEMMA. *If* $x, y \geqq 0$, $p \geqq 2$, *then* $p(x-y)(x^{p-1} + y^{p-1}) \geqq 2(x^p - y^p)$.

*Proof.* Assuming without loss of generality that $1 \geqq t \geqq 0$ where $t = y/x$, we consider the function

$$F(t) \equiv p(1-t)(1+t^{p-1}) - 2(1-t^p).$$

Then,

$$F'(t) = p\{(p-1)t^{p-2} - (p-2)t^{p-1} - 1\},$$

$$F''(t) = p(p-1)(p-2)t^{p-3}(1-t).$$

For $p = 2$, $F(t) \equiv 0$. For $p > 2$, $F'(0) = -p$, $F'(1) = 0$ and $F''(t) \geqq 0$. Thus $F(t)$ is monotonically decreasing in [0, 1] with a minimum value of zero.

The generalization of (12) via (18) is immediate.

REMARKS. Over a year ago, in a private communication, H. Flanders also gave different direct proofs of (1) and (2). Fairly recently and apparently piqued by the not so good hint given with the problem of proving (3) in the 1973 Putnam Competition, E. A. Melzak and J. L. Spouge proved (1) and (2) (with $a = 0$) by establishing the following identities:

$$\left\{\sum_{i=0}^{n} A_i\right\}^2 - \sum_{i=0}^{n} A_i^3 = \sum_{i=0}^{n}\sum_{j=0}^{i} A_i(A_j + A_{j-1})[1 - (A_j - A_{j-1})] \quad (\text{where } A_{-1} = A_0 = 0),$$

$$\left\{\int_0^b f(x)dx\right\}^2 - \int_0^b f^3(x)dx = 2\int_0^b \int_0^y f(y)f(v)[1 - f'(v)]\,dvdy.$$

Additionally, they gave the same inductive proof for (1) as given here.

**References**

1. E. J. Gumbel, Statistics of Extremes, Columbia University Press, N.Y., 1960, p. 43.
2. J. J. Dujmović, Two integrals related to means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Math. Fiz., No. 412—No. 460 (1973), 231–232.

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## COMMUTATIVITY IN FINITE RINGS

DESMOND MACHALE

In a recent paper in this MONTHLY Gustafson [2] considered the problem of finding the probability $\Pr(G)$ that a pair of elements in a finite group $G$ commute with each other. The object of the present paper is to consider the same problem for finite rings. We also mention and give references to a number of additional results in the case of finite groups.

We use the following notation throughout:

$R$ a finite ring with center $Z(R)$.

$G$ a finite group with center $Z(G)$ and commutator subgroup $G'$.

$\mathscr{R}_p$ the collection of finite rings with order divisible by the prime $p$ but by no smaller prime.

$\mathscr{G}_p$ the collection of finite groups with order divisible by the prime $p$ but by no smaller prime.

$C_R(r)$ the subring $\{x \in R \mid xr = rx\}$ of $R$.

$(R: H)$ the index of the ring $H$ in the ring $R$.

We define $\Pr(R)$ to be $1/|R|^2 \Sigma_{x\in R} |C_R(x)|$.

$\Pr(R)$ is clearly the probability that a pair of elements of $R$ will commute with each other. Gustafson [2] proves that $\Pr(G) = 1/|G|^2 \Sigma_{x\in G} |C_G(x)| = k(G)/|G|$, where $k(G)$ is the number of conjugacy classes in the group $G$. Since the concept of conjugacy in groups has no obvious analogue in rings it is to be expected that our methods of proof will be somewhat different from those of Gustafson. Surprisingly, however, the results are very similar.

We begin with an easy lemma.

LEMMA 1. *If $R$ is a non-commutative ring, then $R/Z(R)$ is not a cyclic (additive) group.*

*Proof.* Suppose that $R$ is non-commutative and $R/Z(R)$ is a cyclic group generated by $Z(R) + r$. Typical elements of $R$ are therefore $z_1 + nr$ and $z_2 + mr$ where $z_1, z_2 \in Z(R)$ and $m$ and $n$ are integers. But these elements clearly commute, contrary to hypothesis.

COROLLARY 1. *If $R$ is a non-commutative ring then $(R: Z(R))$ cannot be a prime number.*

The next result concerns the maximum value that $\Pr(R)$ can have.

THEOREM 1. *If $R$ is non-commutative then* $\Pr(R) \leqq 5/8$, *with equality if and only if* $(R: Z(R)) = 4$.

*Proof.* Since $R$ is non-commutative, Corollary 1 gives $(R: Z(R)) \geqq 4$. Hence at least three-quarters of the elements $x$ of $R$ satisfy $(R: C_R(x)) \geqq 2$. Thus,

$$\Pr(R) = 1/|R|^2 \sum_{x \in R} |C_R(x)| \leqq \tfrac{1}{4} + \tfrac{3}{4} \cdot \tfrac{1}{2} = 5/8.$$

If $\Pr(R) = 5/8$ then clearly $(R: Z(R)) = 4$. Assume therefore that $(R: Z(R)) = 4$, in which case $R/Z(R)$ is the direct sum of two cyclic groups $\langle Z + a \rangle$ and $\langle Z + b \rangle$. Now each non-central $x$ has centralizer of index 2 in $R$ and so $\Pr(R) = 5/8$ as required.

The following rings of matrices over GF(2) show that the bound $\Pr(R) \leqq 5/8$ for non-commutative rings is the best possible.

(i) $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ (ii) $\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| \forall\, a, b, c \in GF(2) \right\}$.

Note that (ii) is a ring with identity. The foregoing result is a special case of the following theorem whose proof is exactly analogous:

THEOREM 2. *Let $R$ be a non-commutative ring in $\mathcal{R}_p$. Then* $\Pr(R) \leqq (1/p^3)(p^2 + p - 1)$, *with equality if and only if* $(R: Z(R)) = p^2$.

The corresponding result for groups was first proved by Joseph [4]:

THEOREM 3. *Let $G$ be a non-commutative group in $\mathcal{G}_p$. Then* $\Pr(G) \leqq (1/p^3)(p^2 + p - 1)$, *with equality if and only if* $(G: Z(G)) = p^2$.

We next examine the relation between commutativity in a ring and commutativity in one of its subrings.

THEOREM 4. *If $H$ is a subring of $R$ then* $\Pr(R) \leqq \Pr(H)$.

*Proof.* Since for any $r$ in $R$, $C_H(r)$ is a subring of $C_R(r)$ it follows that

$$|C_R(r)| \leqq (R: H)\,|C_H(r)|.$$

Thus

$$\sum_{r \in R} |C_R(r)| \leqq (R: H) \sum_{r \in R} |C_H(r)| = (R: H) \sum_{h \in H} |C_R(h)| \leqq (R: H)^2 \sum_{h \in H} |C_H(h)|.$$

It follows that $\Pr(R) \leqq \Pr(H)$.

We can express this result intuitively by saying that any subring of a finite ring is at least as commutative as the ring itself.

The corresponding theorem for groups is proved by Joseph [4]. His proof is a direct application of the Frobenius Reciprocity theorem. A more elementary proof can be found in Gallagher [1].

Finally, we list a number of additional results concerning $\Pr(G)$ for finite groups.

(a) [2] If $G$ and $H$ are isoclinie (see Hall [3]) finite groups, then $\Pr(G) = \Pr(H)$.

(b) [2] If $G$ belongs to $\mathcal{G}_p$ then $1 > \Pr(G) > 1/p$ if and only if $|G'| = p$, in which case $\Pr(G) = 1/p + (1/p^{2s+1})$, for some positive integer $s$.

(c) [6] If $G$ belongs to $\mathcal{G}_p$ $(p \neq 2)$ then it is not possible to have $\Pr(G) = 1/p$. However, $S_3$ in $\mathcal{G}_2$ shows that $\Pr(G) = \frac{1}{2}$ is possible. Moreover, $\Pr(A_4) = \frac{1}{3}$.

(d) [5] There are no finite groups $G$ such that $\frac{7}{16} < \Pr(G) < \frac{1}{2}$. There is, however, a group of order 16 with $\Pr(G) = \frac{7}{16}$.

(e) [4] If $N$ is a normal subgroup of $G$ then $\Pr(G) \leqq \Pr(G/N)\Pr(N)$.

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## PERMUTATIONS MOD $m$ IN THE FORM $x^n$

CRAIG M. CORDES

Hewitt [2] showed that the congruence $x^{l+1} \equiv x \pmod m$ has a solution for all $x$ if and only if $m$ is square-free. Moreover, if $m = p_1 \cdots p_r$, then the $l$'s that work are precisely the multiples of $\text{l.c.m}(p_1 - 1 \cdots p_r - 1)$. Hewitt also raised the question of what is the smallest $l$ such that $\{x^{l+1}\}_{x=1}^{m}$ are distinct mod $m$. In this note we answer this question and, in fact, find all $n, m$ such that $x^n$ yields a permutation in $Z_m$.

For such a permutation to exist, it is clear from Hewitt's proof that $m$ must be square-free. Suppose for the moment that $m = p$ is prime. Then the set $\{x\}_{x=1}^{p-1}$ forms a multiplicative cyclic group mod $p$ – see [1, Theorem 111] for example. Hence, there is an $x \not\equiv 1 \pmod p$ with $x^n \equiv 1 \pmod p$ if and only if $\text{g.c.d.}(n, p-1) \neq 1$.

PROPOSITION 1. *Let $p$ be a prime and $n$ a positive integer. Then $x^n$ yields a permutation* mod $p$ *if and only if* $\text{g.c.d.}(n, p-1) = 1$.

*Proof.* $x^n$ fails to yield a permutation if and only if there are $x \not\equiv y \pmod p$ such that $x^n \equiv y^n \pmod p$. At least one of $x, y$, say $x$, must be invertible. So $(y/x)^n \equiv 1 \pmod p$. The proposition now follows from the above remarks.

We want to extend this to a product of distinct primes.

PROPOSITION 2. *Let $m = p_1 \cdots p_r$ where the $p_i$, $1 \leqq i \leqq r$, are distinct primes and let $n$ be a positive integer such that* $\text{g.c.d.}[n, (p_1 - 1) \cdots (p_r - 1)] = 1$. *Then $x^n$ yields a permutation* mod $m$.

*Proof.* The proof is by induction on $r$. From Proposition 1, the case $r = 1$ is true. Assume it holds for $r \leqq k-1$ and consider $m = p_1 \cdots p_r$. Let $\{a_1, \cdots, a_{\varphi(m)}\}$ be a complete reduced residue system mod $m$ (i.e., $\text{g.c.d.}(a_i, m) = 1$). If $a_i^n \equiv a_j^n \pmod m$, then $(a_i/a_j)^n \equiv x^n \equiv 1 \pmod m$ and, hence $x^n \equiv 1 \pmod{p_i}$, $1 \leqq i \leqq r$. So $\text{g.c.d.}(n, p_i - 1) = 1$ and Proposition 1 imply $x \equiv 1 \pmod{p_i}$. But then $x \equiv 1 \pmod m$. Thus $\{a_1^n, \cdots, a_{\varphi(m)}^n\}$ are distinct mod $m$.

Now suppose $a^n p_i^n \equiv b^n p_i^n \pmod m$ where $1 \leqq a, b \leqq m/p_i$. Then $a^n p_i^{n-1} \equiv b^n p_i^{n-1} \pmod{m/p_i}$ and so $a^n \equiv b^n \pmod{m/p_i}$ since $p_i$ is invertible mod $m/p_i$. By the induction hypothesis, $a = b$. Consequently, $\{a^n p_i^n\}_{a=1}^{m/p_i}$ are distinct mod $m$. Moreover, for any $i$, the sets $\{a^n p_i^n\}_{a=1}^{m/p_i}$ and $\{a_1^n, \cdots, a_{\varphi(m)}^n\}$ are disjoint since $a_j^n \not\equiv 0 \pmod{p_i}$, $1 \leqq j \leqq \varphi(m)$. If $a^n p_i^n \equiv b^n p_j^n \pmod m$ for $i \neq j$ with $1 \leqq a \leqq m/p_i$, $1 \leqq b \leqq m/p_j$, then clearly $p_i \mid b$, $p_j \mid a$. Let $a = p_j \bar{a}$, $b = p_i \bar{b}$. It now follows that

$$\bar{a}^n (p_i p_j)^{n-1} \equiv \bar{b}^n (p_i p_j)^{n-1} \left(\text{mod } \frac{m}{p_i p_j}\right),$$

and

$$p_i p_j \text{ invertible mod } \frac{m}{p_i p_j} \quad \text{implies} \quad \bar{a}^n \equiv \bar{b}^n \left(\text{mod } \frac{m}{p_i p_j}\right).$$

By the induction hypothesis and by $1 \leqq \bar{a}, \bar{b} \leqq m/p_ip_j$, $\bar{a} = \bar{b}$, and so $ap_i = bp_j$. The only intersection the sets $\{(ap_k^n)\}_{a=1}^{m/p_k}$, $k = i, j$, have then is at the common multiples of $p_ip_j$. Suppose now that $x^n \equiv y^n \pmod m$ with $x \not\equiv y \pmod m$. From the beginning of this paragraph neither $x$ nor $y$ can be relatively prime to $m$. But then $x = ap_i$, $y = bp_j$ for some $i, j$ with $1 \leqq i, j \leqq r$. Again by the above this implies $x \equiv y \pmod m$. Contradiction. Thus $\{x^n\}_{x=1}^m$ are distinct mod $m$. This completes the induction and the proof of the proposition.

Conversely suppose g.c.d. $[n, (p_1 - 1)\cdots(p_r - 1)] \neq 1$. Then there is at least one $i$ such that g.c.d. $(n, p_i - 1) \neq 1$ and by Proposition 1, $x^n$ does not yield a permutation mod $p_i$. If $x \not\equiv y \pmod{p_i}$ and $x^n \equiv y^n \pmod{p_i}$, then for $z = m/p_i$, $xz \not\equiv yz \pmod m$ and $(xz)^n \equiv (yz)^n \pmod m$. So $x^n$ is not a permutation mod $m$.

THEOREM. *Let $m, n$ be positive integers. Then $\{x^n\}_{x=1}^m$ are distinct* mod $m$ *if and only if $m$ is the product $p_1 \cdots p_r$ of distinct primes $p_i$, $1 \leqq i \leqq r$, and* g.c.d. $[n, (p_1 - 1)\cdots(p_r - 1)] = 1$.

The smallest $n$ that works then is the smallest prime $q$ for which g.c.d. $[q, (p_1 - 1)\cdots(p_r - 1)] = 1$. In [2], it was seen that if $x^{l+1} \equiv x \pmod m$, then $x^{kl+1} \equiv x \pmod m$ for all $k \geqq 1$. We would like to find all $l$ which have this property with respect to permutations. That is, if $x^{kl+1}$ yields a permutation mod $m$ for all $k \geqq 1$, what is $l$? By the theorem, g.c.d. $[kl + 1, (p_1 - 1)\cdots(p_r - 1)] = 1$ for all $k \geqq 1$. But this is true if and only if g.c.d. $(kl + 1, q) = 1$, $k \geqq 1$, for all primes $q$ dividing l.c.m. $(p_1 - 1, \cdots, p_r - 1)$. And g.c.d. $(kl + 1, q) = 1$ for all $k \geqq 1$ is equivalent to $q \mid l$. Combining the above, we obtain the following result.

THEOREM. *Let $m = p_1 \cdots p_r$ be a product of the distinct primes $p_i$, $1 \leqq i \leqq r$. Suppose $\{q_i\}_{i=1}^s$ are the distinct prime factors of* l.c.m. $(p_1 - 1, \cdots, p_r - 1)$. *Then $x^{kl+1}$ yields a permutation* mod $m$ *for all $k \geqq 1$ if and only if $l$ is a multiple of $q_1 \cdots q_s$.*

Hewitt's result in [2] can be obtained directly from this theorem and Fermat's theorem. Moreover, we should point out that the proof of the theorem showed $kl + 1$ works for all $k \geqq 1$ if and only if it works for $k \leqq q - 1$, where $q$ is the largest prime factor of l.c.m. $(p_1 - 1, \cdots, p_r - 1)$.

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## GAMBLING IN $N$ STAGES

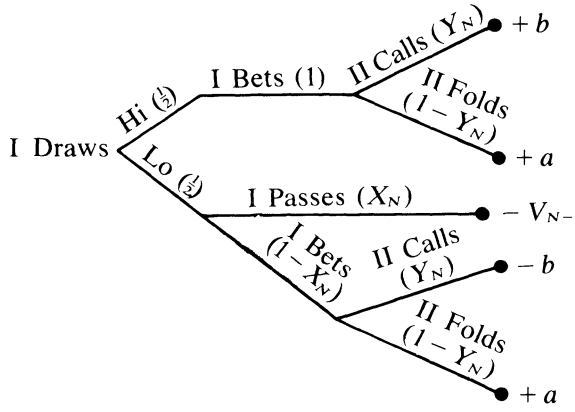
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**I. Statement of the problem and solution.** Two gamblers decide to play the following game: Gambler I draws a Hi or Lo card from a hat. He then has the option of either betting or passing. If Gambler I bets, Gambler II may fold, in which case he pays Gambler I an amount $a > 0$, or call, in which case Gambler I wins or loses $b > a$, depending upon whether he holds Hi or Lo. If Gambler I passes, Gambler II draws a Hi or Lo card from a new hat. Gambler II has the symmetric choice of actions to which Gambler I must respond with the corresponding payoff. If Gambler II elects to pass, Gambler I draws Hi or Lo from a new hat and play continues. A pass by the gambler drawing the card on the $N$th play (if indeed the $N$th play is reached) results in a 0 payoff. Finally, Hi and Lo are equally probable, and both the value of $N$ and the decision as to which gambler will draw first have been determined before the start of the game. We desire to find the value and expected duration of the game, and the optimal strategies of both gamblers.

Define $V_j$ as the value of the $j$-stage game to the gambler drawing the card on the initial play of

this game. $V_j$ is nonnegative for $j = 1, 2, \cdots, N$, and identically 0 when $j = 0$. Our analysis proceeds as follows:

If Gambler I passes on the first play, his expected gain is $-V_{N-1} \leqq 0$. Hence, if I draws Hi, he should always bet, since this ensures a positive gain of either $a$ or $b$. The only choice I has is whether to pass or bet when Lo is drawn; let $x_N$ represent I's probability of passing. Gambler II can respond to I's action by either a call or a fold; suppose he calls with probability $y_N$. The initial play can now be represented by the following tree diagram:




Temporarily dropping the subscripts, we find that I's expected gain $E$ satisfies

$$2E = yb + (1-y)a - xV - (1-x)yb + (1-x)(1-y)a; \tag{1}$$

collecting terms yields equation (2):

$$2E = (a+b)xy - (a+V)x - 2ay + 2a. \tag{2}$$

This expression may be factored to give

$$E = \frac{a+b}{2}(x-2r)\left(y - r - \frac{V}{a+b}\right) + r(b-V), \tag{3}$$

where $r = a/(a+b) < \frac{1}{2}$, since $b > a$.

It is clear from equation (3) that for the optimal strategies,

$$x_N = 2r; \qquad y_N = r + \frac{1}{a+b} V_{N-1} \tag{4}$$

and hence that

$$V_N = r(b - V_{N-1}). \tag{5}$$

Repeated application of this result yields

$$V_N = rb - r^2 b + r^3 b - + \cdots + (-1)^{N-1} r^N b + (-1)^N r^N V_0. \tag{6}$$

Summing the series in equation (6) and remembering that $V_0 = 0$ leads to

$$V_N = rb \cdot \frac{1-(-r)^N}{1-(-r)} = \frac{ab}{2a+b}[1-(-r)^N]. \tag{7}$$

**II. Characteristics of the solution.** Since $x_N = 2r$, independent of $N$, the probability of a pass on any particular play is $\frac{1}{2}x_N = r$. Since the plays are independent, the probability of $k-1$ successive passes followed by a bet is $r^{k-1}(1-r)$. Let $Q_k$ denote the probability that the game will end in $k$

plays. Then:

$$Q_k = r^{k-1}(1-r), \qquad k = 1, 2, \cdots, N-1. \tag{8}$$

If there are $N-1$ passes, the game must end on the $N$th play; hence

$$Q_N = r^{N-1}. \tag{9}$$

The expected duration of the $N$-stage game ($E_N$) is

$$E_N = \sum_{k=1}^{N} kQ_k = Nr^{N-1} + (1-r)\sum_{k=1}^{N-1} kr^{k-1}. \tag{10}$$

This expression can be rewritten as

$$E_N = Nr^{N-1} + (1-r)\frac{d}{dr}\sum_{k=1}^{N-1} r^k. \tag{11}$$

After summing the geometric series in equation (11) and differentiating, one obtains

$$E_N = \frac{1-r^N}{1-r} = \frac{a+b}{b}[1-r^N]. \tag{12}$$

Finally, consider the game where no limit is placed on the maximum number of plays. Since $r < \frac{1}{2}$, letting $N \to \infty$ yields

$$V_\infty = \frac{ab}{2a+b}; \quad y_\infty = r + \frac{1}{a+b} \qquad V_\infty = \frac{2a}{2a+b}; \quad E_\infty = \frac{a+b}{b}. \tag{13}$$

**Acknowledgement.** The author would like to extend his thanks to a referee for his valuable comments and suggestions which helped simplify and improve the exposition.

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## RESEARCH PROBLEMS

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*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

### ARE ALL COMPLETE BINARY TREES GRACEFUL?

I. CAHIT

Let $T$ be a tree on $n$ vertices. Let the integers in $N = \{1, 2, \cdots, n\}$ number the vertices of $T$ so that each vertex of $T$ is associated with a distinct integer in $N$. The weight of an edge of $T$ is defined to be the absolute value of the difference between the vertex numbers at its end points. If all edge weights are distinct and their maximum value is $n-1$, then the tree is said to be gracefully numbered.

It was conjectured that all trees are graceful. Although the word "graceful" is due to Golomb [1], the original conjecture is due to Ringel [2]. Partial results have been reported for trees and graphs in the literature [1], [3], [4], [5], [6].

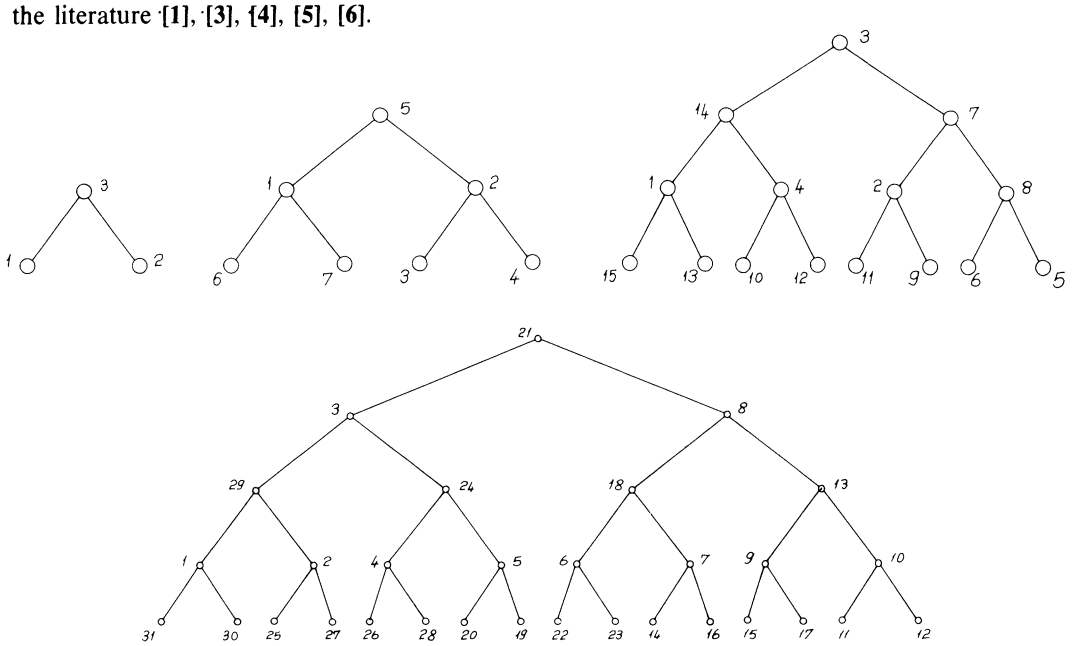


FIG. 1

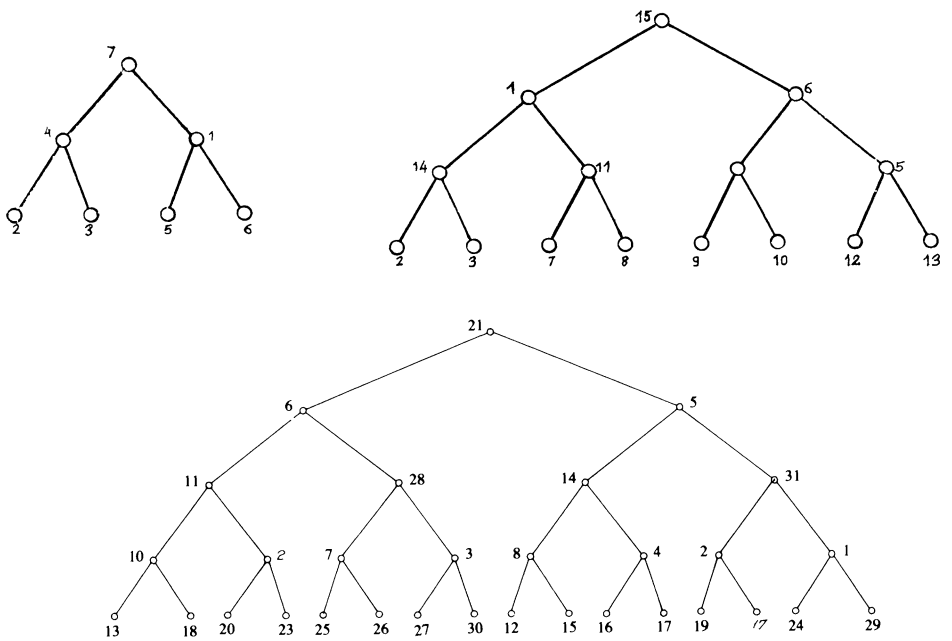


FIG. 2

We ask if all complete binary trees on $n = 2^k - 1$ vertices can be numbered gracefully; we have done this only for $n = 3$, 7, 15 and 31 (Figure 1). The numberings are not unique; other graceful complete binary trees are shown in Figure 2. There are trivial relabellings in which each number is

subtracted from $2^k$. For $n = 3$ and 7, these are the only gracefully numbered complete binary trees. Are there always graceful numberings with the largest number at the apex of the tree? It may also be worth studying graceful numberings of binary trees other than complete ones; there is always the hope of finding an algorithm for combining two binary trees into a larger one so that gracefulness is maintained.

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## CLASSROOM NOTES

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### A THEOREM ON LACUNARY POLYNOMIALS IN A FINITE FIELD

L. CARLITZ

**1.** In a letter to the author, Kai Yiu (of the University of Hawaii) has raised the following question suggested by some work on error correcting codes. Let $k_1, k_2, \cdots, k_s$ denote integers such that

$$q > k_1 > k_2 > \cdots > k_s > 0, \qquad s \geqq 1.$$

Consider the set of polynomials

$$A = \{a_1 x^{k_1} + \cdots + a_s x^{k_s} + 1\},$$

where the $a_j$ are numbers of the finite field GF($q$). Thus the cardinality of $A$ is $q^s$. It is conjectured that there exists at least one polyomial $a(x) \in A$ that has $s$ or more distinct (non-zero) roots in GF($q$). The conjecture has been verified for the case $q = 2^4$.

In the present note we give a proof of the conjecture.

**2.** Let $[k_1, k_2, \cdots, k_s]$ denote the determinant of order $s$:

$$[k_1, k_2, \cdots, k_s] = |c_i^{k_j}| \qquad (i, j = 1, 2, \cdots, s). \tag{2.1}$$

LEMMA. *Let $k_1, k_2, \cdots, k_s$ be integers that satisfy*

$$q > k_1 > k_2 > \cdots > k_s > 0. \tag{2.2}$$

*Then there exist (distinct) $c_1, c_2, \cdots, c_s \in \mathrm{GF}(q)$ such that*

$$[k_1, \cdots, k_s] \neq 0. \tag{2.3}$$

*Proof.* We shall prove the lemma by induction on $s$. For $s = 1$, we have

$$[k_1] = |c_1^{k_1}| = c_1^{k_1}$$

and the lemma is obviously true. Now assume that the lemma holds for $s = 1, 2, \cdots, t-1$.

By the Laplace expansion, we have

$$[k_1, k_2, \cdots, k_t] = c_1^{k_1}[k_2, k_3, \cdots, k_t] - c_1^{k_2}[k_1, k_3, \cdots, k_t] + \cdots + (-1)^{t-1} c_1^{k_t}[k_1, k_2, \cdots, k_{t-1}]. \tag{2.4}$$

By the inductive hypothesis, there exist $c_2, \cdots, c_t \in \mathrm{GF}(q)$ such that

$$[k_2, k_3, \cdots, k_t] = |c_i^{k_j}| \neq 0 \qquad (i, j = 2, 3, \cdots, t).$$

Hence (2.4) becomes

$$[k_1, k_2, \ldots, k_t] = b_1 c_1^{k_1} + b_2 c_1^{k_2} + \cdots + b_t c_1^{k_t} \qquad (b_j \in \mathrm{GF}(q)),$$

where $b_1 \neq 0$. Thus $[k_1, k_2, \cdots, k_t]$ is a polynomial in $c_1$ of degree $k_1 < q$. Hence there exists some $c_1 \in \mathrm{GF}(q)$ such that $[k_1, k_2, \cdots, k_t] \neq 0$. This evidently completes the induction.

REMARK. The $c_j$ of the lemma are obviously all non-zero as well as distinct.

**3.** THEOREM. *Let $k_1, k_2, \cdots, k_s$ be integers that satisfy*

$$q > k_1 > k_2 > \cdots > k_s > 0, \qquad s \geqq 1. \tag{3.1}$$

*Then there exist $a_1, a_2, \cdots, a_s \in \mathrm{GF}(q)$ such that the polynomial*

$$a(x) = a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_s x^{k_s} + 1 \tag{3.2}$$

*has at least $s$ distinct non-zero roots in $\mathrm{GF}(q)$.*

*Proof.* By the lemma, there exist $c_1, c_2, \cdots, c_s \in \mathrm{GF}(q)$ such that

$$[k_1, k_2, \cdots, k_s] = |c_i^{k_j}| \neq 0 \qquad (i, j = 1, 2, \cdots, s).$$

We shall show that we can find $a_1, a_2, \cdots, a_s \in \mathrm{GF}(q)$ such that $a(x)$ has the zeros $c_1, c_2, \cdots, c_s$ (and possibly other zeros as well).

Consider the system of equations in $a_j$:

$$\sum_{j=1}^{s} a_j c_i^{k_j} = -1 \qquad (i = 1, 2, \cdots, s). \tag{3.3}$$

The determinant of the system is evidently

$$[k_1, k_2, \cdots, k_s] \neq 0.$$

Hence Cramer's rule applies and the $a_j$ are uniquely determined. Clearly the $a_j$ are distinct non-zero numbers of $\mathrm{GF}(q)$.

It is evident that the $c_i$ are zeros of $a(x)$. This completes the proof of the theorem.

Supported in part by NSF grant GP-37924X.

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## MORE ON LOWER SEMI-CONTINUITY

OSKAR FEICHTINGER

Whyburn [1] and Smithson [2] presented characterizations of upper and lower semi-continuity of multifunctions in terms of directed families (i.e., filter bases). It is sometimes desirable to use a characterization of semi-continuity in terms of accumulation and adherent points.

Let $f$ be a multifunction on a topological space $X$ onto a space $Y$ and for $B \subseteq Y$, define $f^-(B) = \{x \in X \mid f(x) \text{ meets } B\}$. Then $f$ is lower semi-continuous provided $f^-(B)$ is open in $X$, whenever $B$ is open in $Y$.

THEOREM. *A multifunction $f\colon X \to Y$ is lower semi-continuous (lsc) if and only if whenever $p$ is an accumulation point of $A \subseteq X$, then every point of $f(p)$ is an adherent point of $f[A] = \bigcup_{a \in A} f(a)$.*

*Proof.* Assume $f$ to be lsc. Let $p$ be an accumulation point of $A \subseteq X$. Without loss of generality assume $p \notin A$. Suppose $y \in f(p) - f[A]$ and assume there is an open neighborhood 0 of $y$ such that $0 \cap f[A] = \emptyset$. Because $f$ is lsc, $f^-(0)$ is open. Clearly $p \in f^-(0)$, hence $f^-(0) \cap A \neq \emptyset$. Let $x \in f^-(0) \cap A$; then $f(x)$ meets 0 and $f(x) \subseteq f[A]$, which is a contradiction. Conversely let $V$ be an open subset of $Y$ and suppose $f^-(V)$ is not open in $X$. Hence there exists $p \in f^-(V)$ such that $p$ is an accumulation point of $X - f^-(V)$. According to the hypothesis, every point of $f(p)$ is an adherent point of $B = \cup(f(a) \mid a \in X - f^-(V))$. But there is a point $y \in f(p) \cap V$, and $y$ cannot be an adherent point of $B$, a contradiction. Thus $f^-(V)$ must be open, i.e., $f$ is lsc.

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## A NOTE ON PERIODIC COMPLETELY MULTIPLICATIVE ARITHMETICAL FUNCTIONS

TOM M. APOSTOL

A complex-valued function $f$ defined on the positive integers and not identically zero is called *completely multiplicative* if

$$f(mn) = f(m)f(n) \qquad \text{for all } m \text{ and } n. \tag{1}$$

Such a function satisfies $f(1) = 1$. An arithmetical function $f$ is called *periodic* mod $k$ if there is a positive integer $k$ such that

$$f(m) = f(n), \qquad \text{whenever} \qquad m \equiv n \pmod{k}. \tag{2}$$

Properties (1) and (2) imply that $f(n)$ is a root of unity whenever $n$ is relatively prime to $k$. In fact, if $(n, k) = 1$ then $n^{\varphi(k)} \equiv 1 \pmod{k}$ so $f(n)^{\varphi(k)} = 1$. Here $\varphi(k)$ is Euler's totient.

A completely multiplicative function which is periodic mod $k$ is called a *Dirichlet character* mod $k$ if it has the additional property that

$$f(n) = 0, \qquad \text{whenever} \qquad (n, k) > 1. \tag{3}$$

There are exactly $\varphi(k)$ distinct Dirichlet characters mod $k$. (See [1] or [2].) One of them is the *principal character* defined by (3) and the relation $f(n) = 1$ if $(n, k) = 1$.

The purpose of this note is to show that property (3) follows easily from (1) and (2) if $k$ is the smallest positive period of $f$. This simple fact does not seem to be well known, or at least is not mentioned in the standard texts on number theory which treat Dirichlet characters. We also show that a Dirichlet character mod $k$ has no positive period smaller than $k$ if $k$ is squarefree.

THEOREM 1. *Let $f$ be periodic and completely multiplicative, and let $k$ be the smallest positive period of $f$. Then $f(n) = 0$ whenever $(n, k) > 1$, so $f$ is a Dirichlet character* mod $k$.

*Proof.* Choose $n$ so that $(n, k) > 1$ and let $d = (n, k)$. Then $n = dn_1$ and $k = dk_1$ where $1 \leqq k_1 < k$. Since $k_1$ is not a period of $f$ there exists an $m$ such that

$$f(m + k_1) - f(m) \neq 0.$$

But

$$f(d)\{f(m + k_1) - f(m)\} = f(dm + k) - f(dm) = f(dm) - f(dm) = 0,$$

so $f(d) = 0$. Therefore $f(n) = f(d)f(n_1) = 0$.

The converse of Theorem 1 is not always true. That is, there are Dirichlet characters mod $k$ for which $k$ is not the smallest positive period. For example, if $p$ is prime, the principal character mod $p^2$ also has period $p$. The next theorem provides a partial converse.

THEOREM 2. *Let $f$ be a Dirichlet character* mod $k$. *If $k$ is squarefree then $k$ is the smallest positive period of $f$.*

*Proof.* Let $k'$ denote the smallest positive period of $f$. Then $k = tk'$ for some integer $t \geqq 1$. We assume that $t > 1$ and arrive at a contradiction.

First we note that $f$ is also a character mod $k'$, because $f$ is periodic mod $k'$, and if $(n, k') > 1$ then $(n, k) > 1$ and (3) implies $f(n) = 0$.

Now since $k$ is squarefree, $t$ has a prime factor $p$ which does not divide $k'$. Hence $(p, k') = 1$ so $f(p)$ is a root of unity since $f$ is a character mod $k'$. On the other hand $p \mid k$ so by (3) we have $f(p) = 0$. This contradiction shows that $t = 1$ and $k = k'$.

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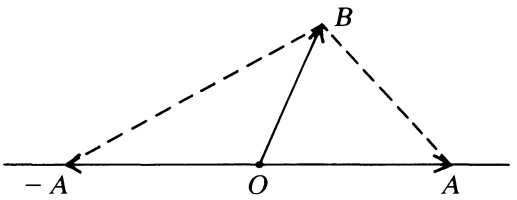
## PYTHAGORAS AND THE CAUCHY-SCHWARZ INEQUALITY

LADNOR GEISSINGER

Most current textbook introductions to coordinate vector algebra appeal to the students' knowledge of Euclidean geometry and geometric intuition to justify the introduction of Cartesian coordinates and the usual distance formula. Similarly, vector addition, scalar multiplication, and length of a vector are accompanied by discussions of their geometric significance. Frequently however, the usual inner (dot) product then appears abruptly, the Cauchy-Schwarz and triangle inequalities are proved as technical exercises, and the cosine of the angle between vectors is defined with no reference to the ratio of sides of a right triangle. The following remarks indicate one way that elementary geometric considerations lead naturally to the introduction of the inner product, to trivial proofs of the inequalities, and to the geometric law of cosines.

We assume that the distance between points (vectors) $A$ and $B$ in $R^n$ and the length of $B - A$ may be computed as $|B - A| = (\Sigma(b_i - a_i)^2)^{1/2}$, and that this has been justified by appeal to the

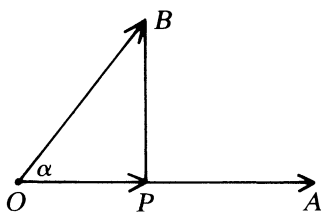
Pythagorean Theorem. Considering the triangle with vertices $A$, $B$, $-A$ it is clear the vector $B$ should be *orthogonal* to vector $A$ if and only if $|B-A|=|B-(-A)|$, so we take this as our definition of orthogonality.

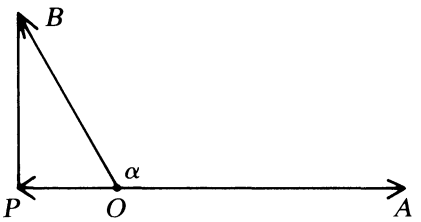



Since

$$|B-A|^2=|A|^2+|B|^2-2\sum a_ib_i \quad \text{and} \quad |B+A|^2=|A|^2+|B|^2+2\sum a_ib_i,$$

$B$ is orthogonal to $A$ if and only if $\Sigma a_ib_i=0$ and also if and only if the Pythagorean identity $|B-A|^2=|A|^2+|B|^2$ holds. Moreover, when $A$ and $B$ are not orthogonal, the quantity $\Sigma a_ib_i$ measures the failure of the Pythagorean identity. Now define the *dot (scalar, inner) product* by $A\cdot B=\Sigma a_ib_i$ and note that, like ordinary number multiplication, it is bilinear and positive.




The point $P$ is the *orthogonal projection of $B$ onto $A$* (onto the line thru the origin $O$ and $A$) if and only if $P=cA$ for some number $c$ and $B-P$ is orthogonal to $A$. From $A\cdot(B-cA)=0$ it follows that $c=A\cdot B/A\cdot A$ and, as we saw above, since $B-P$ is orthogonal to $P$ it follows that $|B|^2=|P|^2+|B-P|^2$. Hence

$$1\geqq\frac{|P|}{|B|}=\frac{|A\cdot B|}{|A||B|} \qquad \textit{(Cauchy-Schwarz inequality)}$$

and equality holds only if $B=P$, that is, $B=cA$. Moreover, in the right triangle $O$, $P$, $B$ the ratio $|P|/|B|$ is the usual cosine of the angle at $O$. Taking into account the sign of $A\cdot B$, it is geometrically clear that the angle $\alpha$ between $A$ and $B$ satisfies

$$\cos\alpha=\frac{A\cdot B}{|A||B|}.$$

From the computation of $|B-A|^2$ above we get $|B-A|^2=|A|^2+|B|^2-2|A||B|\cos\alpha$ (*Law of Cosines*). This equation also makes obvious the *triangle inequality*

$$|A|^2+|B|^2-2|A||B|\leqq|B-A|^2\leqq|A|^2+|B|^2+2|A||B| \quad \text{or}$$
$$\Big||A|-|B|\Big|\leqq|B-A|\leqq|A|+|B|.$$

Finally note that the arguments above work equally well in any space with an inner product.

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### PROBABILITIES IN PROOFREADING

GEORGE PÓLYA

Two proofreaders, $\mathscr{A}$ and $\mathscr{B}$, read, independently of each other, the proofsheets of the same book. As they finished, $A$ misprints were noticed by $\mathscr{A}$, $B$ misprints by $\mathscr{B}$, $C$ misprints by both, and so, as the result of their joint effort, $A + B - C$ misprints were noticed and corrected. We wish to estimate the number of those misprints that remained unnoticed and uncorrected.

Let $M$ denote the number of all misprints, noticed or unnoticed, in the proofsheets examined, $p$ the probability that proofreader $\mathscr{A}$ notices any given misprint, and $q$ the analogous probability for $\mathscr{B}$. It is an essential assumption that these two probabilities are independent. Hence the expected number of misprints that may be noticed

| | by $\mathscr{A}$, | by $\mathscr{B}$, | by both |
|---|---|---|---|
| is: | $Mp$, | $Mq$, | $Mpq$, |

respectively.

In order to arrive at the desired estimate we assume that the expected numbers are approximately equal to the numbers actually found, in symbols

$$Mp \sim A, \qquad Mq \sim B, \qquad Mpq \sim C$$

and so

$$M = \frac{Mp \cdot Mq}{Mpq} \sim \frac{AB}{C}.$$

Hence the number of misprints that remained unnoticed is

$$= M - (A + B - C) \sim \frac{AB}{C} - (A + B - C) = \frac{(A - C)(B - C)}{C}.$$

This is the desired estimate.

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## MATHEMATICAL EDUCATION

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### A COURSE ON THE DEVELOPMENT OF ANALYSIS

E. J. BARBEAU

In 1969, the Mathematics Department of the University of Toronto introduced a sequence of four undergraduate courses with an historical approach. These are described in an article by K. O. May [14]. I have taught the second year course of the group, *Development of Analysis*, to three classes of full-time day students and one class of evening students.

This course studied the growth of analytic concepts from the time of the Greeks until 1900, with some emphasis on the work of Newton and Leibniz. Besides secondary sources [1–11], the students read English paraphrases or translations of original papers, which were drawn from [12, 13] or were prepared by the instructor. In order to give them some intimacy with the materials, they were also required to work problems in the style of certain periods and to prepare an essay.

An expanded version of this note and material used in the course are available from the author at the Mathematics Department, University of Toronto, Toronto, Ontario M5S 1A1.

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## AN INVESTIGATION AND EVALUATION OF GOALS OF MATHEMATICS EDUCATION FOR PROSPECTIVE ELEMENTARY TEACHERS

IRIS MACK DAYOUB

The problem of teacher education is a current issue and one area of concern is the mathematics education program for prospective elementary teachers. Several groups of educators have recommended changes in the mathematics curriculum for prospective elementary teachers. One of the most influential groups, the Panel on Teacher Training of the Mathematical Association of America's Committee on the Undergraduate Program in Mathematics (CUPM), presented new recommendations in 1971 [3]. However, crucial questions still remain which pertain to identifying (1) the goals of mathematics education for elementary teachers, (2) the mathematics an elementary teacher should know, and (3) the curricular offerings designed for them. Even with a detailed course outline, such as those presented in the 1971 CUPM report, persons constructing a curriculum may find it difficult to determine a level of treatment appropriate for the prospective elementary teacher. This paper reports a study which attempted to obtain information pertaining to these questions, to identify important units of content, and to add meaningfulness to some of CUPM's suggested topics.

A selected group of persons responded to test items in the areas of relations and functions, mathematical systems, and geometry. To insure a comprehensive representation in the item pool from the wide range of possible topics and possible treatments of the topics which could be presented in a mathematics program for prospective elementary teachers, several sources [1, 2, 4–15] were reviewed and studied. The main source of topics was the course outlines of the CUPM report [3].

Fifty persons who were considered experts in mathematics and mathematics education were asked to participate in the study. Among these were persons who have written articles for professional journals concerning mathematics education for elementary teachers, authors of textbooks written for use in mathematics and mathematics methods courses for elementary teachers, members of the CUPM Panel on Teacher Training, and members of CUPM.

Responses to the items were received from twenty-two persons, resulting in a 42 percent rate of return. Each person was asked to rate the importance of each item as a unit of content in a mathematics education program for prospective elementary teachers. In addition each participant was asked to determine how accurately the items reflect the CUPM recommendations. The participants' perceptions of what is important in a mathematics curriculum for prospective elementary teachers were obtained through their responses to each item as a unit of content, using the following rating scale:

1—this content is vital in a teacher's preparation
2—this content is important, but not vital to a teacher's preparation
3—this content may or may not be important in a mathematics education program for elementary teachers
4—this content is not important, but could be included in a mathematics education program for prospective elementary teachers
5—this content should not be included in a mathematics education program for elementary teachers.

To report their interpretations of the CUPM recommendations for prospective elementary teachers, the participants responded to each item using the following rating scale:

1—this item definitely reflects the intent of the CUPM recommendations
2—this item appears to reflect the intent of the CUPM recommendations
3—this item may or may not reflect the intent of the CUPM recommendations
4—this item does not appear to reflect the intent of the CUPM recommendations
5—this item definitely does not reflect the intent of the CUPM recommendations.

To analyze the responses of the participants to the items, three levels of importance, $I_1$, $I_2$, and $I_3$, were set. An item was judged at the $I_1$-, $I_2$-, or $I_3$-level of importance if the percentage of responses in the vital or important categories was between 100 and 76, between 75 and 66, or between 65 and 56, respectively. Similarly, three levels of CUPM-reflection, $R_1$, $R_2$, and $R_3$, were determined by the percentage of responses to the items which were in the definitely reflects or appears to reflect categories. Table 1 summarizes the number of items and the percentage of items in each strand which were judged at each level of importance and at each level of CUPM-reflection.

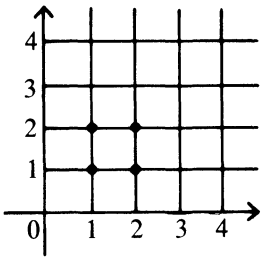
TABLE 1. Number and percentage of items in each strand judged at each level of importance and at each level of CUPM-reflection.

| Strand | Total Number of Items | Number of Items at Level | | | Percentage of Items at Level | | | Number of Items at Level | | | Percentage of Items at Level | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $I_1$ | $I_2$ | $I_3$ | $I_1$ | $I_2$ | $I_3$ | $R_1$ | $R_2$ | $R_3$ | $R_1$ | $R_2$ | $R_3$ |
| Relations and Functions | 52 | 15 | 9 | 12 | 29 | 17 | 23 | 24 | 10 | 8 | 46 | 19 | 15 |
| Mathematical Systems | 26 | 11 | 2 | 7 | 42 | 8 | 7 | 19 | 3 | 2 | 73 | 12 | 8 |
| Geometry | 47 | 19 | 10 | 5 | 40 | 21 | 11 | 38 | 6 | 3 | 81 | 13 | 6 |

Following are items from each of the strands which were rated at both the $I_1$-level of importance and the $R_1$-level of CUPM-reflection:

## RELATIONS AND FUNCTIONS

The graph to the right expresses the same relation as




a. $y = x$
b. $x + y = 2$
c. $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$
d. $\{(1, 2), (2, 1)\}$
e. none of the above.

For any integers $m$ and $n$, if there is a positive integer $p$ such that $m + p = n$, then

a. $m = n$
b. $m < n$
c. $m > n$
d. it is not possible to compare $m$ and $n$ with the given information.

If $R$ is the relation defined by $R = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$, then the inverse of the relation $R$ is

a. $\{(3, 4), (2, 3), (1, 3), (1, 2)\}$
b. $\{(2, 1), (3, 1), (3, 2), (4, 3)\}$
c. $\{(1, 2, 3), (2, 3, 4)\}$
d. does not exist.

Which of the following is implied by the transitivity of the relation "... is a factor of ..."?

a. 3 is a factor of 9, so 9 is a multiple of 3
b. 3 is a factor of 9 and 3 is a factor of 27, so 9 is a factor of 27
c. 3 is a factor of 9 and 9 is a factor of 27, so 3 is a factor of 27
d. all of the above.

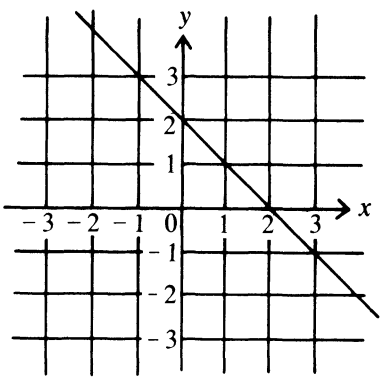
For the set $W = \{0, 1, 2, 3, \cdots\}$ of whole numbers, we say that $m$ divides $n$ if and only if there is a whole number $q$ such that $m = q \cdot n$. The relation "... divides ..." is *not* an equivalence relation because

a. 1 divides every whole number
b. every whole number divides 0
c. it is not reflexive
d. it is not symmetric
e. it is not transitive.

Which of the following is an order relation?

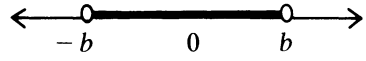
a. "... is less than ..." defined on the set of integers
b. "... is a subset of ..." defined on sets
c. both a and b
d. neither a nor b.

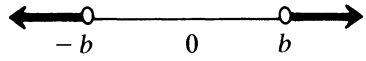
The graph to the right shows that $y$ is greater than 0 for all values of $x$ which are

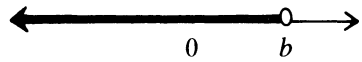



a. greater than 0
b. less than 3
c. greater than 3
d. less than 2.

Which of the following number-line graphs is of the expression $|x| < b$ where $b > 0$?

a. 

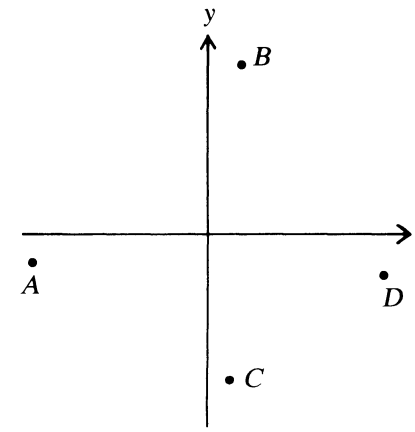

b. 


c. 


d. none of the above.

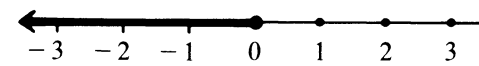
Which of the points labeled on the graph to the right lies in the intersection of the two regions defined by $y > 2x - 2$ and $y < -x$?

a. $A$
b. $B$
c. $C$
d. $D$




Which of the following sets describes the graph to the right?

a. $\{x \in R : |x| \geqq 0\}$
b. $\{x \in R : |x| = -x\}$
c. $\{x \in R : |x| \leqq 0\}$
d. none of the above.



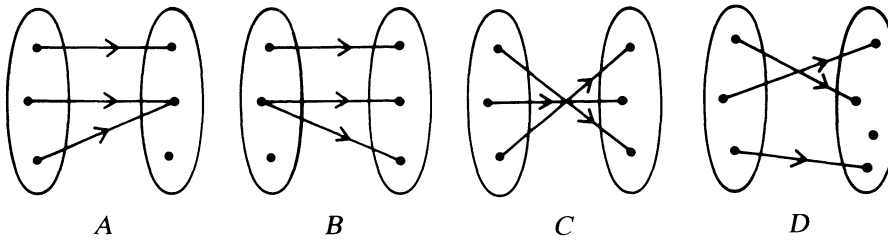

Values for $x$ and $y$ are given in the following table:

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 3 | | | 9 |

The relationship between $x$ and $y$ may be expressed by the equation

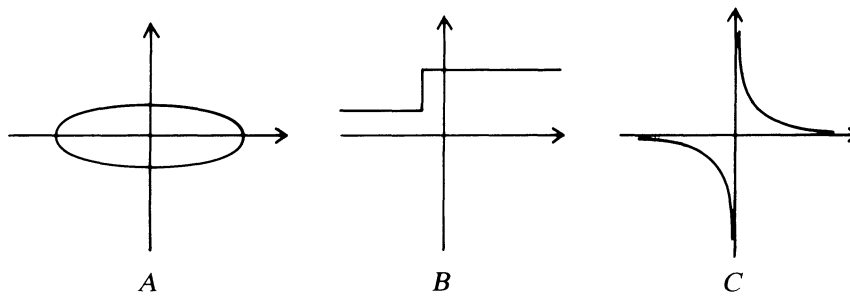
a. $y = x + 2$
b. $y = x + 3$
c. $y = 2x + 1$
d. $y = xy - 1$.

Which of the following representations of correspondences is *not* a function?



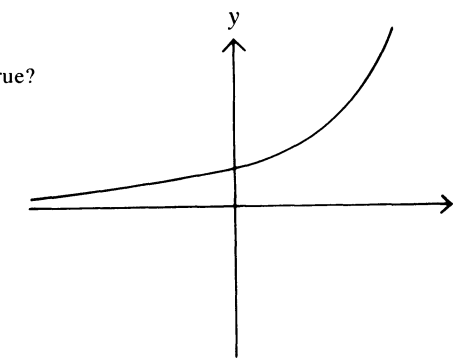

a. $A$ c. $C$
b. $B$ d. $D$.

Which of the following is a graph of a function?




a. $A$ c. $C$
b. $B$ d. none of the above.

For the graph shown, which statement appears to be true?
a. for all $x$, $f(x) = 0$
b. for all $x$, $f(x) > 0$
c. for all $x$, $f(0) = -2$
d. for all $x$, $f(x) < 0$.




## MATHEMATICAL SYSTEMS

To the right is a table for a binary operation * on a set $S = \{a, b, c\}$.

| * | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $c$ | $b$ | $a$ |
| $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ |

There is an identity element for * since

a. $a * a = c$, $b * b = b$, and $c * c = c$
b. $\square * c = c * \square = \square$ for every $\square \in S$
c. $b * \square = b$ for every $\square \in S$
d. $S$ is closed under *.

If $a$ is the identity element for an operation * on a set $S$, then for every $x \in S$
a. $x * a = a * x = 1$ c. $x * x = a$
b. $x * a = a * x = x$ d. $a * a = x$.

The empty set $\varnothing$ is the identity element for the set operation
a. union, since $A \cup \varnothing = A$ for every set $A$
b. intersection, since $A \cap \varnothing = \varnothing$ for every set $A$
c. cross-product, since $A \times \varnothing = \varnothing$ for every set $A$
d. none of the above
e. all of the above.

The addition table for 4-clock addition is shown to the right.

| + | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

The additive inverse for 3
a. is 0 since $3+0=3$
b. is 1 since $3+1=0$
c. is 2 since $3+2=1$
d. does not exist
e. cannot be determined from the table.

A set $A$ is said to be closed under an operation * if for any pair of elements $a$ and $b$ of $A$, $a * b$ is also an element of $A$. Which of the following is *not* true?
a. the set of integers is closed under subtraction
b. the set of whole numbers is closed under addition
c. the set of whole numbers is closed under subtraction
d. the set of positive fractions is closed under multiplication.

Which of the following is true about the set $O$ of odd numbers?
a. $O$ is closed under addition
b. $O$ is closed under multiplication
c. both a and b are true
d. neither a nor b is true.

If $A$ and $B$ are sets such that $n(A)=6$, $n(B)=3$, and $n(A \cap B)=2$, then $n(A \cup B)=$
a. $6+3$
b. $(6+3)-2$
c. $(6-2)+(3-2)$
d. $(6+3)-1$.

If $n(A \times B)=0$, where $A$ and $B$ are sets, then
a. $A=\{0\}$ or $B=\{0\}$
b. either $A$ or $B$ is the empty set
c. $n(A)$ or $n(B)$ is equal to the empty set
d. both $A$ and $B$ must be the empty set
e. none of these.

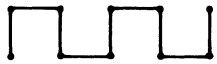
The associative property holds for subtraction on the set of
a. whole numbers
b. integers
c. rational numbers
d. real numbers
e. no choice is correct.

Define an operation $\#$ on the set of rational numbers by $m \# n = 2m + n$. For example $2 \# 3 = 2 \cdot 2 + 3 = 4 + 3 = 7$. Which of the following is true?
a. $\#$ is associative and commutative
b. $\#$ is not associative and not commutative
c. $\#$ is associative but not commutative
d. $\#$ is commutative but not associative.

Assume that the operations $\#$ and * are defined on a set $S$. Letting $a$, $b$, and $c$ represent elements of $S$, which of the following symbolizes that $\#$ is distributive with respect to (or over) * ?
a. $a \# b = b * a$
b. $a \# (b * c) = (a \# b) * (a \# c)$
c. $a \# (b * c) = (a * b) \# (a * c)$
d. $(a \# b) * c = a \# (b * c)$.

## GEOMETRY

Which of the following could be its own image under a reflection (flip)?

a. 

b. 

c. both a and b
d. neither a nor b.

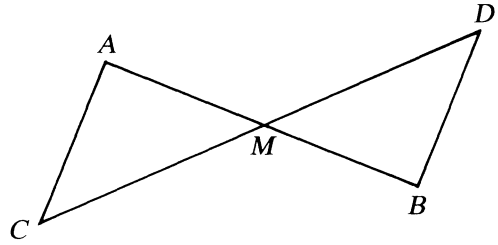
If $P$ is a point on a line $AB$, then the image of $P$ is on the image of the line $AB$ under a
a. reflection
b. rotation (turn)
c. both a and b
d. neither a nor b.

If two lines are perpendicular, then under a rotation (turn) their images
a. are parallel
b. are perpendicular
c. may be neither parallel nor perpendicular
d. may not be lines.

Which of the following figures does *not* determine a region which is symmetric with respect to a point?
a. circle
b. equilateral triangle
c. square
d. rectangle.

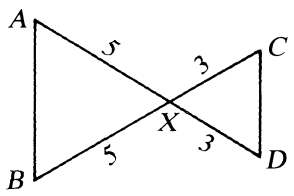
In space, the locus of all points equidistant from two given points is a
a. point
b. line
c. circle
d. plane.

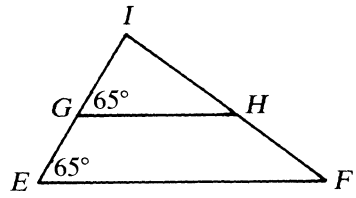
Which of the following is sufficient to show that triangles $AMC$ and $BMD$ are congruent?
a. $M$ is the midpoint of $\overline{AB}$
b. $\angle ACM$ is congruent to $\angle MDB$
c. $\overline{AB}$ and $\overline{CD}$ bisect each other at $M$
d. $\overline{AC}$ is congruent to $\overline{DB}$.



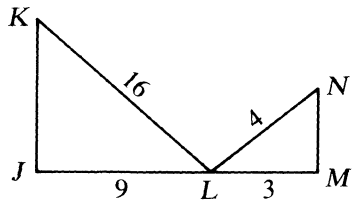

If $\triangle ABC$ is similar to $\triangle A'B'C'$, then the lengths of the corresponding sides $a$, $b$, $c$ and $a'$, $b'$, $c'$ must satisfy
a. $a/b = a'/b'$
b. $a/b' = b/a'$
c. $a/a' = 1$, $b/b' = 1$
d. $aa' = bb' = cc'$.

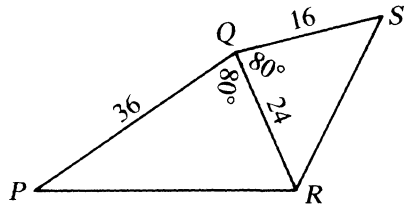
Which of the following pairs of triangles are *not* similar?
a. *AXB* and *CXD* c. *EFI* and *GHI*




b. *JKL* and *MNL* d. *PQR* and *RQS*.

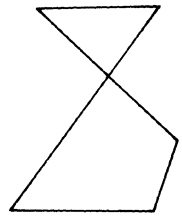



If two hexagons are both regular, they are
a. congruent
b. similar, but not necessarily congruent
c. equal in area, but not necessarily congruent
d. none of the above.

If two lines segments have the same length, then they are
a. parallel
b. congruent
c. perpendicular
d. none of the above.

Which of the following cannot be contained in the interior of a simple closed curve?
a. a line segment
b. a curve
c. a simple closed curve
d. a line.

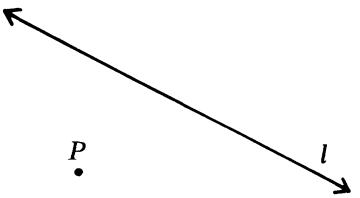


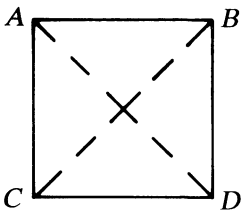
What can you say about the planar figure represented to the right?
a. it is a polygon because it is formed by line segments
b. it is not a polygon because it is not a simple curve
c. it is a polygon because it is a closed curve
d. it is not a polygon because it is not a closed curve.

If points $P$ and $Q$ on the number line have coordinates 2 and $-11$, respectively, then the distance from $P$ to $Q$ is
a. $-13$ c. 9
b. $-9$ d. 13.

What is the length of the line segment in the Cartesian plane joining the points (3, 4) and (7, 7)?
a. 3 c. 5
b. 4 d. 7.

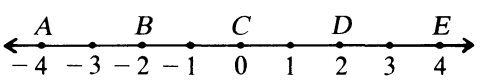
If $Q$ is the image of the point $P(P \notin l)$ under a reflection (flip) about line $l$, then
a. $\overleftrightarrow{PQ}$ is parallel to $l$
b. $\overleftrightarrow{PQ}$ is perpendicular to $l$
c. $m(\overline{PQ}) = 2$
d. none of the above.



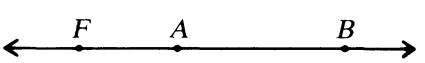

If $ABCD$ is a square, then which of the following best describes the rigid motion that maps $\triangle ABC$ onto $\triangle CDA$ such that $C$ is the image of $A$, $D$ is the image of $B$, and $A$ is the image of $C$?
a. translation (slide)
b. rotation (turn)
c. reflection (flip)
d. none of the above.

Given the line to the right, which of the following represents the solution set for $x$ where $x \leqq 2$?
a. $\overline{AD}$
b. $\overrightarrow{DC}$
c. $\overleftrightarrow{BD}$
d. $AD$

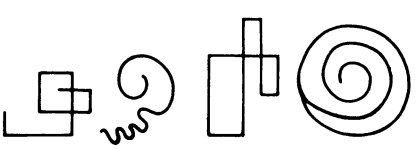



In a line, the set of points whose coordinates satisfy ____ is a ray.
a. $0 \leqq x \leqq 8$
b. $|x| \leqq 0$
c. $|x| \leqq 8$
d. $x \leqq \frac{1}{2}$.

In the figure to the right, $\overrightarrow{FA} \cap \overrightarrow{BA} =$
a. $\overleftrightarrow{FB}$
b. $\{A\}$
c. $\overline{FB}$
d. none of the above.




Which of the following represents a simple curve?




In three-dimensional geometry, the set of points $m$ inches from a point $P$ would be
a. a plane $m$ inches above $P$
b. a cube $m$ inches on an edge with $P$ at its center
c. a line segment $m$ inches long with $P$ at its center
d. a sphere with center at point $P$ and radius of $m$ inches.

The items presented here can be interpreted as an operational version of some of the goals of mathematics education for elementary teachers. These items give more information than a list of topics; and since they are in the multiple-choice form, they provide more information than open-ended test questions. The items also imply a level of treatment appropriate for the prospective elementary teacher. While the items do not indicate methods which should be used in presenting the material in the classroom, they do suggest some ideas which can be used in planning instruction. For example, the items suggest that experiences with geo-paper (dot paper) can be used not only in the usual manner to learn about area and perimeter, but can also be used to draw out concepts related to transformations.

The main purpose of this research was to prepare materials which would give assistance to those involved in the improvement of the mathematics education of elementary teachers. A few mathematicians and mathematics educators have voiced a concern shared by many—a concern that the school mathematics curriculum should not be determined solely by the behavioral psychologists or professional test-writers. They have urged that the opinions of mathematicians be enlisted to determine the content-goals of school mathematics. Perhaps one of the most important aspects of school curricular change is the influence exerted through the mathematics courses taken by preservice teachers. Few mathematicians have the opportunity to actually teach mathematics courses for prospective elementary teachers; but by engaging in projects and studies, their advice can be significant. Through sharing these findings, this study intended to continue the influence that mathematicians have in setting goals for school mathematics.

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## PROBLEMS AND SOLUTIONS

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

### ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before April 30, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 1822* [1965, 903]. *Proposed by Necdet Ucoluk*

Let $A$, $A_1$ and $B$, $B_1$ be any two pairs of given points in the plane. Consider the locus of points $N$ such that the angles $ANA_1$ and $BNB_1$ (with measures having absolute values $\alpha$ and $\beta$ respectively) satisfy the condition $\alpha = k\beta$, where $k$ is a given positive real number. (a) Determine the differentiability properties of this locus, and (b) when the tangent line exists give a geometric procedure (finite) for its construction.

E 2570. *Proposed by J. G. Sunday, University of Guelph, Canada*

Let $(m_1, n_1), \cdots, (m_k, n_k)$ be distinct lattice points with $n_i \geqq 2m_i > 0$ for each $i$, and suppose that no two of them lie on any line through the origin. Show that the least common multiple of $n_1, \cdots, n_k$ is not less than $2k$. When can equality occur?

E 2571. *Proposed by Sidney Kravitz, Dover, New Jersey*

It is well known that if $2^p - 1$ is prime, then $n = 2^{p-1}(2^p - 1)$ is a perfect number (i.e., $\sigma(n) = 2n$ where $\sigma(n)$ is the sum of the divisors of $n$) and that every even perfect number is of this form. A number $n$ is *perfect-plus-one* (pp1) if $\sigma(n) = 2n - 1$; see Problem E 1445* [1960, 1028; 1975, 73] and the following references; R. P. Jerrard and Nicholas Temperley, *Almost perfect numbers*, Math. Mag. 46 (1973), 84–87, and J. T. Cross, *A note on almost perfect numbers*, Math. Mag. 47 (1974), 230–231. It is known that if $n = 2^k$ then $n$ is pp1, but it is not known if there are any other pp1 numbers.

Discuss the situation for pp2 numbers, i.e., numbers $n$ for which $\sigma(n) = 2n - 2$.

E 2572. *Proposed by C. D. H. Cooper, Macquarie University, North Ryde, Australia*

Prove, or give a counterexample: If $f: \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and $f'$ is differentiable at some point $a$, then $f'$ is continuous in some neighborhood of $a$.

E 2573. *Proposed by Murray S. Klamkin, University of Waterloo*

If $n$ positive real numbers vary such that the sum of their reciprocals is fixed and equal to $A$, find the maximum value of the sum of the reciprocals of the $\binom{n}{j}$ sums of the $n$ numbers taken $j$ at a time.

E 2574*. *Proposed by F. David Hammer, Stockton State College, Pomona, New Jersey*

Let $N = \{0, 1, 2, \cdots\}$ and let $p$ be a prime. There is a binary operation $*$ on $N$ satisfying $x * y \leqq x + y$ for all $x$, $y$ in $N$ such that $(N, *)$ is an Abelian group with every element (except 0) of order $p$: for example, write $x$ and $y$ to base $p$ and add individual digits mod $p$. Prove, or disprove, that this gives the only such operation.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### Squares in Long Products

E 1075 [1953, 418; 1974, 1111]. *Proposed by H. S. Shapiro*

A sequence of $N$ positive integers contains precisely $n$ distinct numbers. If $N \geqq 2^n$, show that it is possible to find a block of consecutive terms whose product is a square, and that this result need not hold if $N < 2^n$.

*Solution by Joseph Zaks, University of Haifa, Israel.* (1) Let $\{a_1, \cdots, a_n\}$ denote the $n$ distinct integers. Let $S = (b_1, \cdots, b_N)$ be a sequence in which $b_i \in \{a_1, \cdots, a_n\}$ for all $i$, $1 \leqq i \leqq N$, and $N \geqq 2^n$. Let $V(n)$ denote the vertex set of the $n$-dimensional unit cube. For every $j$, $1 \leqq j \leqq N$, let $V_j = (k_1, \cdots, k_n) \in V(n)$ be defined by taking $k_i$ to be 0 (resp. 1) if $a_i$ occurs an even (resp. odd) number of times in the subsequence $(b_1, \cdots, b_j)$ of $S$. If some $V_j = (0, 0, \cdots, 0)$ then each one of $a_1, \cdots, a_n$ appears an even number of times in the subsequence $(b_1, \cdots, b_n)$ of $S$, hence $\prod_{i=1}^{j} b_i$ is a square. Suppose therefore that every $V_j \neq (0, 0, \cdots, 0)$. Since $N \geqq 2^n$, it follows that $V_j = V_m$ for some $j$ and $m$, $1 \leqq j < m \leqq N$. Thus from the definition of $V_j$ each one of the $a_1, \cdots, a_n$ appears an even number of times in the consecutive subsequence $(b_{k+1}, \cdots, b_m)$, hence $\prod_{i=k+1}^{m} b_i$ is a square.

(2) The result need not hold for $N < 2^n$, as can be seen by considering the sequence $S_n$ of length $N = 2^n - 1$ defined inductively by $S_1 = p_1$, $S_{n+1} = (S_n, p_{n+1}, S_n)$ where $p_n$ denotes the $n$th prime.

(3) The above argument is easily extended to show that if $N \geqq m^n$ then it is possible to find a block of consecutive terms whose product is an $m$th power, and that this result need not hold if $N < m^n$.

Also solved by forty-seven other contributors.

*Editor's comments.* Several solvers proved the necessity of $N \geqq 2^n$ using the fact that it is possible to arrange the subsets of a finite set in a list with the empty set first, every subset occurring exactly once in the list, and so that each subset differs from the preceding subset in the list by the addition or deletion of one element (a problem on the 1968 Putnam examination: a solution is given in this MONTHLY, 76 (1969), p. 913). Several solvers also pointed out that such arrangements are called Gray codes.

### Fitting a Cube in a Tetrahedron

E 2349 [1972, 393]. *Proposed by C. S. Ogilvy, Hamilton College*

Find the side of the largest cube that can be wholly contained within a regular tetrahedron of side 1.

*Solution by D. P. Robbins, The Phillips Exeter Academy, Exeter, New Hampshire.* First we restate the problem. Instead of fitting a cube inside a fixed tetrahedron (all tetrahedra mentioned are assumed to be regular), we find the smallest tetrahedron containing a fixed cube, the cube whose vertices are $(\pm 1, \pm 1, \pm 1)$ in the standard Cartesian system.

To describe the tetrahedron we will make use of the four vectors of unit length, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$, pointing outward from the four faces. We will need to use the easily established fact that

$$\mathbf{u}_i \cdot \mathbf{u}_j = -1/3 \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, 3, 4. \tag{1}$$

Also any set of four unit vectors satisfying (1) are the unit normals for some tetrahedron. We shall call such a set of unit vectors a tetrahedral set.

Our first task will be to find an algebraic expression for the size of the smallest tetrahedron containing the cube with a given tetrahedral set of unit normals, $\mathbf{u}_i$, $i = 1, 2, 3, 4$. (Except where otherwise noted the index $i$ will run over 1, 2, 3, 4.) A convenient measure of the size of the tetrahedron turns out to be its altitude. To calculate the altitude we make use of the fact that for a tetrahedron the altitude is the sum of the distances from any interior point to the four faces. We shall use this fact with the center of the cube, (0, 0, 0), as the interior point.

The face with the unit normal $\mathbf{u}_i$ has an equation of the form $\mathbf{u}_i \cdot (x, y, z) = a_i$ where $a_i$ is the distance from the origin to this face. In order for the tetrahedron to contain the cube we must have all the vertices on the same side of the plane as the origin and hence the $a_i$ must satisfy

$$\mathbf{u}_i \cdot (\pm 1, \pm 1, \pm 1) \geqq a_i$$

for every choice of signs. Since we wish to minimize the altitude of the tetrahedron which is, by our previous remark, the sum of the $a_i$, our best choice for a given tetrahedral set of $\mathbf{u}_i$ is to take

$$a_i = \max \mathbf{u}_i \cdot (\pm 1, \pm 1, \pm 1),$$

where the maximum is to be taken over all choices of signs. If we introduce the notation $\mathbf{u}_i = (u_{i1}, u_{i2}, u_{i3})$, it follows easily from the equation above that

$$a_i = \sum_{j=1}^{3} |u_{ij}|.$$

Hence the altitude of the smallest tetrahedron containing the cube and having the unit normals $\mathbf{u}_i$ is given by the formula

$$\text{altitude} = \sum_{i=1}^{4} \sum_{j=1}^{3} |u_{ij}|. \tag{2}$$

Thus we have reduced the problem to the determination of the minimum value of the sum in (2) over all tetrahedral sets of unit vectors. (The minimum certainly exists since the sum in (2) is a continuous function of the compact set of all tetrahedral sets of unit vectors.)

LEMMA. *Among all tetrahedral sets of* $\mathbf{u}_i$ *minimizing the sum* (2) *there is at least one satisfying the conditions*:

(i) $u_{11} = u_{22} = 0$, (ii) $u_{12}, u_{13}, u_{21} \geqq 0$.

*Proof.* We use rotations to prove part (i). Suppose a tetrahedral set of $\mathbf{u}_i$ minimizes the sum (2). If we rotate the $\mathbf{u}_i$ by an amount $\theta$ around the $z$-axis, the $z$-components of the $\mathbf{u}_i$ remain the same but the sum of those terms in (2) with $j = 1$ or 2 is replaced by

$$\sum_{i=1}^{4} (|u_{i1} \cos\theta - u_{i2} \sin\theta| + |u_{i1} \sin\theta + u_{i2} \cos\theta|). \tag{3}$$

Now suppose for a moment that all the $u_{i1}$ and $u_{i2}$ were $\neq 0$. Then for $\theta$ near 0 the sum (3) would have the form $A \cos\theta + B \sin\theta$ with $A > 0$. Regarding the sum as a function of $\theta$, we would then have a negative second derivative at $\theta = 0$, contradicting the fact that $\theta = 0$ is a minimum. Thus we may conclude that one of the $u_{i1}$ or $u_{i2}$ is 0. By renumbering the $\mathbf{u}_i$ and renaming the coordinate axes as necessary we may assume that $u_{11} = 0$.

Next we apply a similar argument with rotation about the $x$-axis to show that one of the $u_{i2}$ or $u_{i3}$ is zero.

To finish the proof of (i) we need to consider two cases: First, if $u_{12}$ or $u_{13}$ is 0, we may, by renaming the coordinate axes if necessary, take $u_{12} = 0$. Then, in the sum (3), the term with $i = 1$ is identically zero. We again use rotation about the $z$-axis to conclude that one of the $u_{i1}$ or $u_{i2}$ is 0 for $i = 2, 3$, or 4. Then by suitable renaming of axes and the $\mathbf{u}_i$ we may take $u_{22} = 0$. Second, if $u_{i2}$ or $u_{i3}$ is 0 for $i = 2, 3$, or 4, we can renumber the last three $\mathbf{u}_i$ and switch the $y$- and $z$-axes as necessary to obtain $u_{22} = 0$. This proves (i).

Now suppose we have a tetrahedral set of $\mathbf{u}_i$ minimizing the sum in (2) and satisfying the condition (i). If $u_{12} \leqq 0$, we can reflect all the $\mathbf{u}_i$ in the $xz$-plane. This will change the sign of $u_{12}$. In a similar way the signs of $u_{13}$ and $u_{21}$ can be adjusted. This proves the lemma.

Next we shall obtain the family of all tetrahedral sets of unit vectors satisfying the conditions of the lemma in terms of a single parameter and then express the altitudes of the corresponding tetrahedra in terms of this parameter.

Let $\mathbf{u}_i = (u_{i1}, u_{i2}, u_{i3})$ be a tetrahedral set of unit vectors satisfying the conditions of the lemma. Introduce the parameter $t = \sqrt{3}\, u_{13}$. Then, since $\mathbf{u}_1$ is a unit vector and $u_{11} = 0$, $u_{12} \geqq 0$, we have

$$\sqrt{3}\,\mathbf{u}_1 = (0, \sqrt{3 - t^2}, t). \tag{4}$$

Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = -1/3$ and $u_{21} \geqq 0$, $u_{22} = 0$, we obtain

$$\sqrt{3}\,\mathbf{u}_2 = (\sqrt{3t^2 - 1/t}, 0, -1/t). \tag{5}$$

Before proceeding we consider what restrictions may be placed on $t$. In order for the square roots to be defined we must have $1/3 \leqq t^2 \leqq 3$. But since $t = \sqrt{3}\, u_{13} \geqq 0$, we know that $1/\sqrt{3} \leqq t \leqq \sqrt{3}$. Now if we replace $t$ by $1/t$ and then switch the $x$- and $y$-axes and change the signs of the $z$-components, the two vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ will be exchanged. Thus we may be sure that there is a tetrahedral set of unit vectors minimizing the sum in (2) that has $\mathbf{u}_1$ and $\mathbf{u}_2$ as in (4) and (5) with $1 \leqq t \leqq \sqrt{3}$.

Given unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ satisfying $\mathbf{u}_1 \cdot \mathbf{u}_2 = -1/3$, it is easily proved that the other two vectors in a tetrahedral set must be (in some order) given by

$$\mathbf{u}_3 = (\sqrt{3}/2)\mathbf{u}_1 \times \mathbf{u}_2 - (1/2)(\mathbf{u}_1 + \mathbf{u}_2),$$

$$\mathbf{u}_4 = (\sqrt{3}/2)\mathbf{u}_2 \times \mathbf{u}_1 - (1/2)(\mathbf{u}_1 + \mathbf{u}_2).$$

Substituting our expressions above for $\mathbf{u}_1$ and $\mathbf{u}_2$, we obtain

$$2\sqrt{3}\,\mathbf{u}_3 = ((-1/t)\sqrt{3 - t^2} - (1/t)\sqrt{3t^2 - 1},\ \sqrt{3t^2 - 1} - \sqrt{3 - t^2},$$

$$(-1/t)\sqrt{3 - t^2}\sqrt{3t^2 - 1} - t + (1/t))$$

and

$$2\sqrt{3}\,\mathbf{u}_4 = ((1/t)\sqrt{3 - t^2} - (1/t)\sqrt{3t^2 - 1},\ -\sqrt{3t^2 - 1} - \sqrt{3 - t^2},$$

$$(1/t)\sqrt{3 - t^2}\sqrt{3t^2 - 1} - t + (1/t)).$$

Next we use the formula (2) to express the altitude of these tetrahedra in terms of $t$. We need to determine the signs of each of the $u_{ij}$. It is assumed that $u_{12}, u_{13}, u_{21} \geqq 0$ while it is obvious that $u_{23}, u_{31}, u_{42} \leqq 0$. Using the assumption $t \geqq 1$ it is not difficult to conclude that $u_{32} \geqq 0$ and $u_{33}, u_{41} \leqq 0$. For $u_{43}$ the situation is more complicated; $u_{43} \geqq 0$ for $1 \leqq t \leqq (1 + \sqrt{5})/2$ and $u_{43} \leqq 0$ for $(1 + \sqrt{5})/2 \leqq$

$t \leqq \sqrt{3}$. Using this information and the formula (2) we have

$$\sqrt{3}(\text{altitude}) = t + (1/t) + (1 + (2/t))\sqrt{3t^2 - 1} + \sqrt{3 - t^2}$$

$$+ (1/t)\sqrt{3 - t^2}\sqrt{3t^2 - 1},\ 1 \leqq t \leqq (1 + \sqrt{5})/2;$$

$$\sqrt{3}(\text{altitude}) = 2t + (1 + (2/t))\sqrt{3t^2 - 1} + \sqrt{3 - t^2},$$

$$(1 + \sqrt{5})/2 \leqq t \leqq \sqrt{3}.$$

In each case rather laborious calculations show that these functions have negative second derivatives in the interior of the relevant interval; hence no minima occur there. It follows that the minimum altitude occurs when $t$ has one of the critical values, $t = \sqrt{3}$, $(1 + \sqrt{5})/2$ or 1. By direct evaluation we find that the minimum occurs at $t = \sqrt{3}$. The minimum value for the altitude is $(2/3)(3 + \sqrt{6} + 2\sqrt{2})$ which yields a side of the tetrahedron of $(1/3)(6 + 4\sqrt{3} + 3\sqrt{6})$ compared to the edge of 2 for the cube. Scaling down this figure so that the edge of the tetrahedron is 1, we see that the largest cube fitting inside the tetrahedron has

$$\text{side} = 6/(6 + 4\sqrt{3} + 3\sqrt{6}) \doteq 0.295907.$$

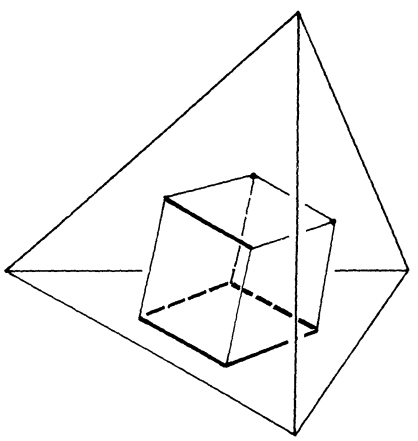


FIG. 1. Edge length ≐ .295907

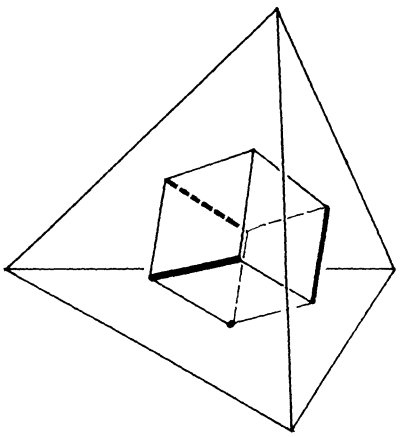


FIG. 2. Edge length ≐ 291344

It is not difficult to see that this solution corresponds to the obvious placement of the cube with one face lying on a face of the tetrahedron (Figure 1). The other critical values $t = 1$ and $t = (1 + \sqrt{5})/2$ are also local extrema and correspond to interesting geometrical placements of the cube. When $t = (1 + \sqrt{5})/2$, a main diagonal of the cube lies along an altitude of the tetrahedron in such a way that each lateral face of the tetrahedron contains an edge of the cube (Figure 2). When the cube is placed this way inside the unit tetrahedron, its side is given by

$$\text{side} = (\sqrt{10} - \sqrt{2})/6 \doteq 0.291344.$$

In the case $t = 1$, each face of the tetrahedron contains one edge of the cube (Figure 3). When the cube is placed this way in the unit tetrahedron, we have

$$\text{side} = (2 - \sqrt{2})/2 \doteq 0.292893.$$

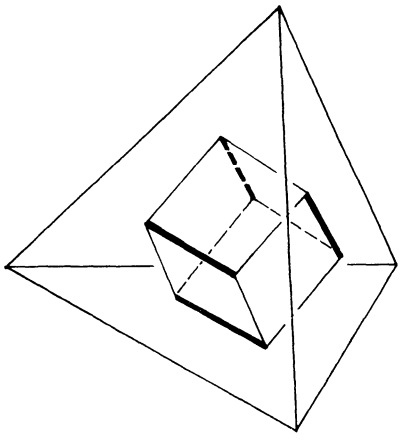


FIG. 3. Edge length ≐ .292893

Also solved by L. E. Mattics.

*Editor's comment.* There are two rather obvious and one perhaps not so obvious ways of positioning the cube, corresponding respectively to Robbins' first, third and second critical points (Figures 1, 3, and 2, respectively — We are indebted to Leon Bankoff for Figures 1 and 3.) Oddly enough, most attempts to solve the problem started by assuming the correct position (Figure 1) without adequate justification. In Figure 2, note that not all vertices of the cube touch the tetrahedron.

**A Geometric Characterization of $\zeta(4)$**

E 2503 [1974, 1027]. *Proposed by R. F. Jackson, University of Toledo*

A fixed disk $C_0$ of unit radius is centered at $(-1, 1)$. Beginning with the disk $C_1$, centered at $(1, 1)$ and tangent to the $x$-axis and to $C_0$, an infinite chain of disks $\{C_k\}$ is constructed, each tangent to the $x$-axis, to $C_0$, and to $C_{k-1}$. Find the sum of their areas.

I. *Solution by L. E. Mattics, University of South Alabama.* Let $r_k$ be the radius of $C_k$. The horizontal distance from the center of $C_0$ to the center of $C_k$ can be given both by $2\sqrt{r_k}$ and by $2\sqrt{r_{k+1}}(1+\sqrt{r_k})$, using the Pythagorean theorem on three triangles whose hypotenuses connect the centers of $C_0$, $C_k$, and $C_{k+1}$. Hence $\sqrt{r_{k+1}} = \sqrt{r_k}/(1+\sqrt{r_k})$. Since $r_1 = 1$, we obtain $r_k = 1/k^2$, and the sum of the areas is

$$\sum_{k=1}^{\infty} \pi/k^4 = \pi\zeta(4) = \pi^5/90.$$

II. *Solution by P. K. Garlick, University of Michigan.* The similarity transformation $x' = (x+1)/2$ and $y' = y/2$ of ratio $\frac{1}{2}$ maps $C_0$ to the Ford circle $C(0/1)$ which has center $(0, \frac{1}{2})$ and radius $\frac{1}{2}$ and maps $C_k$, $k \geq 1$, to the Ford circle $C(1/k)$ which has center $(1/k, 1/2k^2)$, radius $1/2k^2$, and area $\pi/4k^4$. [Hans Rademacher, *Lectures on Elementary Number Theory*, Blaisdell, 1964.] The series is summed as in Solution I.

III. *Solution by Jordi Dou, Barcelona, Spain.* Una inversion de centro $(-1, 0)$, potencia $2^2$, transforma los circulos $C_k$ en circulos de radio 1 y centros $(2k-1, 1)$. Los radios $r_k$ serán $r_k = 1 \cdot (4/2k)/2k = 1/k^2$. La suma de áreas será

$$\pi \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^5}{90}.$$

IV. *Solution by J. G. Mauldon, Amherst College.* As was first pointed out (in verse) by F. Soddy (H. S. M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1969, p. 15) the relation between the radii $a, b, c, d$ of four coplanar mutually externally tangent circles can be expressed in the beautiful form

$$(a^{-1}+b^{-1}+c^{-1}+d^{-1})^2 = 2(a^{-2}+b^{-2}+c^{-2}+d^{-2}).$$

If the fourth circle is a straight line, $d^{-1}=0$ and the condition reduces to

$$\prod (a^{-1/2} \pm b^{-1/2} \pm c^{-1/2}) = 0.$$

Thus if we denote the radius of $C_k$ by $u_k^{-2}$, then for $k \geqq 2$ we have $u_k = u_{k-1} + u_0$ and hence $u_k = k$. (The series is then summed as in Solution I.)

Also solved by seventy-three other contributors, and the proposer.

*Editor's comment.* A number of these solvers proved by induction that the center of the $k$th circle is $((2-k)/k,\ 1/k^2)$, or noticed that all the centers lie on the parabola $(x+1)^2 = 4y$.

**Extended Medians of a Triangle**

E 2505 [1974, 1111]. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York*

Let $a$, $b$, $c$ be the sides of a triangle $ABC$, and let $m_a$, $m_b$, $m_c$ be the medians to sides $a$, $b$, $c$ respectively. Extend the medians so as to meet the circumcircle again, and let these chords be $M_a$, $M_b$, $M_c$ respectively. Show that

$$M_a + M_b + M_c \geqq \frac{4}{3}(m_a + m_b + m_c) \tag{1}$$

$$M_a + M_b + M_c \geqq \frac{2}{3}\sqrt{3}(a+b+c). \tag{2}$$

When does equality occur?

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* Define $k>0$ by $a^2+b^2+c^2=4k^2$. Then, using $4m_a^2 = 2b^2+2c^2-a^2$ and two similar formulas, we have $m_a^2+m_b^2+m_c^2=3k^2$. The well-known inequality $x^2+y^2+z^2 \geqq (x+y+z)^2/3$ gives $3k^2 = \Sigma m_a^2 \geqq (\Sigma m_a)^2/3$ and $4k^2 = \Sigma a^2 \geqq (\Sigma a)^2/3$. Therefore

$$4k \geqq \frac{4}{3}\sum m_a \quad \text{and} \quad 4k \geqq \frac{2\sqrt{3}}{3}\sum a. \tag{i}$$

Eliminating $a$ from the equations $4m_a^2 = 8k^2 - 3a^2$ and $m_a(M_a - m_a) = (\frac{1}{2}a)^2$, we have $M_a m_a = 2(k^2+m_a^2)/3$, whence

$$M_a = \frac{2k}{3}(k/m_a + m_a/k) \geqq \frac{4k}{3}. \quad \text{Therefore}$$

$$\sum M_a \geqq 4k. \tag{ii}$$

Combining (i) and (ii) we have the required result. It is clear that equality holds if and only if the triangle is equilateral.

II. *Solution to* (1) *by Paul Erdös and M. S. Klamkin, University of Waterloo, Ontario.* Since

$m_a(M_a - m_a) = a^2/4$, etc., (1) can be rewritten as

$$3\sum a^2/m_a \geqq 4\sum m_a. \tag{1'}$$

It is known that one can form a triangle having sides $m_a$, $m_b$, $m_c$ with its respective medians being $3a/4$, $3b/4$, $3c/4$. Thus (1') is equivalent to

$$4\sum m_a^2/a \geqq 3\sum a. \tag{1''}$$

Since $4m_a^2 = 2b^2 + 2c^2 - a^2$, etc., (1'') reduces to $\Sigma(b^2 + c^2)/a \geqq 2\Sigma a$ or

$$\sum (b + a)(b - a)^2/ab \geqq 0.$$

Also solved by Anders Bager (Denmark), G. E. Bennett, M. G. Greening (Australia), and I. I. Kolodner. Partial solutions by Leon Bankoff and the proposer.

### Limits of Differences of Square Roots

E 2506 [1974, 1111]. *Proposed by Solomon W. Golomb, University of Southern California*

*Let* $N$ denote the set of natural numbers. What is the set of limit points of the set $\{\sqrt{a} - \sqrt{b} : a, b \in N\}$?

I. *Solution by Julian H. Blau, Antioch College.* The set contains all integral multiples of each of its members. Since it has arbitrarily small members $\sqrt{n+1} - \sqrt{n}$, every real number is a limit point of it.

II. *Solution by Arthur Freund, University of Wisconsin.* Given an arbitrary real number $r$, let $r_n = \sqrt{[(n+r)^2]} - \sqrt{n^2}$, where [ ] denotes the greatest integer function. Then $\lim_{n\to\infty} r_n = r$ since $\sqrt{(n+r)^2 - 1} - n < r_n \leqq r$ for all $n \geqq 1 - r$ and the left-hand side converges to $r$ as $n \to \infty$. Thus the required set of limit points is the real line (the extended real line if $\pm\infty$ are admitted).

III. *Generalization by Michael Somos, State University of New York at Albany.* THEOREM. *If* $a_1, a_2, \cdots$ *and* $b_1, b_2, \cdots$ *are unbounded monotone increasing sequences of real numbers and* $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$, *then* $\{a_n - b_m : n, m \in N\}$ *is dense in the reals.*

*Proof.* Let $r$ be any real and let $\varepsilon > 0$. Then there exists $k$ such that $a_{n+1} - a_n < \varepsilon$ for all $n \geqq k$. Pick $m$ large enough so that $b_m \geqq a_k - r$ and then let $n \geqq k$ be determined (uniquely) by the condition $a_n \leqq b_m + r < a_{n+1}$. This gives $|(a_n - b_m) - r| < \varepsilon$.

In the problem posed, $a_n = b_n = \sqrt{n}$ satisfies the hypothesis of the theorem and therefore every real number is a limit point of $\{\sqrt{a} - \sqrt{b} : a, b \in N\}$.

Also solved by 72 other solvers, and the proposer.

*Editor's comments.* Many solvers prove first that every rational number $p/q$ is a limit point of the given set, in most cases using the fact that

$$\lim_{n\to\infty}(\sqrt{q^2n^2 + \lambda pn} - \sqrt{q^2n^2 - \mu pn}) = p/q,$$

where $\lambda + \mu = 2$. The error in the resulting approximations to $p/q$ is $O(1/n)$, except if $\lambda = \mu = 1$ when it is $O(1/n^2)$. (The rate of convergence in solutions such as II above may be similarly improved if one uses instead

$$\sqrt{[(n + r/2)^2]} - \sqrt{[(n - r/2)^2]}$$

say.) The proposer remarks on the closeness of the approximations obtained in the case $\lambda = \mu = 1$ when in addition one writes $p/q$ as $p^2/pq$. One could ask for a simple formula for the closest $\sqrt{a} - \sqrt{b}$ to $p/q$ with $a, b \leqq k$.

Of the various generalizations received, Somos's (III) is the most far-reaching, the others being easy consequences of it. As pointed out by the proposer, the fact that if $a_{2n-1} = 2^n$ and $a_{2n} = 2^n + 2^{-n}$ then 0 is the only limit point of $\{a_n - a_m : n, m \in N\}$, shows that one cannot replace lim by $\underline{\lim}$ in the hypothesis of Somos's result. Weber asks for the set of limit points of $\{\sqrt{p} - \sqrt{q}: p \text{ and } q \text{ primes}\}$; it is a difficult and as yet unsolved problem whether $\lim_{n\to\infty}(\sqrt{p_{n+1}} - \sqrt{p_n}) = 0$ or not ($p_n$ = the $n$th prime), so Somos's result cannot be used. Nevertheless, Paul Erdős informs us that $\{\sqrt{p} - \sqrt{q}: p \text{ and } q \text{ primes}\}$; is dense in $\mathbf{R}$; his proof uses the theorem of A. Selberg that for all but $o_\epsilon(N)$ values of $n \leqq N$ the interval $[n, n + n^{\epsilon+1/5}]$ contains a prime.

**Summing Minima**

E 2507 [1974, 1111]. *Proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey*

Show that

$$\sum \min(L_1, \cdots, L_k) = \sum_{m=0}^{n} m^k,$$

where the left-hand sum is over all $(L_1, \cdots, L_k)$ with $0 \leqq L_i \leqq n$ for $i = 1, 2, \cdots, k$.

*Solution by Peter G. de Buda, Undergraduate, University of Toronto.* In general, if one is summing integer terms $a_i \geqq 0$, one can rearrange the summation to obtain

$$\sum a_i = (\text{number of terms} \geqq 1) + (\text{number of terms} \geqq 2) + \cdots$$

$$= \sum_{m=1}^{\infty} (\text{number of terms } \geqq m).$$

In the present case, if $j > n$, then the number of terms $\geqq j$ is zero, and if $1 \leqq j \leqq n$ then the number of terms $\geqq j$ is $(n - j + 1)^k$. Thus the required sum is

$$\sum_{j=1}^{n} (n - j + 1)^k = \sum_{m=0}^{n} m^k.$$

Also solved by fifty-four other contributors.

*Editor's Comment.* R. C. Grimson suggests a number of generalizations and gives many references including his papers: *The generating function for* $\{\min(n_1, \cdots, n_k)\}^m$, (Elemente der Mathematik, 29 (1974)); *The evaluation of certain arithmetic sums*, (The Fibonacci Quarterly 12 (1974), 373–380); *A summation formula and some properties of Eulerian functions* (Proc. AMS, to appear) and *Summation methods*, I, (The Institute of Statistics, Memo Service no. 989, March 1975); M. S. Klamkin and D. J. Newman note that generalizations of the present result are given in *Inequalities and identities for sums and integrals*, page 26 in this issue of the MONTHLY.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before April 30, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6066*. *Proposed by The late C. W. Anderson, University of California, Berkeley*

For $n = 3$ and $x \in (0, 1)$ rational, show that $f_n(x) = (1 - x^n)^{1/n}$ is algebraic of degree $n$.

6067. *Proposed by Ron Evans, University of Wisconsin, Madison*

Prove that for each real $\sigma$, there exist infinitely many $t > 0$ for which $\Gamma(\sigma + it) < 0$, where $\Gamma$ denotes the Gamma function.

6068. *Proposed by Seth Warner, Duke University*

Let $A$ be an algebra over a commutative ring $K$, and let $A_+$ be the $K$-algebra $K \times A$ where addition and scalar multiplication are defined componentwise and multiplication by $(x, a)(y, b) = (xy, x \cdot b + y \cdot a + ab)$. Let $N$ and $R$ be respectively the (Jacobson) radicals of $K$ and $A$. It is standard that if $N = (0)$, $N \times R$ is the radical of $A_+$. What are the necessary and sufficient conditions for $N \times R$ to be the radical of $A_+$?

6069. *Proposed by A. R. Charnow, California State University at Hayward*

Let $R$ be an integral domain, $G$ a torsion free group, and $R[G]$ the group ring of $G$ over $R$. Let $x = r_1 g_1 + r_2 g_2$, $r_i \in R$, $r_i \neq 0$, $g_i \in G$, $g_1 \neq g_2$. Prove that $x$ is neither a zero divisor nor a unit in $R[G]$.

6070. *Proposed by Paul Erdös and the late C. W. Anderson*

Where $\phi(n)$ is Euler's totient function, let $\Phi(n) = \phi(n)/n$, $\Phi$: $N \to (0, 1]$ densely. For given $a$, demonstrate that there are only a finite number of $b$ (coprime with $a$) such that $\Phi(n) = a/b$ has solutions.

6071. *Proposed by J. G. Milcetich, Federal City College, Washington, D.C.*

The set of analytic functions defined on the unit disk, $U$, with the topology of uniform convergence on compact subsets of $U$ forms a locally convex, linear topological space. In such a space, $\overline{\text{co}}B$ denotes the closed convex hull of a subset $B$. Let $K$ denote the set of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which map $U$ onto a convex domain. Show that for $k \geq 2$, $z + a_k z^k \in \overline{\text{co}}K$ if and only if $|a_k| < \frac{1}{2}$.

## SOLUTIONS OF ADVANCED PROBLEMS

### Linear Programming with Random Selections

5723 [1970, 313]. *Proposed by H. D. Ruderman, Hunter College Campus School*

Let $M = xy + zt - uv$. A random integer is selected uniformly from the numbers 0 to 9 inclusive and is used for some one of the six variables $x$, $y$, $z$, $t$, $u$, or $v$. Then a second random integer is selected independently from the same uniform distribution and is used for some other of the remaining variables. This selection and assignment is continued in the same way until six integers have been selected and assigned to the six variables. What is the best strategy for assigning the integers so as to maximize the expectation of $M$?

I. *Solution by David Sankoff, Université de Montréal, Canada.* It suffices to find a strategy which

maximizes the conditional expectations $E(M \mid A_n)$ where $A_n$ is an arbitrary selection and assignment of $n < 6$ random integers. For the random variables we will employ $a$, $b$, $c$, $d$, $e$, $f$ and for $M$ take $ab + cd - ef$. Since some configurations $A_n$ are merely relabellings of one another, we need consider only $a$ and $e (n = 1)$; $ab$, $ac$, $ae$ and $ef (n = 2)$; $abc$, $abe$, $ace$ and $aef (n = 3)$; $abcd$, $abce$, $abef$ and $acef (n = 4)$; $abcde$ and $abcef (n = 5)$.

For any strategy, $E(M \mid abcde) = ab + cd - 4.5e$, $E(M \mid abcef) = ab + 4.5c - ef$ and, by independence, $E(M \mid abcd) = ab + cd - (4.5)^2$ and $E(M \mid abef) = ab + (4.5)^2 - ef$.

Suppose the values $E(M \mid B_{n+1})$ and $E(M \mid C_{n+1})$ are known under an optimal strategy, and the assignment of an integer $x \in [0, 9]$ to $A_n$ can only result in configuration $B_{n+1}$ or $C_{n+1}$. Then a best strategy is one which maps $A_n$ to $B_{n+1}$ when $x$ is such that $E(M \mid B_{n+1}) > E(M \mid C_{n+1})$. For example, $abce \to abcde$ for $d = x > 4$ and $abce \to abcef$ for $f = x \leqq 4$. For such a strategy, $E(M \mid A_n) = 0.1 \sum_{x=0}^{9} E(M \mid A_{n+1}(x))$ where the strategy maps $A_n$ to $A_{n+1}(x)$. For example

$$E(M \mid abce) = 0.1 \left[ \sum_{d=0}^{4} E(M \mid abcde) + \sum_{f=5}^{9} E(M \mid abcef) \right] = ab + 5.75c - 3.25e.$$

When $A_{n+1}$ may result from $A_n$ in two different ways, as in $acef \to abcef$; $acef \to acdef$ (relabel), these must be considered different strategies. Thus $E(M \mid acef) = 5.75 \max(a, c) + 3.25 \min(a, c) - ef$.

| $A_n$ | Assignment of $(n+1)$st integer $x$, R indicates $A_{n+1}$ must be relabelled. | $E(M \mid A_n)$ |
|---|---|---|
| $abcde$ | $f$ | $ab + cd - 4.5e$ |
| $abcef$ | $d$ | $ab + 4.5c - ef$ |
| $abcd$ | $e$ | $ab + cd - 20.25$ |
| $abce$ | $d, x > 4$; $f, x \leqq 4$ | $ab + 5.75c - 3.25e$ |
| $abef$ | $c$ | $ab + 20.25 - ef$ |
| $acef (a \geqq c)$ | $b, x > 4$; $d, x \leqq 4$, R | $5.75a + 3.25c - ef$ |
| $abc$ | $d, x > 5$; $e, x \leqq 5$ | $ab + 6.45c - 12.975$ |
| $abe$ | $c, x > 3$; $f, x \leqq 3$ | $ab + 30.525 - 2.55e$ |
| $ace (a \geqq c)$ | $b, x > 5$; $d, 3 < x \leqq 5$, R; $f, x \leqq 3$ | $6.45a + 4.5c - 2.55e$ |
| $aef (a \leqq 3)$ | $b, \begin{cases} x \leqq 3 \\ x = 4 \\ x = 5 \\ x > 5 \end{cases}$; $c$ elsewhere | $2.55a + 30.525 - ef$ |
| $(a = 4)$ | | $41.4 - ef$ |
| $(a = 5)$ | | $45.9 - ef$ |
| $(a > 5)$ | | $6.45a + 12.975 - ef$ |
| $ab$ | $c, x > 4$; $e, x \leqq 4$ | $ab + 28.8$ |
| $ac (a \geqq c)$ | $b, x > 6$; $d, 4 < x \leqq 6$, R; $e, x \leqq 4$ | $6.915a + 5.285c - 9.0375$ |
| $ae (a = 3)$ | $f, \begin{cases} 2 < x \leqq 4 \\ 2 < x \leqq 5 \\ 3 < x \leqq 5 \\ x \leqq 2; 5 < x \leqq 7; \\ x > 5 \\ x > 6 \\ x > 6 \end{cases}$ $c$ elsewhere | $48.9825 - 2.085e$ |
| $(a = 4)$ | | $52.9275 - 2.085e$ |
| $(a = 5, e > 0)$ | | $57.3 - 2.085e$ |
| $(a = 6)$ | | $62.7825 - 2.085e$ |
| $(a = 7)$ | | $69.5925 - 2.085e$ |
| $(a = 8)$ | | $76.47 - 2.085e$ |
| $(a = 9)$ | | $83.385 - 2.085e$ |
| $(a = 5, e = 0)$ | $f, x \leqq 3$; $b, 3 < x \leqq 5$; $c$ elsewhere | $57.315$ |
| $ef$ | $a$ | $47.01 - ef$ |
| $a \ (a = 3)$ | $b, \begin{cases} 3 < x \leqq 5 \\ 3 < x \leqq 5 \\ x = 5 \\ x = 6; \\ x > 6 \\ x > 6 \\ x > 6 \end{cases} \begin{matrix} \\ \\ \\ e, x \leqq 4; \\ \\ \\ \\ \end{matrix}$ $c$ elsewhere | $50.274$ |
| $(a = 4)$ | | $54.866$ |
| $(a = 5)$ | | $59.6465$ |
| $(a = 6)$ | | $65.07175$ |
| $(a = 7)$ | | $71.83825$ |
| $(a = 8)$ | | $79.06$ |
| $(a = 9)$ | | $86.3005$ |
| $e \ (e = 0)$ | | $59.2485$ |
| $(e > 0)$ | $a, x > 2$; $f, x \leqq 2$ | $59.247 - 1.7595e$ |
| start | $a, x > 3$; $e, x \leqq 3$ | $64.32155 = E(M)$ |

We continue using the conditional expectations at the different $A_{n+1}$ to find the best strategy at $A_n$. In calculating this solution, certain conditional expectations and assignment strategies are computed which are later found to be irrelevant, because the overall strategy will never involve certain configurations (e.g., $ac$ where $a = c = 0$). These are not listed.

II. *Solution by Uriel Rothblum, Courant Institute, New York University.* The method we use to compute the optimal strategy is based on dynamic programming. We start by computing the optimal strategy for maximizing the expected reward for each of the simple objective functions which can

| Objective Function | Remark | Case | Optimal assignment* for first withdrawal number | | | | | | | | | | Optimal Value of Objective Function |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| $ax + by$ | $b \leqq a$ | | $y$ | $y$ | $y$ | $y$ | $y$ | $x$ | $x$ | $x$ | $x$ | $x$ | $5.75a + 3.25b$ |
| $ax + by + cz$ | $c \leqq b \leqq a$ | | $z$ | $z$ | $z$ | $z$ | $y$ | $y$ | $x$ | $x$ | $x$ | $x$ | $6.45a + 4.5b + 2.55c$ |
| $-xy + az$ | $0 \leqq a \leqq 9$ | $0 \leqq a \leqq 3$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $6.425a - 12.9$ |
| | | $4 \leqq a \leqq 9$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $z$ | $6.45a - 12.975$ |
| $xy + az$ | $-9 \leqq a \leqq 9$ | $-9 \leqq a \leqq 3$ | $z$ | $z$ | $z$ | $z$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $2.55a + 30.525$ |
| | | $a = 4$ | $x$ | $x$ | $x$ | $x$ | $z$ | $x$ | $x$ | $x$ | $x$ | $x$ | 41.4 |
| | | $a = 5$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $x$ | $x$ | $x$ | $x$ | 45.9 |
| | | $6 \leqq a \leqq 9$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $z$ | $6.45a + 12.975$ |
| $xy + zt$ | | | arbitrary | | | | | | | | | | 47.01 |
| $xy - zt$ | | | $z$ | $z$ | $z$ | $z$ | $z$ | $x$ | $x$ | $x$ | $x$ | $x$ | 28.8 |
| $xy + az - bt$ | $0 \leqq a, b \leqq 9$ | $0 \leqq a \leqq 3$ | $t$ | $t$ | $t$ | $z$ | $z$ | $x$ | $x$ | $x$ | $x$ | $x$ | $3.715a - 2.085b + 37.83$ |
| | | $a = 4$ | $t$ | $t$ | $t$ | $z$ | $z$ | $z$ | $x$ | $x$ | $x$ | $x$ | $-2.085b + 52.9275$ |
| | | $a = 5, b = 0$ | $t$ | $t$ | $t$ | $t$ | $z$ | $z$ | $x$ | $x$ | $x$ | $x$ | 57.315 |
| | | $a = 5, 1 \leqq b \leqq 9$ | $t$ | $t$ | $t$ | $x$ | $z$ | $z$ | $x$ | $x$ | $x$ | $x$ | $-2.085b + 57.3$ |
| | | $a = 6, 7$ | $t$ | $t$ | $t$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $z$ | $6.87a - 2.085b + 21.502$ |
| | | $a = 8, 9$ | $t$ | $t$ | $t$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $6.915a - 2.085b + 21.15$ |
| $-xy + az + bt$ | $0 \leqq b \leqq a \leqq 9$ | $b = 0$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $z$ | $6.87a - 8.985$ |
| | | $1 \leqq b \leqq 3, b \leqq a \leqq 3$ | $x$ | $x$ | $x$ | $x$ | $x$ | $t$ | $z$ | $z$ | $z$ | $z$ | $6.8675a + 5.32b - 9$ |
| | | $1 \leqq b \leqq 3, 4 \leqq a \leqq 9$ | $x$ | $x$ | $x$ | $x$ | $x$ | $t$ | $z$ | $z$ | $z$ | $z$ | $6.87a + 5.32b - 9.0075$ |
| | | $4 \leqq b \leqq a \leqq 9$ | $x$ | $x$ | $x$ | $x$ | $x$ | $t$ | $z$ | $z$ | $z$ | $z$ | $6.87a + 5.33b - 9.0375$ |
| $xy + zt - bu$ | $0 \leqq b \leqq 9$ | $b = 0$ | $u$ | $u$ | $u$ | $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | 59.24725 |
| | | $b \neq 0$ | $u$ | $u$ | $u$ | $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | $t$ | $-1.7595b + 59.25575$ |
| $-xy + zt + by$ | $0 \leqq b \leqq 9$ | $b = 0$ | $x$ | $x$ | $x$ | $x$ | $x$ | $u$ | $z$ | $z$ | $z$ | $z$ | 36.72975 |
| | | $b = 1$ | $x$ | $x$ | $x$ | $x$ | $x$ | $u$ | $z$ | $z$ | $z$ | $z$ | 41.20626 |
| | | $b = 2$ | $x$ | $x$ | $x$ | $x$ | $x$ | $u$ | $z$ | $z$ | $z$ | $z$ | 45.69175 |
| | | $b = 3$ | $x$ | $x$ | $x$ | $x$ | $u$ | $u$ | $z$ | $z$ | $z$ | $z$ | 50.193 |
| | | $b = 4$ | $x$ | $x$ | $x$ | $x$ | $u$ | $u$ | $z$ | $z$ | $z$ | $z$ | 54.803 |
| | | $b = 5$ | $x$ | $x$ | $x$ | $x$ | $x$ | $u$ | $z$ | $z$ | $z$ | $z$ | 59.6015 |
| | | $b = 6$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $z$ | $z$ | $z$ | 64.9465 |
| | | $b = 7$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $u$ | $u$ | $u$ | 71.82475 |
| | | $b = 8$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $u$ | $u$ | $u$ | 79.0375 |
| | | $b = 9$ | $x$ | $x$ | $x$ | $x$ | $x$ | $z$ | $z$ | $u$ | $u$ | $u$ | 86.269 |
| $xy + zt - uv$ | | | $u$ | $u$ | $u$ | $u$ | $x$ | $x$ | $x$ | $x$ | $x$ | $x$ | 64.290975 |

* The optimal assignment is not necessarily unique. Whenever there is symmetry between two variables either of t two can be assigned.

result when all but two of the variables have been assigned. We also compute the optimal value of each of these objective functions. We then use this information to compute, recursively, the optimal strategies and values for the more complicated functions which result when all but three, all but four, all but five and all but six variables have been assigned. The results of our computations are summarized in the accompanying table, which shows the optimal strategy and the resulting maximum expected value.

Throughout, $a$, $b$, $c$, $d$, $e$ and $f$ denote fixed values of integers, whereas $x$, $y$, $z$, $t$, $u$ and $v$ denote variables to which integers have to be assigned.

Also solved by Haig Bohegian and by R. E. Kibler.

**Range of a Holomorphic Function in $|z|<1$**

5936 [1973, 1067; 1975, 185]. *Proposed by David Styer, University of Cincinnati*

Is there a function $f$, bounded and holomorphic in $|z|<1$, and a polynomial $p$ such that, in $|z|<1$, $f/p$ assumes every complex value infinitely often with at most one exception?

II. *Solution by M. J. Pelling, University of Benin, Nigeria.* There are solutions which assume every complex value infinitely often without exception. An example is

$$f(z) = (1+z)\sin\log((1+z)/(1-z)), \quad p(z) = 1-z.$$

To see this, note first that $w = (1+z)/(1-z)$ maps $|z|<1$ into the half plane $\operatorname{Re} w > 0$ and that $\log w$ is analytic in this half plane if we define it by $\log(re^{i\theta}) = \log r + i\theta$, $|\theta| < \pi/2$. The map $s = \log w$ then transforms this half plane into the infinite strip $|\operatorname{Im} s| < \pi/2$ and $f/p$ finally transforms into $e^s \sin s$, while $f$ transforms into $2(e^s \sin s)/(e^s+1)$ which is bounded in the strip.

Let $N$ be a bounded neighborhood of 0 contained within the strip such that under the mapping $s \to e^s \sin s$ the image of $N$ contains the disk $|s|<r$ for some $r>0$. Then, since

$$e^{(s+2n\pi)}\sin(s+2n\pi) = e^{2n\pi}e^s \sin s$$

for an integer $n$, it follows that the image of the translate $N+2n\pi$ contains the disk $s < re^{2n\pi}$. Taking $n$ increasingly large and positive, we conclude that $e^s \sin s$ assumes every complex value infinitely often for $s$ in the strip $|\operatorname{Im} s| < \pi/2$, and hence $f/p$ does also in the disk $|z|<1$.

**Spanning Ideals**

5970 [1974, 524]. *Proposed by Albert Wilansky, Lehigh University*

Let $A$ be an algebra with identity and let $I$, $J$ be two distinct ideals of codimension 1. Show that $IJ \cup JI$ spans $I \cap J$. (That is, every member of $I \cap J$ is a linear combination with scalar coefficients of members of $IJ \cup JI$.) Show also that $IJ$ need not span $I \cap J$.

*Solution by L. J. Dickson, University of New South Wales, Australia.* (a) Let $F$ be the base field, $e$ the identity of $A$. Since $I$ and $J$ are both subalgebras of codimension 1 and $I \neq J$, we have $J \subsetneq I + J = A$. Thus the following generalizes the given problem.

THEOREM. *Let $A$ be an algebra with identity, $I$ and $J$ two-sided ideals of $A$ such that $I+J=A$. Then $IJ \cup JI$ spans $I \cap J$.*

*Proof.* The claim is equivalent to the statement $K = I \cap J$, where $K$ is the 2-sided ideal $IJ + JI$. Clearly $K \subset I \cap J$. But by the hypothesis there exist $e_1 \in I$, $e_2 \in J$ such that $e = e_1 + e_2$. Thus, if $b \in I \cap J$ then $b = be_1 + be_2$, where $be_1 \in JI$ and $be_2 \in IJ$.

(b) We note that $A$, $I$, $J$ yield the desired counterexample if and only if $IJ \not\supset JI$. For instance, if

$$A = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in F \right\}, \qquad I = \left\{ \begin{pmatrix} 0 & y \\ 0 & z \end{pmatrix} \right\}, \quad \text{and} \quad J = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right\},$$

then $0 = IJ \neq JI = I \cap J$.

Also solved by J. A. Dentinger, Jr., Larry Eifler, F. L. Flanigan, A. A. Jagers (Netherlands), Surjeet Singh (India), Seth Warner, and the proposer.

*Editorial Note.* Flanigan's solution is built on the following theorem: *Let $A$ be a unital algebra, not necessarily associative, of arbitrary dimension over a field $k$. Let the (two-sided) ideal $I$ have codimension 1, and let $L$ be any left ideal with $L \not\subseteq I$. Then* (i) $I \cap L \subset IL + LI$ *(left ideals in $A$)*; (ii) *moreover, $I \cap L = IL + LI$ if and only if $L$ is a two-sided ideal of $A$.*

### Partitioning $R_+$

5971 [1974, 524]. *Proposed by William Sánchez and David Spear, New York City*

Can the positive reals $R_+$ be partitioned into two sets both of which are closed under addition? Can these sets be Lebesgue measurable?

*Solution by D. Borwein and P. B. Borwein, University of Western Ontario.* The answer to the first question is YES and to the second NO. (i) Let $H$ be a Hamel basis of the reals over the rationals, let $h \in H$, and let $A$ be the set of all positive real numbers expressible as

$$rh + \sum_{i=1}^{n} r_i h_i,$$

where $r$ is a positive rational, $r_i$ is rational and $h_i \in H \setminus \{h\}$. The non-empty sets $A$, $R_+ \setminus A$ are closed under addition and partition $R_+$.

(ii) Suppose that $R_+$ is partitioned by non-empty sets $A, B$ which are closed under addition. Assume that $A, B$ are Lebesgue measurable. Then one of the sets, say $A$, must have non-zero measure. Also

$$A \supseteq A + A = \{x \mid x = x_1 + x_2;\ x_1, x_2 \in A\}.$$

It is known (see W. Rudin, *Real and Complex Analysis*, chapter 8, problem 12) that this implies that there is an open interval $I \subseteq A$. Let $b \in B$. Then there are positive integers $p, q$ such that $pb/q \in I$, whence $pb \in A$. But this is impossible since $pb \in B$. Hence neither $A$ nor $B$ can be Lebesgue measurable.

Also solved by Richard Amundson, P. R. Chernoff, George Crofts, L. J. Dickson (Australia), Daniel Gallin, Peter Garst, M. B. Gregory, A. A. Jagers (Netherlands), O. P. Lossers (Netherlands), S. P. Marin, L. E. Mattics, L. F. Meyers, William Nuesslein, and the proposers. Partial solution, part (i) by Mark Hunacek, part (ii) by J. C. Morgan II.

### Mihimum $n$, $x^n = x$ for all $x$ in a Ring

5972 [1974, 534]. *Proposed by Gregory Wene, University of Iowa*

Let $R$ be a ring and let * be the statement: There exists a positive integer $n \geq 2$ such that $a^n = a$ for each $a \in R$.

For each ring $R$ satisfying *, define the $\mu$-value of $R$ to be $\mu = \min\{m \geq 1 \mid m$ a positive integer and $a^m = a$ for each $a \in R\}$. It is easily shown that 18 is not the $\mu$-value for any ring. Find all positive integers $k$ such that $k$ is the $\mu$-value for some ring and show that there are infinitely many positive integers which are not $\mu$-values for any ring.

*Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands.* (1) For any ring $R$ and $x \in R$, let $N(x)$ be the set of all positive integers $n$ such that $x^n x = x$, and let $N(R)$ be the intersection of all $N(x)$, $x \in R$. If $N(x)$ is empty, put $\nu(x) = \infty$. Similarly if $N(R)$ is empty, put $\nu(R) = \infty$. Otherwise set $\nu(x) = \min N(x)$ and $\nu(R) = \min N(R)$. Obviously $\nu(R) + 1$ is just the $\mu$-value of $R$, if $R$ satisfies *. The following holds under the usual conventions concerning the symbol $\infty$.

(i) If $\nu(x) < \infty$, then $N(x) = \{\nu(x), 2\nu(x), 3\nu(x), \ldots\}$.

(ii) $\nu(R) = \mathrm{LCM}\{\nu(x) \mid x \in R\}$.

(iii) If $R$ is a subdirect sum of rings $R_i$, $i \in I$, then

$$\nu(R) = \mathrm{LCM}\{\nu(R_i) \mid i \in I\}.$$

*Proof.* Clearly $k\nu(x) \in N(x)$. Conversely, if $n \in N(x)$, then $x^r x = x$ where $r = n - k\nu(x)$ such that $0 \leqq r < \nu(x)$. Thus $r = 0$ and (i) follows. Further (ii) is a direct consequence of (i) and (iii) follows from (i) and (ii).

Now by a result of Jacobson (*Structure of Rings*, AMS (1964) p. 217), any ring $R$ satisfying * is a subdirect sum of fields. This fact, together with (iii) and the observation that $\nu(F) = p^t - 1$ for any field $F$ of order $p^t$, yields the following theorem.

THEOREM. (A) *A ring $R$ satisfies * if and only if $R$ is a subdirect product of finite fields $F_i$, say, involving only finitely many nonisomorphic $F_i$.* (B) *A number $k$ is the $\mu$-value of some ring if and only if $k$ is of the form*

$$\mathrm{LCM}(p_1^{t_1} - 1, p_2^{t_2} - 1, \cdots, p_n^{t_n} - 1) + 1,$$

*where $n = 1, 2, \cdots$, the $p_i$ are primes (not necessarily distinct) and the $t_i$ are positive integers.*

(2) By Dirichlet's theorem the arithmetic sequence $5, 9, 13, \cdots$ contains infinitely many primes, $p_1, p_2, p_3, \cdots$, say. Then $p_i + 1$ is never the $\mu$-value of any ring. In fact, it easily follows from (B), that, if $m > 3$ is the $\mu$-value of a ring such that $m - 1$ is prime, then $m$ is necessarily a power of 2; but in this case $m - 1 \not\equiv 1 \pmod 4$.

Also solved by L. J. Dickson (Australia), F. J. Flanigan, Robert Gilmer (Australia), R. C. Lyndon, L. E. Mattics, Surjeet Singh (India), and the proposer.

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## REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
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COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield MN 55057.*

*Shapes and Perceptions: An Intuitive Approach to Geometry.* By Gail S. Konkle. Prindle, Weber and Schmidt, Boston, Massachusetts, 1974. viii + 248 pp. $9.95. (Telegraphic Review, February 1975.)

This book was used at Winthrop College in a one semester sophomore level course. Many of the students had a poor understanding of the space that includes them. The author's fresh approach to the teaching of geometry involves the students in the spatial experiences they need, to gain an

intuitive feeling for the ideas in geometry. In spite of a "play" method of introducing geometry the author often presents its more subtle aspects. For instance the chapter called "Rubber Sheet Geometry" includes not only the traditional intuitive approach to "topologically equivalent curves" but also distinguishes among curves, surfaces, and solids, and discusses separation concepts.

Chapters one through seven, nine, and ten, which deal with shapes, size, symmetry, topology, tessellations, maps, networks, and projections, are the best organized and give the students many insights into geometry. Less good are the chapters about measure, area, and volume, where the fragmentation of topics is confusing and makes it necessary to pick and choose exercises carefully from several chapters for any one presentation.

The exercises involve the students with physical reality, observation, drawing, and building models. A teacher should be able to select an adequate number of exercises appropriate for his students from the many provided. The exercises are distributed unevenly. Sometimes they are clustered in the text, sometimes clustered at the end of a section or chapter, and sometimes spread uniformly throughout the text.

There are several errors of fact or concept in this book. For example, in the answer to a problem on page five the reader is told that it isn't possible to make a polyhedron with faces that are copies of a given right triangle unless at least two lie in the same plane. This statement may be demonstrated to be false with a model of an octahedron made from eight copies of the triangle. Another example occurs on page 126, where the author says that the reason the sun's rays appear essentially parallel is that the sun is so vast, when in fact it is because the sun is so distant.

The writing occasionally becomes unnecessarily complicated or difficult to follow for a person just learning concepts. Here are just two examples. "Symmetry can be intuited well through examination of earthly objects" (p. 23), and "Definition: a *curve* is a set of points topologically equivalent to a line, a portion of a line (other than isolated points), or the union of any finite number of these," (p. 33). The first quotation could be stated more simply as "Many common objects illustrate symmetry," whereas the latter quotation is too complicated to be an intuitive description of a curve and incorrect as a technical definition. The book's open size of 18 cm high by 51 cm wide makes it inconvenient to use on student desks, although ideally the student using this text ought to be at a work table. Its length also makes it awkward to shelve.

In spite of its faults, I recommend this text to anyone who teaches a course in geometry for prospective elementary teachers. The errors can be pointed out by the instructor, the exercises are more useful for the geometrically naive than those in any other geometry book using the intuitive approach that I've seen, and the ideas presented are fundamental to the understanding of geometry.

DONALD APLIN, Winthrop College

*Complex Variables and Applications*, Third Edition. By Ruel V. Churchill, James W. Brown, and Roger F. Verhey. McGraw-Hill, New York, 1974. x + 332 pp. $12.50. (Telegraphic Review, November 1974.)

*Complex Analysis with Applications.* By Richard A. Silverman. Prentice-Hall, Englewood Cliffs, N.J., 1974. x + 274 pp. $12.95. (Telegraphic Review, November 1974.)

*Introductory Complex Analysis and Applications.* By William R. Derrick. Academic Press, New York, 1972. xi + 218 pp. $9.95. (Telegraphic Review, April 1972.)

Each of these books is appropriate for an introductory, one term course in functions of a complex variable for scientists and engineers. The book by Ruel V. Churchill *et al.* is a standby in the field, and the other two books compare favorably to it. In fact, all three books are worth considering if you are looking for a textbook at this level.

An introductory course of this kind is based on only three or four important theorems. The core

of the course usually consists of the theory of the Cauchy-Riemann equations and differentiability (including conformal mapping), Cauchy's Theorem, and Cauchy's Integral Formula. A surprising amount of analytic function theory follows from the last two theorems. Other important theorems such as the maximum modulus theorem and Rouché's theorem are mentioned. However, they are not used extensively in the introductory course, and their importance only becomes apparent in later work.

The most innovative of the three books is the one by William R. Derrick. It is refreshing to find new techniques used to prove some standard results. There is, for example, an interesting proof (beginning on page 38) of the fundamental theorem of integration. As the title of the book suggests, the emphasis is on applications. Some of his applications, mainly those based on the concept of conformal mapping, can be found in most books. Additional topics include the Fourier transform, the Laplace transform, and the method of steepest descent. In terms of applications, this book is better than the other two books being reviewed here. In fact, by comparison they seem dated. Alas, Derrick's book has a serious flaw. The book could use a considerable amount of editorial work. This book requires more diligence on the part of the teacher, but for certain programs its virtues (in terms of applications) should outweigh the drawbacks.

In sharp contrast, the book by Richard A. Silverman is beautifully written. It is a book worthy of its author, who is particularly well known for his work as a translator of books in mathematics, including translations and revisions of the treatise on functions of a complex variable by A. I. Markushevich. Compared to Derrick's book, Silverman's might be described as a book on the classical theory of analytic functions. Each topic is carefully thought through, definitions are clear, and there are ample comments and illustrations. At the end of each chapter, a page or two of comments is added. These comments clarify and expand upon the material in the main section of the chapter. The book is sprinkled with references that will aid the student who wants to look more deeply into the subject. It is also the only one of these books to provide "Selected Hints and Answers."

The book by Ruel V. Churchill *et al.* is probably known to most readers. This, its third edition, has been revised very successfully. The book is similar to the earlier editions and none of the flavor which made the earlier editions so impressive has been lost. Yet, the revision is nonetheless thorough. In terms of applications, the books by Churchill and Silverman only provide applications based on the theory of conformal mapping. A chapter on Laplace transformations would have added much to the value of these two books.

The books by Derrick and Silverman proceed at a faster pace. For example, integration is first considered on page 28 in Derrick, page 48 in Silverman, and not until page 102 in Churchill. On the other hand, the slow pace of the Churchill book may be a clue to its success.

There are two parts in the Churchill book which I consider weak. Using too much space, the authors develop both the concept of a multiple-valued function and the idea of a function defined on a Riemann surface. The concept of the Riemann surface is one of the most beautiful topics in complex function theory. And furthermore, it works, in contrast to the theory of multiple-valued functions. Secondly, I wish the authors had omitted the route method of finding residues on pages 178–9. This reviewer has never seen an example that can not be done more easily and with more insight by some other method. Otherwise, hats off to Churchill, Brown, and Verhey for producing this fine book.

In summary, all three of these books are excellent. As a teaching tool, Derrick's book is probably the least effective. Still, it is the most modern and innovative of the three. The book by Churchill is a thorough revision of a book that even in its earlier form was considered a classic. Although its pace is slower than that of the other two books, it is very effective. Silverman's book is a gem that any serious student of mathematics will enjoy.

D. H. TRAHAN, Naval Postgraduate School

*Graphs, Groups and Surfaces.* By Arthur T. White. Math. Stud., vol. 8. North-Holland, 1973. x + 142 pp. $9.50 (P). (Telegraphic Review, August-September 1974.)

*Graphs, Groups and Surfaces* is the first book that leads a novice easily through the title topics to an understanding of graph imbedding theory and its main construction, the current graph. True to the foreword, it assumes only an introductory knowledge of group theory (light experience with a few definitions, e.g. cosets, index) and of point-set topology (less than what is taught in an undergraduate real analysis course). Well-chosen examples throughout illustrate the content of the main theorems. These merits and numerous precise historical attributions in context make White's book a top priority choice for an undergraduate library.

A highly attractive feature to me is that after a brief historical survey in Chapter 1 and some basic (Michigan style) graph theoretic definitions in Chapter 2, White immediately gets to groups. Chapter 3 is about graph automorphism groups and Chapter 4 is about Cayley "color" graphs, providing some early substance with a nice proof of R. Frucht's theorem that every finite group is the automorphism group of some graph.

The early problem sets adequately reinforce the main ideas. Students are sometimes asked to supply omitted proofs. I wish White had used these early exercises as an opportunity to develop some of the graph-theoretic concepts, like traversibility, that he mentions only briefly, or to introduce additional mathematical connections, with matrices, for example.

Chapter 5 begins development of the book's main interest, imbedding graphs in surfaces. It includes a classification of the platonic solids and an outline of an elementary proof (i.e., without mention of homology) of the invariance of Euler characteristic. Some unfortunate vagueness in this chapter contrasts sharply with White's carefulness elsewhere. A beginner might get the mistaken impression that projective planes, Klein bottles, and other closed nonorientable surfaces imbed in euclidian 3-space. It should be emphasized that two-sidedness (p. 36) of surfaces is an imbedding property, not an intrinsic property. (For example, the "middle" Möbius band $\mathrm{Mb} \times \{\frac{1}{2}\}$ has two sides in the cartesian product $\mathrm{Mb} \times I$ of a Möbius band with the interval [0, 1].) The definition of "dual pseudograph" (p. 46) fails to specify the number of edges between two vertices of the dual.

Chapter 6 continues the discussion of graph imbeddings, including the Heffter-Edmonds permutation technique, mentions thickness and crossing numbers, and gives some important formulas, mostly without proof. I would have defined an "imbedding" (p. 53) to be a function, not its image.

Chapter 7 discusses the genus of a group, including some of White's own valuable research contributions. Chapter 8 first obtains familiar upper bounds on the chromatic numbers of surfaces and lower bounds on the genus of graphs and then describes some relatively recent results on colorings.

Chapter 9 is about the constructive graph imbedding techniques associated with the expression "current graphs." Ringel's use of generating rows for schemes provided the first major breakthrough in the solution of the Heawood map-coloring problem, and a device (popularly called a current graph) for constructing generating rows was invented by W. Gustin and augmented by J. W. T. Youngs. After describing the Gustin-Youngs theory, White presents the unifying approach of A. Jacques for Cayley graph imbeddings.

The exercises for the later chapters are often quite interesting. A few new research problems are posed.

In teaching a graduate course on this material, I would prefer to emphasize group presentations and covering spaces at the expense of some other topics. I would use different terminology at times. "Suspension" seems to me a more descriptive word than "join" (p. 10), for example, and "brin" (p. 121) sounds harshly artificial.

In general, I strongly recommend White's book, not only for an independent course, but also as supplementary reading in an elementary algebraic topology course. No other book takes a newcomer so well into topological graph theory.

J. L. Gross, Columbia University

## FILMS

For general information about this series of films see the introduction on page 416 of this MONTHLY, vol. 82, number 4.

*Inversion*: A film produced by the College Geometry Project at the University of Minnesota. Mathematician: Daniel Pedoe. 16 mm sound and color; 12 minutes. Available for rent or purchase from International Film Bureau—sale $160; rental $10. Also available for rent from numerous University Film Libraries.

The film before us, titled "Inversion," deals with a particular geometric transformation which gave rise to a branch of geometry called *inversive geometry*. The nature of the problem introduced here and the level of the treatment are such as to highly recommend this film for courses in geometry, complex variables, and mathematics appreciation.

What is the film all about?

Consider a circle $C_1$ and a circle $C_2$ enclosed by it. Let $D_1, D_2, \cdots, D_k$ be a sequence of pairwise tangent circles which are each tangent to $C_1$ and $C_2$. It may happen that such a sequence forms a closed ring in which $D_1$ and $D_k$ are also tangent, and the question is this: When can a closed ring of circles $D_1, \cdots, D_k$ be constructed? If the answer in any particular situation is yes, does it depend on the position of $D_1$?

A moment's reflection shows that the problem should be simplest when $C_1$ and $C_2$ are concentric, and with this case the film begins. It is clear at once that the radii of $C_1$ and $C_2$ must be related in a certain way if the ring of inscribed circles is to close. It is also clear that in this case the answer is independent of the choice of $D_1$.

The viewer is now presented with the classical procedure of reducing a general case to a simpler special case. Students with a calculus background, for example, could be reminded of a similar procedure in proving the mean value theorem from Rolle's theorem. The film proceeds now in a step by step construction of the inversion transformation, and in the end the viewer sees that, indeed, if there is one solution, then there are infinitely many.

A class discussion ranging from 30 minutes to two or three hours is quite necessary in most cases. In a way, an important lesson is learned and the film has real dramatic impact when it is presented after the class tried to solve the problem the hard way. The important lesson is that the student sees a situation in which simple graphics make a difficult problem transparent and lead to its solution. With the right class, the instructor may like to use a modified "discovery" approach by presenting the film with practically no preliminary discussion, and then asking the class to explain the problem and discuss the solution.

On the technical side, the film is extremely well done. It has about it artistic simplicity and great elegance which, to may taste, are very appealing. The film; both mathematically and aesthetically, offers an enriching experience to beginning freshmen and advanced students alike.

DAVID A. SPRECHER, University of California, Santa Barbara

*Infinity*, *Look Again*, *Curves*, and *Dimensions*. Four films by David Nulsen Enterprises. Live and animated, color and sound, 14–20 minutes each. Distributor: AIMS Instructional Media Services.

Each of the four films is designed to develop and illustrate several mathematical concepts, and all are well made from the point of view of film technique. Some instances occur, however, where the examples or generalizations chosen to illustrate the mathematical ideas are either inappropriate or tend to mislead; but none of the films is without its merits.

*Infinity* stands out from the other three. Using an animated Georg Cantor, it discusses the concepts of countable and uncountable infinities, exhibits the proof of countability of the positive

rationals, and outlines the proof that (0, 1) is uncountable. The animation is quite good; but when the film was shown to an audience of college students, some objected to the burlesque heaviness of the Cantor character's German accent. The film does a very good job in focussing in on the essential concept of one-to-one correspondence (calling it "one-for-one"). The only distracting flaw is the narrator's habit of saying "integers" and "rationals" when he only means to refer to "positive integers" and "positive rationals."

*Look Again* is a profitable trip through mathematics, urging the viewer to look for the mathematical in the world of everyday experience. It discusses in interesting fashion sphere packings, minimal surfaces, Euler's formula for polyhedra, Möbius strips, coin tossing, random walks (drunkard and lamppost), and continuity vs. discreteness of time and space (unfortunately dismissed as "best left to the realm of philosophy"; Zeno's paradoxes are not mentioned). The narration, though, is a little too low-key, and there are one or two minor mathematical inaccuracies: for instance, in discussing the Möbius strip the narrator is unwilling to say that it has one side but seems to imply that the number of sides is an open question, while in discussing probability he says indirectly that a 75% chance is the same as "almost certain."

*Curves* has the best visuals and the best music (a Brandenburg concerto). Although the definition given of a curve as "anything formable from a piece of wire" strikes us as oversimplified, a plethora of examples help make up for it. The straight line, circle, ellipse, and spiral are photographed as they are drawn on a drafting table; then the sine curve is demonstrated as the shadow of a point on a rotating disk, with applications to sound mentioned. The conic sections are treated, with the parabola as the trajectory of a rocket and the ellipse as the path of a planet. The description of curves by means of polar coordinates is alluded to (no equations are given in the film), followed by presentation of cycloids (with the brachistochrone property) and hypocycloids. The film climaxes with a beautiful series of hypocycloids of different colors superimposed on one another.

*Dimensions* is a bad film on a difficult topic. The aspects of two- and three-dimensionality which the filmmakers have attempted to generalize are difficult mathematical concepts that do not admit of easy explanations. The film begins with overheld establishing shots of examples in nature of one-, two-, and three-dimensional objects. Then, after generating a line by motion of a point, a square by motion of a line, and a cube by motion of a square, a cube is used to generate a tesseract. Up to this point, the film is fine. Next, the fourth dimension is "used" to change the orientation of three-dimensional objects (handedness of gloves), to enter and leave a jammed telephone booth, and to see "the inside of the inside of the inside" of nested three-dimensional objects. The accompanying two- and three-dimensional examples work fine, but the analogies in the fourth dimension are made to appear as some sort of sleight of hand. A further analogy, that of seeing all sides of an object at once, might have been more effectively presented by the use of split screen than by the whirling superposition actually used. Serious flaws, however, lie in the failure to discuss co-ordinate systems or the distinction between mathematical and physical dimensions, and in passing fleetingly over time as a fourth dimension (always a top question with students).

In summary, *Infinity*, *Curves*, and *Look Again* are good films which will not require much comment from the teacher in the way of corrections, and could be used profitably at the high school or college levels. *Dimensions*, on the other hand, could lead to some serious misconceptions without detailed comment by a skilled instructor.

PAUL J. CAMPBELL, St. Olaf College
PIERRE J. MALRAISON, Carleton College

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
P = professional reading
L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = approximate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(13-15), L**. *Concepts of Modern Mathematics*. Ian Stewart. Penguin, 1975, ix + 315 pp, $3.50 (P). A sweeping, informal survey of modern mathematics (e.g., networks, fundamental group, catastrophe theory, undecidability) to combat "feelings of unease" that afflict most laymen when viewing the "new math." A bold sequel to Courant and Robbins' classic *What is Mathematics?* LAS

GENERAL, L**, P. *A Compendium of CUPM Recommendations: Discussions and Recommendations by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America*. MAA. *V. I:* 457 pp; *V. II:* 298 pp, $9 set. A reprinting (with only slight editorial changes for cross-referencing) of the more recent CUPM curricula reports--many of which had gone out of print. Volume I contains the reports on core mathematics, including teacher training and two-year college programs; Volume II is devoted to statistics, computing and applied mathematics. A valuable collection of documents that have been (and still are) shaping undergraduate mathematics education. LAS

GENERAL, S. *Mathematik für Ingenieure, Naturwissenschaftler, Ökonomen und Landwirte, B. 1-5, 7/1, 14, 16, 18*. N. Sieber, H.-J. Sebastian, G. Zeidler. B.G. Teubner, 1973, (P). Classical calculus and advanced calculus in the first five volumes, differential equations, linear optimization, optimal processes, and numerical methods in the others. All the material is presented without fanfare for direct use. No exercises, just numerical examples. JAS

GENERAL, P. *Lecture Notes in Mathematocs-431: Séminaire Bourbaki Vol. 1973-74, Exposés 436-452*. Springer-Verlag, 1975, iv + 347 pp, $12.90 (P).

BASIC, T(13). *Continuous Sequence in Basic Mathematics, Course 2: Beginning Algebra*. Melvin Poage, Carl Arendsen, Delano Wegener. Prindle, 1975, viii + 580 pp, $10 (P). A textbook most suited to individualized instruction. Short units contain study guides with objectives, lots of exercises, display and diagrams, laboratory activities, and self tests. Topics include exponents, polynomials, linear equations and systems, radicals and quadratic equations. CEC

BASIC, T(13; 1, 2). *Technical Mathematics*. Jacqueline Austin, Margarita Alejo de Sanchez Isern. Saunders, 1975, ix + 590 pp, $12.50 (P); *Instructor's Guide*, 142 pp, (P). A workbook-type text which is designed to teach manipulative skills to students in technological programs. It is suitable for a self-paced method of instruction. It is even less inspiring than most books of this type. CEC

PRECALCULUS, T*(13; 1). *Fundamentals of College Algebra, Third Edition*. Earl W. Swokowski. Prindle, 1975, ix + 388 pp, $10.95. Significant revision, but retaining the same basic content, of the author's 1971 *Second Edition*. Includes many more solved examples and 40% more exercises. Same as the author's *Fundamentals of Algebra and Trigonometry, Third Edition* (TR, August-September 1975) with the trigonometry removed. RSK

PRECALCULUS, S(13). *Programmed Guide to Accompany Fundamentals of College Algebra, Third Edition, by Earl W. Swokowski*. Roy A. Dobyns. Prindle, 1975, v + 260 pp, $4.50 (P); *Programmed Guide to Accompany Fundamentals of Algebra and Trigonometry, Third Edition, by Earl W. Swokowski*. Roy A. Dobyns. Prindle, 1975, v + 346 pp, $4.50 (P); *Programmed Guide to Accompany Fundamentals of Trigonometry, Third Edition, by Earl W. Swokowski*. Roy A. Dobyns. Prindle, 1975, v + 153 pp, $3.95 (P). Must be used in conjunction with the texts (*College Algebra*, TR above; *Algebra and Trigonometry*, TR, August-September 1975; *Trigonometry*, TR, December 1975). The *College Algebra Guide* and the *Trigonometry Guide* are essentially subsets of the *Algebra and Trigonometry Guide*. They contain a disturbing number of typographical errors, but should be helpful aids for some students. RSK

PRECALCULUS, T*(13; 1). *Fundamentals of Algebra with Trigonometry and Applications*. Gary Musser, Alan Hoffer, Sam Sperry. Prindle, 1975, xi + 532 pp, $12.95. The standard college algebra topics are treated in a more contemporary fashion. There are goals stated at the beginning of chapters, self tests, lots of problems and examples, and many very interesting word problems. CEC

PRECALCULUS, T*(13; 1). *Modern College Algebra, Third Edition*. Elbridge P. Vance. A-W, 1975, 310 pp, $9.95. Rigorous presentation. Includes chapters on matrices and determinants, permutations, combinations and the binomial theorem, and complex numbers and vectors, in addition to the standard topics. RSK

EDUCATION, P, L. *Overview and Analysis of School Mathematics Grades K-12*. CBMS, 1975, xiv + 157 pp, free (P). Report of the National Advisory Committee on Mathematical Education on the status of school mathematics after two decades of reform. Major conclusions: "new math" is a paper tiger that no longer really exists (if it ever did); decline in mathematics scores correlates not to changes in mathematics curricula but to broad declines in reading and other areas; hand calculators should be used regularly in classes beyond eighth grade level. LAS

EDUCATION, S(5-8). *Numerous Numerals*. James M. Henle. NCTM, 1975, viii + 40 pp, $1.50 (P). Enrichment topics for junior high, designed to test an "exciting, challenging, unpredictable, and hazardous" mode of teaching: teach something you do not know. LAS

EDUCATION, S**(9-14), L*. *Elements of Mathematics, Book B: Problem Book.* Ed: Edward Martin. CEMREL, 1975. *V. I:* v + 73 pp; *V. II: Answer Key*, vi + 161 pp, $2.10 (P) each. Part of a 16 volume series of books designed to supplement high school mathematics, this one consists of 200 challenging problems (in graph theory, algorithms, combinatorics, especially), enough for a whole high school career at the recommended rate of one per week. The *Answer Key* provides one or, more often, several different solutions to each problem. LAS

HISTORY, P. *Historical Studies in the Physical Sciences.* Ed: Russell McCormmach. Princeton U Pr. *V. 4,* 1974, 235 pp, $12.50; *V. 5,* 1975, 185 pp, $12.50. Annual volumes devoted to papers in the history of physics. LAS

HISTORY. *A History of Vector Analysis: The Evolution of the Idea of a Vectorial System.* Michael J. Crowe. U of Notre Dame Pr, 1967, xvii + 270 pp, $5.95. Detailed analysis (with very extensive footnote references to original sources) of the nineteenth century roots of vectorial analysis. LAS

HISTORY, P, L. *Science Awakening II: The Birth of Astronomy.* Bartel L. van der Waerden. Noordhoff, 1974, xv + 347 pp, $35. Thoroughly revised and updated translation of the 1965 German edition, a detailed examination of one of the three classical roots of modern science (the other two: mathematics and mechanics) which were brought together in the work of Isaac Newton. Egyptian, Assyrian and Babylonian astronomy, carefully referenced to sources. LAS

HISTORY, L**. *The Teaching and History of Mathematics in the United States.* Florian Cajori. Scholarly Res, 1974, 400 pp, $21.50. A reprint of the 1890 classic, originally published by the Government Printing Office. A thorough treatment of mathematics curricula in schools and colleges (treated individually) from colonial times to the late nineteenth century, its reprinting is an appropriate mathematical contribution to the bicentennial. LAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-454: Forcing, Arithmetic, Division Rings.* Joram Hirschfeld, William H. Wheeler. Springer-Verlag, 1975, vii + 266 pp, $10.80 (P). Survey of various metamathematical analogues of algebraically closed fields with applications to a wide variety of other algebraic systems. Primary focus is on the unexpected new perspective provided by forcing techniques of model theory. LCL

FOUNDATIONS, P. *Lecture Notes in Mathematics-453: Logic Colloquium.* Ed: R. Parikh. Springer-Verlag, 1975, 251 pp, $10.80 (P). Papers, some of them updated, from the Boston Logic Colloquium of 1972-1973. JAS

FOUNDATIONS, P. *The Logical Enterprise.* Ed: Alan Ross Anderson, Ruth Barcan Marcus, R.M. Martin. Yale U Pr, 1975, x + 261 pp, $15. A collection of papers on mathematical logic, philosophical logic and ontology in honor of the logician and philosopher Frederic B. Fitch. LAS

FOUNDATIONS, P. *Proceedings of the Third Scandinavian Logic Symposium.* Ed: Stig Kanger. Stud. in Logic and Found. of Math., V. 82. North-Holland, 1975, v + 214 pp, $24.95. Papers from Uppsala, April 1973. LAS

COMBINATORICS, P. *Graph Theory, Coding Theory and Block Designs.* P.J. Cameron, J.H. van Lint. London Math. Soc. Lect. Notes, No. 19. Cambridge U Pr, 1975, v + 114 pp, $7.95 (P). Purpose of the book is "to present to an audience already familiar with the theory of designs some of the connections and applications with other branches of mathematics." Includes, however, a very short introduction to design theory. PJM

NUMBER THEORY, S(13), L. *Caractères de Divisibilité: Suite de Fibonacci.* N. Vorobiev. MIR, 1973, 214 pp. An elementary introduction to characters of divisibility and Fibonacci sequences translated from Russian to French. A very interesting exposition with lots of good problems. CEC

NUMBER THEORY, S(13), L. *Number Systems.* S.V. Fomin. Trans: Joan W. Teller, Thomas P. Branson. M.I.T. Press Pr, 1974, vii + 39 pp, $2.50 (P). A brief study of different number bases and their applications. Suitable for the mathematical novice. CEC

NUMBER THEORY, L?. *Prime Period Lengths.* Samuel Yates (104 Brentwood Drive, Mt. Laurel, N.J. 08057), 525 pp, $10 (P). The period length of a prime, p, is the number of digits in the period of the decimal representation of a fraction whose denominator is p. This book is a table which gives the period lengths of all primes up to 1,370,471. A must for the number mystic who has everything.CEC

LINEAR ALGEBRA, T(14-16: 1), S, L. *Computational Methods in Linear Algebra.* R.J. Goult, *et al.* Halsted Pr, 1974, vii + 204 pp, $11.95. Solves matrix problems and sets of linear equations by techniques appropriate for use on a computer. Solution of linear equations by elimination and decomposition and by iteration. Computation of eigenvalues and eigenvectors. Appendix on essential concepts from linear algebra. Exercises. References. Index. RJA

ALGEBRA, T(18: 1), P. *Lecture Notes in Mathematics-443: Commutative Formal Groups.* Michel Lazard. Springer-Verlag, 1975, 236 pp, $9.90 (P). A systematic development of the author's notes on the fundamental theorem of formal commutative groups, along with discussions of formal groups in characteristic p, and the extending and lifting of some formal groups. CEC

ALGEBRA, P. *Lecture Notes in Mathematics--450: Algebra and Logic.* Ed: J.N. Crossley. Springer-Verlag, 1975, viii + 307 pp, $12.10 (P). Papers from the 1974 Summer Research Institute of the Australian Mathematical Society held at Monash University in Clayton, Victoria, Australia (N.B.: summer means January and February). JAS

ALGEBRA, T(18: 1), P. *Grundbegriffe der Kategorientheorie.* Gerhard Preuss. Hochschultaschenbücher, B. 739. Bibliographisches Inst, 1975, 105 pp, (P). A short, general (non-additive) solid introduction to categories for mature students. Examples come from the usual sources and demand considerable background. JAS

ALGEBRA, T(17-18), P. *Maximal Orders.* I. Reiner. Acad Pr, 1975, xii + 395 pp, $36.50. A superb book, well-suited for a second graduate course in algebra. Topics: extensive review of basic algebra; maximal order in skew fields over discrete valuation rings, over Dedekind domains; Morita equivalence, crossed-product algebras, simple algebras, hereditary orders. Most distinguishing features are the many, fine exercises and the excellent mathematical exposition. SG

ALGEBRA, P. *Lecture Notes in Mathematics-462: Construction de Séries Discrètes p-adiques.* Paul Gérardin. Springer-Verlag, 1975, 180 pp, $8.60 (P). Extension of results of Harish-Chandra on algebraic groups over R to certain algebraic groups over p-adic fields. Topics: Heisenberg groups over finite fields; Chevalley groups over p-adic fields; maximal tori, representations. SG

ALGEBRA, P*. *Lecture Notes in Mathematics-445: Model Theory and Topoi.* Ed: F.W. Lawvere, C. Maurer, G.C. Wraith. Springer-Verlag, 1975, 354 pp, $12.90 (P). A collection of lectures by various authors including a concise, excellent introduction to topoi by Wraith. PJM

ALGEBRA, P. *Lecture Notes in Mathematics-455: Kommutative algebraische Gruppen und Ringe.* Hanspeter Kraft. Springer-Verlag, 1975, 163 pp, $8.60 (P).

ALGEBRA, P. *Lecture Notes in Mathematics-460: Jordan Pairs.* Ottmar Loos. Springer-Verlag, 1975, xvi + 218 pp, $9.90 (P). The theory of Jordan pairs provides a unifying framework for both the theory of Jordan algebras and Jordan triple systems. A bibliography is included. CEC

ALGEBRA, S, P. *Einführung in die Transfinite Algebra.* Andor Kertész. Birkhauser, 1975, 74 pp, Sfr. 24 (P). An interesting and carefully written book, intended for students but containing some results not known to all professional mathematicians, illustrating the use of transfinite methods (chiefly Zorn's Lemma) in algebra. JD-B

CALCULUS, T(13-14; 3). *Mathematical Analysis: A Brief Course for Engineering Students.* A.F. Bermant, I.G. Aramanovich. Trans: V.M. Volosov, I.G. Volosova. MIR, 1975, 781 pp. Elementary calculus. Does Fourier series and integrals but not linear algebra. Many worked-out examples and "review questions" but no exercises. Some English usage is outdated ("Definition. A function $\alpha(x)$ tending to zero as $x \to x_0$ is called an *infinitesimal* as $x \to x_0$."). DFA

COMPLEX ANALYSIS, P. *Analytic Functions and Manifolds in Infinite Dimensional Spaces.* G. Coeuré. Math. Stud., V. 11. North-Holland, 1974, 85 pp, $7.70 (P). Provides a systematic approach for analytic continuation of analytic mappings in infinitely many variables. Develops the properties of the simultaneous continuation of some natural Fréchet spaces of analytic mappings. I-CH

DIFFERENTIAL EQUATIONS, P. *Complexes of Partial Differential Operators.* Aldo Andreotti. Math. Mono., No. 6. Yale U Pr, 1975, 49 pp, $5.95 (P). This study extends some classical results of the theory of Cauchy-Riemann equations to general complexes of linear partial differential operators. Boundary value problems of the Dirichlet and Cauchy type are formulated in terms of cohomology theory. A bibliography is included. CEC

DIFFERENTIAL EQUATIONS, T(17-18), P. *Stochastic Differential Equations and Applications, V. 1.* Avner Friedman. Prob. and Math. Stat., V. 28. Acad Pr, 1975, xiii + 231 pp, $24.50. Study of systems of stochastic differential equations and applications to probability, partial differential equations and stochastic control problems. (Volume 1 is devoted to basic theory, and some applications; Volume 2 will be devoted entirely to applications.) Self-contained (modulo elementary probability); well-written; several exercises included. SG

DIFFERENTIAL EQUATIONS, T*(17-18; 2), S*, P. *Basic Linear Partial Differential Equations.* Francois Treves. Pure and Appl. Math., V. 62. Acad Pr, 1975, xvii + 470 pp, $29.50. Exploits classical results (the three basic types of PDE's, their Dirichlet and Cauchy problems, etc.) using non-traditional methods, at a level of abstraction appearing in modern research. Fourier transformation of distributions is used systematically. Truly introductory, it nonetheless touches on various important topics such as potential theory, Dirac equations, random walks, continuous semigroups of operators. An excellent book for modern analysts who appreciate rich roots in the past. I-CH

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-448: Spectral Theory and Differential Equations.* Ed: W.N. Everitt. Springer-Verlag, 1975, xii + 321 pp, $12.90 (P). Proceedings of a symposium held at Dundee, Scotland in July 1974. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-451: Probabilistic Methods in Differential Equations.* Ed: M.A. Pinsky. Springer-Verlag, 1975, vii + 162 pp, $8.60 (P). Proceedings of the conference held at the University of Victoria in August 1974. JAS

NUMERICAL ANALYSIS, T*(16-17; 2), S*. *Methods of Numerical Integration.* Philip J. Davis, Philip Rabinowitz. Comp. Sci. and Appl. Math. Acad Pr, 1975, xii + 459 pp, $34.50. Presents the major methods of approximate integration (over finite and infinite intervals, and in higher dimensions), error analysis and automatic integration. Most topics are well motivated, clearly explained and efficiently referenced by citations immediately after each section. Very informative appendices: FORTRAN programs, bibliography of tables, of ALGOL, FORTRAN, and PL/1 procedures. I-CH

NUMERICAL ANALYSIS, P*. *Mathematical Functions and their Approximations.* Yudell L. Luke. Acad Pr, 1975, xvii + 568 pp, $14.50. Can be viewed as an updated supplement to the immensely popular *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* (Bureau of Standards, 1964). Main thrusts of the present volume are of an analytic character (e.g., expansions and approximations of functions for universal use) as opposed to numerical curve fits such as "best" Chebyshev polynomial and rational approximations. Contains virtually no proofs, but extensive references and an immense up-to-date bibliography. I-CH

FUNCTIONAL ANALYSIS, T(18: 2), S, P. *Integration in Hilbert Space*. A.V. Skorohod. Ergebnisse der Math., B. 79. Springer-Verlag, 1974, xii + 177 pp, $19.70. Familiar chapters (measures, measurable functions, absolute continuity, etc.) yet studied in infinite-dimensional Hilbert spaces where many interesting unsolved problems exist. Largely an account of the author's research that provided the measure and integration theory in Hilbert spaces with a rigorous foundation. I-CH

FUNCTIONAL ANALYSIS, T(18: 1), S, P*. *Calkin Algebras and Algebras of Operators on Banach Spaces*. S.R. Caradus, W.E. Pfaffenberger, Bertram Yood. Lect. Notes in Pure and Appl. Math., V. 9. Dekker, 1974, viii + 146 pp, $13.75 (P). Introduction to the connection between the theory of Banach algebras and the space B(X) of bounded linear operators on a Banach space containing, e.g., the "remarkable result" of Gohberg, Markus and Fel'dman (1960) about closed ideals in B(X). Some of the material on Riesz-Schauder operators appears here for the first time. I-CH

OPTIMIZATION, S(17-18), P. *Numerical Methods for Constrained Optimization*. Ed: P.E. Gill, W. Murray. Acad Pr, 1974, xiv + 283 pp, $23.25. Based on the proceedings of a 1974 symposium of the same title held at the National Physical Laboratory. Good introduction and survey of methods used on constrained optimization problems. References. Author and subject indices. RJA

OPTIMIZATION, T*(16-17: 2), S*, L*. *Optimization Problems*. L. Collatz, W. Wetterling. Trans: P. Wadsack. Appl. Math. Sci., V. 17. Springer-Verlag, 1975, x + 356 pp, $14.50 (P). Provides an overview of optimization problems and relates them to graph theory, approximation problems, and game theory. Formulates convex and non-convex optimization for problems in physics. Contains a first-rate coverage of the simplex method and its algorithmic implementations, including an illustration of the duality principle in statics (due to W. Prager, 1962). The original German edition first appeared in 1966. I-CH

ANALYSIS, S(17-18), P, L. *Some Modern Mathematics for Physicists and Other Outsiders*. Paul Roman. Pergamon Pr, 1975. *V. 1: An Introduction to Algebra, Topology, and Functional Analysis*, xxv + 398 pp, $17.50 (P), $25; *V. 2: Functional Anlaysis with Applications*, xxii + 322 pp, $17.50 (P), $25. Volume 1 covers algebra (groups, rings, fields, vector spaces and Lie algebras), topology (metric spaces, connectivity and compactness), and measure theory (Lebesgue-Stieltjes integration). All of the above is applied in Volume 2 to do functional analysis up to spectral theory. Position and momentum operators appear as applications to quantum theory (although "quantum mechanics" is absent from the index). Very concise, good exercises, good bibliography. An ambitious one-year course, but possible with good students (at the graduate level). PJM

ANALYSIS, P. *Sidon Sets*. Jorge M. López, Kenneth A. Ross. Lect. Notes in Pure and Appl. Math., V. 13. Dekker, 1975, v + 193 pp, $14.50 (P). In classical analysis, a lacunary set is the set of points on the boundary of the region of convergence where a power series converges. This monograph studies a generalization of lacunary sets called Sidon sets. PJM

ANALYSIS, P. *Lie Groups and Their Representations: Summer School of the Bolyai János Mathematical Society*. Ed: I.M. Gelfand. Halsted Pr, 1975, 726 pp, $79. Advanced papers from the 1971 summer school in Budapest. Introductory lectures will be published separately. LAS

ANALYSIS, P. *Algebras in Analysis*. Ed: J.H. Williamson. Acad Pr, 1975, xi + 312 pp, $28.50. Proceedings of a NATO "instructional" institute in Birmingham, September 1973. Six lengthy expository papers plus some research abstracts. LAS

GEOMETRY, T(13), S*, L. *Analytic Geometry, A Refresher*. Karl J. Smith. Brooks/Cole, 1975, xi + 131 pp, $4.95 (P). A standard course in analytic geometry with lots of good drawings and problems. Includes cartoons. CEC

GEOMETRY, P. *Lecture Notes in Mathematics-473: Groups of Automorphisms of Manifolds*. Dan Burghelea, Richard Lashof, Melvin Rothenberg. Springer-Verlag, 1975, vii + 156 pp, $7.80 (P).

GEOMETRY, P. *Comparison Theorems in Riemannian Geometry*. Jeff Cheeger, David G. Ebin. Math. Lib., V. 9. North-Holland, 1975, viii + 174 pp, $20.95. Idea is to compare the geometry of a complete Riemannian manifold with that of a simply connected space of constant curvature. Under added assumptions about sectional curvature, topological properties of the original space can be derived. Topics include: homogeneous spaces, Morse theory, sphere theorems, complete manifolds of non-negative curvature. SG

GEOMETRY, P. *Differential Geometry*. Ed: S.S. Chern, R. Osserman. Proc. of Symp. in Pure Math., V. XXVII., AMS, 1975. *Part 1:* x + 451 pp; *Part 2:* x + 443 pp, $55.40. Proceedings of the 1973 AMS summer research institute at Stanford. Covers Riemannian geometry, submanifolds, foliations, algebraic and piecewise linear topology (in Part 1); complex differential geometry, partial differential equations, homogeneous spaces and relativity (in Part 2). Author and subject indices in each volume. LAS

GEOMETRY, S(17-18), P. *Projektive Ebenen*. Günter Pickert. Grund. math. Wissenschaften, B. 80. Springer-Verlag, 1975, ix + 371 pp, $42.20. The second edition of a standard work, first published in 1955. No attempt has been made to bring the book up to date, but there are a few additions and corrections to the text and four new appendices, one of them to the bibliography. JD-B

GEOMETRY, P. *Manifolds-Tokyo 1973*. Ed: Akio Hattori. U of Tokyo Pr, 1975, x + 433 pp, $40. Proceedings of the international conference at Tokyo in April 1973. Includes a substantial list of open problems. JAS

TOPOLOGY, P. *Knots, Groups, and 3-Manifolds: Ppaers Dedicated to the Memory of R.H. Fox*. Ed: L.P. Neuwirth. Annals of Math. Stud., No. 84. Princeton U Pr, 1975, xi + 335 pp, $17.50. This collection of papers contributed by friends and colleagues includes a complete bibliography of R.H. Fox's published papers. JAS

TOPOLOGY, T(17-18: 2), P. *Homotopy Theory: An Introduction to Algebraic Topology.* Brayton Gray. Pure and Appl. Math., V. 64. Acad Pr, 1975, xiii + 368 pp, $22. A presentation from a new and different viewpoint. Homotopy theory is used on the category of compactly generated spaces as a tool in itself and to give homology and cohomology theories via spectra. But, probably for the first time, these ideas are introduced in a basic text as first principles. The list of references would be enhanced if it led better to some of the roots of these branches of algebraic topology. JAS

TOPOLOGY, T*(13-14), S, L. *What is Distance?* Yu. A. Shreider. Trans: Leslie Cohn, Harvey Edelberg. U of Chicago Pr, 1974, ix + 71 pp, $2.95 (P). Starting with an intuition of distance, proceeding through a discussion of properties of the Euclidean metric, the author develops the notion of an abstract metric space and applies it to information theory and generalizing distance to higher dimensional spaces. An excellent book, a good text for an interim (3-4 week) course. PJM

TOPOLOGY, T*(16-17), S, P, L. *An Introduction to Differentiable Manifolds and Riemannian Geometry.* William M. Boothby. Pure and Appl. Math., V. 63. Acad Pr, 1975, xiv + 424 pp, $27.50. This is an excellent text for a first course in differential topology. Includes a review of the relevant multivariable analysis, vector and tensor fields, integration and differentiation, and curvature. Plentiful exercises, notes for further reading after each chapter, good bibliography. Prerequisites: topology, linear algebra and advanced (multivariable) calculus. Probably suitable only for very good undergraduates, but perfect for a first year graduate course other than the usual ones. PJM

TOPOLOGY, P. *Lecture Notes in Mathematics-469: Continuous Convergence on C(X).* Ernst Binz. Springer-Verlag, 1975, ix + 140 pp, $7.80 (P). A continuous convergence structure defined on C(X) yields a complete convergence algebra $C_c(X)$. Properties of $C_c(X)$ are studied. Bibliography. Index. RJA

TOPOLOGY, P. *On the Groups JO(G).* Chung-Nim Lee, Arthur G. Wasserman. Memoirs No. 159. AMS, 1975, ii + 62 pp, $3.20 (P). A study of equivalence classes of representations of a compact Lie group. PJM

TOPOLOGY, T(18), P. *Hewitt-Nachbin Spaces.* Maurice D. Weir. Math. Stud., V. 17 (Notas de Matematica (57)). North-Holland, 1975, vii + 270 pp, $15.50 (P). A compact space can be embedded as a subspace of a product of unit intervals. A Hewitt-Nachbin (real compact) space can be imbedded in a product of copies of the real line. A text for a second, graduate level, course in general topology. PJM

TOPOLOGY, P. *Continuous Pseudometrics.* W.W. Comfort, S. Negrepontis. Lect. Notes in Pure and Appl. Math., V. 14. Dekker, 1975, v + 126 pp, $12.75 (P). A curious, expensive book. Curious: advanced general topology, structure of compactification, basic sets, but no explicit mention of pseudometrics outside of the introduction. Expensive: 10¢ a page for an offset of typescript. PJM

TOPOLOGY, S(18), P*. *Applications of Algebraic Topology: Graphs and Networks, The Picard Lefschetz Theory and Feynman Integrals.* S. Lefschetz. Appl. Math. Sci., V. 16. Springer-Verlag, 1975, viii + 189 pp, $9.50 (P). This book, written shortly before the author's death and published posthumously, is far more applicable topology than a handbook of electrical networks (the principal application).JAS

TOPOLOGY, T(18: 1, 2), P, L. *Fibre Bundles, Second Edition.* Dale Husemoller. Grad. Texts in Math., V. 20. Springer-Verlag, 1975, xv + 327 pp, $16.80. Corrected and enlarged edition (with a chapter on the Adams conjecture and an appendix on the suspension theorems) of the 1966 original. It is good to see this handy source of so much material renewed. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-438: Geometric Topology.* Ed: L.C. Glaser, T.B. Rushing. Springer-Verlag, 1975, x + 459 pp, $16 (P). Papers from a conference at Park City, Utah, February 1974. LAS

STATISTICS, T*(13: 1). *STATLAB. An Empirical Introduction to Statistics.* J.L. Hodges, Jr., David Krech, Richard S. Crutchfield. McGraw, 1975, vi + 378 pp, $11.95 (P). In the McGraw Hill Series in Probability and Statistics. Novel approach. All student assignments involve analysis of actual sample data obtained (using dice) from the "StatLab Population", a collection of 1296 ($=6^4$) real families on each of which 32 measures were made (medical, socioeconomic, physical and physiological). Also contains several interesting historical sidelights. RSK

STATISTICS, P*. *SPSS: Statistical Package for the Social Sciences, Second Edition.* Norman H. Nie, *et al.* McGraw, 1975, xxiv + 675 pp, $10.95 (P). A complete instructional guide to SPSS, "an integrated system of computer programs designed for the analysis of social science data." Contains "almost double the amount of statistical procedures and data-management facilities as were documented in the first (1970) edition." Currently operational only in a batch system, but a conversational version is being developed. RSK

STATISTICS, T*(15-16: 1, 2). *Probability and Statistics.* Morris H. DeGroot. A-W, 1975, xiv + 607 pp, $15.95. A sound introduction for students who have had calculus. Considers both classical and Bayesian methods. Makes good use of elementary linear algebra in dealing with linear models. FLW

STATISTICS, P. *Proceedings of the Twentieth Conference on the Design of Experiments.* US Army Research Office, Durham, N.C., 1975. *Part 1:* xviii + 462 pp; *Part 2:* viii + 522 pp, (P). Papers from a conference at Fort Belvoir, October 1974. LAS

COMPUTER SCIENCE, S(17-18), P. *On Programming: An Interim Report on the SETL Project, Revised June 1975.* Jacob T. Schwartz. *Installment I: Generalities; Installment II: The SETL Language and Examples of Its Use,* Courant Inst, 1975, xii + 675 pp, $17.25 (P). Second of three volumes on a new programming language, SETL, based on mathematical set theory. This volume contains descriptions of the SETL language definition, code optimization algorithms, and implementation of set-theoretic primitives provided by SETL. Bibliography. Index. RJA

COMPUTER SCIENCE, T(13: 1), S. *Basic Programming in Real Time.* Don Cassel. Reston, 1975, ix + 190 pp, $9.95; $7.95 (P). Written mainly for BASIC programming on a real time system, yet flexible enough that most of the material is still applicable on a batch BASIC system. The theme is essentially business applications for students with no prior computer experience. I-CH

COMPUTER SCIENCE, T(15-17: 1), L. *Operating Systems.* Dionysios C. Tsichritzis, Philip A. Bernstein. Comp. Sci. and Appl. Math. Acad Pr, 1974, xviii + 298 pp, $14. Presents fundamental concepts and principles governing operating systems. Includes processes, processor allocation, memory management, I/O, files, security, protection, design, implementation and examples of various operating systems. References. Index. RJA

COMPUTER SCIENCE, S(17-18), P. *Foundations of Computer Science.* Ed: J.W. de Bakker. Math. Centre Tracts, No. 63. Math Centrum, 1975, i + 215 pp, Dfl. 23 (P). A collection of the lecture notes of five of the courses given at the University of Amsterdam, May 20-31, 1974 on the Foundations of Computer Science. Some references are provided. RJA

COMPUTER SCIENCE, T(13: 1), *Practical Fortran: An Applied and Simplified, Problem-Solving Approach.* Harold S. Engelsohn. Macmillan, 1975, xii + 434 pp, $8.95 (P). A slowly-paced, wordy introduction to Fortran. Numerous elementary examples. Not precise. The approach is to teach by programming examples. RWN

COMPUTER SCIENCE, S, P. *Rechnernetze.* Elmar Holler, Oswald Drobnik. Bibliographisches Inst, 1975, 195 pp, (P). A discussion of computer networks, of interest chiefly to specialists. Treats, among other subjects, communications, incompatibility, and the organization and management of processing and data. Describes three networks now in operation. JD-B

COMPUTER SCIENCE, T(13-16: 1), S, L. *Computer Mathematics.* Clifford L. Conrad, Nancy J. Conrad, Harry B. Higley. Hayden, 1975, v + 210 pp, $13.95. Uses three formal languages to present the mathematics necessary for computer technology students: (1) "Numeric Calculus", to perform arithmetic operations; (2) "Logic Calculus", to discuss list and formula functions; and (3) "Array Calculus", to perform array operations. Numerical operations in different number systems. Fixed-point and floating-point operations. Base conversion and modular arithmetic. Operations on arrays. Exercises. Index. RJA

COMPUTER SCIENCE, P, L. *Computer Chess.* Monroe Newborn. Acad Pr, 1975, xii + 200 pp, $15. History of serious chess programs (from Shannon's seminal 1950 paper to the 1974 world computer chess tournament in Stockholm) related to general theories of artificial intelligence via detailed discussion of minimax algorithms, tree pruning and heuristics. A fascinating sourcebook for both computer and chess buffs, although it reveals no secrets about details of successful programs. LAS

APPLICATIONS (INFORMATION THEORY), S(18), P. *Lecture Notes in Mathematics-376: Information, Weight of Evidence, the Singularity between Probability Measures and Signal Detection.* David Bridston Osteyee, Irving John Good. Springer-Verlag, 1974, xi + 156 pp, $6.60 (P). Expected mutual information, weight of evidence related to singularity between probability measures and errorless discrimination between random processes. Concepts expressed as functionals and integral operators to include generalized Gaussian processes. LH

APPLICATIONS (PHYSICS), T(14: 1), *Vector Field Theory with Applications.* L. Sowerby. Longman, 1974, 252 pp, $7.50 (P). Vector calculus with applications to electromagnetic theory and hydrodynamics. Integral theorems are derived using examples and diagrams. Gradient, divergence, and curl are discussed in separate chapters. Treatment is classical, and not unified with notion of exterior derivative. RSK

APPLICATIONS (PHYSICS), S(16-18), *Practical Quantum Mechanics.* Siegfried Flügge. Springer-Verlag, 1974, xvi + 287 pp, $14.80 (P). Unaltered reprint in one volume of the 1971 two volume translation (TR, April 1974) of the 1947 German original. Consists exclusively of 219 carefully solved problems plus a mathematical appendix. LAS

APPLICATIONS (SOCIOLOGY), T(15-17: 1), L. *Mathematical Sociology.* Robert K. Leik, B.F. Meeker. P-H, 1975, xii + 242 pp, $12.95. A survey of mathematical models in sociology--including models of status, small groups, kinship, occupational mobility, value conflict--presuming calculus, linear algebra, probability and statistics. Introduces such modelling tools as computer simulation, graphs, Markov processes and partial differential equations. Problems conclude each chapter. Serious omission: no references or bibliography. LAS

APPLICATIONS (SOCIAL SCIENCE), S(13-18), P, L. *Social Information Processing and Statistical Systems--Change and Reform.* Edgar S. Dunn, Jr. Wiley, 1974, x + 246 pp, $12.50. Problems in design and processing of social data--e.g., adequate symbolic representation of "poverty." Sketch of a general theory, specific issues of federal statistical reform, individual privacy versus social control. LH

APPLICATIONS (TRANSPORTATION), T(16-17: 2), P. *Highway Traffic Analysis and Design.* R.J. Salter. A-W, 1974, vi + 378 pp, $17.50. Covers the fundamental principles of land use, transport planning techniques and the subsequent economic evaluation of highway schemes. Reviews the analytical and practical aspects of highway traffic flows. Treats in detail the traffic signal control, ranging from isolated signal-controlled intersections to area-wide signal control. I-CH

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; Frank L. Wolf, Carleton.

## NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

### PERSONAL ITEMS

*Oklahoma State University*: Associate Professor Shair Ahmad has been promoted to Professor; Assistant Professors J. R. Choike, M. S. Keener, and J. W. Maxwell have been promoted to Associate Professors.

*University of Hawaii*: Mr. L. T. Ramsey, Louisiana State University, has been appointed Acting Assistant Professor; Dr. L. C. Wilson, Brandeis University, has been appointed Assistant Professor; Assistant Professors R. S. Freese, T. B. Hoover, Dorothy I. Koehler, D. W. Myers, and J. K. M. Siu have been promoted to Associate Professors; Associate Professor H. M. Hilden has been promoted to Professor.

*Virginia Polytechnic Institute and State University*: Assistant Professor Daniel Anderson, University of Iowa, has been appointed Assistant Professor; Dr. Daniel Farkas, Brandeis University, has been appointed Assistant Professor; Dr. Robert Olin, Indiana University, has been appointed Assistant Professor; Assistant Professor J. K. Shaw has been promoted to Associate Professor; Associate Professor J. R. Holub has been promoted to Professor.

Assistant Professor J. T. Annulis, University of Arkansas at Monticello, has been promoted to Associate Professor.

Dr. E. D. Gibson, University of Illinois, has been appointed Associate Professor at Southern University.

Assistant Professor Don Hill, Florida A & M University, has been promoted to Associate Professor.

Associate Professor J. E. Kuczkowski, Indiana University — Purdue University at Indianapolis, has been promoted to Professor.

Assistant Professor C. A. Loch, Duquesne University, has been promoted to Associate Professor.

Professor A. M. White, Harvey Mudd College, has been awarded a Danforth Faculty Fellowship as the nominee of the Claremont Colleges.

Associate Professor K. S. Williams, Carleton University, Ottawa, Canada, has been promoted to Professor.

Dr. Kenneth W. Anderson, SUNY at Binghamton, died on June 13, 1975. He was a member of the Association for thirteen years.

Dr. Anthony F. Bartholomay, Somerset, New Jersey, died on March 21, 1975, at the age of 55. He was a member of the Association for five years.

Professor Emeritus Moffatt Grier Boyce, Vanderbilt University, died on July 10, 1975, at the age of 75. He was a member of the Association for fifty-one years.

Professor Dan E. Christie, Bowdoin College, died on July 18, 1975, at the age of 59. He was a member of the Association for thirty-three years.

Professor Emeritus H. B. Curtis, Lake Forest College, died on March 3, 1975, at the age of 90. He was a member of the Association for fifty-two years.

Professor Richard F. DeMar, University of Cincinnati, died on February 11, 1975, at the age of 50. He was a member of the Association for fifteen years.

Professor Walter Jennings, Naval Postgraduate School, died on July 10, 1975, at the age of 64. He was a member of the Association for thirty-seven years.

Professor Harry Lighthall, Jr., University of Vermont, died on January 11, 1975, at the age of 51. He was a member of the Association for eighteen years.

# An Invitation To Participate*

*Change* Magazine, under a grant from the Fund for the Improvement of Postsecondary Education, has launched a program of undergraduate teaching recognition. The purpose of the program is to identify, with the cooperation of the disciplinary associations, examples of some of the best efforts to improve undergraduate teaching. Faculty in each discipline so identified will receive national recognition through publication of a detailed description of their work in a series of reports on teaching.

In order to participate, faculty members must be willing to share their experiences, through written material and in person, with other teachers, with students, and with outside evaluators. The learning experience developed by the teacher must be acceptable to a panel of disciplinary association staff and outside evaluators well known in the discipline. The experience should be adaptable to other institutions and if possible to other disciplines.

The examples of improved teaching will be selected on the basis of the criteria listed below. Improved teaching is defined *relatively*, rather than absolutely in the sense of "good" or "best" teaching, or different in the sense of "innovative" teaching. Although the focus is on outcome, it will not be essential that the learning experience meet all of the ultimate criteria. It will be expected, however, that most of the intermediate and some of the ultimate criteria will be met, either through measurement (empirical evidence) or through assessment (consensus of judgment). The term "learning experience" is used here instead of course, activity, or some other designation in order to be open to the widest possible range of settings and events.

### I. Ultimate Criteria

Improved teaching is defined by the extent to which a particular experience meets these criteria better than the typical experience with the same purpose:

The students learned what the instructor was trying to teach
- -in cognitive, affective, and/or psychomotor development
- -in rate and/or absolute achievement

Learning continued after the formal experience ended
- -rate and/or extent of later learning known

Students retained what was learned

Teacher goals and/or outcomes for the learning experiences were met

Student goals and/or outcomes for the learning experience were met

The learning experience related to other learning experiences students might have had (congruence, continuity, sequence)
- -prior learning was capitalized upon
- -learning increased in other formal experiences the students may have had at the same time
- -learning improved in the rest of a sequential series the students may have had afterward

The learning experience positively affected the attitudes and/or behavior of other faculty members and of administrators within the immediate learning community

The learning experience cost less than traditional ways of delivering the same learning experience
- -the same learning for less money
- -more learning for the same money
- -more learning for less money

Enrollment levels were sustained or increased in subsequent offerings of the learning experience

### II. Intermediate Criteria

Improved teaching is defined by the extent to which a teacher has influenced or provided the following conditions for learning better than the typical experience with the same purpose:

Students were motivated to learn

The structure of the learning experience was determined by the goals of the experience

The content was well ordered, comprehensive, and appropriate to the abilities of the learners

Teacher involvement in the learning experience was in harmony with the goals of the experience

Time was provided for students to contemplate and respond

Rewards and sanctions were appropriate to the goals of the learning experience

Students understood what they were doing, why they were doing it, and how they would be evaluated in the learning experience

Goals and/or outcomes were clearly specified at the outset

Evaluation criteria, standards, and methodologies were clear and appropriate to the goals of the experience

Student products reflected the goals of the learning experience

The kind and variety of instructional resources were congruent with the goals of the experience and abilities of the learners

Methodology was appropriate to the goals of the experience and the abilities of the learners

* This announcement is reprinted from the November issue of CHANGE magazine. Faculty members interested in submitting applications for consideration should send a brief description of their work, specifying how it meets the criteria listed above, to the MAA Washington Office by March 1. These descriptions will be forwarded to the Director of the CHANGE Undergraduate Teaching Program. Final selections for publication will be made by CHANGE.

## MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifty-second meeting of the Louisiana-Mississippi Section was held at Centenary College, Shreveport, Louisiana, on February 21–22, 1975. Chairman Russell Whittington presided over the meeting.

The invited speaker for the Friday session was Knox Millsaps. Professor Millsaps' afternoon address was entitled "What is Applied Mathematics?" His address delivered at the Friday evening banquet was "Theodore von Karman, an Engineer for All Seasons."

During the business meeting on February 22, the following officers were elected: Chairman, Carol Ottinger, Mississippi State College for Women; Mississippi Vice-Chairman, Ray Wilson; Louisiana Vice-Chairman, James Keisler; Secretary-Treasurer, Joe Foote.

The following papers were presented in sessions Friday afternoon and Saturday morning:

1. *Algorithmically defined functions*, by R. D. Anderson, Louisiana State University, Baton Rouge.
2. *Recent advances in classification of bundle representations of $C^*$-algebras*, by M. J. Dupré, Tulane University.
3. *Proximity in probabilistic metric spaces*, by R. T. Fritsche, Northeast Louisiana University.
4. *Another race between the tortoise and the hare*, by J. R. Dorroh, Louisiana State University, Baton Rouge.
5. *A numerical filtering technique*, by Edward Anders, Northwestern State University.
6. *Some continua can be the boundary cluster sets*, by Stan Chadick, Northwestern State University.
7. *Approximate solutions of the 5th order Runge-Kutta equations*, by G. J. Landry, Nicholls State University.
8. *Regular $\Sigma$-trees*, by Margaret M. LaSalle, University of Southwestern Louisiana.
9. *Principles of graph theory applied to eigenvector-eigenvalue problems*, by Alton Crawley and Karen Jehn, Northeast Louisiana University.
10. *Freeness in valued vector spaces*, by David Thomas, Centenary College.
11. *Semigroup multiplications*, by P. A. Grillet, Tulane University.
12. *Multitopological spaces*, by H. W. Brewer, McNeese State University.
13. *Linear programming: The simplex method and applications to game theory*, by B. E. Mitchell, Louisiana State University, Baton Rouge.
14. *An interesting area result*, by Sam Knox, Millsaps College.
15. *Functions satisfying a u-Lipschitz condition on the interval $[a, b]$*, by Katheen A. Drude, Northeast Louisiana University.
16. *Oscillatory criteria for certain n-th order functional differential equations*, by J. B. Garner, Louisiana Tech University.
17. *Some symmetries in the Wilcoxon and Mann-Whitney tests*, by Joe Albree, Northwestern State University.

J. R. FOOTE, *Secretary-Treasurer*

## CALENDAR OF FUTURE MEETINGS

Fifty-ninth Annual Meeting, San Antonio, Texas, January 24–26, 1976.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A&M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Franklin College, Franklin, May 1, 1976.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13–14, 1976.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Bowie State College, Bowie, Maryland, April 24, 1976.

METROPOLITAN NEW YORK, Hofstra University, Hempstead, N.Y., April 25, 1976.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY, Montclair State College, Upper Montclair, May 8, 1976.

NORTH CENTRAL, St. Cloud State University, St. Cloud, Minnesota, April 30–May 1, 1976.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, University of California, Davis, February 21, 1976.

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OKLAHOMA-ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, Fort Lewis College, Durango, Colorado, April 30–May 1, 1976.

SEAWAY, College of St. Rose, Albany, N. Y., April 30–May 1, 1976.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, N. Carolina, March 26–27, 1976.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Texas A&M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), April or May 1976.

## FUTURE MEETINGS OF OTHER ORGANISATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 22–25, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, University of Nevada, Reno, April 23–24, 1976.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Antonio, Texas, January 22–23, 1976.

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MU ALPHA THETA, West Chester State College, West Chester, Pennsylvania, August 8–11, 1976..

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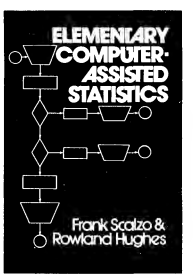
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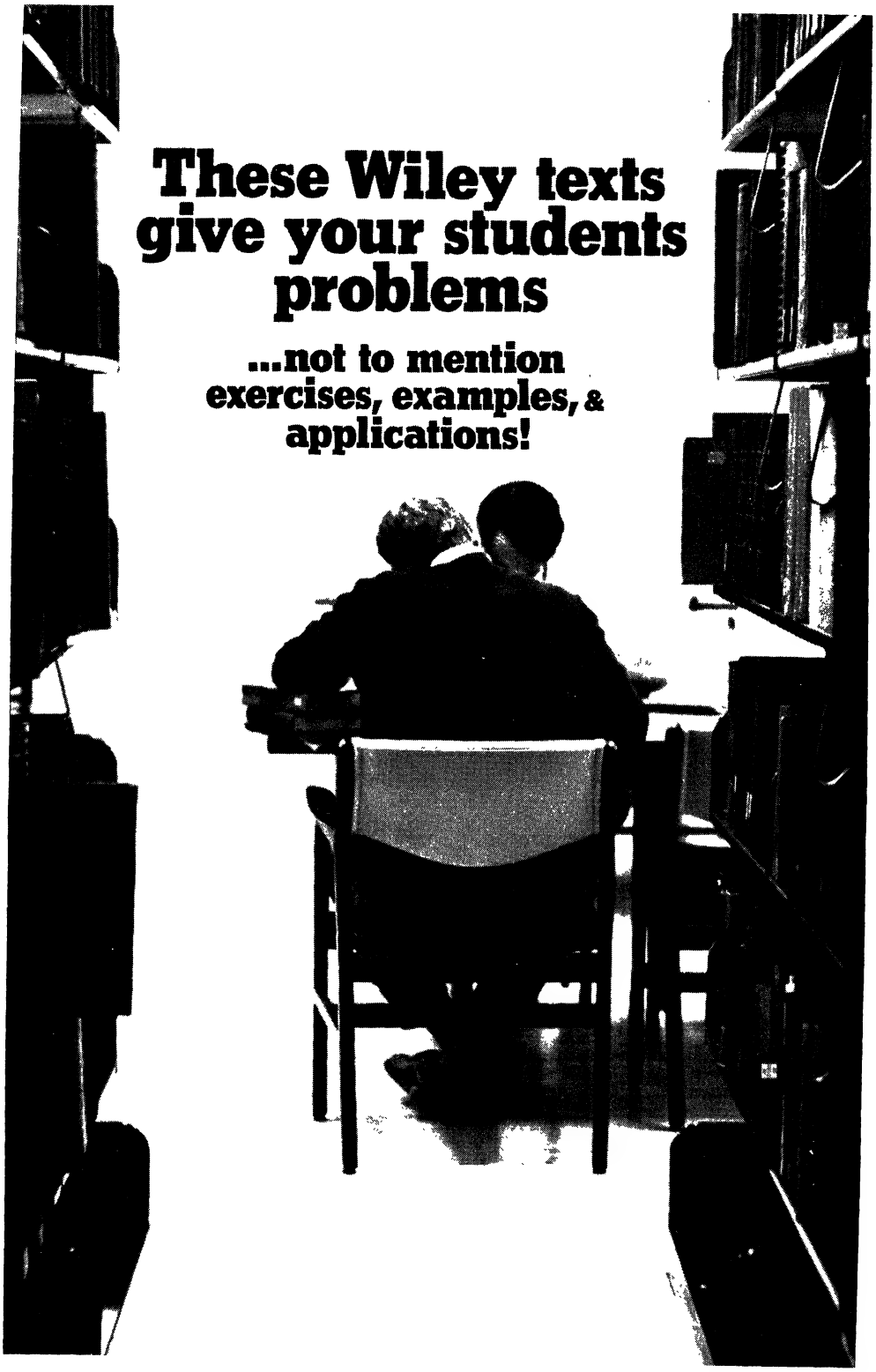
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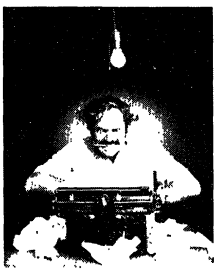
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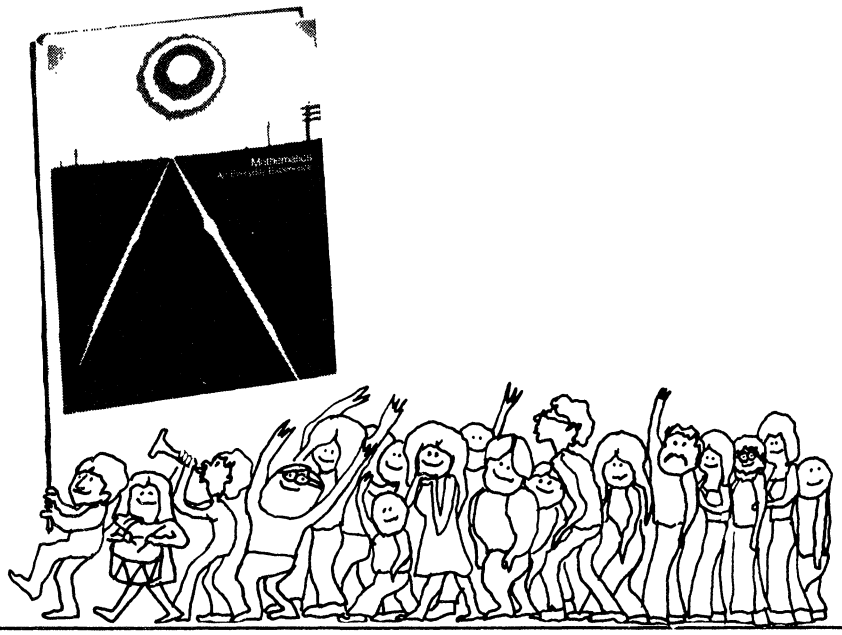
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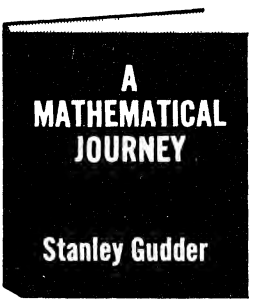
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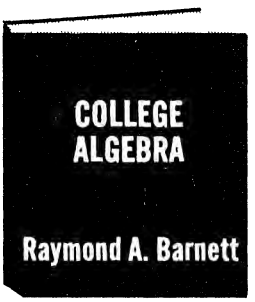
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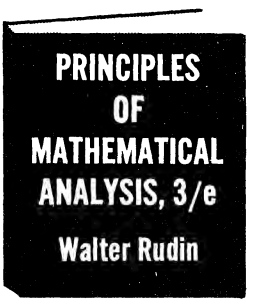

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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 2

CODEN: AMMYAE


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PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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## AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR LEON WALTER COHEN

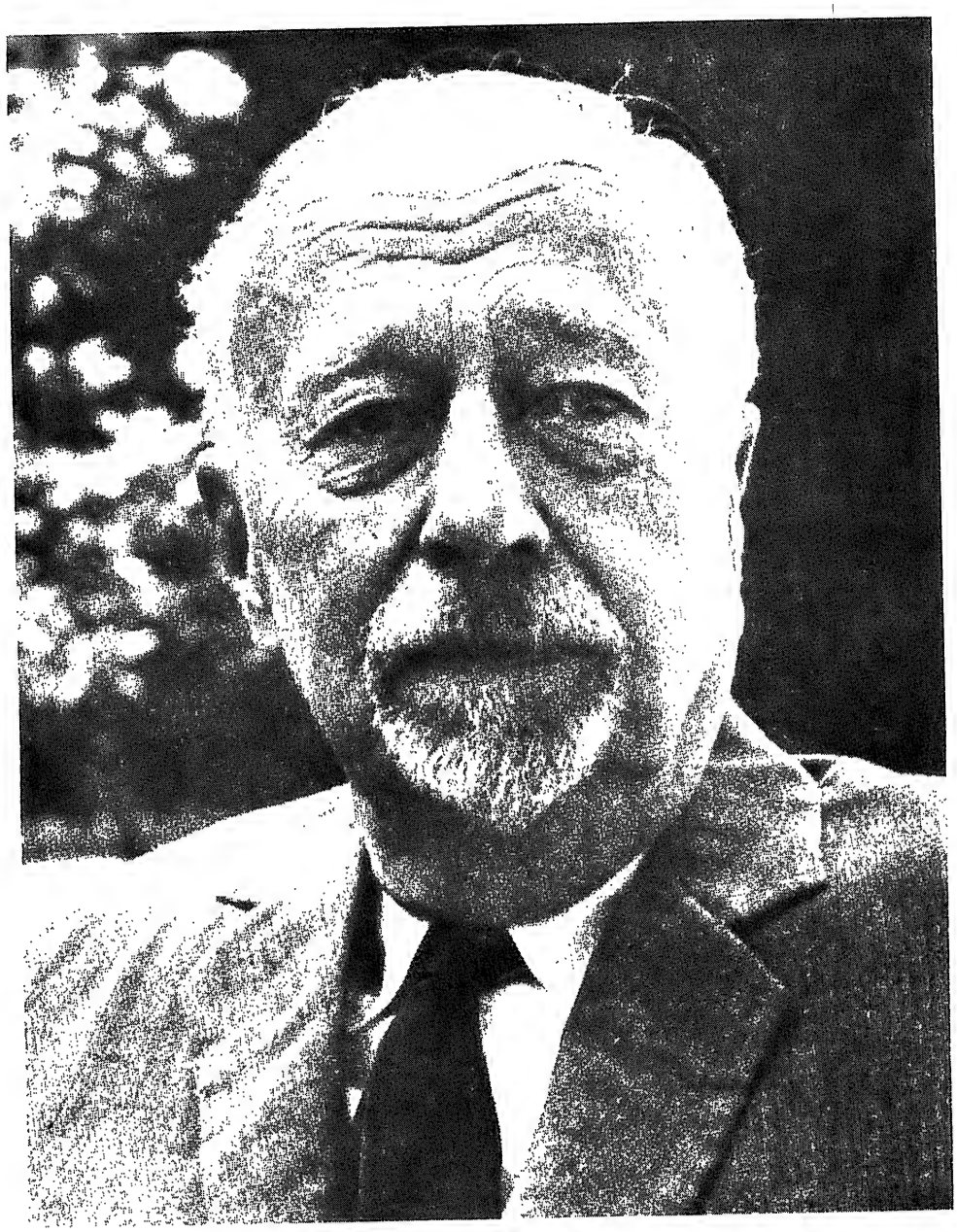
Mathematics, as a human activity, has become complicated in the last few generations. At the beginning of this century the set of mathematicians in the United States (hardly to be thought of as a mathematical community) consisted almost exclusively of a few professors in colleges and universities, a very few of whom tried to get some research done in the time that their teaching did not fill. Now there are thousands of persons engaging in a multitude of activities of a mathematical nature, and with an assortment of sources of funds. The quiet disorganization of 1890 is unsuited to 1976. But the need for organization implies a need for organizers. There exist mathematicians who have added greatly to the knowledge of mathematics by their research contributions, although they are averse to teaching and to all group activities. There are mathematicians who have contributed to mathematics by inspiring students to enthusiastic research. But there is also a need for mathematicians who are willing to take time off from their research, even though they are active researchers, and from their teaching, even though they are dedicated teachers, in order to organize the community efforts of all mathematicians. When such a person is not only willing to do the work, but is able to do it well, that person performs a great service to the mathematical community. Such a person is Leon Walter Cohen, to whom the Mathematical Association of America today extends its Award for Distinguished Service.

Leon Cohen was born in New York City on April 24, 1903. From Columbia University he received his A.B. degree in 1923 and his A.M. in 1925. In 1928 he received the degree of Ph.D. from the University of Michigan, being Ray Wilder's first doctoral student. Today's award is a second-generation honor. Three years ago Ray Wilder received the Award for Distinguished Service. Among his numerous services listed on that occasion we find his training of mathematicians, Leon Cohen being one of those named. Today we are reaffirming that Wilder performed a distinguished service to mathematics by training Leon Cohen.

Cohen quickly received his first distinction; he was awarded one of the prized and scanty National Research Council fellowships for 1929 to 1931, which time he spent at Princeton. In those depression days such a fellowship was also desirable for reasons in addition to the honor and the opportunity for mathematical development. From 1931 to 1947 he was a member of the faculty at the University of Kentucky, and from 1947 to 1955 he was a member of the Department of Mathematics of Queens College in his home city, New York. For the last three years of this period he was on leave, first for a year as Ford Fellow at the Institute for Advanced Study, then as Program Director for Mathematics in the National Science Foundation.

During the years, 1953 to 1958, of Cohen's service as Program Director, the then young National Science Foundation was in a period of rapid growth. Funds were increasing rapidly, and requests for them even more rapidly. For the good of mathematics, it was important that convincing arguments be made for sufficient support for mathematical proposals, and even more important that the funds be wisely used. One eminent mathematician remarked at about that time that if sufficient funds were available, it would be possible to put a complete stop to mathematics in this country. The work of the Program Director called for Cohen's ability as an organizer and for his good judgement in apportionment of support. It probably also called frequently for Cohen's ability to retain his calmness and urbanity under stress.

There have always been two schools of thought concerning optimal use of funds in support of mathematics. One view is that the best work is produced by a relatively few outstanding mathematicians at a few institutions, and that the development of first-class mathematics and mathematicians is best served by concentrating support at these centers. The other view is that in the long run it is better to spread the support with the aim of bringing more institutions up to a state of excellence. Cohen held the latter view, which would seem to be the one most accordant with the goals of this Association.



LEON WALTER COHEN

After leaving the National Science Foundation in 1958, Cohen became a professor at the University of Maryland, where he was chairman of the Department of Mathematics from 1958 to 1968 and used his wisdom to build up the department. But this did not prevent him from performing many other services, somehow managing to keep several teacups in the air without ever dropping one. The list of his activities is too long to give in detail. Suffice it to say that in the last sixteen years he has served on a round dozen committees of the Mathematical Association, and that in 1964 (a peak year) he was chairman of the department of mathematics of the University of Maryland, was a member of the Association's committee on educational media, was on that committee's advisory panel on individual lecturers, was on the CUPM panel on pregraduate training, was on the Science Manpower Commission, was executive secretary of the Conference Board of Mathematical Sciences, and was executive secretary of the Division of Mathematics of the National Research Council. It speaks well for Leon and for Isabel too that he came through such years in good health and in good spirits.

The Mathematical Association of America extends to Leon Walter Cohen its Award for Distinguished Service in recognition of and with thanks for long years of distinguished service to mathematics.

E. J. McSHANE

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## AWARD OF THE 1976 CHAUVENET PRIZE TO PROFESSOR LAWRENCE ZALCMAN

The Board of Governors of the Mathematical Association of America voted to award the 1976 Chauvenet Prize to Professor Lawrence Zalcman of the University of Maryland for his paper "Real Proofs of Complex Theorems (and vice versa)," this MONTHLY, 81 (1974) 115–37.

A certificate and a monetary award in the amount of five hundred dollars was forwarded to Professor Zalcman following the announcement of the award at the Annual Business Meeting of the Association on January 25, 1976, in San Antonio, Texas. At the time of the meeting, Professor Zalcman was on leave at the Technion (Israel Institute of Technology) and thus unable to be in attendance.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English, which comes within the range of profitable reading for members of the Association. The purpose of the Prize is to stimulate the writing of expository and survey articles. The 1976 Prize, awarded for a paper published in the three year period 1972–74, is the twenty-fourth award of the Chauvenet Prize since its institution by the Association in 1925. For the list of names of the previous winners, see this MONTHLY, 71 (1964), p. 589; 72 (1965), pp. 2–3; 74 (1967), p. 3; 75 (1968), pp. 3–4; 77 (1970), pp. 117–118; 78 (1971), pp. 112–113; 79 (1972), pp. 112–113; 80 (1973), pp. 120; 81 (1974), pp. 113–114; and 82 (1975), pp. 108–109.

Lawrence Zalcman was born on June 9, 1943, in Kansas City, Missouri, and attended the public schools of that city. He attended Dartmouth College as a General Motors National Scholar, receiving his A.B. in 1964 *summa cum laude* with valedictory rank. He studied at M.I.T. on a National Science Foundation Graduate Fellowship and received his Ph.D. (under the direction of Kenneth Hoffman) in 1968. From 1968 to 1972 he was an assistant professor at Stanford University. In 1972 he joined the University of Maryland mathematics department, where he has been a professor since 1974.

Zalcman has been a visiting lecturer at the University of California at San Diego, the University of Michigan, Northwestern University, and Wayne State University. During the academic year 1970–71, he was a NATO Post-doctoral Fellow at the Hebrew University of Jerusalem and the Weizmann Institute of Science in Rehovoth, Israel. Currently, he is doing research abroad at the Weizmann Institute, the Technion (Israel Institute of Technology, Haifa), and the Forschungsinstitut für Mathematik of the ETH (Zürich).

Zalcman is a member of the Editorial Board of the *Proceedings of the American Mathematical Society*. He is also a member of a committee of the Association charged with editing a volume of selected papers in analysis, published in the MONTHLY and MATHEMATICS MAGAZINE.

He is a past winner of the William Lowell Putnam Mathematical Competition and received the Association's Lester R. Ford Award in 1975 for the paper for which he has now been awarded the Chauvenet Prize.

Zalcman's research centers on complex analysis, especially on the interaction of function theory with related branches of mathematics (including approximation theory, Banach algebras, differential equations, and harmonic analysis). He has strong interest in the history and philosophy of mathematics and is currently occupied with problems which stress the intellectual (as opposed to purely technical) content of mathematics.

In his acceptance, Professor Zalcman expressed his delight and very great honor for having been awarded the 1976 Chauvenet Prize. The acceptance was read for Professor Zalcman at the Annual Business Meeting by D. P. Roselle, Secretary of the Mathematical Association of America.

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## DIRECTIONS IN MATHEMATICAL PSYCHOLOGY, I

ANATOL RAPOPORT

One way of inquiring into what mathematical psychology is about is to peruse the literature published under that rubric. The material is readily available in concentration, for instance in the three volume *Handbook of Mathematical Psychology* [31] and in the *Journal of Mathematical Psychology*, a quarterly published, since 1964.

This approach to the question "What is mathematical psychology?" is analogous to the semi-facetious answer to the question "What is physics?" — "What physicists do." If one attempts to answer the more searching question, "What sort of investigations deserve to be subsumed under 'mathematical psychology'?" one is forced to take a stand on the nature of mathematics and the nature of psychology. An explicit stand on this question will immediately push the discussion into the philosophical domain. Attempting to postpone the inevitable, we shall instead offer some criteria for subsuming an investigation under mathematics, under psychology, and under both (i.e., under mathematical psychology), on which a minimum of disagreement is expected.

An investigation is mathematical if it makes use of deductive techniques developed in mathematics. Some of these techniques are so widely known that no further elucidation is needed. Other techniques, less familiar to non-mathematicians, such as axiomatization of a theoretical framework, are presumably familiar to the readers of this journal. It turns out that these latter techniques are assuming an important role in the development of mathematical psychology, supplementing and, at times, competing with the traditional emphasis on the construction of specific mathematical models, relating dependent to independent variables. Below we shall address ourselves to the distinction between the two approaches.

As for the scope of psychology, traditionally it included the study of the "inner states" of sensate beings, either introspectively appreciated or inferred to exist in other beings more or less like ourselves. The strict behaviorists have challenged this definition of the scope of psychology, preferring to confine themselves to observed regularities of overt behavior. However, if inferred subjective experiences are regarded as theoretical constructs, connected to observables by strictly operational definitions, the self-imposed restrictions of the orthodox behaviorists appear as reflections of extreme positivist prejudices, which, by and large, have not proved to be viable in the practice of science. By a "positivist prejudice," I mean the attitude that eschews concepts not directly linked to observable referents, such as the early behaviorists' lack of interest in what goes on inside the sentient being, mediating between a stimulus and a response. There is, to be sure, also a constructive aspect of this attitude that directs efforts toward *making* observable what had not been previously observed, say for lack of techniques.

There is, for example, a branch of psychology called *physiological* aimed at discovering relations between neural events and behavior. The simplest of these relations was established with the discovery of the reflex arc. The discovery of the conditioned reflex permitted the inference that some structural changes in synaptic connections between neural elements account for systematic changes in behavioral patterns. If this is indeed the case, it may be possible to link learning to specific neural events. Further, the concept of the nervous system as serving an "integrating" or "organizing" function suggests structural models of neural complexes that could account for behavioral and perceptual patterns, both fixed and systematically modifiable.

Ablation experiments on animals and experiments with drugs have shed some light on the events underlying some grossly described functions of neural activity. The same can be said of some experiments on perception for example, those involving adjustments to distorting lenses. The results, often quantitatively recorded, suggest inferences about the structure of innervation that transmits and organizes visual stimuli. For the most part, however, concrete anatomical or physiological evidence of *specific* structures related to any but the simplest overtly observed patterns of behavior or perception is still lacking, and so the gap between physiological and psychological investigations remains large.

*Psychophysics* typically by-passes the problem of investigating the mediating physiological links between the physical stimuli impinging on the organism and the latter's behavioral or perceptive responses. It seeks to establish directly relations between the external world and its subjective "map".

Finally, there are areas of psychology where no attempt is made to relate "psychological matters" to either physiological events within the organism or to the structure of the physical world outside it. Application of quantitative or mathematical methods in these areas of "pure" psychology presents special problems, which will be briefly outlined below.

In its most general sense, mathematical psychology can be said to comprise investigations in which mathematical deductive techniques are utilized in the construction of theories aimed at explaining regularities of behavior and/or subjective experience and in some cases (but by no means always) at relating these to physical events impinging on the organism or to neurophysiological events within the organism.

The earliest applications of mathematics to psychology having been in psychophysics, we shall first briefly discuss some examples of mathematically oriented psychophysical theories. Then we shall go on to more general formulations, with few, if any, connections to events describable in the language of physics.

**Psychophysics.** What has since been known as "Weber's Law" was formulated in 1846 in Wagner's *Handbuch der Physiologie.* It is a presumably empirically established relationship of the "just distinguishable difference" (jnd) between the intensities of two stimuli to the magnitudes of the one or the other of these:

$$\text{jnd } (s)/s = K \text{ (constant).} \tag{1}$$

If we further assume that the "subjective magnitudes" associated with jnd's are all equal, we can write

$$\Delta s = k s \Delta u, \quad k \text{ constant}, \tag{2}$$

where $\Delta s$ is the "objective" increment of the intensity and $\Delta u$ the "subjective" increment. If $\Delta s$ and $\Delta u$ are interpreted as infinitesimals, (2) becomes a differential equation, which, when integrated, yields what is commonly referred to as Fechner's Law or the Weber-Fechner Law: the subjective perception of stimulus intensity is logarithmically related to the objective intensity.

As R. D. Luce and E. Galanter remark [32], Fechner's Law is not a "law" in any usual sense, because the method of measuring the "subjective intensity" is not specified. Nor, for that matter, are "objective intensities" or, more generally, objective magnitudes of stimuli unambiguously defined. For example, the pitch of a tone is a "magnitude" that can be represented by the corresponding frequency of vibration. The subjectively perceived magnitude of an interval between two tones can be established by recognition of two intervals as "equal." In this interpretation, the logarithmic relation between the "objective" and the "subjective" difference can be established with great precision: any moderately musical person recognizes a fifth as a fifth, a third as a third, etc., regardless of the pitch of the reference tone; that is, identifies the "difference between two pitches" with the ratio of their frequencies. There is, however, no compelling reason to measure the "magnitude of pitch" by frequency; it can also be measured by the logarithm of frequency, in which case the relation between "objective" and "subjective" difference intervals turns out to be linear.

In contrast to Fechner's Law, Weber's Law can be in principle subjected to direct experimental test. If the physical units of stimulus intensity are defined and if the jnd's for the various intensities can be established, the constancy of jnd $(s)/s$ would constitute a corroboration of the law. Most data show departures from constancy, typically at low intensities, and generalizations of Weber's Law have been proposed. These lead, of course, to different functional relations between subjective and objective magnitudes, and the neat theoretical underpinning of Fechner's logarithmic "law" is replaced by a collection of purely empirical formulae.

Experimental determination of jnd's is further complicated by typical inconsistencies in discrimination. For the experimenting psychophysicist, such inconsistencies are a fundamental fact of life. The contemporary mathematical psychophysicist often makes a virtue of necessity by incorporating the inconsistencies into his mathematical models. In a typical experiment aimed at estimating jnd's, a subject is asked to judge which of two stimuli (a standard stimulus, $s$, and a variable stimulus, $x$) is the larger (stronger, heavier, etc.). The mathematical psychophysicist posits a probability $p(s, x)$ that $x$ will be judged larger than $s$. Clearly if $x$ is sufficiently larger or sufficiently smaller than $s$, $p$ should be 1 or 0 respectively, reflecting a consistent judgment. The range of interest, however, is where $0 < p < 1$, for in that range the jnd can be defined as, say, $\frac{1}{2}(x_{3/4} - x_{1/4})$, where $x_\theta$ is defined by $p(s, x_\theta) = \theta$.

More generally, a function $\Delta(s, \pi)$ is defined such that

$$p[s, s + \Delta(s, \pi)] = \pi \qquad (0 < \pi < 1). \tag{3}$$

This function $\Delta(s, \pi)$ is called the Weber function. The problem is that of finding a suitable transformation of a physical magnitude to a subjective one, the modern version of Fechner's problem, that of linking a mathematically deduced law to actual observables, in this case to frequencies of judgments concerning relative magnitudes of stimuli. Namely, for each fixed $\pi$ $(0 < \pi < 1)$, find "smooth" monotonic functions $u$ such that for all $s > s_0$

$$u[s + \Delta(s, \pi)] - u(s) = g(\pi), \tag{4}$$

where $g$ is a strictly monotonic increasing function of $\pi$ and is independent of $s$.

The functional equation (4) has a plethora of solutions. For example, it is known that if $u'(s)$ is one solution then

$$u(s) = u'(s) + F[u'(s)] \tag{5}$$

is another if $F$ is a periodic function with period $g$. If, however, a generalized form of Weber's Law holds:

$$\Delta(s, \pi) = K(\pi)s + C(\pi) \tag{6}$$

with $C(\pi)/K(\pi) = \gamma$, a constant independent of $\pi$, then

$$u(s) = A \log(s + \gamma) + B, \qquad A, \gamma, B \text{ constants}, \tag{7}$$

a generalized form of "Fechner's Law". The example illustrates an application of rigorous mathematical analysis to the earlier, somewhat loose formulations of a psychophysical model.

Another example is provided by the examination, in the light of mathematical arguments, of S. S. Stevens' contention **[52, 53]** to the effect that the relation between objective and subjective magnitudes, particularly of intensities, should be expressed by a power law:

$$\psi(s) = \alpha s^{\beta}, \qquad \alpha, \beta \text{ constants}, \tag{8}$$

where $\psi(s)$ is the subjective intensity associated with the objective intensity $s$. An impressive volume of experimental evidence has been amassed in corroboration of such a psychophysical law. Unlike the experiments centered on determining jnd's, Steven's experiments typically elicit from the subject direct attempts to assign numerical or ratio values to stimulus intensities. In some experiments, the subject is even asked to match the intensity of one stimulus with another of a different modality, for instance the loudness of a sound with the intensity of induced vibration in the skin **[51]**. If the psychophysical laws governing the two subjective intensities are

$$\psi_1(s) = \alpha_1 s^{\beta_1} \quad \text{and} \quad \psi_2(t) = \alpha_2 t^{\beta_2} \tag{9}$$

then a matching of intensities occurs when $\psi_1(s) = \psi_2(t)$. Thus

$$s = \left(\frac{\alpha_2}{\alpha_1}\right)^{1/\beta_1} t^{\beta_2/\beta_1}, \tag{10}$$

so that the matching relation is also a power function with $\beta_2/\beta_1$ as the exponent. Consequently, the law predicts not only a power function relation but also the value of the exponent, assuming that $\beta_1$ and $\beta_2$ had been previously estimated. Equation (10) has been corroborated in many instances.

Now it can be and has been pointed out that the logarithmic law also predicts a power relation connecting matched intensities. For if

$$\psi_1(s) = a_1 \log(s/b_1); \qquad \psi_2(t) = a_2 \log(t/b_2), \tag{11}$$

then

$$a_1 \log(s/b_1) = a_2 \log(t/b_2); \qquad s = ct^{a_2/a_1}, \tag{12}$$

where $c = b_1 (b_2)^{-a_2/a_1}$. A counter-argument is that in the case of the power law, the matching relation predicts the ratio of two estimable parameters, $\beta_1$ and $\beta_2$, whereas the logarithmic law does not, since $a_1/a_2$ is merely the ratio of two *arbitrary* units of the two scales.

The possibility of establishing quantitative subjective comparisons of stimuli of different modalities raises the question concerning the mechanism that underlies such comparisons. To see the problem in perspective, consider the class of experiments in which subjects compare magnitudes of

stimuli either of the same or of different modalities. In one type of procedure a subject is presented with two stimuli and is instructed to judge the "ratio" of the induced sensations ("pair estimation"). In another, he is presented with stimuli one at a time and is asked to assign numbers "in proportion to the sensations evoked" ("magnitude estimations"). The responses generated by these two procedures are positive numbers. In a third procedure ("cross modality matching"), stimuli are again presented singly, and the subject is instructed to choose a stimulus of another modality that "matches" the stimulus presented.

In the magnitude estimation procedure, the subject is sometimes informed that a particular number, $p$, is to be assigned to a particular stimulus $x$. The magnitude estimation function $N(y \mid x, p)$ is the average estimate of the magnitude of $y$, given the pair $(x, p)$, called the *modulus*. By definition, $N(x \mid x, p) = p$. The pair estimation function $P(x, y)$ is the number assigned by the subject to the "ratio" of $x$ to $y$.

Experimental results are sufficiently consistent to warrant the following (idealized) empirical generalizations.

(13) *Magnitude consistency*: $N(y \mid x, p)/N(y \mid x, q) = k(p, q)$,

where $k$ is independent of $x$ and $y$. Putting $x = y$, in view of $N(x \mid x, p) = p$, $N(y \mid y, q) = q$, we have $k = p/q$.

(14) *Pair consistency*: $P(x, y)P(y, z) = P(x, z)$

(15) *Magnitude pair consistency*: $P(x, y) = N(x \mid z, p)/N(y \mid z, p)$

for any fixed $z$.

The above regularities hold with respect to stimuli of the same modality. To state the result on cross-modality matching, let $C_{ji}(y_j \mid x_j, x_i)$ denote the average match of a stimulus of modality $i$ to a particular stimulus $y_j$ of modality $j$, given modulus $(x_j, x_i)$. By definition, $C_{ji}(x_i \mid x_j, x_i) = x_i$. Having established $C_{ij}$ for each pair of modalities, the following empirical generalization appears to be warranted:

*Consistency of magnitude estimation and cross-modality matching*:

$$\frac{N_i[C_{ji}(y_j \mid x_j, x_i) \mid z_i, p]}{N_i(x_i \mid z_i, p)} = \frac{N_j(y_j \mid z_i, q)}{N_j(x_j \mid z_j, q)}. \tag{16}$$

Finally, if the $x_i$ are energies associated with physical stimuli, we have the *Power law* [cf. (8)]:

$$N_i(y_i \mid x_i, p) = (p)(y_i/x_i)^{B_i}, \tag{17}$$

where $B_i$ is a positive constant.

Theories proposed to deal with these regularities are of two types. According to a *mapping theory*, the magnitude of any modality is mapped by the subject on a "sensation continuum," a theoretical constraint serving as sort of common denominator of all modalities. Thereby the perceptive apparatus is enabled to "calculate" all ratios of magnitudes in terms of ratios induced in this continuum. According to a *relation theory*, the subject perceives a magnitude associated with a pair of stimuli rather than a quantity associated with each.

The history of psychological theories has been largely a history of disputes on the verbal level which could only occasionally be settled by decisive experiments. A task of mathematical psychology in its modern version is to formulate various theories in terms of mathematically rigorous sets of axioms in order to pinpoint the issues of whatever controversies arise concerning the status of rival theories.

Accordingly, D. H. Krantz [24] undertook an axiomatization of both types of theory. As a result, he was able to put his finger on the issues between the theories and spell out in concrete terms his reasons for preferring the relation theory.

To account for the generalizations (13)–(16), mapping theory must postulate functions $\psi_i(x_i)$ from stimuli to the "sensation" continuum and also functions from the latter to real numbers. Krantz finds the latter assumption unreasonable, since the subject "does not have a unit sensation, nor any means of taking a unit sensation and reduplicating it until he builds up to match a given sensation." To get around this difficulty, Krantz suggests that some concrete continuum, say length (henceforth designated as modality 1), plays a mediating role. What must be assumed, then, are mappings $\psi_i(x_i)$ to the length continuum and a mapping $\phi_1(x_1)$ from this modality to real numbers.

The introduction of the mediating modality does not remove the principal difficulty of the mapping theory, according to Krantz. Recall that the magnitude estimation function for each modality, $N_i(y_i \mid x_i, p)$, depends on the modulus $p$. Therefore, to account for magnitude consistency [cf. (13)], an axiom must be assumed that essentially states this explicitly:

$$N_i(y_i \mid x_i, p) = \frac{p}{\phi \circ \psi_1^{-1} \circ \psi_i(x_i)} \circ \phi \circ \psi_1^{-1} \circ \psi_i(y_i), \tag{18}$$

implying that the subject performs the indicated mapping operations, takes the ratio, and multiplies the result by the modulus parameter $p$. Similar difficulties attend the remaining axioms of mapping theory, which turn out to be hardly more than restatements of the empirical generalizations.

On the other hand, the axioms of relation theory, formulated by Krantz, involve essentially only the ordering of stimulus pairs and certain "calibrations" of pairs from different modalities. The critical axioms can be easily tested experimentally and demand less tacit calculation from the subject. For example, whereas in mapping theory, the subject is assumed to remember a fixed ratio, determined by a modulus and to multiply every natural scale value $p$ by it before responding, in relation theory, he need only remember the last response $p$ and find a $q$ so that $q/p$ yields a suitable ratio. This ratio "requires knowledge of length, in the sense of mental estimates of length ratios, but it does not require any actual arithmetic." It is in this role that the mediating modality (length) enters the axiomatization of relation theory. It is noteworthy that the estimation of length *ratios* is veridical, that is, corresponds more or less to the objective ratios. This is the justification of the axiom $P(x_i, y_i)P(y_1, z_1) = P(x_i, z_i)$ in relation theory, which accounts, via the calibration of cross-modality ratios, for consistency of magnitude estimation and cross-modality matching.

The theoretical dividends of the above axiomatization are obtained in the form of leverage provided in connecting a theory of magnitude estimation with other psychophysical theories. As one of several situations discussed by Krantz in this connection, consider the greater discriminability exhibited by a subject when he is asked to detect an increment $\Delta x$ imposed on the background stimulus $x$, than when he is asked to decide whether $x$ or $x + \Delta x$ has been presented. A possible explanation is in terms of the subject's sensitivity to the increment that involves "transient" neural responses, present in the first but not in the second mode of presentation. The relation theory outlined above generates a more parsimonious explanation in terms of measures that yield an ordering of pairs of stimuli. (One could, of course, offer a counter argument to the effect that the assumption about the role of transient neural responses may suggest neurophysiological investigations not suggested by the relation theory.) Another set of interesting observations relates to the situation where subjects are asked to judge sensation "differences" instead of "ratios." The question to be investigated is that of the way the relation $\succsim$ on the union of the Cartesian products $X_i \times X_i$ where $X_i$, the $i$th modality continuum, is altered by the instructions. In particular if $\succsim_1$, $\succsim_2$ both satisfy the axioms of relation theory, scaling functions can be constructed so that the same scales account for "ratio" and "difference" judgments. The question then becomes that of deriving the conditions that makes this possible. The key condition turns out to be [**25**].

$$\text{If } x_iu_i \succsim_2 y_iv_i \succsim_2 z_iw_i, \text{ then } x_iy_i \succsim_1 y, z, \tag{19}$$

$$\text{if and only if } u_iv_i \succsim_1 v_iw_i,$$

where $\succsim_i$, $i = 1, 2$ are distinct ordering relations on pairs corresponding to different instructions.

As we have seen, the principal problem of psychophysics is to map some "objective," e.g., physical scale of magnitudes onto a corresponding "subjective" scale that reflects a subject's perception of these magnitudes. The earliest attempts of this sort were based on purely inductive methods, such as obtaining in some way the subject's estimates of magnitudes and proposing an equation relating the physical magnitudes to these. The so-called Weber-Fechner Law (cf. p. 87) introduces a single theoretical "step" by assuming that the differential increment of the subjective magnitude is proportional to the ratio of the differential increment of the objective magnitude to the magnitude that suffers the increment. Integrating, one obtains the "law." Thurstone's theory of discrimination [**56**] introduces a theoretical (i.e., explanatory) component by postulating a distribution of instantaneous subjective magnitudes and deriving the probabilities of judgments about relative magnitudes of pairs of stimuli. There is usually little or no opportunity to test the underlying assumptions by independent evidence.

It goes without saying that psychophysical theories would be much more securely grounded if the models were based on direct physiological evidence. Experimental and observational techniques developed comparatively recently have made such evidence available with the result that the stochastic processes in the nervous system heretofore postulated *in abstracto* could be formulated on the basis of some physiological data. A theory of auditory discrimination and of reaction times may serve as an example.

The physiological evidence concerns the responses of peripheral nerve fibers to auditory stimuli of varying frequencies and intensities. The phenomena to be explained are those of discrimination of these magnitudes and of reaction times to auditory stimuli. It has been long established that the "firing" of a neuron is a propagated pulse whose amplitude is practically independent of impinging conditions. At the time this finding suggested models of neural action analogous to that of a digital computer, where information is coded in logical functions involving conjunctions, disjunctions, and negations of these atomic events at neural synapses [**38**]. Whatever may be taking place in the brain (where a detailed structure of these events is practically inaccessible to direct observation) it is now fairly clear that this picture does not fit the coding and the transmission of information in the peripheral nerves. Rather, information, at least that concerning the frequency and the intensity of auditory stimuli, appears to be coded by spatial distributions and temporal patterns of *trains* of impulses. Specifically, each fiber responds most readily to a signal of a particular frequency. When the signal is below a certain threshold of intensity, the fiber produces a train of impulses at a certain basic rate. This rate increases in response to superthreshold signals up to a certain maximal rate (the upper threshold). If the frequency of the stimulus deviates from the characteristic ("resonant") frequency of the fiber, both thresholds are raised. In this way, changes of frequency result in a changing composition of the populations of responding fibers. Changing intensity results in increasing or decreasing the number of participating fibers as well as changes in the rates of impulses produced.

In addition to these responses to changes of frequency and intensity, changes in the temporal patterns of the pulse trains also play a part, as evidenced by the changes of distributions of inter-arrival times of the pulses (IAT). A histogram of inter-arrival times displays a sequence of regularly spaced peaks (modes), the first one being at 0.5 to 1 msec, presumably representing a refractory period, and the others at multiples of the period of the input signal. Moreover, the magnitudes of these modes decrease in geometric progression, suggesting that the probability of a neuron firing exactly $i$ periods after the last firing is $p(1-p)^{i-1}$, where $p$ is some function of frequency and intensity.

The problem of constructing a theoretical model of intensity and frequency discrimination and of reaction times is that of formulating hypotheses about how the information, carried by the peripheral fibers, is transmitted to decision centers and of postulating specific decision rules of response. The first task in attacking this problem is to propose a stochastic model to account for the observed (multi-modal) frequency distribution of inter-arrival times of pulses.

D. M. Green and R. D. Luce [16] assume that IAT is the sum of two random variables:

$$\text{IAT} = J/f + X, \tag{20}$$

where $J$ is geometrically distributed:

$$\Pr[J = j] = p(1-p)^{j-1}, \qquad j = 1, 2, \cdots, \tag{21}$$

and $p$ is some unknown function of the intensity $I$ and frequency $f$ of the signal. As for $X$, the data suggest that it has a symmetric distribution bounded on some interval $(-\varepsilon/f, \varepsilon/f)$, $0 < \varepsilon < \frac{1}{2}$, i.e.,

$$\Pr[X = x] = f\varepsilon^{-1} g(f\varepsilon^{-1}x), \qquad 0 < \varepsilon < \tfrac{1}{2}, \tag{22}$$

where $g$ is a symmetric density function on $(-1, 1)$ with $g(1) = g'(1) = 0$.

We have accordingly a renewal process (with independent inter-arrival times):

$$\Pr[\text{IAT} = x] = \psi(x) = \begin{cases} p(1-p)^{i-1} f\varepsilon^{-1} g[(x - if^{-1})f\varepsilon^{-1}], & i = 1, 2, \cdots \\ \qquad \text{when } |x - if^{-1}| \leqq f\varepsilon^{-1} \\ 0 \quad \text{otherwise.} \end{cases} \tag{23}$$

In their basic model of intensity discrimination, Green and Luce assume that only the temporal aspect of the information carried by the peripheral channels is utilized, spatial patterns being ignored. Thus, inter-arrival times determine the decision, for instance, which of two signals is more intense. Two such decision rules suggest themselves: (1) the decision center counts the number of impulses in a fixed time interval (a counting model); or (2) it times the interval in which a fixed number of pulses arrive (a timing model). Mathematically derived consequences of each model compared with experimental data suggest the appropriateness of the one or the other in a given experimental situation.

Specifically, pressures to be fast and accurate can be exerted on a subject by imposing a deadline for response (with penalties if exceeded) and by a payoff matrix for accuracy. In comparing intensities of two stimuli the counting model predicts the same mean reaction time (MRT) in all four cells of the signal response matrix. On the other hand, the timing model predicts

$$\text{MRT}_i = r + h(Y, \kappa, V_i^{1/2}/M_i) M_i, \qquad i = 1, 2, \cdots \tag{24}$$

where $r$ is the mean delay aside from decision time, $\kappa$ is the number of IATs collected per channel, $Y$ the number of active channels, $M_i$ and $V_i$ respectively the expectation and the variance of $\text{IAT}_i$ and $h(Y, \kappa, \sigma)$ is the mean decision time when $M_i = 1$, $V_i = \sigma^2$. If $M_1 > M_2$, assuming the difference in intensity small so that differences in $Y$ and $\sigma$ can be neglected, eliminating $h(Y, \kappa, \sigma)$, gives

$$\text{MRT}_i = (M_1/M_2)\text{MRT}_2 - r(M_1/M_2 - 1) \tag{25}$$

which can be tested experimentally by varying the deadline.

Differential predictions are also obtained with respect to the ROC curves (probability that stimulus $s_1$ will be guessed correctly vs. the probability that it will be guessed incorrectly). Assuming approximate normality of sums of IATs (justifiable for large $Y\kappa$) and converting to $z$ scores, the ROC curve in $z$ scores predicted by the timing model is given by

$$z_2 = \left(\frac{V_1}{V_2}\right)^{1/2} z_1 + \frac{M_1 - M_2}{V_2^{1/2}} (Y\kappa)^{1/2}. \tag{26}$$

On the other hand, the counting model predicts

$$z_2 = \left[\frac{V_1}{V_2}\left(\frac{M_2}{M_1}\right)^3\right]^{1/2} z_1 + (Y\delta)^{1/2}\left(1 - \frac{M_2}{M_1}\right)\left(\frac{M_2}{V_2}\right)^{1/2}, \tag{27}$$

where $\delta$ is the fixed time interval sampled. For details see [16], p. 387.

It is such theoretical, that is, deductive arguments that characterize the approach to psychophysical research as mathematical rather than just quantitative.

On the more "organized" level of perception, some theoretical psychophysicists have utilized the apparatus of transformation groups in constructing models of perceptual space. For instance, W. C. Hoffman [18] discusses Lie groups that "account" for certain invariants of perception and suggest hypotheses concerning the presence of structures in the central nervous system that carry out the corresponding transformations. Specifically, the shape of an object is presumably recognized regardless of its position in the visual field and of its orientation; a moving object is perceived as "identical to itself," etc. Hoffman presents the following table, relating perceptual invariants to corresponding Lie groups.

TABLE 1

Perceptual Invariants and Corresponding Groups of Transformations (after Hoffman)

| | *Perceptual Invariants* | | *Groups of Transformations* |
|---|---|---|---|
| A. | Shape constancy | A. | Affine groups |
| | (a) Location in the field of vision | | (a) Translation |
| | (b) Orientation | | (b) Rotation |
| | (c) Binocular vision | | (c) Pseudo-Euclidean (hyperbolic) rotation |
| | (d) Form memory | | (d) Time translation |
| B. | Size constancy | B. | Dilatation group |
| C. | Motion | C. | Two-dimensional Lorentz group |
| D. | Efferent binocularity | D. | Hyperbolic rotation in plane-time |
| E. | Circulating ("motion") memory | E. | Rotations in plane-time |

It is interesting to note that in relation to "plane-time," a "relativistic" metric is considered:

$$ds^2 = dx^2 + dy^2 - c^2\,dt^2, \tag{28}$$

where $c$ is presumably the upper bound on the velocity of a moving object perceived as such. As in relativity theory, the metric implies a perceived contraction of length in the direction of motion. It was reported that such a contraction effect has actually been observed [5].

That the geometry of visual space is non-Euclidean (specifically, hyperbolic with constant negative curvature) has found extensive experimental corroboration. R. K. Luneburg [35] deduced this effect from the theory of binocular vision, where the perceived distance of an object is determined by the angle of convergence of the axes of the two eyeballs focusing on the object. (Actually, of course, perceived distance is also strongly affected by prior experience with the sizes of objects compared with their apparent sizes in the visual field. This factor, however, can be presumably removed in controlled experiments where corresponding cues are absent.)

Luneburg introduced bi-polar coordinates for the visual field, namely the binocular convergence angle $\gamma$ and the quasi-azimuth angle $\psi$. Objects are perceived at equal distances from the viewer if they induce equal angles of binocular convergence. Thus, equations, $\gamma = \text{constant}$, represent circles (so called Vieth-Mueller circles) through the two eyes and the object viewed. Similarly, visual perception transforms the hyperbolas, $\psi = \text{constant}$, into polar rays. The problem remains to determine the transformations relating the polar coordinates $(r, \theta)$ of physical space to the bipolar coordinates $(\psi, \gamma)$ of visual space. Luneburg determined $r$ as a function of $\gamma$ empirically:

$$r = e^{-\gamma\sigma}, \qquad \sigma \text{ constant} \tag{29}$$

which makes the mapping of the physical upon visual space conformal.

Mathematical modes of thought immediately suggest a general problem. Given that the visual

manifold for some specified metric (say, hyperbolic) can be determined, find the class of conformal mappings, $r = \Phi(\gamma, \psi)$; $\theta = \Psi(\gamma, \psi)$ that satisfy conditions imposed by empirical correspondences between physical and subjectively perceived curves.

R. Lake [27] showed that the solution of this problem involves a general integral of the system of partial differential equations:

$$\Phi_\gamma^2 + \Psi_\gamma^2 = \rho\Psi^2$$

$$\Phi_\gamma \Phi_\psi + \Psi_\gamma \Psi_\psi = 0 \tag{30}$$

$$\Phi_\psi^2 + \Psi_2 = \rho\Phi^2, \quad \rho \text{ constant.}$$

To my knowledge, this problem has not been successfully attacked.

**A Survey of directions in mathematical psychology.** Beyond the scope of psychophysics, mathematical psychology turns attention to quantitative or structural aspects of perception, cognition, or behavior without reference to physical magnitudes. An idea of the scope of such problems can be formed in perusing the contemporary literature of mathematical psychology.

TABLE 2

A Rough Categorization of Papers in Luce, Bush, and Galanter's *Handbook of Mathematical Psychology*

| | *Method* | | | |
|---|---|---|---|---|
| *Content* | Axiomatic | Classical | Stochastic and Statistical | Structural |
| Measurement and scaling | 4 | | | |
| Psychophysics | 1 | 3 | 3 | |
| Conditioning and learning | | | 17 | |
| Choice, preference, decision | 2 | | 8 | 1 |
| Computer models | | | | 5 |
| Formal language analysis | | | | 10 |
| Social interaction | | 2 | 3 | 3 |

Table 2 presents a categorization of the papers included in the *Handbook of Mathematical Psychology* and the supplementary *Readings* [31]. Several purely experimental papers on sensory processes, apparently included as background material in the *Readings*, are not represented in the table.

The rows of Table 2 represent content categories. Measurement and scaling theory plays a rather special role in mathematical psychology outside the scope of psychophysics. Conditioning and learning, as these phenomena are treated in mathematical psychology, deal with quantitative changes in responses attributable to experience. We have lumped under "choice, preference, and decisions" studies of responses that involve singling out or ordering of stimuli on the basis of some aspects of the

stimuli themselves (rather than being modified by the effects of choices, as in typical learning experiments). Studies of signal detection, although, perhaps properly subsumed under psychophysics, sometimes also fall into this category, for they involve decisions as to whether a stimulus was present or not. Computer models are presumably relevent to mathematical psychology as analogues of neural organization or of organized functioning of the nervous system. Formal language analysis deals with models of languages characterized by certain axiomatized syntactic rules, which in their totality constitute the "grammar" of a language. A task of the analysis is to specify a set of structural rules by means of which it is possible to decide whether or not a particular utterance (a string of units) is or is not a sentence in the language. Another task is to specify rules by means of which, given a set of primitive units, it would be possible to generate all conceivable sentences in the language and only those. Social interaction models deal with patterns of behavior elicited, reinforced, or inhibited in a group of subjects interacting with one another.

The columns of Table 2 represent a rough categorization of mathematical methods predominantly used in the studies. The role of axiomatization will be discussed below in connection with measurement and scaling theory. "Classical" methods are the well known ones extensively used in the mathematicized physical sciences, for the most part various branches of analysis. In stochastic and statistical methods random variables and probability distributions play a central role. Under structural methods we have subsumed those of so-called non-quantitative mathematics, for example, set theory, graph theory, algebraic systems like groups, semi-groups, lattices, etc.

TABLE 3

A Rough Categorization of a Random Sample of Papers in *The Journal of Mathematical Psychology*

| | *Method* | | | |
|---|---|---|---|---|
| *Content* | Axiomatic | Classical | Stochastic and Statistical | Structural |
| Measurement and scaling | 19 | | 2 | |
| Psychophysics | 2 | 2 | 6 | 1 |
| Conditioning and learning | 2 | 2 | 25 | |
| Choice, preference, decision | 8 | 2 | 13 | 2 |
| Computer models | | | 1 | 1 |
| Social interaction | | | 2 | 3 |

Table 3 presents a similar categorization of some 90 odd papers selected at random from several issues of the *Journal of Mathematical Psychology*. Except for the absence of papers on formal language analysis (which continue to be published in journals on linguistics), the picture looks about the same as in the *Handbook*. We note the preponderance of papers on conditioning and learning and on choice and decision and of stochastic methods applied in those areas. (It should be noted, however, that extensive work on psychophysics is published in *Perception and Psychophysics*, *J. Optical Society*,

and *J. Acoustical Society of America.*) We also note a strong representation of papers on measurement and scaling and of the axiomatic approach associated with them. It is reasonable to conclude, then, that these three areas, and the two methodological approaches associated with them, are dominant in on-going research in mathematical psychology. An inquiry into the reasons for this emphasis will shed some light in the nature of mathematical psychology and its tasks, as they are conceived today.

Mathematical deductive techniques can be defined as those that prescribe absolutely unambiguous rules for constructing legitimate, i.e., "grammatical" strings of symbols and for rewriting such strings to obtain other equivalent ones. The rules are called axioms, and the totality of "sentences" so deduced, that is, the totality of theorems are called a mathematical system. It stands to reason that if a mathematical system is to be of use as a sector of applied mathematics, there must be correspondence rules connecting the symbols to referents in the "real world" and the rules of deduction to relations among these referents. In the earliest applications of mathematics these correspondence rules were intuitively obvious and so did not need to be explicitly stated. The simplest rules of arithmetic, governing the addition of positive integers, found perfect application in the counting of clearly recognizable objects. The extension of the number system to the field of rational numbers connected mathematical deduction to measurements of lengths, volumes, weights, etc. Further extensions to complex numbers laid the foundations of analysis, the principal tool of the mathematicized physical sciences.

Traditionally, then, applied mathematics has dealt with operations on quantities, either real numbers or ordered sets or arrays of real numbers such as ordered pairs (complex numbers), ordered $n$-tuples (vectors), arrays (tensors), sets of ordered pairs (function). etc. The referents of the real numbers were quantities in the real world (distances, time intervals, masses, currents, angular momenta ) that were either directly observable or defined by mathematical operations. Relations among these quantities reflected "physical laws."

The unprecedented success of the mathematicized physical sciences was a consequence of the fact that no matter how far the deductive techniques carried a mathematical theory, the ultimate conclusions were still anchored through strings of rigorous deduction to the very few postulates representing the "physical laws," which, for practical purposes, could be considered as immutable. On this remarkably solid foundation an amazingly complex mathematical superstructure was erected. Witness the catalogue of tools of the mathematical physicist with its special functions representing the solutions of fundamental differential equations, with its "operators" facilitating complex mathematical deductions, with its refined analytic techniques legitimizing deductive leaps into the infinite (passage to limit), etc.

The mathematician perusing the literature of mathematical psychology will find very little of this grandiose superstructure. He will find no Bessel functions, no Hermitian operators, no complex numbers. Only very occasionally he will see a differential equation, although he will occasionally encounter functional equations. For the most part, he will encounter the symbolism of set theory, of the algebra of probability theory, and of stochastic processes. If he extends his reading to all quantitative psychology, he will see a great deal of mathematics used in statistical inference.

The reasons for the absence in mathematical psychology of a complex mathematical superstructure, comparable to that of mathematical physics, and for the heavy reliance on statistical inference in quantitative psychology, are not far to seek. Nothing resembling the axiomatic foundations of mathematical physics underlies the multichanneled development of psychological theory. One can, if one is so inclined, hold on to the reductionist belief that all overt behavior and even the "psychic states" are determined by configurations of events in the nervous system and that these events, being physico-chemical, are subject to the same physical laws that govern the properties and behavior of all matter. Nevertheless, a *program* of deriving events of central interest in psychology from physico-chemical laws, or even from observed or postulated properties of neural elements and of their interactions, is, at least for the time being, a Utopian dream. In a practical sense, the mathematical psychologist must confine himself to the construction of mathematical models that are for the most

part *ad hoc*, that is, models of specific situations. The starting point of an investigation in a model of this sort, is a set of postulates relating to a special context. The end result is a set of deductions, usually only a very few steps removed from the postulates, relating to some observations on behavior. The "landscape" of mathematical psychology resembles several low hills rather than the imposing mountain range of classical physics.

Another fundamental difference between psychology and physics is that quantification in the former depends far more than in the latter on counting recognizable events rather than on measuring basic categories of which physical reality is made up. This accounts for the predominance of probabilistic and stochastic approaches in psychology, where the tools for dealing with frequencies of occurrence are developed. Measurement, as opposed to counting, presents special problems in psychology, with which, in fact, much of mathematical psychology is concerned. We shall now turn to these problems.

**Problems of measurement.** The act of measurement is that of assigning numbers to observations. Measurement theory is concerned with the way such numbers can be legitimately assigned and with the sort of operations that can be performed on them. In the case of physical magnitudes, both questions are answered in an obvious manner. In our space, of which Euclidean geometry of three dimensions is an excellent model, three real numbers can be assigned to each position corresponding to coordinates related to an orthogonal system of axes. Once assigned, the coordinates relative to any other system vary in a prescribed manner, whereby the function representing Euclidean distance remains invariant. Further, the ratio of two masses is invariant with respect to choice of units of mass. In fact, it is legitimate to form products and quotients of powers of physical quantities when and only when they are measured on the so-called ratio scale, in which the zero point is "naturally" given and only the unit of measurement can be arbitrarily chosen. It is *not* permissible to form products or ratios of quantities for which the zero point can be arbitrarily chosen. All these rules are obvious to the physicist, for whom quantities are inseparable from the methods of measurement that reveal them. The psychologist, on the other hand, is faced with a serious problem of ascertaining the sort of scale that is proper for the magnitudes ascribed to the observations. To take a typical example, a psychologist may wish to assign a number to each of a set of objects such that the numbers reflect a subject's preferences among the objects. An obvious procedure seems to be to ask the subject to arrange the objects in the order of preference. In practice, however, problems arise. The task may be too difficult if the set is large. Even if the set is not large, and the subject does offer an arrangement, the question of consistency rears its ugly head. Will the subject present the same arrangement on another occasion or if the objects are presented in different orders?

One way to test consistency is by asking the subject to make paired comparisons. A necessary condition for consistency is that the subject always reports the same preference order for each pair presented on different occasions. Another necessary condition is that the relation $\succ$ (is strictly preferred to) be transitive. If "indifference" is allowed, the ordering relation $i \precsim j$ ($j$ is not preferred to $i$) must also be transitive with the implication that $i \precsim j$ and $j \not\precsim i$ means $i \succ j$ ("$i$ is strictly preferred to $j$") and $i \precsim j$, $j \precsim i$ means $i \sim j$ ("$i$ is indifferent to $j$"). If both $\succ$ and $\sim$ are transitive relations, a weak order is induced on the objects. It can happen, however, that $\succ$ is a transitive relation but $\sim$ is not. This situation is often observed in experiments on discrimination. For instance, a subject may well not be able to discriminate between the pitches of tones $i$ and $j$, so that $i \sim j$, and between $j$ and $k$, so that $j \sim k$, but may well be able to discriminate between $i$ and $k$, so that $i \succ k$.

It appears, therefore, that even the problem of assigning the "weakest" quantitative scale, the so-called ordinal scale, which would supposedly reflect some subjective ordering on a set of objects, is beset with difficulties. First, the data on paired comparisons may be inconsistent in different presentations. Further, even if paired comparisons are consistent, and the relation $\succ$ is transitive, the relation $\sim$ may not be. Finally, even the relation $\succ$ may not be transitive. Attempts to get around

these difficulties lead to re-definitions of order relations. The new definitions are made rigorous by axiomatization.

Consider the case where the only obstacle to establishing a weak order is the intransitivity of $\sim$, as in discrimination experiments. The psychologist already has a concept that provides an intuitive explanation for the phenomenon, namely that of a "threshold." If there is a minimum difference between the "objective" magnitudes that allows discrimination, the subject will fail to discriminate between adjacent pairs of magnitudes that differ by less than that minimum, yet may distinguish between the extremes, if their difference exceeds the threshold. The mathematician provides a set of axioms that leads to the "emergence" of the threshold concept as a consequence [**29**].

Let a countable set $A$ (of stimuli) be given together with an empirically established relation $>$, representing the subject's responses to pairs of stimuli. Assume that the following axioms are satisfied.

For every $a, b, c, d$ of $A$:

$S_1$: Not $a > a$.

$S_2$: If $a > b$ and $c > d$, then either $a > d$ or $c > b$.

$S_3$: If $a > b$ and $b > c$, then either $a > d$ or $d > c$.

It will be readily seen that $S_3$ together with $S_1$ establishes the transitivity of $>$. For if $d = c$, then $a > b$ and $b > c$ implies $a > c$, since $c \not> c$ by $S_1$. However, an equivalence relation is not yet incorporated into these axioms. The familiar "indifference" relation $\sim$ could be incorporated in the usual way: $a \sim b$ if and only if $b \not> a$ and $a \not> b$; but the transitivity of this relation, required if it is to play the role of an equivalence relation in a weak order, is not implied by the above axioms. Therefore, another relation, $E$, is introduced together with $\sim$ as defined above. Namely, $aEb$ if and only if for every $c \in A$, $a \sim c$ if and only if $b \sim c$.

It is shown that $E$ is indeed an equivalence relation, reflexive, symmetric, and transitive.

A system with a countable number of equivalence classes satisfying $S_1$—$S_3$ is called a semi-order. The equivalence classes induced by $E$ form a relational system isomorphic to a *numerical* semi-order, represented by a set of real numbers $N$ and a binary relation $>$ with the property that for $x$ and $y \in N$, $x > y$ if and only if $x > y + \delta$, $\delta$ constant.

Thus, the semi-order is an axiomatization of conditions on the data necessary and sufficient to justify an inference that a jnd (just noticeable difference) exists in terms of which a subject's acuity of discrimination among a set of stimuli can be expressed. Note that this jnd or threshold of discrimination need not represent a difference between a pair of physical magnitudes. The jnd derived from the semi-order axioms is a difference between a pair of subjectively perceived "magnitudes," not necessarily related to objectively measurable ones.

A particular representation by a numerical semi-order is not unique, and the problem of determining the set of permissible transformations on the representation is not a simple one. It may be intrinsically interesting to the mathematician, although its psychological import is not immediately apparent. If a psychological interpretation of the problem does eventually suggest itself, we shall have an instance of a feedback circuit: a psychological situation presents a mathematical problem, which, when solved provides an additional, possibly useful psychological concept.

Next, suppose that the results of paired comparisons are inconsistent. The inconsistencies can be attributed to random fluctuations in a set of magnitudes that reflect the "true" subjective magnitudes associated with the objects compared. Thus, the subjective magnitude associated with each object in a particular instance is considered as a random variable. The pair is ordered by the subject in accordance with the values assumed by the corresponding random variables at the time of comparison. This assumption suggests the problem of determining the frequency distributions of the random variables, which would account for the probabilities of the results of paired comparisons, as reflected in their frequencies in many presentations. To put the problem into practical perspective, special assumptions are made about the presumed frequency distributions, for instance, normality; further, in the interest of simplicity, a common variance, etc. [**56**].

Another approach is to ask under what conditions can a magnitude $v(i)$ be assigned to each object in the set $A = \{i, j, \cdots\}$ under consideration so that if $p_{ij}$ is the probability that $i > j$, as reflected in the relative frequencies of that datum in paired comparisons, then

$$p_{ij} = \frac{v(i)}{v(i) + v(j)}? \tag{31}$$

Note that such an assignment has intuitively acceptable properties. As $v(i)$ and $v(j)$ become widely disparate, $p_{ij}$ approaches 0 or 1. As they approach equality, $p_{ij}$ approaches 1/2. It is easy to see that the following condition is necessary and sufficient for the existence of a function $v(\ ): A \to R$:

$$p_{ik}/p_{ki} = (p_{ij}/p_{ji})(p_{jk}/p_{kj}) \tag{32}$$

for all $i, j, k \in A$. If (32) is satisfied, we can assign $v(i)$ arbitrarily for some $i$ and let $v(j) = v(i)(p_{ji}/p_{ij})$ for every $j \in A$. The magnitudes $v$ are then given on a ratio scale, invariant up to a similarity transformation (choice of unit) [**54**].

It seems natural to generalize (31) for the case where the subject chooses the most preferred alternative $x$ from a finite set $A = x, y, \cdots$. Let $P_A(x)$ be the probability that he does so. Now, assuming $p_{ij} + p_{ji} = 1$, we have

$$p_{ij} = \frac{p_{ij}}{p_{ij} + p_{ji}}. \tag{33}$$

The suggested generalization for several alternatives of a set $A$ is

$$P_{xy} = \frac{P_A(x)}{P_A(x) + P_A(y)}. \tag{34}$$

However, difficulties arise. To take the simplest case, let $A = (x, y, z)$. Changing the notation slightly, let $p(x, y)$ be the probability that $x$ is preferred to $y$. Dropping the subscript denoting the set of alternatives, let $P(x)$ be the probability that the subject names $x$ as the most preferred of $x, y, z$.

Finally, let $p(x, y, z)$ be the probability that when presented with all three, the subject names them in that order from the most preferred to the least preferred. The following axioms related to the structure of the choices seem innocuous:

$$p(x, y, z) = P(x)p(y, z), \tag{35}$$

$$P(x) = p(x, y, z) + p(x, z, y), \tag{36}$$

$$p(x, y) = p(x, y, z) + p(x, z, y) + p(z, x, y). \tag{37}$$

Assuming (35)–(37), it is easy to prove that

$$p(x, y) = P(x)[P(x) + P(y)]^{-1} \tag{38}$$

which is a special case of Luce's Choice Axiom. The relation (38) is corroborated by considerable experimental evidence, which, in turn, supports (35)–(37) as a reasonable model of choice structure.

Suppose now that the subject is asked to rank the three alternatives in reverse order, i.e., from the least preferred to the most preferred. Accordingly, define $p^*(x, y, z)$. Analogously $p^*(x, y)$ is the probability that when presented with the pair $\{x, y\}$ the subject will name $x$ as the less preferred, and $P^*(x)$ is the probability that when asked to name the least preferred, the subject will name $x$. Then assuming the validity of (35)–(37), the following axioms seem compelling:

$$p^*(x, y, z) = p(z, y, x) \tag{39}$$

$$p^*(x, y) = p(y, x) \tag{40}$$

$$p^*(x, y, z) = P^*(x)p^*(y, z) \tag{41}$$

$$P^*(x) = p^*(x, y, z) + p^*(x, z, y) \tag{42}$$

$$p^*(x, y) = p^*(x, y, z) + p^*(x, z, y) + p^*(z, x, y). \tag{43}$$

An initially surprising result can now be proved: if (35)–(43) hold then $P(x) = P(y) = P(z) = P^*(x) = P^*(y) = P^*(z) = 1/3$. In other words, the "reasonable" assumptions concerning the structure of choice probabilities are consistent only in the trivial case when the three alternatives are "completely indistinguishable." One is reminded of K. Arrow's Impossibility Theorem, where the apparently reasonable assumptions concerning the rule of deriving the social ranking of alternatives from individual rankings are inconsistent except when the rule is "trivial," e.g., when the social ranking is always identical to that of a particular individual (a dictator).

The counter-intuitive result on the structure of individual choice behavior has naturally stimulated further analysis [**55**]. Consider the six equations

$$p(x, y, z) + p(z, y, x) + p(z, x, y) = \tfrac{1}{2} + \alpha_i \qquad (i = 1, \cdots, 6) \tag{44}$$

representing the six permutations of $a$, $b$, and $c$, where the $\alpha_i$ are the "excesses" or "deficiencies" of the sums with respect to $\frac{1}{2}$. It can be easily shown that if $\alpha_i = 0$ $(i = 1, \cdots, 6)$, then all rankings are equally probable. Moreover, in every case the $\alpha_i$ come in pairs with opposite signs. Hence if the rankings are not all equiprobable, at least one $\alpha_i$ is positive. Call it $\alpha$. An elementary calculation shows that in that case

$$\begin{aligned} &p(x, y, z) \leqq (\tfrac{1}{4} + \alpha/2) - (\alpha/2)^{1/2} \\ \text{or} \quad &p(x, y, z) \geqq (\tfrac{1}{4} + \alpha/2) + (\alpha/2)^{1/2}. \end{aligned} \tag{45}$$

An even stronger result is that only two probability values satisfying (45) need be known in order to compute all the remaining probability values of the model, whereas for an arbitrary structure, of course, five values of $p$ would need to be known in order to determine the sixth and so the remaining ones.

The paradox dissolves somewhat when we interpret the results psychologically. *Formally speaking*, we can assign arbitrary values to five of the probabilities $p(x, y, z)$. However, in practice these probabilities will be usually strongly interdependent. Thus, for a given subject the probabilities $p(x, y, z)$ and $p(z, y, x)$ can be expected to be rather disparate, representing, as they do, reverse preference orders. These psychologically generated interdependencies are not taken into account in the abstract model. It is, on second thought, not surprising that either the magnitudes of some probabilities must constrain the magnitudes of others or the subject's probabilities of choices must violate some of the axioms above. Similar considerations apply to Arrow's Impossibility Theorem. It derives from the demand that the rule governing social choice be applicable to *all* possible individual rankings of a set of alternatives. In practice, however, both the individuals' rankings and the alternatives are usually strongly interdependent. For instance, in the simplest case when both the individuals and the alternatives can be ranked on a single political spectrum (e.g., Right-Left), non-trivial social decision rules exist that satisfy the axioms of social choice.

In apparently paradoxical situations the mathematical psychologist is stimulated to re-examine the fundamental assumptions of a proposed model of a psychological phenomenon by stating them as formal axioms so as to pinpoint the probable points of vulnerability, where, presumably, paradoxes or inconsistencies have their roots. A. Tversky and J. E. Russo [**58**] examined the so called ***simple scalability*** assumption, which underlies most probabilistic theories of choice. In what follows, $x, y, \cdots$, elements of a finite set $S$, represent the alternatives among which choices are to be made.

If $p(x, y)$ is the probability that $x$ is chosen over $y$, so that $p(x, y) + p(y, x) = 1$, the principle of simple scalability can be stated as follows:

$$p(x, y) = F[v(x), v(y)], \tag{46}$$

where $F$ is a strictly increasing function of the first argument and strictly decreasing in the second. In spite of its generality, the principle has testable consequences. Normally, for all $x, y, z$ in $S$

$$\begin{aligned} &\text{If } p(x, y) \geqq \tfrac{1}{2} \text{ and } p(y, z) \geqq \tfrac{1}{2}, \\ &\text{then } p(x, z) \geqq \max[p(x, y), p(y, z)] \end{aligned} \tag{47}$$

(strong stochastic transitivity). The latter implies, in turn

$$p(x, z) \geqq p(y, z) \text{ if and only if } p(x, y) \geqq \tfrac{1}{2} \tag{48}$$

(substitutability). Finally, for all $x, y, z, w \in S$,

$$p(x, z) \geqq p(y, z) \text{ if and only if } p(x, w) \geqq p(y, w), \tag{49}$$

(independence).

It is shown that simple scalability, strong stochastic transitivity, substitutability, and independence are all equivalent conditions. In several experiments violations of one or another of the conditions have been observed, suggesting a choice-determining factor not taken into account in the models based on simple scalability. This factor has to do with similarity of stimuli. For a fixed difference of psychological scale values, $v(x) - v(y)$, the more similar $x$ and $y$, the easier is the choice between them (in other words, the more salient is the difference of values). The implication is that substitutability is violated because alternatives may be substituted in some contexts but not in others.

The independence principle can also be violated by similarity. Tversky and Russo asked subjects to judge which of two geometric figures had the larger area, where each stimulus was paired with two standard stimuli that were maximally dissimilar from each other with respect to shape. The independence principle states that orders of choice probabilities should coincide. If, on the other hand, similarity facilitates judgment, then discrepancies should appear in specified pairs of stimuli. The results confirmed the latter hypothesis.

On some reflection, the similarity hypothesis conforms with our intuition about choices, at least in the context of preference. In the simplest, idealized case, consider three alternatives, $x$, $x'$, and $y$. Assume that the subject can distinguish between $x$ and $y$ but is indifferent between them and that he cannot distinguish between $x$ and $x'$. We would expect that the subject would prefer $x$ to $x'$ with probability 1/2, $x$ to $y$ with probability 1/2, and $x'$ to $y$ with probability 1/2. Simple scalability or any of the equivalent conditions would imply that given the choice among all three, the subject would choose any one with probability 1/3. However, this would hardly happen in a concrete situation. Rather, he would choose $x$ (or $x'$) with probability 1/2 and $y$ with probability 1/2. That is he would lump the indistinguishable $x$ and $x'$ into one alternative.

These considerations led Tversky to propose a different probabilistic model of choice, based on elimination by aspects (the EAB model) [57]. To each alternative $x \in S$ there is associated a finite set of aspects $x' = \{\alpha, \beta, \cdots\}$. Faced with a choice, the subject picks an aspect with a probability proportional to the value (importance) of the aspect to him and eliminates from consideration all the alternatives that do not have that aspect. He continues the procedure with the reduced set. If an aspect picked is shared by all the alternatives, another aspect is picked. Thus the set of alternatives is further reduced until either a single alternative remains or until the last aspect is shared by all the remaining alternatives. In the latter case, the choices among them become equiprobable. Thus, the model

assumes a value scale $v(\alpha)$ that is associated with the aspects instead of with the alternatives. Formally, let $A$ be the set of alternatives considered, $A^0$ the set of aspects shared by all the alternatives in $A$, and $A_\alpha = \{x \mid x \in A \text{ and } \alpha \in x'\}$. Then, according to the EAB model

$$P_A(x) = \frac{\sum_{\alpha \in x' - A^0} v(\alpha) P_{A_\alpha}(x)}{\sum_{\beta \in x' - A^0} v(\beta)}. \tag{50}$$

The testable consequences of the EAB model are weaker than those of models that assume simple scalability. For instance,

$$\text{If } p(x,y) \geqq \tfrac{1}{2} \text{ and } p(y,z) \geqq \tfrac{1}{2}, \text{ then } p(x,z) \geqq \min[p(x,y), p(y,z)], \tag{51}$$

which is clearly a weaker property than (46).

Experiments on estimating numbers of dots in a frame, on choices among gambles, and on assessments of job applicants from score profiles yielded results corroborating the EAB model.

Numerical measures of "utility" are usually envisaged as given on interval scales, determined up to a positive linear transformation (choice of unit and origin). Following von Neumann and Morgenstern's suggestion [**60**], the empirical determination of a subject's utilities for each of a set of objects can be derived from his preferences among so called risky choices. In particular, suppose a subject prefers \$10 to \$5 and \$5 to \$0. We want to estimate the ratio of the two utility intervals (\$10, \$5) and (\$5, \$0); in other words, we want to get an idea of the relative strengths of these preferences. The subject is offered a choice between two alternatives:

$L_1$: \$5 with probability 1.

$L_2$: A lottery ticket that awards \$10 or \$0 with equal probabilities.

If the subject prefers $L_1$, the odds in favor of \$10 in $L_2$ are increased until the subject is indifferent between $L_1$ and $L_2$. In the contrary case, the odds are adjusted in favor of \$0. At some odds the subject must switch his preference; otherwise he would eventually prefer \$5 to \$10 or \$0 to \$5. Let the odds that elicit indifference between $L_1$ and $L_2$ be $p$: $q$ $(p + q = 1)$. We set

$$u(\$10) = 1, \quad u(\$0) = 0, \quad u(\$5) = pu(\$10) + qu(\$0) = p. \tag{52}$$

In general, a utility can be assigned to each of a set of objects $A_1, A_2, \cdots, A_r$, listed in the order of preference, if of every pair of lotteries involving them, $L^{(i)} \equiv (p_1^{(i)}, p_2^{(i)}, \cdots, p_r^{(i)})$ and $L^{(j)} \equiv (p_1^{(j)}, p_2^{(j)}, \cdots, p_r^{(j)})$, the subject always prefers the one which, after the assignment of utilities, turns out to have the larger expected utility. (The $p$'s are understood to be the probabilities with which a given lottery awards the corresponding object.) Such utilities can be assigned if the following axioms are satisfied [**33**].

$U_1$: A transitive ordering $\gtrsim$ holds between every two objects: $A_1 \gtrsim A_2, \cdots \gtrsim A_r$.

$U_2$: Lotteries can be compounded. If

$$L^{(i)} = (p_1^{(i)}, p_2^{(i)}, \cdots, p_r^{(i)}) \qquad (i = 1, \cdots, s)$$

are lotteries on $A$ and $L' = (q_1, q_2, \cdots, q_s)$ is a lottery that awards a ticket to lottery $L^{(i)}$ with probability $q_i$, then there exists a lottery on $A$, $L = (p_1, p_2, \cdots, p_r)$ such that the subject is indifferent between $L'$ and $L$, namely, the lottery in which $p_k = \sum_{i=1}^{s} q_i p_k^{(i)}$.

$U_3$: The subject is indifferent between each object $A_i \in A$ and some lottery involving only the most preferred $A_1$ and the least preferred $A_r$: $A_i \sim (u_1, 0, 0, \cdots, u_r)$ $(u_1 + u_r = 1)$.

$U_4$: The ticket to the above lottery can be substituted for $A_i$ in any lottery for the object to be awarded with probability $p_i$.

$U_5$: A weak preference ordering exists on the lotteries.

Utilities so considered can be extended to an arbitrary set of objects or situations with a countable, order dense subset, provided the axioms are satisfied by the data. It is to be expected that the more "abstract" or qualitatively different the prizes are, the more unlikely are the axioms to be satisfied, since in comparing them, the subject's attention may be turned now to one aspect of a situation, now to another and not always consistently.

The so-called additive conjoint measurement model **[34, 48]** was designed to deal with preferences among objects that differ on "several dimensions," as it were. Consider a set of objects, each characterized by two attributes, say appearance and price, that is a product set $A \times P$, and assume that the subject is able to arrange the objects in an order of preference. This ordering permits the assignment of an ordinal measure to each object (invariant up to a monotone transformation). Let $M(a, p)$ with $a \in A$, $p \in P$ be such a measure: $M(a, p)$: $A \times P \to R$. We want to know whether the preference order can be accounted for by a sum of two utilities $f(a)$ and $g(p)$, representing respectively the utility of each "appearance" and the utility of each price. In other words, we seek a function $\phi(a, p) = f(a) + g(p)$, such that $M(a, p) \geqq M(a', p')$ if and only if $f(a) + g(p) \geqq f(a') + g(p')$. Such a function exists if the following axioms are satisfied.

$A_1$: For all $a, b, c \in A$ and $p, q, r$ in $P$, if $M(a, q) \geqq M(b, p)$ and $M(b, r) \geqq M(c, q)$, then $M(a, r) \geqq M(c, p)$.

$A_2$: If $M(a, p) \geqq t \geqq M(a, q)$ for some real number $t$, then there exists an $r \in P$ such that $M(a, r) = t$. Analogously, if $M(a, p) \geqq t \geqq M(b, p)$, then there exists a $c \in A$ such that $M(c, p) = t$.

Roughly speaking, $A_2$ guarantees that the sets $A$ and $P$ are sufficiently "dense" to allow interpolation with respect to both aspects.

If $A_1$ and $A_2$ are satisfied, $\phi(a, p)$, as specified above, exists. Moreover, $f(a)$ and $g(p)$ can be represented on an interval scale with a common unit, so that if $f_1(a)$ and $g_1(p)$ also satisfy $\phi(a, p) = f_1(a) + g_1(p)$, then there exist constants $m, n$, and $s > 0$ such that $f_1(a) = sf(a) + m$; $g_1(p) = sg(p) + n$. In other words, it is possible to determine (up to a positive linear transformation) the subject's utilities for each of the aspects so as to account for the utility of each object as the sum of the former utilities, and moreover, these two utilities have a common unit.

Needless to say, the axioms relating to the determination of utilities by preferences among risky choices and axiom $A_2$, relating to additive conjoint measurement, make very stringent demands on the subject's acuity of preference and on his consistency of choices. In practice, these axioms will hardly ever be completely satisfied. They can, however, serve as points of anchorage for a systematic study of preferences. The question of practical interest is: *to what extent* a subject behaves *as if* he were governed by utilities of the sort defined?

A conjoint measurement model is a special case of a more general class of models called composite measurement models. The central problem in constructing these models is that of assigning magnitudes (on whatever scale) in situations where the properties of the objects investigated are in product sets. For example, a lottery involving $n$ prizes can be conceived as a point in $R^n$, $(p_1, \cdots, p_n)$ corresponding to the probabilities associated with the prizes. The problem is to associate a utility to each lottery so as to generate a prediction about the subject's preferences among the lotteries.

To take another example, consider $n$ students attempting to solve $m$ problems. Under what conditions is it possible to rank both students and problems in the order of ability and difficulty respectively, so that if student $i$ solves problem $j$, then he must have solved all the "easier" problems and all the "more able" students have solved problem $j$? Such a composite scale, if it exists, is called a Guttman scale (after L. Guttman).

Or consider a set of objects and a set of subjects, where each subject can rank the objects in the order of preference. Under what conditions is it possible to assign to each object and to each subject a position on a one-dimensional continuum so that the distance from each object to each subject will reflect his order of preference for the objects? Situations that permit such a representation are called Coombs systems (after C. H. Coombs).

Finally consider problems of classification involving more than one class of categories. Specifically, consider $n$ patients and $m$ symptoms. Under what conditions is it possible to assign a disease (defined as a set of symptoms) to each patient in such a way that every patient exhibits all the symptoms corresponding to his disease and only those? Situations where this is possible are called bi-classificatory systems.

An axiomatization of various instances of composite measurement was undertaken by A. Ducamp and J. S. Falmagne [**13**].

The examples discussed may serve to illustrate the contention that the problems of measurement that arise in psychology are more complex than those that arise in the physical sciences, where, for instance, the additivity of extensive quantities (weight, volume, time intervals, electric current, etc.) is taken for granted and where ratio scales are appropriate for these quantities, allowing the greatest latitude in the formulation of "physical laws." In this connection, see [**30**]. Systematic treatments of the problems of measurement with special reference to the issues raised in mathematical psychology are presented in [**25**] and [**46**].

It is interesting to observe that the axiomatizations associated with composite measurements are generalizations of those associated with the measurement of composite physical quantities such as pressure, kinetic energy, density, etc. Originally, the legitimacy of the mathematical operations associated with those concepts was simply taken for granted. Problems of mathematical psychology have brought out the necessity of rigorous justification of mathematical operations in more general contexts and so have illuminated the logical foundations of physical measurement.

In fact, the thrust in the work discussed so far has been in the direction of generalizing the theory of measurement so as to make it applicable to quantities, which, unlike physical quantities, do not suggest "obvious" measurement operations with meter sticks, balances, or clocks. Mathematization is still thought of as dependent on given unidimensional or multidimensional continua of real numbers constituting "psychological spaces." Thus the basic paradigm of classical physics (with "given continua") is not lost sight of.

Other directions in mathematical psychology represent departures from this paradigm. Stochastic models, for example, are based essentially on counting occurrences of presumably identical events (intuitively recognized as such) rather than on generalizations of physical measurements. Further, a class of models, which we have called descriptive, play a more important part in mathematical psychology than in mathematized natural science. This is because in psychology a structural description of a situation, which is a pre-requisite of any theory with predictive pretensions, presents its own problems. Problems of multi-dimensional scaling are of this sort.

Finally, some purely conceptual developments, not yet formulated in any substantive theory can be subsumed under "mathematical psychology" to the extent that they have been suggested by some recent developments in "non-physical mathematics" (e.g., information theory and game theory).

The above mentioned directions will be discussed in Part II.

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# INVERSE PROBLEMS

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**1. Introduction.** We call two problems **inverses** of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former is called the **direct problem,** while the latter is called the **inverse problem.** As illustrations, we present the following three inverse problems. The corresponding direct problems, which are their solutions, are given in the appendix.

1. What is the question to which the answer is "Washington Irving"?
2. What is the question to which the answer is "Nine W"?
3. What is the question to which the answer is "Chicken Sukiyaki"?

These examples demonstrate that inverse problems often have many solutions, and also that some particular solution is preferable to the others.

Some examples of inverse problems in mathematics are the following:

4. Find a polynomial $p(x)$ of degree $n$ with the roots $x_1, \cdots, x_n$. This is inverse to the direct problem of finding the roots $x_1, \cdots, x_n$ of a given polynomial $p(x)$ of degree $n$. In this case the inverse problem is easier, having the solution $p(x) = c(x - x_1) \cdots (x - x_n)$, which is not unique because $c \neq 0$ is an arbitrary constant.

5. Find a polynomial $p(x)$ of degree $n$ with given values $y_1, \cdots, y_n$ at $x_1, \cdots, x_n$. The corresponding direct problem is to find the values $y_1, \cdots, y_n$ of a given polynomial $p(x)$ at $x_1, \cdots, x_n$. The inverse problem is called the Lagrange interpolation problem, while the direct problem is that of evaluation of a polynomial.

6. Given a real symmetric matrix $A$ of order $n$, and $n$ real numbers $\lambda_1, \cdots, \lambda_n$, find a diagonal matrix $D$ so that $A + D$ has eigenvalues $\lambda_1, \cdots, \lambda_n$. This is inverse to the direct problem of finding the eigenvalues $\lambda_1, \cdots, \lambda_n$ of a given real symmetric matrix $A + D$.

A common inverse problem used on intelligence tests is this:

7. Given the first few members $a_1, a_2, a_3, a_4$ of a sequence, find the law of formation of the sequence, i.e., find $a_n$ for all positive integers $n$. Usually only the next few members $a_5, a_6, a_7$ are asked for as evidence that the law of formation has been found. The direct problem is to evaluate the first few members of a sequence $a_n$, given the law of formation. A well-known instance of this inverse problem is to find the next few members of the sequence which begins 4, 14, 34, 42. The solution is 59, 125, 145, since the sequence consists of the express stops on the 8th Avenue subway in New York. It is clear that such inverse problems have many solutions, and for this reason their use on intelligence tests has been criticized.

The main sources of inverse problems are science and engineering. Often these problems concern the determination of the properties of some inaccessible region from observations on the boundary of the region. Here are some examples:

8. Find the mass distribution $\rho(x)$ of matter within the earth, given the gravitational potential $\phi(x)$ for $x$ on the surface of the earth. This is inverse to the direct problem of finding $\phi(x)$ given $\rho(x)$, which has the solution

$$\phi(x) = \frac{G}{4\pi} \int \rho(x')/|x - x'|\, dx'.$$

Here $G$ is the gravitational constant and $|x - x'|$ is the distance from $x$ to $x'$. The inverse problem is

important in locating high or low density regions within the earth, which may contain ore or oil, from observations of anomalies in the gravitational potential on the earth's surface.

9. Find the intermolecular potential $V(r)$ between two molecules a distance $r$ apart, given the equation of state of a gas composed of such molecules. This is inverse to the direct problem of finding the equation of state, given the potential, which is a basic problem of statistical mechanics.

10. Find the intermolecular potential $V(r)$ from scattering data; i.e., from information about the angle through which a molecule is scattered when it collides with another molecule. The direct problem is that of calculating the scattering angle, given the potential. These problems can be analyzed by either classical mechanics or quantum mechanics, and the two kinds of analysis are applicable in different physical situations.

11. Find the shape of a scattering object, given the intensity of light, of radar waves, or of sound waves it scatters in any direction. The direct problem is that of calculating the scattered light, radar or sound intensity in any direction from a given illuminated object. This problem is important in identifying objects in space from radar observations of them, and of identifying objects in the ocean from sonar observations.

In each of the seven following sections of this paper, an inverse problem which has arisen in physics will be analyzed.

**2. Determination of the shape of a hill from travel time, and probing the ionosphere.** Suppose we slide a particle up a frictionless hill with initial energy $E$, and measure the time $T(E)$ required for it to return. If we vary $E$ and measure $T(E)$, can we determine the shape of the hill from it? This problem was formulated and solved by Abel in 1826. To analyze it we shall first formulate and solve the **direct problem:** Given the shape of the hill, find the travel time $T(E)$.

Let $s$ denote arclength along the hill and let the height of the hill at $s$ be $h(s)$ with $h(0) = 0$. We denote by $m$ the mass of the particle, by $g$ the acceleration of gravity, and by $s(t)$ the position of the particle at time $t$. Then the equation of motion of the particle is

$$m\frac{d^2s(t)}{dt^2} = -\frac{dV(s)}{ds}. \tag{2.1}$$

Here $V(s) = mgh(s)$ is the potential energy of the particle at $s$, so $V(0) = 0$. We shall measure $s$ and $t$ from the initial position and from the instant of release of the particle, respectively, and denote by $v_0$ its initial velocity. Then the initial conditions are

$$s(0) = 0, \qquad \frac{ds(0)}{dt} = v_0. \tag{2.2}$$

We shall assume that $v_0 > 0$.

We now multiply (2.1) by $ds/dt$ and integrate to get the energy equation

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 + V(s) = E. \tag{2.3}$$

In (2.3) $E$ is the total energy of the particle, given by

$$E = \tfrac{1}{2}mv_0^2. \tag{2.4}$$

Next we solve (2.3) for $(ds/dt)^{-1}$ choosing the positive square root because $v_0 > 0$, and integrate using (2.2) to get

$$t = \left(\frac{m}{2}\right)^{1/2}\int_0^s [E - V(s')]^{-1/2}\,ds', \qquad 0 \leqq t \leqq \tfrac{1}{2}T(E). \tag{2.5}$$

This equation yields the solution in the form $t = t(s)$ from $s = 0$ until $s$ reaches the value $s_1(E)$, which is the smallest value of $s \geqq 0$ at which $E - V(s) = 0$. The corresponding value of $t$, which we shall call

$\frac{1}{2}T(E)$, is given by setting $s = s_1(E)$ in (2.5). Thus

$$T(E) = (2m)^{1/2} \int_0^{s_1(E)} [E - V(s')]^{-1/2}\, ds'. \tag{2.6}$$

At the instant $T(E)/2$ the particle velocity vanishes, and then the velocity becomes negative. Thus the solution for $t \geqq T(E)/2$ is obtained by using the negative square root in solving (2.3). The result can be written in the form

$$t = \frac{1}{2}T(E) + \left(\frac{m}{2}\right)^{1/2} \int_s^{s_1(E)} [E - V(s')]^{-1/2}\, ds', \qquad \frac{1}{2}T(E) \leqq t. \tag{2.7}$$

By setting $s = 0$ in (2.7), we see that the time at which the particle returns to its initial position $s = 0$ is just $T(E)$ given by (2.6). Thus (2.6) yields the solution of the direct problem. We see that it is defined only for those values of $E$ for which $s_1(E)$ exists. These are just the values of $E$ in the interval $0 \leqq E \leqq V_m \equiv \sup_{s \geqq 0} V(s)$. For $E > V_m$ the particle never returns to $s = 0$, so we can set $T(E) = \infty$ for $E > V_m$.

Now we can consider the inverse problem: Given the travel time $T(E)$ for $E \geqq 0$, find the shape of the hill, i.e., find the potential $V(s)$ for $s \geqq 0$. Once we know $V(s)$ then the height of the hill is given by $h(s) = V(s)/mg$. To find the equation of the hill in cartesian coordinates we write it parametrically as $x(s), y(s)$. Then $y(s) = h(s)$, while $x(s)$ is related to $y(s)$ by the arclength condition $(dx/ds)^2 + (dy/ds)^2 = 1$. Solving this equation for $dx/ds$ and integrating yields

$$x(s) = x(0) + \int_0^s \left(1 - \left[\frac{dh(s')}{ds}\right]^2\right)^{1/2} ds', \qquad s \geqq 0. \tag{2.8}$$

This shows that the shape of the hill can be determined readily once $V(s)$ is found, so it suffices to find $V(s)$.

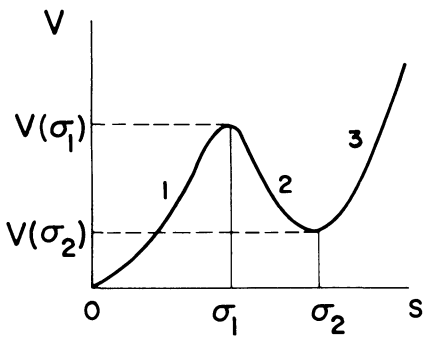



FIG. 1. A potential $V(s)$ which is increasing in the interval $0 \leqq s < \sigma_1$, decreasing in $\sigma_1 < s < \sigma_2$ and increasing again in $\sigma_2 < s$. The inverse function is called $s_1(V)$ in the first interval, $s_2(V)$ in the second interval and $s_3(V)$ in the third interval.

To find $V(s)$ we shall start with (2.6), which relates $T(E)$ to $V(s)$. It is a nonlinear integral equation for $V$. We can convert it into a linear integral equation by introducing $V$ instead of $s$ as the integration variable. This change of integration variable must be made separately in each interval within which $V(s)$ is monotonic. Therefore we first consider the first such interval, $0 \leqq s \leqq \sigma_1$ and suppose that $V(s)$ is monotone increasing within it. (See Fig. 1.) Let $s = s_1(V)$ denote the inverse of $V(s)$ within this interval, within which $0 \leqq V \leqq V(\sigma_1)$. Then we can rewrite (2.6) in this interval in the form

$$T(E) = (2m)^{1/2} \int_0^E [E - V]^{-1/2} \frac{ds_1(V)}{dV}\, dV, \qquad 0 \leqq E \leqq V(\sigma_1). \tag{2.9}$$

Eq. (2.9) is a Volterra integral equation of the first kind for $ds_1/dV$, and with the kernel $(E-V)^{-1/2}$ it is called an Abel equation. Its solution for $ds_1(V)/dV$ is

$$\frac{ds_1(V)}{dV} = \frac{T(0)}{(2mV)^{1/2}\pi} + \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-1/2}T'(E)dE. \tag{2.10}$$

This can be verified by substitution into (2.9). Finally integrating (2.10) from 0 to $V$ yields, after an integration

$$s_1 = \frac{1}{(2m)^{1/2}\pi}\int_0^V (V-E)^{-1/2}T(E)dE, \qquad 0 \leqq V \leqq V(\sigma_1). \tag{2.11}$$

This is the solution of the inverse problem for $V(s)$ in the interval $0 \leqq s \leqq \sigma_1$. It was obtained by Abel. If $V(\sigma_1) = \infty$, it is the complete solution. This solution applies to a particle in any potential, not only a gravitational one.

A method like that described here is used to find the electron density $n(s)$ at height $s$ above the earth in the ionosphere. The method is to send up a radio wave of angular frequency $\omega$, and to measure the time $T(\omega)$ required for it to return to the ground from the ionosphere. The group velocity $c_g$ of a wave of frequency $\omega$ in an electron plasma, like the ionosphere, is $c_g = c\omega^{-1}[\omega^2 - \omega_p^2(s)]^{1/2}$. Here $c$ is the velocity of light in vacuum and $\omega_p^2(s) = 4\pi e^2 n(s)/m$ is the plasma frequency, where $e$ and $m$ are the charge and mass of an electron, respectively. Thus the time for a wave to travel from $s = 0$ up to the first point $s_1(\omega)$, where $\omega^2 = \omega_p^2(s_1)$, and then back down to $s = 0$ is

$$T(\omega) = 2\int_0^{s_1(\omega)} \frac{ds}{c_g} = \frac{2\omega}{c}\int_0^{s_1(\omega)} [\omega^2 - \omega_p^2(s)]^{-1/2}ds. \tag{2.12}$$

This is of the form (2.6) and therefore its solution is of the form (2.11):

$$s_1(\omega) = \frac{c}{\pi}\int_0^{\omega_p^2} [\omega_p^2 - \omega^2]^{-1/2}T(\omega)\,d\omega. \tag{2.13}$$

**3. The nonmonotonic case.** Now we suppose that $V(\sigma_1)$ is finite, and that $V$ is decreasing in the interval $\sigma_1 < s < \sigma_2$ and then increasing in the next interval $\sigma_2 < \sigma < \sigma_3$. (See Fig. 1.) We denote by $s_2(V)$ and $s_3(V)$ the inverses of $V(s)$ in these two intervals, respectively. Then we can write (2.6) in the form

$$T(E) = (2m)^{1/2}\left[\int_0^{V(\sigma_1)} (E-V)^{-1/2}\frac{ds_1}{dV}\,dV - \int_{V(\sigma_2)}^{V(\sigma_1)} (E-V)^{-1/2}\frac{ds_2}{dV}\,dV + \int_{V(\sigma_2)}^{E} (E-V)^{-1/2}\frac{ds_3}{dV}\,dV\right]. \tag{3.1}$$

Next we define $w(V)$, the width at depth $V$ of the "potential well," by

$$w(V) = s_3(V) - s_2(V), \qquad V(\sigma_2) \leqq V \leqq V(\sigma_1). \tag{3.2}$$

In writing (3.2) we assume that $E \geqq V(\sigma_1)$. Then we can rewrite (3.1) as

$$T(E) = (2m)^{1/2}\left[\int_0^{V(\sigma_1)} (E-V)^{-1/2}\frac{ds_1}{dV}\,dV + \int_{V(\sigma_1)}^{E} (E-V)^{-1/2}\frac{ds_3}{dV}\,dV + \int_{V(\sigma_2)}^{V(\sigma_1)} (E-V)^{-1/2}\frac{dw}{dV}\,dV\right]. \tag{3.3}$$

The formula (3.3) shows that $T(E)$ involves only the width function $w(V)$ of the well, and does not depend upon its two sides $s_2(V)$ and $s_3(V)$ separately. Therefore it is not possible to obtain these sides from $T(E)$. However, as we shall soon see, it is not even possible to determine $w(V)$ from $T(E)$.

Let us regard (3.3) as an integral equation for $ds_3/dV$ in the range $V \geqq V(\sigma_1)$. Then it is of the same form as (2.9) with $T(E)$ replaced by $T(E)$ plus the integrals involving $ds_1/dV$ and $dw/dV$, and with the lower limit replaced by $V(\sigma_1)$. Therefore the solution is given by (2.10) with the same replacement. After integration, the solution becomes

$$(3.4)\qquad s_3(V) = s_3[V(\sigma_1)] + \frac{1}{\pi}\int_{V(\sigma_1)}^{V} (V-E)^{-1/2}\left[(2m)^{-1/2}T(E) - \int_0^{V(\sigma_1)} (E-z)^{-1/2}\frac{ds_1}{dz}\,dz - \int_{V(\sigma_2)}^{V(\sigma_1)} (E-z)^{-1/2}\frac{dw}{dz}\,dz\right] dE, \qquad V(\sigma_1) \leqq V.$$

This form of the solution can be simplified with the aid of the integral formula

$$(3.5)\qquad \int_{V(\sigma_1)}^{V} (V-E)^{-1/2}(E-z)^{-1/2}dE = 2\sin^{-1}\left(\frac{E-z}{V-z}\right)^{1/2}\Bigg|_{V(\sigma_1)}^{V} = \pi - 2\sin^{-1}\left[\frac{V(\sigma_1)-z}{V-z}\right]^{1/2}.$$

When (3.5) is used in (3.4), the solution can be written in the simpler form

$$(3.6)\qquad s_3(V) = \frac{1}{(2m)^{1/2}\pi}\int_{V(\sigma_1)}^{V} (V-E)^{-1/2}T(E)dE + \frac{2}{\pi}\int_0^{V(\sigma_1)} \frac{ds_1}{dz}\sin^{-1}\left[\frac{V(\sigma_1)-z}{V-z}\right]^{1/2} dz + \frac{2}{\pi}\int_{V(\sigma_2)}^{V(\sigma_1)} \frac{dw}{dz}\sin^{-1}\left[\frac{V(\sigma_1)-z}{V-z}\right]^{1/2} dz, \qquad V(\sigma_1) \leqq V.$$

The result (3.4) or (3.6) shows that $s_3(V)$ is not unique, since $w(V)$ can be chosen arbitrarily. For each choice of $w(V)$, $s_3(V)$ is determined uniquely in the range $V \geqq V(\sigma_1)$, up to the next local maximum of $V(s)$. The function $ds_1/dV$ in (3.4) is given by (2.10) in terms of $T(E)$.

The preceding considerations can be applied at once to a potential $V(s)$ with any number of local maxima and minima, but we shall not do that.

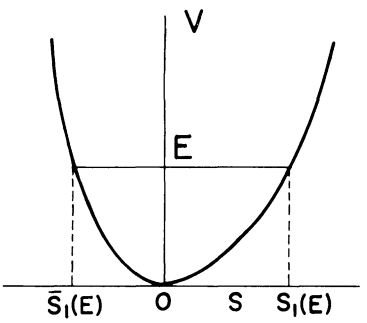



FIG. 2. A potential $V(s)$ which increases as $|s|$ increases. The intersections of the graph with the horizontal line at height $E$ are the roots $s_1(E)$ and $\bar{s}_1(E)$ of the equation $V(s) = E$.

**4. Determination of a potential from the period of oscillation.** Let us now consider a particle in a potential well. (See Fig. 2.) Its motion is still governed by (2.1) with the initial conditions (2.2). Therefore the results (2.3)–(2.7) apply, so the particle travels to the right until it reaches the point $s_1(E)$ at the time $T(E)/2$ and then it travels to the left, returning to the origin at the time $T(E)$. Its velocity at the origin is $-v_0$, so it will continue past the origin until it reaches a point $\bar{s}_1(E)$ where $E - V(s) = 0$, at a time we shall call $T(E) + \frac{1}{2}\bar{T}(E)$. Then its velocity reverses sign and it travels to the right, reaching the origin again at the time $T(E) + \bar{T}(E)$ with the velocity $v_0$. Thereafter the motion repeats periodically, since the position and velocity have returned to their initial values $s = 0$ and $ds/dt = v_0$.

The period $P(E)$ of this periodic motion is $P(E) = T(E) + \bar{T}(E)$, since the initial conditions first recur at this time. Now $T(E)$ is given by (2.6), and $\bar{T}(E)$ is given by the corresponding expression with $s_1(E)$ replaced by $\bar{s}_1(E)$, and with the negative square root. Thus the period is given by

$$(4.1)\quad P(E) = T(E) + \bar{T}(E) = (2m)^{1/2} \int_0^{s_1(E)} [E - V(s)]^{-1/2} ds - (2m)^{1/2} \int_0^{\bar{s}_1(E)} [E - V(s)]^{-1/2} ds$$

$$= (2m)^{1/2} \int_{\bar{s}_1(E)}^{s_1(E)} [E - V(s)]^{-1/2} ds.$$

Here $s_1(E)$ and $\bar{s}_1(E)$ are respectively the smallest (in absolute value) positive and negative roots of the equation $V(s) = E$.

The result (4.1) is the solution of the following problem, which we shall call the **direct problem:** Given a potential well $V(s)$, find the period $P(E)$ of oscillation of a particle with energy $E$. It is important to note that in general the period does vary with $E$, since in the often studied case of small amplitude oscillations, the period is independent of $E$. We shall return to this point later. Now we shall pose the **inverse problem:** Given the period of oscillation $P(E)$ of a particle with energy $E$ in a potential $V(s)$, find the potential.

To solve the inverse problem, let us consider first the case in which $V(s)$ is symmetric, $V(s) = V(-s)$. In this case it is clear that $\bar{s}_1(E) = -s_1(E)$ and then $\bar{T}(E) = T(E)$. Thus the period is just $P(E) = 2T(E)$, where $T(E)$ is given by (2.6). The problem of finding $V(s)$ was solved in Section 2 for the case in which $V(s)$ is monotonic increasing for $s > 0$. The solution is given by (2.11). The inverse problem for this symmetric monotonic case was first posed and solved by B. F. Kimball (1932). The symmetric nonmonotonic case is treated in Section 3.

Next, let us consider the nonsymmetric case. For simplicity we shall assume that $V(s)$ is monotone increasing as $|s|$ increases. Then we can introduce the two inverses $s_1(V) > 0$ and $\bar{s}_1(V) < 0$ of $V(s)$, and write (4.1) in the form

$$(4.2)\quad P(E) = (2m)^{1/2} \int_0^E (E - V)^{-1/2} \frac{ds_1}{dV} dV - (2m)^{1/2} \int_0^E (E - V)^{-1/2} \frac{d\bar{s}_1}{dV} dV$$

$$= (2m)^{1/2} \int_0^E (E - V)^{-1/2} \frac{dw}{dV} dV.$$

In writing the last form of the expression for $P(E)$, we have again introduced $w(V)$, the width of the well at height $V$, defined by $w(V) = s_1(V) - \bar{s}_1(V)$. This form shows that $P(E)$ depends only upon $w(V)$, and not upon the separate sides of the well, $s_1(V)$ and $\bar{s}_1(V)$. Therefore it is not possible to obtain these sides from $P(E)$, but only the width.

Eq. (4.2) is an Abel equation for $dw/dV$ which is of the form (2.9) with $w$ and $P$ replacing $s_1$ and $T$, respectively. Therefore the solution is given by (2.11) with the same replacements, i.e.,

$$(4.3)\quad w(V) = \frac{1}{(2m)^{1/2}\pi} \int_0^V (V - E)^{-1/2} P(E) dE, \qquad 0 \leqq V.$$

In the symmetric case $\bar{s}_1(V) = -s_1(V)$ so $w(V) = 2s_1(E)$ and $P(E) = 2T(E)$. Then (4.3) becomes identical with (2.11). It is always possible to *assume* that the well is symmetric, and then (2.11) yields the unique symmetric solution. However, whether or not the solution is symmetric is not determined by $P(E)$; only the width is determined.

As an application of (4.3), we shall find $w(V)$ when the period $P(E)$ is constant, independent of $E$. Then (4.3) yields

$$(4.4)\quad w(V) = \frac{P}{\pi} \left[\frac{2V}{m}\right]^{1/2}.$$

Solving (4.4) for $V$ yields

$$V = \frac{m}{2}\left[\frac{\pi w}{P}\right]^2. \tag{4.5}$$

Thus, if and only if $V$ is proportional to $w^2$ is the period constant, independent of $E$. In the symmetric case when $w = 2s_1$, (4.5) is just the quadratic potential which arises in the analysis of small amplitude oscillations.

The results of this section are due to J. B. Keller (1962). They can be extended to the case in which $V(s)$ is not a monotone function of $|s|$ by the method of Section 3, but we shall not carry out that extension.

**5. Inverse transit time problem.** We shall again consider a particle moving in the potential $V(s)$, governed by (2.1) with the initial condition (2.2). We now pose the following **direct problem:** Find the transit time $\tau(E)$ required for a particle of energy $E$ to travel from $s = 0$ to $s = L > 0$. From the analysis of Section 3, we know that the particle will reach $L$ if and only if $E$ satisfies the condition

$$E > V_m = \sup_{0 \le s \le L} V(s). \tag{5.1}$$

When (5.1) is satisfied, $\tau(E)$ is obtained by setting $s = L$ in the solution (2.5), which yields

$$\tau(E) = \left[\frac{m}{2}\right]^{1/2} \int_0^L [E - V(s)]^{-1/2} ds, \qquad E > V_m. \tag{5.2}$$

From the solution (5.2) of the direct problem, we see that $\tau(E)$ is a decreasing function of $E$. A series expansion of $\tau$ can be obtained by expanding the integrand with the aid of the binomial theorem, and integrating term by term. This yields with binomial coefficients $c_n$

$$\tau(E) = \left[\frac{m}{2E}\right]^{1/2} \sum_{n=0}^{\infty} c_n E^{-n} \int_0^L V^n(s) ds, \qquad E > V_m. \tag{5.3}$$

Now we can formulate the **inverse problem:** Given the transit time $\tau(E)$ for $E > V_m$, find the potential $V(s)$. In order that this problem have a solution, the given $\tau(E)$ must possess an expansion of the form (5.3), which we shall write as follows:

$$\tau(E) = \left[\frac{m}{2E}\right]^{1/2} \sum_{n=0}^{\infty} c_n E^{-n} \tau_n, \qquad E > V_m. \tag{5.4}$$

By comparing (5.3) with (5.4), we see that the coefficients $\tau_n$ in the expansion of the given function $\tau(E)$ are related to the unknown function $V(s)$ by the relations

$$\tau_n = \int_0^L V^n(s) ds, \qquad n = 0, 1, \cdots. \tag{5.5}$$

To find $V(s)$ from (5.2) or (5.5), we shall assume that $V(s)$ is monotone increasing. Then we can rewrite (5.2) and (5.5) as follows:

$$\tau(E) = \left(\frac{m}{2}\right)^{1/2} \int_0^{V(L)} (E - V)^{-1/2} \frac{ds}{dV}\, dV, \qquad E > V(L) = V_m. \tag{5.6}$$

$$\tau_n = \int_0^{V(L)} V^n \frac{ds}{dV}\, dV, \qquad n = 0, 1, \cdots. \tag{5.7}$$

Eq. (5.6) is a linear integral equation for $ds/dV$, called a Fredholm equation of the first kind. It differs from the Volterra equation (2.9) by having a fixed upper limit of integration, rather than a variable limit. Because of this difference we cannot solve it explicitly, as we did (2.9). Therefore we shall instead try to find $ds/dV$ from (5.7).

The integral in (5.7) is called the $n$th moment of $ds/dV$. Thus the problem of finding $ds/dV$ is equivalent to that of finding a function from all its moments. This classical problem is called the **moment problem.** One aspect of it is to determine whether a given set of numbers $\tau_n$ are the moments of some function. We shall assume that they are, because of the manner in which they are determined. Then it remains to find $ds/dV$ from them. Because (5.7) is a linear relation between $ds/dV$ and $\tau_n$, it follows that $ds/dV$ is a linear function of the $\tau_n$. Thus we shall seek $ds/dV$ in the form

$$\frac{ds}{dV} = \sum_{n=0}^{\infty} \tau_n g_n(V). \tag{5.8}$$

The functions $g_n(V)$ are to be determined in such a way that (5.7) holds. This leads to the condition

$$\int_0^{V(L)} V^n g_j(V) dV = \delta_{nj}. \tag{5.9}$$

Eq. (5.9) can be satisfied by choosing for $g_n(V)$ an appropriate polynomial of degree $n$. To find this polynomial it is convenient to introduce the orthonormal polynomials $\phi_n(V)$ of degree $n$, which satisfy

$$\int_0^{V(L)} \phi_n(V)\phi_j(V) dV = \delta_{nj}. \tag{5.10}$$

Then we express $V^n$ and $g_n(V)$ in terms of the $\phi_n$ as follows:

$$V^n = \sum_{j=0}^{\infty} c_{nj}\phi_j(V), \qquad c_{nj} = \int_0^{V(L)} V^n \phi_j(V) dV, \tag{5.11}$$

$$g_k(V) = \sum_{j=0}^{\infty} g_{kj}\phi_j(V). \tag{5.12}$$

Substitution of (5.11) and (5.12) into (5.9), and then using (5.10), yields for the $g_{kj}$ the equations

$$\sum_{j=0}^{\infty} c_{nj} g_{kj} = \delta_{nk}. \tag{5.13}$$

It follows from (5.11) that the matrix $c_{nj}$ is triangular, i.e., $c_{nj} = 0$ for $j > n$, and therefore (5.13) can be solved recursively for the $g_{kj}$. Then $g_n(V)$ is given by (5.12) and $ds/dV$ is given by (5.8). Integrating (5.8) yields finally

$$s(V) = \sum_{n=0}^{\infty} \tau_n \int_0^V g_n(V') dV'. \tag{5.14}$$

The foregoing analysis yields the unique monotone increasing solution $V(s)$ of (5.2). It is clear from (5.2), however, that any function equimeasurable with this solution is also a solution. Thus the inverse problem has many solutions, and we have found a particular one of them.

**6. Inverse eigenvalue problem.** In quantum mechanics the wave function $\psi(x)$ of a particle of mass $m$ in a potential $V(x)$ satisfies the Schrödinger equation

$$\psi_{xx} + \frac{2m}{\hbar^2}[E - V(x)]\psi = 0, \qquad -\infty < x < \infty. \tag{6.1}$$

Here $\hbar$ is Planck's constant divided by $2\pi$, and $E$ is the energy of the particle. Let us suppose that $V(x)$ is a monotonic increasing function of $|x|$, tending to $+\infty$ as $|x|$ becomes infinite. Then (6.1) has a quadratically integrable solution, not identically zero, if and only if $E$ is equal to one of the discrete set of eigenvalues $E_n$, $n = 0, 1, \cdots$ of the equation (6.1). The eigenvalues tend to $+\infty$ as $n$ increases, and they can be labelled so that $E_{n+1} \geqq E_n$.

The direct problem is that of finding the $E_n$, given $V(x)$. This problem can be solved asymptotically for large values of $E_n$ by the so-called WKB method. In this method one seeks a solution of (6.1) of the form $\psi(x) \sim A(x)\sin s(x)$. By first ignoring derivatives of $A$, one gets an equation for $s$. Then by keeping $A_x$ but ignoring $A_{xx}$, one gets an equation for $A$. This leads to the following asymptotic form of $\psi$:

$$\psi(x) \sim [E - V(x)]^{-1/4} \sin\left\{\frac{\pi}{4} + \int_{\bar{x}_1(E)}^{x} \left[\frac{2m}{\hbar^2}[E - V(x')]\right]^{1/2} dx'\right\}, \quad \bar{x}_1(E) < x < x_1(E). \tag{6.2}$$

This result holds only between the two roots $\bar{x}_1(E)$ and $x_1(E)$ of $V(x) = E$, and the constant $\pi/4$ results from matching (6.2) to a boundary layer solution valid around $\bar{x}_1(E)$. A similar matching to a boundary layer solution valid around $x_1(E)$ leads to the condition

$$(2m)^{1/2} \int_{\bar{x}_1(E)}^{x_1(E)} [E - V(X)]^{1/2} dx = \left(n + \frac{1}{2}\right)\pi\hbar, \qquad n = 0, 1, \cdots. \tag{6.3}$$

This equation (6.3) can be viewed as an equation for the determination of $E_n$, and thus provides an asymptotic solution to the **direct problem**. The **inverse problem** is that of finding the potential $V(x)$ given the eigenvalues $E_n$, $n = 0, 1, \cdots$. To solve it, we shall begin with (6.3), which holds asymptotically with $E = E_n$ and $n$ a large integer. We interpolate the given values $E_n$ by a smooth function $E(n)$, defined for all real $n \geqq 0$. Then we assume that (6.3) holds for all $n \geqq 0$ with $E = E(n)$, and (6.3) becomes an integral equation for $V(x)$. Differentiating (6.3) with respect to $n$ and dividing by $dE/dn$ yields

$$2\pi\hbar\left[\frac{dE}{dn}\right]^{-1} = (2m)^{1/2} \int_{\bar{x}_1(E)}^{x_1(E)} [E - V(x)]^{-1/2} dx. \tag{6.4}$$

If we set $P(E) = \hbar(dE/dn)^{-1}$, (6.4) becomes identical with (4.1). Therefore (6.4) has the solution (4.3) for the width $w(V)$ of the potential, which is all that can be determined from $E(n)$. This nonuniqueness is to be expected from the general theory of the inverse eigenvalue problem, and it is not a consequence of our method of approximate solution. If the potential is even, $V(-x) = V(x)$, then $w(V) = 2x_1(E)$ and the potential is unique. When the eigenvalues $E_n = (n + \frac{1}{2})\hbar\omega$ are used in (4.3), and $V(x)$ is assumed to be even, the potential $V(x) = \omega^2 m x^2/2$ is obtained, which is exactly that which yields these eigenvalues.

The fact that (6.4) coincides with (4.1) when $P(E) = 2\pi\hbar(dE/dn)^{-1}$ is an instance of the correspondence principle, according to which certain results of quantum mechanics are asymptotically equal to the corresponding results of classical mechanics at high energies. In the present case, the quantum mechanical frequency of oscillation $\nu$ associated with the energies $E_{n+1}$ and $E_n$ is given by $\hbar\nu = E_{n+1} - E \sim dE_n/dn$. Thus the period $\nu^{-1} \sim \hbar(dE/dn)^{-1}$ is asymptotic to the classical period $P(E)$.

**7. Inverse scattering problem.** Suppose a moving particle is repelled from a fixed scattering center by a force derivable from a potential $V(r)$, where $r$ is distance from the center. Then the path of the moving particle is a curve which lies in the plane containing the center, the initial position of the moving particle, and its initial velocity. This path can be found by solving Newton's equations of motion for the particle. The result is that the path is like a hyperbola, with one asymptote along which it comes in from infinity and another asymptote along which it goes out again to infinity.

To describe the path, it is convenient to introduce polar coordinates $r, \theta$ in the plane of the path, with origin at the center. Let the incoming asymptote be the line $y = b$, $x < 0$ so that the particle comes in from $x = -\infty$ parallel to the $x$-axis and at distance $b$ above it. The distance $b$ is called the **impact parameter** of the particle. Then the direction of the outgoing asymptote, determined by solving the equations of motion, is found to be

$$\theta(b) = \pi - 2\int_{r_0}^{\infty} [b^{-2} - r^{-2} - V(r)E^{-1}b^{-2}]^{-1/2} r^{-2}\, dr. \tag{7.1}$$

Here $E < V(0)$ is the energy of the particle, and $r_0$ is the largest root of the bracketed expression in the integrand.

Suppose a uniform beam of particles of energy $E$ is incident from $x = -\infty$. The number scattered in the directions from $\theta$ to $\theta + d\theta$ is denoted $-\sigma(\theta)2\pi \sin\theta d\theta$ apart from a constant factor, and $\sigma(\theta)$ is called the differential scattering cross-section. These particles are incident in the annular ring of area $2\pi b db$ bounded by the impact parameters $b$ and $b + db$ and their number is proportional to this area. By equating the incoming and outgoing numbers of particles we get $-\sigma(\theta)2\pi \sin\theta d\theta = 2\pi b db$, from which

$$\sigma(\theta) = -\frac{b}{\sin\theta}\frac{db}{d\theta}. \tag{7.2}$$

By using (7.1) to eliminate $b$ from (7.2), we obtain the solution of the **direct scattering problem**, which is to find $\sigma(\theta)$ given $V(r)$ and $E$.

The **inverse scattering problem** is to find $V(r)$, given $\sigma(\theta)$ and $E$. To solve it we follow Keller, Kay and Shmoys (1956). First we integrate (7.2) to get

$$\int_\theta^\pi \sigma(\theta)\sin\theta d\theta = b^2/2. \tag{7.3}$$

Here we have used the fact that $\sigma(0) = \pi$, which follows from (7.1). Now (7.3) determines $\theta(b)$. Therefore we may now consider the inverse problem of finding $V(r)$, given $\theta(b)$ and $E$. To solve it, we shall consider (7.1) to be an integral equation for $V(r)$.

Let $x = b^{-2}$ and let us consider $\theta(x)$ to be a function of $x$. Similarly, let $u = r^{-1}$ and let us consider $V(u)$ to be a function of $u$. Then we can write (7.1) as

$$\theta(x) = \pi - 2\int_0^{u_0} \{x[1 - V(u)E^{-1}] - u^2\}^{-1/2}du, \qquad u_0 = r_0^{-1}. \tag{7.4}$$

Now we define $v(u)$, $w(u)$ and $g(w)$ by

$$v(u) = 1 - V(u)E^{-1}, \qquad w(u) = u^2v^{-1}(u), \qquad g(w) = v^{-1/2}\frac{du}{dw}. \tag{7.5}$$

Then we can put (7.4) in the form

$$\frac{1}{2}[\pi - \theta(x)] = \int_0^x (x - w)^{-1/2}g(w)dw. \tag{7.6}$$

The Abel equation (7.6) has the solution

$$g(w) = \frac{d}{dw}\left[\frac{1}{2\pi}\int_0^w \frac{\pi - \theta(x)}{(w - x)^{1/2}}\,dx\right]. \tag{7.7}$$

From the definitions (7.5) of $v$ and $g$ we obtain by integration

$$v = \exp\int_0^w [2g(w)w^{-1/2} - w^{-1}]dw. \tag{7.8}$$

Finally by using (7.7) in (7.8) we get

$$v = \exp\left\{\frac{1}{\pi}\int_0^w (w')^{-1/2}\int_0^{\theta(w')} [w' - x(\theta)]^{-1/2}d\theta dw'\right\}. \tag{7.9}$$

This result, together with the definitions (7.5) of $v$ and $w$, determines the potential $V(r)$.

As an example let us apply our result to the Rutherford scattering cross-section

$$\sigma(\theta) = \frac{e^2}{16E^2 \sin^4(\theta/2)}, \tag{7.10}$$

where $e$ is a constant. We obtain $V = e/r$, the Coulomb potential from which (7.10) comes. As a second example, we consider

$$\sigma(\theta) = \frac{e\left[1 - \frac{\theta}{\pi}\right]}{\pi E \sin\theta \left(\frac{\theta}{\pi}\right)^2 \left[2 - \frac{\theta}{\pi}\right]^2}. \tag{7.11}$$

From the equations above we get the inverse square potential $V = e/r^2$, which gives rise to (7.11).

**8. Inversion of thermodynamic data.** A problem of statistical mechanics is to determine the equation of state of a gas in terms of the potential $V(r)$ between the molecules of the gas. If the number density $n$ of the gas is small, the equation of state is found to be

$$\frac{p}{nRT} = 1 - 2\pi b(kT)n + O(n^2). \tag{8.1}$$

Here $p$ is the pressure, $T$ is the temperature, $R$ is the gas constant, $k$ is Boltzmann's constant and

$$b(kT) = \int_0^\infty [1 - e^{-V(r)/kT}] r^2 dr. \tag{8.2}$$

The expansion (8.1) is called the virial expansion of the equation of state, and $b$ is called the second virial coefficient.

The inverse problem which we consider, following J. B. Keller and B. Zumino (1959), is to determine $V(r)$ given the equation of state. Since the equation of state can be obtained by thermodynamic measurements, this problem is that of finding the intermolecular potential from thermodynamic data. If the equation of state is known, then in particular $b(kT)$ is known. Therefore we shall consider the **inverse problem** of finding $V(r)$ given $b(kT)$. The solution of the **direct problem** is given by (8.2), which we shall use as an integral equation for $V(r)$.

Let us set $\mu = (kT)^{-1}$ and $v = r^3$, and regard $b = b(\mu)$ as a function of $\mu$ and $V = V(v)$ as a function of $v$. Then we can write (8.2) in the form

$$b(\mu) = \frac{1}{3}\int_0^\infty [1 - e^{-\mu V(v)}]\, dv. \tag{8.3}$$

Differentiation of (8.3) with respect to $\mu$ yields

$$b'(\mu) = \frac{1}{3}\int_0^\infty V(v) e^{-\mu V(v)} dv. \tag{8.4}$$

We first suppose that $V(v)$ is monotone decreasing from $V(0) = \infty$ to $V(\infty) = 0$, and introduce $V$ as integration variable. Then we write (8.4) as

$$-3b'(\mu) = \int_0^\infty e^{-\mu V} V (dV/dv)^{-1}\, dV. \tag{8.5}$$

Equation (8.5) expresses $-3b'(\mu)$ as a Laplace transform of $V(dV/dv)^{-1}$. Thus if $L^{-1}$ denotes the inverse Laplace transform, the solution of (8.5) is

$$V(dV/dv)^{-1} = L^{-1}[-3b'(\mu)]. \tag{8.6}$$

The solution of (8.6) which vanishes at $v = \infty$ is given by

$$v = 3\int_V^\infty L^{-1}[b'(\mu)] \frac{dV'}{V'}. \tag{8.7}$$

Thus when $V(r)$ is monotonically decreasing, it is determined uniquely by $b(kT)$, and (8.7) yields it explicitly.

If $V$ is not monotone then it is not uniquely determined by $b(kT)$. In fact, all equimeasurable functions $V(v)$ yield the same function $b(\mu)$, as (8.3) shows. Let us suppose that $V$ is not monotonic, but has the form shown in Fig. 3, which is typical of intermolecular potentials. Then we proceed as above, introducing $V$ separately as the integration variable on each of the two monotonic branches of $V(v)$. We find that (8.7) holds as before for $V>0$. However for $V<0$ we find instead

$$v_2(V)-v_1(V)=3\int_{V_0}^{V} L^{-1}[b'(\mu)]\,\frac{dV'}{V'}. \tag{8.8}$$

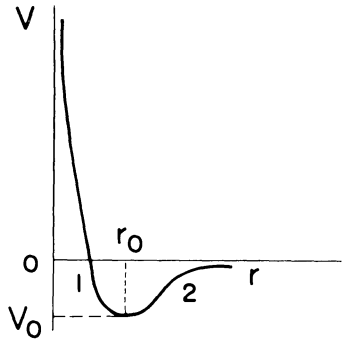



FIG. 3. A typical intermolecular potential $V(r)$ which is decreasing in the interval $0<r<r_0$ and increasing for $r>r_0$. The inverse is $r_1(V)$ in the first interval and $r_2(V)$ in the second interval.

Here $v_2(V)$ and $v_1(V)$ are the larger and smaller inverses of $V(v)$, and $V_0$ is the minimum of $V$. As (8.8) shows, only the volume $4\pi/3[r_2^3(V)-r_1^3(V)]$ of the region where the potential is less than $V$ is determined.

**Appendix.**

1. What is the capital of the United States, Max?
2. Do you spell your name with a "V," Herr Wagner?
3. What is the name of the sole surviving Kamikaze pilot?

Supported by the Office of Naval Research under Contract No. N00014-67-A-0467-0006 and at the Applied Mathematics Summer Institute under N00014-67-A-0467-0027.

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# THE FOURTH U.S.A. MATHEMATICAL OLYMPIAD

S. L. GREITZER

The U.S.A. Mathematical Olympiad, now in its fourth year, has, to judge from the correspondence we have received, attained an enviable position in the educational world of the secondary schools. Recommendations from school officials on behalf of superior mathematical students increased markedly this year. Participation is by invitation only.

This year, as in previous years, most students were selected on the basis of their performance on the Annual High School Mathematics Examination. As before, several students from Michigan and Wisconsin were also invited to participate. In addition, a very few students were invited this year on the recommendation of regional school supervisors. Thus, 112 students were asked to take part, and 103 finally took part in the Olympiad on May 6, 1975.

As before, the Olympiad consisted of five essay-type problems requiring mathematical ingenuity and power to solve. The time allowed for the Olympiad was three hours. The problems appear at the end of this article.

All test papers were returned by May 16, and graded on May 19 by a committee of the mathematics faculty at Rutgers University. The top thirty papers were then regraded at Memphis State University. The results of the grading are shown in Table I below. Final results, listing the top twenty-five scorers, were mailed to the schools involved on May 26.

TABLE I

| Olympiad / H. S. Exam | 0 | 1 – 10 | 11 – 20 | 21 – 30 | 31 – 40 | 41 – 50 | 51 – 60 | 61 – 70 | 71 – 80 | 81 – 90 |
|---|---|---|---|---|---|---|---|---|---|---|
| 131 — 140 | | 2 | 1 | | 1 | 1 | | | | 1 |
| 121 — 130 | 1 | 2 | | 1 | 4 | 2 | 1 | | | |
| 111 — 120 | 1 | 6 | 4 | 9 | 2 | | 1 | | | |
| 101 — 110 | 3 | 17 | 9 | 8 | 8 | 1 | | 2 | | |
| Other | | 3 | 6 | 3 | 1 | | 2 | | | |
| Totals | 5 | 30 | 20 | 21 | 16 | 4 | 4 | 2 | | 1 |

Seven of the eight finalists had scores indicated in the rectangle at the right in Table I. Again, the spread in scores on the Annual High School Examination is considerable. The two students assigned to the "Other" row were both recommended highly by their supervisors. Both did well enough to be selected as members of this year's International Team. A list of the eight students who scored highest is given below:

| | | |
|---|---|---|
| Paul Herdeg | Hamilton-Wenham Reg. H. S. | South Hamilton, Mass. |
| Miller Puckette | Sewanee Academy | Sewanee, Tenn. |
| Steven Tschantz | Irvington High School | Fremont, Cal. |
| Paul Vojta | Southwest Secondary School | Minneapolis, Minn. |
| Russell Lyons | Lexington High School | Lexington, Mass. |
| Bernard Beard | Keystone School | San Antonio, Texas |
| Reed Kelly | Stuyvesant High School | New York, N.Y. |
| Alan Geller | Ridgewood High School | Ridgewood, N.J. |

The correlation between pairs of scores in Table I is so small as to be negligible. The listing of grades attained by participants on each problem is given in Table II. Since a similar table presented last year aroused no surprise, perhaps a closer examination is indicated. This writer believes that it has great significance, and suggests directions for investigation for the teacher.

TABLE II

| No. / Score | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 21 — 25 | 1 | 3 | 1 | | 4 |
| 16 — 20 | 1 | 16 | 3 | 6 | 23 |
| 11 — 15 | 3 | 4 | 1 | 3 | 6 |
| 6 — 10 | 15 | 2 | 1 | 11 | 7 |
| 1 — 5 | 52 | 37 | 30 | 9 | 24 |
| 0 | 31 | 41 | 67 | 74 | 39 |

For example, examine each of the problems to determine what area of mathematics it covers. Compare the scores for each problem. Why are there so many of the scores between 0 and 5? Is what is contained in the problem of possible use in the future training of the student in mathematics? Might it perhaps be advisable to place different stress on what is now being taught? These and many more questions arising from a study of Table II, and which the teacher should ask himself, may be very helpful.

At any rate, the writer would like to hear from readers, including their opinions and suggestions concerning the Olympiad problems and about any conclusions they reach on studying Tables I and II.

The eight finalists were honored at ceremonial exercises which were held in Washington, D.C. on June 5. These exercises included an awards ceremony at the National Academy of Sciences, a reception in the John Quincey Adams Room of the Department of State, and dinner in the Thomas Jefferson Room. These ceremonies were made possible through the generosity of International Business Machines. Among the awards to the students were engraved silver trays, from IBM, an HP-35 Calculator per student from Hewlett-Packard Company, and books from the M.A.A. and N.C.T.M.

The Olympiad Committee gratefully acknowledges the help and support of the many individuals and organizations that made the Fourth Olympiad possible. The problems were prepared by a committee consisting of Murray Klamkin, C. C. Rousseau and Tom Griffiths. The Annual High School Mathematics Examination Committee, consisting of R. Artino, A. Gaglione and N. Shilkret, all of City College of CUNY, provided the data needed for selecting and inviting the participants. The papers were graded by Michael Aissen, John Bender, Phil Guza, Sol Leader, Ben Muckenhoupt, Barbara Osefsky, Elazar Teitz and Hyman Zimmerberg, all on the staff of the Mathematics Department of Rutgers University. The top thirty papers were regraded by Cecil C. Rousseau at Memphis State University.

This year, the U.S.A. has been invited to participate in the Seventeenth International Mathematical Olympiad in Burgas, Bulgaria. The result of this participation will be duly reported to the reader. Last year, after undergoing an intensive three-week training session, the U.S.A. team scored an impressive second behind the U.S.S.R. but ahead of the sixteen other nations. It will be difficult to do better, but we hope not to do much worse.

The Fifth U.S.A. Mathematical Olympiad will take place on Tuesday, May 4, 1976.

**FOURTH U.S.A. MATHEMATICAL OLYMPIAD—MAY 6, 1975**

1. (a) Prove that $[5x]+[5y] \geqq [3x+y]+[3y+x]$, where $x, y \geqq 0$ and $[u]$ denotes the greatest integer $\leqq u$ (e.g., $[\sqrt{2}]=1$).

(b) Using (a) or otherwise, prove that

$$\frac{(5m)!(5n)!}{m!\,n!(3m+n)!(3n+m)!}$$

is integral for all positive integral $m$ and $n$.

2. If $A$, $B$, $C$, $D$ denote four points in space and $AB$ denotes the distance between $A$ and $B$, etc., show that

$$AC^2+BD^2+AD^2+BC^2 \geqq AB^2+CD^2.$$

3. If $P(x)$ denotes a polynomial of degree $n$ such that $P(k)=k/(k+1)$ for $k=0,1,2,\cdots,n$, determine $P(n+1)$.

4. Two given circles intersect in two points $P$ and $Q$. Show how to construct a segment $AB$ passing through $P$ and terminating on the two circles such that $AP \cdot PB$ is a maximum.

5. A deck of $N$ playing cards which contains three aces is shuffled at random (it is assumed that all possible card distributions are equally likely). The cards are then turned up one by one from the top until the second ace appears. Prove that the expected (average) number of cards to be turned up is $(N+1)/2$.

DEPARTMENT OF MATHEMATICS, RUTGERS — THE STATE UNIVERSITY, NEWARK, N.J. 07102.

## QUERIES

EDITED BY A. C. ZITRONENBAUM

*This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Mathematisches Institut, D8 München 2, Theresienstrasse 39, West Germany.*

**Reply to Query 25:** In this query B. R. Ust asked for references illustrating the use of mathematics in Cartography. The following list has kindly been supplied by Detlef Laugwitz, Fachbereich Mathematik der Technischen Hochschule Darmstadt, Germany: G. Scheffers, K. Strubecker: *Wie findet und zeichnet man Gradnetze von Land-und Sternkarten*, Stuttgart 1965; J. Moschek: *Mathematische Grundlagen der Kartographie*, Mannheim 1969; K. Wagner: *Kartographische Netzentwürfe*, Mannheim 1962; D. Laugwitz: *Differential and Riemannian Geometry*, New York 1965, pp. 147–163.

## MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

*Material for this Department should be sent to Richard A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706.*

### REMARKS ON THE ZEROS OF BESSEL POLYNOMIALS

A. WRAGG AND C. UNDERHILL

**1. Introduction.** In a recent note, Barnes [1] discussed the properties of Bessel polynomials and their relation to the exponential function, and also claimed a new result concerning the zeros of the Bessel polynomials.

Our purpose in the present note is to draw attention to established properties of the zeros of Bessel polynomials (which include that claimed by Barnes), and to point out the connection between Bessel polynomials and Padé approximants to the exponential function. The latter feature is important because of its applications in numerical analysis. Finally we illustrate the derivation of properties of Bessel polynomials by matrix methods by presenting a new proof that their zeros have negative real parts.

**2. Location of the zeros of Bessel polynomials.** We follow the notation used by Barnes [1], that is, we denote the Bessel polynomial of order $n$ by $y_n(x)$, where

$$y_n(x) - (2n-1)xy_{n-1}(x) - y_{n-2}(x) = 0 \tag{1}$$

for $n \geqq 2$, with $y_0(x) = 1$, $y_1(x) = 1 + x$.

The fact that $y_n(x)$ and $y_{n+1}(x)$ have no zeros in common and that $y_n(x)$ has no repeated zeros was established by Krall and Frink [9]. Grosswald [6] proved that all zeros of $y_n(x)$ for $n \geqq 2$ lie within $|x| < 1$, that $y_{2n}(x)$ has no real zero, that $y_{2n-1}(x)$ has only one real zero and that the real zeros of $y_{2n-1}(x)$, $n = 1, 2, \cdots$, satisfy

$$-1 = r_1 < r_3 < \cdots < r_{2n-1} < \cdots < 0. \tag{2}$$

The negativity of the real part of the zeros of the Bessel polynomials $y_n(x)$ was established by Wimp [12] as a corollary to a result on the zeros of a confluent hypergeometric function. A much stronger result on the location of the zeros of $y_n(x)$ can, however, be derived from the results of Dočev [4], namely that they lie within the circle $|x| \leqq 2/(n+1)$. Obviously this can be combined with the negativity property to locate the zeros in the semicircle $x = 2e^{i\theta}/(n+1)$, $\pi/2 \leqq \theta \leqq 3\pi/2$. An immediate consequence of this is, of course, that the origin is the limit point of the zeros of the Bessel polynomials.

**3. Bessel polynomials and Padé approximants.** An important reason for studying the properties of Bessel polynomials is their relation to the denominators of Padé approximants to the exponential function. If we denote the $(p, q)$ Padé approximant to $e^x$ by

$$g_{p,q}(x) = \frac{n_{p,q}(x)}{d_{p,q}(x)}, \tag{3}$$

where

$$n_{p,q}(x) = \sum_{k=0}^{q} \frac{(p+q-k)!\,q!}{(p+q)!\,k!\,(q-k)!}\,x^k, \qquad d_{p,q}(x) = \sum_{k=0}^{p} \frac{(p+q-k)!\,p!}{(p+q)!\,k!\,(p-k)!}\,(-x)^k, \tag{4}$$

then we can write

$$d_{p,q}(x) = \frac{(-x)^p q!}{(p+q)!}\,Y_p^{(q-p)}\left(-\frac{2}{x}\right). \tag{5}$$

The function $Y_n^{(\alpha)}(x)$ occurring in (5) is the generalized Bessel polynomial, that is, the solution of the differential equation

$$x^2y'' + \{(\alpha+2)x + 2\}y' - n(n+\alpha+1)y = 0 \tag{6}$$

under appropriate boundary conditions. (The function $Y_n^{(0)}(x)$ is, in fact, identically equal to $y_n(x)$.)

Padé approximants to the exponential function have been studied extensively in their own right and because of their importance in numerical analysis. See, for example, Ehle [5] for their rôle in the numerical solution of initial value problems, and Wragg and Davies [13] for their use in the computation of matrix exponentials.

The case $p = q = n$ in (5) is probably of most interest since it corresponds to a relation between the denominators of diagonal Padé approximants to $e^x$ and ordinary Bessel polynomials $y_n(x)$. Birkhoff and Varga [3] refer to the positivity of the real parts of the zeros of $d_{n,n}(x)$ (which is equivalent, of course, to the negativity of the real parts of the zeros of $y_n(x)$) as being a well-known theorem from the theory of passive networks (see Guillemin [7]), and outline a proof which is virtually identical to that given by Barnes [1].

**4. Bessel polynomials and matrices.** We conclude by giving a proof, which we believe to be new, that the zeros of the Bessel polynomials have negative real parts and lie in the region $|x| < 1$, for $n \geqq 2$. The proof will depend only on matrix properties and will make use of two fundamental theorems on the localization of eigenvalues of matrices. The first of these, which we shall refer to as the Bendixson-Hirsch theorem, can be enunciated as follows.

THEOREM (Bendixson-Hirsch). *Let $A$ be an $n \times n$ complex matrix and let $B$ and $C$ denote, respectively, the Hermitian matrices $(A + A^*)/2$ and $(A - A^*)/2i$. Let $\lambda_j$, $\mu_j$, and $\nu_j$, $j = 1, 2, \cdots, n$, denote the eigenvalues of $A$, $B$, and $C$ respectively; then*

$$\min_{1 \leqq j \leqq n}(\mu_j) \leqq \mathrm{Re}\{\lambda_k\} \leqq \max_{1 \leqq j \leqq n}(\mu_j) \quad \text{and} \quad \min_{1 \leqq j \leqq n}(\nu_j) \leqq \mathrm{Im}\{\lambda_k\} \leqq \max_{1 \leqq j \leqq n}(\nu_j)$$

*for $k = 1, 2, \cdots, n$.*

This theorem follows immediately from the original work carried out by Bendixson [2] and Hirsch [8]. A more accessible discussion of their results is given by Marcus and Minc [10], where a proof of the following theorem can also be found.

THEOREM (Gershgorin). *Let $A$ be an $n \times n$ complex matrix with elements $a_{jk}$, then all the eigenvalues of $A$ lie in the union of the disks*

$$|x - a_{jj}| \leqq \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{jk}|, \qquad j = 1, 2, \cdots, n.$$

We are now in a position to establish results for zeros of the Bessel polynomials of order $n \geqq 3$ which we state in the form of a theorem. (The cases $n = 1$ and $n = 2$ are trivial, since the zeros are known explicitly.)

THEOREM 1. *If $\xi$ and $\eta$ denote the real and imaginary parts, respectively, of a zero of the Bessel polynomial $y_n(x)$, then for $n \geqq 3$*

$$-\frac{2}{3} \leqq \xi \leqq -\frac{2}{(2n-3)(2n-1)} < 0 \tag{7}$$

*and*

$$|\eta| \leqq \frac{8}{15}. \tag{8}$$

*Proof.* Following Parodi [11], and using the recurrence relation (1) together with the expressions for $y_0(x)$ and $y_1(x)$, we form a system of $n$ linear equations for the $n$ unknowns $y_1(x), y_2(x), \cdots, y_n(x)$:

$$
\begin{aligned}
&\frac{y_n}{2n-1} - xy_{n-1} - \frac{y_{n-2}}{2n-1} = 0,\\
&\frac{y_{n-1}}{2n-3} - xy_{n-2} - \frac{y_{n-3}}{2n-3} = 0,\\
&\qquad\cdots\\
&\frac{y_2}{3} - xy_1 = \frac{1}{3},\\
&y_1 = 1 + x.
\end{aligned}
\tag{9}
$$

Cramer's rule applied to the equations (9) gives

$$
y_n(x) = (-1)^n\, 1\cdot 3\cdot 5\cdots(2n-1)\det
\begin{bmatrix}
-x & -\frac{1}{2n-1} & 0 & & \\
\frac{1}{2n-3} & -x & -\frac{1}{2n-3} & & \\
 & \ddots & \ddots & \ddots & \\
 & \frac{1}{3} & -x & -\frac{1}{3} & \\
 & 0 & 1 & -1-x &
\end{bmatrix}.
\tag{10}
$$

Hence the zeros of $y_n(x)$ are the eigenvalues of the matrix $A$ defined by

$$
A = \begin{bmatrix}
0 & -\frac{1}{2n-1} & 0 & \cdots & 0 & 0 & 0\\
\frac{1}{2n-3} & 0 & -\frac{1}{2n-3} & \cdots & 0 & 0 & 0\\
 & & & \cdots & & & \\
0 & 0 & 0 & \cdots & \frac{1}{3} & 0 & -\frac{1}{3}\\
0 & 0 & 0 & \cdots & 0 & 1 & -1
\end{bmatrix}.
\tag{11}
$$

Now the eigenvalues of $A$ are invariant under similarity transformations; in particular, therefore, the eigenvalues of $A$ will be the same as those of the matrix $W$, where $W = P^{-1}AP$, and $P$ is the matrix

$$
P = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1\\
0 & 0 & \cdots & 1 & 1\\
 & & \cdots & & \\
0 & 1 & \cdots & 1 & 1\\
1 & 1 & \cdots & 1 & 1
\end{bmatrix}
\tag{12}
$$

with inverse

$$P^{-1} = \begin{bmatrix} 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & \cdots & & \\ -1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{13}$$

That is, the zeros of $y_n(x)$ are equal to the eigenvalues of the matrix

$$W = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{2}{15} & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & -\frac{1}{2n-3} & -\frac{2}{(2n-3)(2n-1)} & \frac{1}{2n-1} \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2n-1} & -\frac{1}{2n-1} \end{bmatrix}. \tag{14}$$

We now write $W$ in the form $B + iC$, where $B$ and $C$ are the Hermitian matrices

$$B = \begin{bmatrix} -\frac{2}{3} & 0 & \cdots & 0 & 0 \\ 0 & -\frac{2}{15} & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & \frac{-2}{(2n-3)(2n-1)} & 0 \\ 0 & 0 & \cdots & 0 & -\frac{1}{2n-1} \end{bmatrix}, \tag{15}$$

$$C = \begin{bmatrix} 0 & -\frac{i}{3} & 0 & \cdots & 0 & 0 & 0 \\ \frac{i}{3} & 0 & -\frac{i}{5} & \cdots & 0 & 0 & 0 \\ & & & \cdots & & & \\ 0 & 0 & 0 & \cdots & \frac{i}{2n-3} & 0 & \frac{-i}{2n-1} \\ 0 & 0 & 0 & \cdots & 0 & \frac{i}{2n-1} & 0 \end{bmatrix}$$

so that the Bendixson-Hirsch theorem is immediately applicable. The result (7) follows by direct observation of the eigenvalues of $B$ and the result (8) follows by straightforward application of Gershgorin's theorem to the matrix $C$.

It is perhaps worth noting that it is not difficult to obtain marginal improvements to the above results by invoking, for example, refinements of Gershgorin's theorem but, so far, we have not been able to reproduce Dočev's results using this matrix approach.

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## SUBGROUPS OF HOLOMORPHS OF GROUPS

FREDERICK HOFFMAN

*The concept of the holomorph of a group is useful in the description of groups of small order, and has many other applications in the study of finite groups — frequently in the construction of counterexamples.*

Recently, Nakano, in a series of papers ([9], [10], [11]) has investigated some questions related to the holomorph of an arbitrary group. It is the purpose of this note to provide a brief proof for the Characterization Theorem of [11], restated in more standard terminology. The holomorph of a finite group $G$ was first defined by Burnside in the 1897 edition of his book [1], although the concept is somewhat older. A reasonably full treatment of the holomorph of a finite group is found in most of the early texts in group theory ([1], [2], [6], [13] among others); the coverage in recent texts is somewhat briefer. Rotman [12], Kurosh [5], and Hall [3] define the holomorph for an arbitrary group. The statements about the holomorph which we use in the proof can be found in Hall ([3] pp. 86–7) or Mills ([7], [8]).

Let $G$ be a group. A *holomorphism* of $G$ is an element $\theta$ of the group $S[G]$ of 1–1 transformations on the set of elements of $G$ which can be written in the form $\theta = \phi\rho_a$, where $\phi$ is an automorphism of $G$, and $\rho_a$ is right multiplication by an element $a \in G$. The *holomorph* $H(G)$ of $G$ is the set of all holomorphisms of $G$. Alternatively, ([12], [4]) a holomorphism of $G$ can be defined as a 1–1 transformation $\theta$ on $G$ satisfying $(ab^{-1}c)\theta = (a\theta)(b\theta)^{-1}(c\theta)$, for all triples $\{a, b, c\}$ of elements of $G$. $G$ can be identified with the subgroup $\bar{G}$ of $H$ of all right multiplication operations, $\rho_a$, for $a \in G$. The holomorph of $G$ is the normalizer of $\bar{G}$ in the group of all 1–1 transformations of $G$. (This, in fact, is the definition in [1].) The subgroup $G_l$ of all left multiplications $\lambda_a : G \to G$, for $a \in G$, is isomorphic to $G$ under the map, sending each $a \in G$ to $\lambda_a{}^{-1}$. $G_l$, sometimes called the *conjoint* of $\bar{G}$, is the centralizer of $\bar{G}$ in $S(G)$, and conversley. If $A(G)$ is the automorphism group of $G$, we have

$H(G) = \bar{G} \cdot A(G)$ and $\bar{G} \cap A(G) = \{id_G\}$ (the identity subgroup of $S(G)$). Further,

$$(*) \qquad \text{for} \quad \theta \in A(G) \quad \text{and} \quad a \in G, \quad \rho_{a\theta} = \theta^{-1}\rho_a\theta.$$

THEOREM (Nakano). *Let $M$ be a group. The following are equivalent:*

(1) *There is a group $G$ such that $M$ is isomorphic to a subgroup $\bar{M}$ of $H(G)$ containing $\bar{G}$ and $G_l = \{\lambda_a \mid a \in G\}$.*

(2) *There are subgroups $R$, $L$ and $B$ of $M$ such that $R \trianglelefteq M$, $L \trianglelefteq M$, $R = C_M(L)$, $L = C_M(R)$, $M = BR = BL$, and $B \cap R = B \cap L = \{e\}$.*

(3) [*Here we identify $G$ and $\bar{G}$, and consider $H(G)$ as a split extension of $G$ by $A(G)$.*] *There is a group $G$ such that $M$ is isomorphic to a subgroup $\bar{M}$ of $H(G)$ containing $G \cdot I(G)$, where $I(G)$ is the group of inner automorphisms.*

[Nakano proved the equivalence of (1) and (2). The use of (3) to simplify the proof and give more insight into the structure of the holomorph was suggested by the referee. We note that the condition $L \trianglelefteq M$ in (2) is clearly redundant. It will be seen from the proof that $R = C(L)$ is also redundant.]

*Proof.* (1) implies (2): We may as well assume $M = \bar{M}$. Then, let $R = \bar{G}$, $L = G_l$ and $B = A(G) \cap M$. The conclusion follows.

(2) implies (3): Let $G = R$. Let $\Phi$: $B \to A(R)$ be given by $b\Phi =$ the restriction to $R$ of the inner automorphism of $M$ induced by $b$. $\Phi$ is clearly a homomorphism of $B$ into $A(R)$. $\operatorname{Ker}\Phi = B \cap C_M(R) = B \cap L = \{e\}$, so $\Phi$ is 1-1. If $d \in R$, then $d = bg$, for some $b \in B$, $g \in L$. Then the inner automorphism induced on $R$ by $d$ is the same as $b\Phi$. Thus $I(G) \subseteq \operatorname{Im}\Phi$. Clearly, the function $\Psi$: $M \to H(G)$ given by (for $d \in R$, $b \in B$) $db\Psi = d(b\Phi)$ is well-defined (since $B \cap R = \{e\}$), and is a 1-1 function from $M$ onto a subgroup of $H(G)$ containing $G \cdot I(G)$. The homomorphism property follows from $(*)$:

$$\begin{aligned}(d_1b_1)\psi(d_2b_2)\psi &= d_1(b_1\Phi)d_2(b_2\Phi) = d_1(b_1\Phi)d_2(b_1\Phi)^{-1}(b_1b_2)\Phi \\ &= [(d_1b_1d_2b_1^{-1})(b_1b_2)]\Phi = (d_1b_1d_2b_2)\psi.\end{aligned}$$

(3) implies (1): This is obvious, since a subgroup contains $G$ and $G_l$ if and only if it contains $G$ and $I(G)$.

### References

No attempt has been made to offer a complete list of references on holomorphs; on the other hand, no attempt has been made to make the list minimal with respect to the property of providing the background for reading the paper. This is a small sampling of material on holomorphs.

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## RESEARCH PROBLEMS

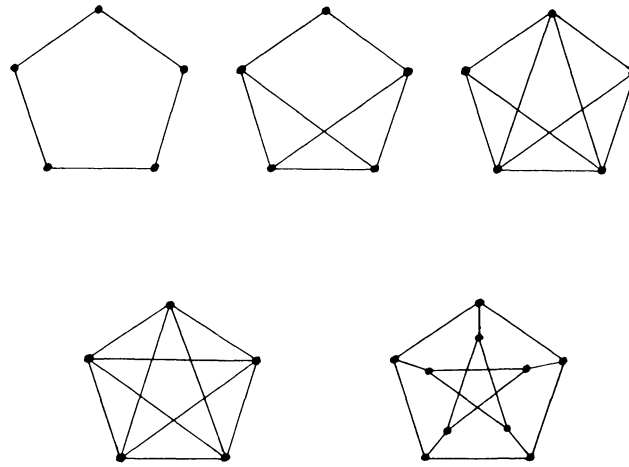
EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.*

### THREE CONJECTURES ON CRITICAL GRAPHS

ROBIN J. WILSON AND LOWELL W. BEINEKE

If $G$ is a simple graph (i.e., a finite, undirected graph without loops or multiple edges), we define its **chromatic index** (or **edge-chromatic number**) $\chi'(G)$ to be the minimum number of colors needed to color the *edges* of $G$ in such a way that any edges which share a common vertex are assigned different colors. Vizing [**8, 9**] has shown that if $\rho$ is the maximum valency (vertex-degree) in $G$, then either $\chi'(G) = \rho$ (in which case we say that $G$ is **of class 1**), or $\chi'(G) = \rho + 1$ (in which case $G$ is **of class 2**). For example, all of the graphs shown in the figure are of class 2.



In trying to decide which graphs are of class 1 and which are of class 2, one is quickly led to the concept of a **critical graph**; this is defined to be a graph of class 2 with the property that the removal of any edge leaves a graph of class 1. For example, the first three graphs shown in the diagram are critical graphs. Such graphs have been studied by several authors [**4, 6, 9, 10**], and various methods for constructing them have been given. Surprisingly, all critical graphs so far discovered have an odd number of vertices, and this led Beineke and Wilson [**2**] and Jakobsen [**7**] to formulate the following conjecture:

CONJECTURE 1. *There are no critical graphs with an even number of vertices.*

In support of this conjecture, Jakobsen [**7**] and Beineke and Fiorini [**1**] have shown that there are no critical graphs with 2, 4, 6, 8 or 10 vertices, and none with 12 vertices and maximum valency 3.

The second and third conjectures were formulated by Vizing. In [**9**], he proved that every planar graph with maximum valency $\rho \geqq 8$ is of class 1; on the other hand, one can easily construct planar graphs of class 2 with maximum valency 2, 3, 4 and 5. This suggests the following conjecture:

CONJECTURE 2. *Every planar graph with maximum valency* 6 *or* 7 *is of class* 1.

Using critical graphs, Fiorini and Wilson [5] have shown that if there exists a planar graph of class 2 with maximum valency 6 (resp. 7), then it must contain at least 4 (resp. 6) vertices of maximum valency.

Vizing also observed [11] that the truth of Conjecture 2 for $\rho = 7$ follows from the truth of the following conjecture:

CONJECTURE 3. *If G is a critical graph with n vertices and maximum valency* $\rho$, *then G contains at least* $\frac{1}{2}\{n(\rho-1)+3\}$ *edges.*

The best lower bound obtained so far for the number of edges is $\frac{1}{4}n(\rho+1)$, which is due to Fiorini [3].

The authors of this note offer a prize of \$10 for the first correct proof of, or counter-example to, any of these conjectures.

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## CLASSROOM NOTES

EDITED BY RICHARD A. BRUALDI

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### REAL SOLUTIONS OF $y'' = Ay^2 + 2By + C$


A. HECKENBACH AND K. HEIMES


**1. Introduction.** Elementary techniques from ordinary differential equations suggest a power series approach for solving the initial value problem

$$y'' = Ay^2 + 2By + C, \qquad A > 0, \tag{1}$$

$$y(0) = \alpha, \qquad y'(0) = m, \tag{2}$$

where $A, B, C$ are real constants. Unfortunately, there results a multiple term recursion relation for determining an arbitrary coefficient of the series in terms of the initial data and the parameters $A$, $B$ and $C$. Consequently, one must be content with successively computing a finite number of coefficients.

Standard texts in differential equations and complex analysis ([**1**]–[**4**]) point out that the general complex valued solution of equation (1) can be given in terms of the Weierstrass $\wp$-function. However, it is not apparent how the real solution with initial data (2) can be obtained in terms of the $\wp$-function. We will show how to make this connection below.

**2. The Weierstrass $\wp$-function.** We recall here only those properties of $\wp(z)$ which are essential to the current discussion; details and proofs can be found in [**1**]–[**3**].

Given complex constants $g_2$, $g_3$, define

$$\wp(z) = z^{-2} + \sum_{k=2}^{\infty} C_k z^{2k-2}, \tag{3}$$

$$\begin{cases} C_2 = g_2/20 \\ C_3 = g_3/28 \\ C_k = \dfrac{3}{(2k+1)(k-3)} \displaystyle\sum_{j=2}^{k-2} C_j C_{k-j} \qquad \text{for } k > 3. \end{cases} \tag{4}$$

It is clear from equations (3) and (4) that when $g_2$, $g_3$ and $z$ are real, $\wp(z)$ is also real. The function $\wp(z)$ is doubly periodic and if $2w_1, 2w_3$ are its fundamental periods and $w_2 = -(w_1 + w_3)$, then $\wp(w_i) = e_i$ and $\wp'(w_i) = 0$, where $e_1, e_2, e_3$ are the appropriately ordered roots of the equation

$$4z^3 - g_2 z - g_3 = 0. \tag{5}$$

Finally, we remind the reader that $\wp(z)$ satisfies the differential equation

$$[\wp'(z)]^2 = 4\,\wp^3(z) - g_2\,\wp(z) - g_3 \tag{6}$$

and the addition formula

$$\wp(z+w) = \tfrac{1}{4}[\wp'(z) - \wp'(w)]^2[\wp(z) - \wp(w)]^{-2} - [\wp(z) + \wp(w)]. \tag{7}$$

**3. The intial value problem.** The initial value problem (1), (2) is seen to be equivalent to solving equation (6) with appropriate choices of $g_2$, $g_3$ by making a change of variable. Let $y(t)$ solve (1), (2) and set $w(t) = \frac{1}{6}[Ay(t) + B]$, $g_2 = (B^2 - AC)/3$, $a = (A\alpha + B)/6$, $b = Am/6$. Then $w(t)$ is a solution of the initial value problem

$$w'' = 6w^2 - \tfrac{1}{2} g_2, \qquad w(0) = a, \qquad w'(0) = b. \tag{8}$$

Multiplication of equation (8) by $w'$ and an integration shows that $w(t)$ solves equation (6) with $g_3 = 4a^3 - g_2 a - b^2$. Conversely, if $w(t)$ is a real solution of (6) for these values of $g_2$ and $g_3$, then $y(t) = (1/A)[6w(t) - B]$ solves (1), (2).

THEOREM. *Let $e_j$ be the largest real root of equation* (5) *with $e_j \geqq a$ and $\tau = \pm \int_{e_j}^{a} (4s^3 - g_2 s - g_3)^{-1/2} ds$, where the sign of the integral agrees with the sign of $b$. Then the real solution of* (1), (2) *is given by*

$$y(t) = \frac{1}{A}\left\{\frac{3}{2}[\wp'(t+\tau)]^2[\wp(t+\tau) - e_j]^{-2} - 6[\wp(t+\tau) + e_j] - B\right\}.$$

*Proof.* The function $w_0(t) = \wp(t + w_j)$ is a solution of (6) satisfying $w_0(0) = e_j$, $w_0'(0) = 0$. An application of the addition formula (7) gives

$$w_0(t) = \tfrac{1}{4}[\wp'(t)]^2[\wp(t) - e_j]^{-2} - [\wp(t) + e_j],$$

which is real since $g_2$, $g_3$ and $t$ are real. The trajectory of the solution $w_0(t)$ in the $(w, w')$ phase plane

passes through the points $(e_j, 0)$ and $(a, b)$. If we can determine the number $\tau$ for which $w_0(\tau) = a$, $w_0'(\tau) = b$, then $w_1(t) = w_0(t + \tau)$ will solve equation (6) with $w_1(0) = a$, $w_1'(0) = b$. If $b > 0$, then $\tau > 0$ and equation (6) gives

$$\frac{dw_0}{dt} = (4w_0^3 - g_2 w_0 - g_3)^{1/2} \quad \text{for} \quad 0 < t < \tau,$$

so that

$$\tau = \int_{e_j}^{a} \frac{dt}{dw_0} = \int_{e_j}^{a} (4s^3 - g_2 s - g_3)^{-1/2} ds.$$

When $b < 0$, it is easily seen that $\tau$ is given by the negative of this integral.

Thus, in every case, $w_0(t + \tau)$ solves equation (6) with $w_0(0) = a$, $w_0'(0) = b$. By our earlier remarks, this translates back to the solution of (1), (2) in terms of the formula given in the theorem.

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## MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

*Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.*

### THE MATHEMATICIANS' ROLE IN EDUCATIONAL PLANNING

ELAINE L. TATHAM AND HAROLD L. FINCH

Dr. John Kemeny expressed during his invited address at a summer MAA meeting that "a college president either needs to be a mathematician or he needs someone first-rate on his planning staff who has considerable mathematical sophistication." Most college presidents are not mathematicians. Yet, planning is a career option considered by very few mathematicians despite the fact that planners need mathematical skills. Included among these desirable skills is the ability to use mathematical modeling.

The current mismatch between student enrollments and facilities for many public schools and for many colleges has called increased attention to the need for planning tools to be used as a sound basis for projects such as:

1. Planning future staff, facility and budget requirements.
2. Planning non-credit continuing education and community service programs.
3. Assisting public, private and commercial agencies to plan for the future.

A planning model was used by Johnson County Community College in order to determine the amount to be spent on a college commons (student union). The critical variables were enrollment, total building cost, student service income, period of bond retirement, student fee, coverage factor and interest rate. The model provided an effective means for quantitatively assessing the impact of user alternatives. The effects of these alternatives as illustrated by the model formed the basis for deciding the amount of revenue bonds for the college commons.

As is the case with most planning models used by colleges and school districts, student enrollment is one of the critical variables. However, post-secondary enrollment projections have frequently

proven to be overly optimistic. This is due in part to superficial techniques, wishful thinking and an unawareness of the importance of good projections in educational planning and administration. Enrollment projections for a particular institution depend heavily upon the size and composition of the population from which the students are drawn. In order to meet the educational needs of the population, its size and composition must be known.

As a result, in 1972 the Johnson County Community College and the Shawnee Mission School District commissioned the development of an enrollment model. The model is capable of projecting population by age and sex for fixed geographic areas. It has two important features that allow it to be a planning tool and not just a number generator. First it can take into account possible future social, economic and political forces which would affect such population factors as fertility and death rates, migration patterns and land use. The second feature enables the model to be used in *a priori* planning. That is, it provides a means of quantitatively assessing the impact of possible user alternatives and of choosing optimum courses of actions. Further, it allows the user to make assumptions and to test the sensitivity, reasonableness and appropriateness of these assumptions.

Regression techniques were used to develop (1) number of births per number of women of specified age groups and (2) number of deaths per number of people of specified age groups (separate equations were developed for males and females). Exponential curves such as the logistic two-point curve were used to develop net migration factors for the terminal year. A modified cohort procedure was used to obtain population projections as a function of births, deaths and net migration by age, sex and geographic area.

The computer model has demonstrated that the declines in overall elementary-secondary enrollments which began in the late 1960's were largely brought about by second generation effects of the low birth rate during the depression. Thus the current decline which caught most educators by surprise was predictable to an important degree as early as the late 1930's. Likewise, the current and forthcoming declines in post-secondary enrollments were predictable. Of more importance are the projected increases in elementary-secondary education beginning around 1980. Educational planners who have a demographic modeling capability can be prepared for this projected upturn. Those who overreact to the current decline may find their shortsightedness to be intolerably costly in terms of public confidence as well as educational dollars.

These are examples of a relatively untouched area in which mathematicians can apply their knowledge and skill outside the typical, traditional college classroom. The field of educational planning tends to be controlled by non-mathematically trained persons. Students need to be made aware of the potential areas in which mathematics can be applied outside the realm of engineering, teaching and science.

INSTITUTE FOR COMMUNITY/COLLEGE DEVELOPMENT, JOHNSON COUNTY COMMUNITY COLLEGE, COLLEGE BOULEVARD AND QUIVIRA ROAD, OVERLAND PARK, KANSAS 66210.

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## PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

## ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before May 31, 1976.*

E 2575. *Proposed by David Shelupsky, City College of New York*
Solve the functional equation

$$f\left(\frac{x-y}{\log x-\log y}\right)=\frac{1}{2}f(x)+\frac{1}{2}f(y),$$

this to hold for all distinct $x$, $y$ in $(0,\infty)$ and $f:(0,\infty)\to R$ to be continuous.

E 2576. *Proposed by Robert L. Helmbold, Northridge, California*
What is the area $A(\mathbf{n})$ of the orthogonal projection of the ellipsoid $(x/a)^2+(y/b)^2+(z/c)^2=1$ onto a plane perpendicular to the unit vector $\mathbf{n}=(n_1,n_2,n_3)$?

E 2577. *Proposed by F. W. Light, Jr., Jonestown, Pennsylvania*
Given the $2\times n$ Latin rectangle

$$\begin{array}{cccccc} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1, \end{array}$$

find the number of ways $f(k)$ in which a $3\times n$ Latin rectangle can be built up from it by adding a third row starting with $k$, where $k$ is one of the numbers $3,4,\cdots,n$. (If the initial number $a$ is not prescribed, the number of possible third rows is, of course, the already known ménage number $U_n$.)

E 2578. *Proposed by Carl Pomerance, University of Georgia*
Prove that $x^4+1$ is reducible over every field of prime characteristic. Do the same for $x^4-x^2+1$.

E 2579. *Proposed by Benjamin Klein and Brian White, Davidson College, North Carolina*
Let $0<\theta<\frac{1}{2}\pi$ and let $p$, $q$ be arbitrary distinct points in the Euclidean plane $E$. Define $f_\theta(p,q)$ to be the unique point $r$ in $E$ such that triangle $pqr$ is in the counterclockwise sense and $\measuredangle rpq=\measuredangle rqp=\theta$ radians. Show that $f_{\pi/3}(p,q)$ can be written as an expression involving only $f_{\pi/6}$, $p$, $q$, and parentheses.

E 2580. *Proposed by Clark Kimberling, University of Evansville, Indiana*
Show that

$$\frac{d}{dx}\left[\prod_{k=0}^{n-1}\left(a-2\sqrt{x}\cos\frac{(2k+1)\pi}{2n}\right)\right]=-n\prod_{k=1}^{n-2}\left(a-2\sqrt{x}\cos\frac{k\pi}{n-1}\right).$$

## SOLUTIONS OF ELEMENTARY PROBLEMS

### Miquel Point—A Tough Nut has been Cracked

E 585 [1943, 454; 1974, 1026]. *Proposed by A. H. Stone, University of Rochester*
Let a circle with center $O$ meet the sides $BC$, $CA$, $AB$ of a triangle in the pairs of points $X$ and $X'$,

$$R[\xi x(y^2-z^2)+\eta y(z^2-x^2)+\zeta z(x^2-y^2)]=xyz(a\xi+b\eta+c\zeta)\tan\theta$$

(the trilinear equation of line $DEF$). Considering the area of triangle $DMM'$, we see that

$$\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)(MM')d=4\Delta\tan\theta,$$

the symmetry of which shows that $d$ is also the directed distance from $E$ and $F$ to $MM'$, so that the lines $DEF, MM'$ are parallel, coinciding if $\theta=0$.

If $M$ lies on the circumcircle of triangle $ABC$, the inscribed conic $\Sigma$ with focus $M$ is a parabola having the Simson $S$ of $M$ as vertex tangent; $XYZ$ is an isopedal line at angle $\theta$ to $S$ and tangent to $\Sigma$; $M', X', Y', Z'$ are at infinity on the axis of $\Sigma$ and on $BC, CA, AB$; and $DEF$ is perpendicular to $S$ and passes through the reflection of $M$ in $XYZ$ and the contact point of $\Sigma$ with $XYZ$. If $XYZ, X'Y'Z'$ in (i) are also the pedal triangles of an isopair $M$ and $M'$, then $D, E, F, M, M', P, P'$ are collinear on an axis passing through the reflection of the orthocenter of triangle $ABC$ in its circumcenter [**10**].

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Also solved by the proposer. M. G. Greening (Australia) provided a solution to the first part of the problem.

*Editor's comment.* Walker's solution refers to two papers, [**2**] and [**3**], which make a valuable contribution to the subject and which seem to have been overlooked by recent writers. Walker invites comments on the first solution of (ii).

**A Point in a Spiral**

E 1073 [1953, 417; 1975, 72]. *Proposed by G. W. Walker*

A polygonal spiral $A_1A_2A_3\cdots$ of unit segments winds counterclockwise, and is constructed in the following manner: Point $A_1$ is at the origin, point $A_2$ is at $(1,0)$, $\measuredangle A_{n-1}A_nA_{n+1}=2\pi/n$ for all $n\geqq 2$. Is there a point lying within the interior of each $\measuredangle A_{n-1}A_nA_{n+1}$? If so, what are its coordinates?

*Solution by Peter Ungar, New York University.* We prove the following more general statement: Let $A_1A_2A_3\cdots$ be a polygon with sides $|A_nA_{n+1}|=1$ which winds counterclockwise, and suppose the interior angles $\alpha_n=\measuredangle A_{n-1}A_nA_{n+1}$ are strictly decreasing: $\pi>\alpha_2>\alpha_3>\cdots$. Then the open interiors of these angles have a common point.

*Proof.* Let $M_n$ be the midpoint of $A_nA_{n+1}$ and let $P_n$ be the intersection of the perpendicular bisectors of $A_{n-1}A_n$ and $A_nA_{n+1}$. Then the circle $C_n$ with center $P_n$ and radius $P_nM_n$ will touch the sides $A_{n-1}A_n$ and $A_nA_{n+1}$ at their midpoints. The radius of $C_n$ is $\frac{1}{2}\tan\frac{1}{2}\alpha_n$. Since $C_{n+1}$ and $C_n$ both touch $A_nA_{n+1}$ at $M_n$ and $C_{n+1}$ has smaller radius, the circle $C_{n+1}$ is inside $C_n$. By the nested set theorem the closed circular disks have a point $L$ in common. Since no point lies on three consecutive circles, $L$ must be in the interior of all the disks and hence it must be in the interior of each angle $A_{n-1}A_nA_{n+1}$.

$L$ can be obtained as the limit of the midpoints of the intervals $A_nA_{n+1}$ or as the limit point of the centers $P_n$ of the circles. We use the latter approach because it yields an absolutely convergent series.

Using complex numbers to describe vectors in the plane, and taking $A_1A_2$ parallel to the real axis, we get

$$\overline{P_nP_{n+1}} = \frac{1}{2i}(\tan\tfrac{1}{2}\alpha_n - \tan\tfrac{1}{2}\alpha_{n+1})e^{i((\pi-\alpha_2)+(\pi-\alpha_3)+\cdots+(\pi-\alpha_n))}.$$

In the particular example given by the proposer, the first angle is $\pi$ and no circle corresponds to it. In effect, the construction starts with the point $P_3 = 1.5 + \frac{1}{2}i\sqrt{3}$ which is the intersection of the perpendicular bisectors of $A_2A_3$ and $A_3A_4$. It follows that the common point of all the sectors is

$$L = 1.5 + \frac{\sqrt{3}}{2}i + \tfrac{1}{2}\sum_{n=3}^{\infty}\left(\tan\frac{\pi}{n} - \tan\frac{\pi}{n+1}\right)i(-1)^{n-1}e^{-2\pi iS}$$

$$\approx 1.832 + .792i, \qquad S = \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

The probability of $L$ being expressible in closed form is zero; the numerical approximation to $L$ is by courtesy of Mr. Michael Williams.

Also solved by Aage Bondesen (Denmark) and by M. J. Pelling (Nigeria). Bondesen observes that for the proposed problem only one point is in the interior of all the angles.

### Using a Calculator Efficiently

E 2509 [1974, 1112]. *Proposed by C. W. Dodge, University of Maine at Orono*

The Texas Instruments' SR-10 calculator has no memory, but it does have a squaring key $[X^2]$. To calculate $n^2$ for a given $n$ one must either enter $n$, $[\times]$, $n$, $[=]$ (which requires entering the value of $n$ twice) or enter $n$, $[X^2]$ (which is more efficient since the value of $n$ is entered only once); it is not possible to calculate $n^2$ by entering $n$, $[\times]$, $[=]$ as it is on some calculators.

To find $n^3$ efficiently, one can either enter $n$, $[X^2]$, $[\times]$, $n$, $[=]$ or enter $n$, $[X^2]$, $[X^2]$, $[\div]$, $n$, $[=]$. In either case, the value of $n$ must be entered twice. It is not possible to calculate $n^3$ without entering $n$ twice.

Find the smallest power of $n$ that cannot be computed without entering the value of $n$ at least $k$ times in the keyboard.

*Solution compiled by the editors from several submissions.* Let the minimum number of entries of $n$ necessary to compute $n^j$ be $E(j)$ and let $f(j)$ be the expansion in terms of $[X^2]$, $[\times]$, $[\div]$, and $n$, used in the most efficient calculation of $n^j$. To find $E(j)$ we shall use these four facts:

$$E(2j) = E(j) \tag{1}$$

$$E(2j+1) = \min\{E(2j), E(2j+2)\} + 1 \tag{2}$$

$$E(4j-1) = E(j) + 1 \tag{3}$$

$$\forall_i((\forall_{j<i}E(j) \prec t) \Rightarrow (\forall_{j<4i-1}E(j) < t+1)). \tag{4}$$

To prove (1) we see that $E(2j) \leqq E(j)$ since $n^{2j} = (n^j)^2$, and if we assume that $E(2j) < E(j)$, this would imply that $f(2j)$ does not end in $[X^2]$, otherwise dropping the $[X^2]$ would leave a better expansion than $f(j)$. Therefore following the last $[X^2]$ of $f(2j)$ (or following the initial entry of $n$ if there is no $[X^2]$) there must be an even number of $([\times]n)$'s say $2i$ (or equivalently $2i$ $([\div]n)$'s, but not a mixture of both). But then we can construct a better expansion, replacing the $[X^2]$ followed by $2i$ $([\times]n)$'s with just $i$ $([\times]n)$'s followed by $[X^2]$. This contradicts minimality of $E(2j)$ so that $E(2j) \nleq E(j)$ and (1) follows.

To prove (2) we see that $E(2j+1) \leqq \min\{E(2j), E(2j+2)\}+1$ since $n^{2j+1} = n^{2j}[\times]n = n^{2j+2}[\div]n$. Since $2j+1$ is odd, we see that $f(2j+1)$ must end with $[\times]\, n$, or $[\div]n$ and thus either $n^{2j}$ or $n^{2j+2}$ respectively can be calculated with $E(2j+1)-1$ entries of $n$, so we have also $\min\{E(2j), E(2j+2)\} \leqq E(2j+1)-1$ and (2) follows.

To prove (3) we make use of (1) and (2):

$$\begin{aligned} E(4j-1) &= \min\{E(4j), E(4j-2)\}+1 \\ &= \min\{E(2j), E(2j-1)\}+1 \\ &= \min\{E(j), \min\{E(2j), E(2j-2)\}+1\}+1 \\ &= \min\{E(j), \min\{E(j), E(j-1)\}+1\}+1. \end{aligned}$$

Now if $E(j) \leqq E(j-1)$, then clearly (3) follows. If $E(j) > E(j-1)$, then since $E(j) \leqq E(j-1)+1$ (because $n^j = n^{j-1}[\times]n$) again (3) follows.

To prove (4) we need

(i) $E(4j-1) \leqq E(j)+1$ (iii) $E(4j+1) \leqq E(j)+1$

(ii) $E(4j) \leqq E(j)+1$ (iv) $E(4j+2) \leqq E(j)+1$.

We see that (i) follows from (3), (ii) from (1), (iii) from the fact that $n^{4j+1} = n^j[X^2][X^2][\times]n$, and (iv) from the fact that $n^{4j+2} = n^j[X^2][\times]\, n[X^2]$. Thus (4) is established.

Finally, if $t_k$ is the smallest power of $n$ that requires at least $k$ entries of $n$, we see from (3) and (4) that $t_{k+1} = 4t_k - 1$. Since $t_1 = 1$ we have $t_k = (2 \cdot 4^{k-1}+1)/3$.

Also solved by Peter de Buda (Canada), John Doyle, Milton Eisner, Thomas Foregger, Sylvan Greene, Bruce Hajek, George Harrison, Carl Hurd, Dennis Jesperson, O. P. Lossers (Netherlands), Carolyn MacDonald, L. E. Mattics, Gillian Valk, and an anonymous solver. Partial solutions by David Farnsworth, Michael Goldberg, M. Kothmann, Richard Sielaff, and the proposer.

**Power to the Integers**

E 2510 [1975, 73]. *Proposed by Saul Singer, Brooklyn College*

If $n$ is a natural number, let

$$Q(n) = \prod_{k=1}^{n-1} k^{2k-n-1}.$$

(1) Show that $Q(n)$ is an integer whenever $n$ is prime.

(2) For which composite $n$, if any, is $Q(n)$ an integer?

I. *Solution of* (1) *by Paul S. Bruckman, University of Illinois, Chicago Circle.* We have

(a) $$\prod_{k=1}^{n-1} k! = \prod_{k=1}^{n-1} (n-k)! = \prod_{k=1}^{n-1} k^{n-k}.$$

Here, the first equality in (a) follows by a change of variable of $k$ into $(n-k)$; the second equality may be seen by expanding the first product and counting the number of times $k$ occurs in the resulting product, $k = 1, 2, 3, \cdots, n-1$. It follows that

$$\begin{aligned} Q(n) &= \prod_{k=1}^{n-1} \frac{k^{n-1}}{k^{2n-2k}} = \left\{\prod_{k=1}^{n-1} k\right\}^{n-1} \div \left\{\prod_{k=1}^{n-1} k^{n-k}\right\}^2 \\ &= \{(n-1)!\}^{n-1} \div \prod_{k=1}^{n-1} k!(n-k)!. \end{aligned}$$

Hence

(b) $$Q(n) = \prod_{k=1}^{n-1} \left\{\binom{n}{k} \Big/ n\right\}.$$

It is well known that, if $n$ is prime, then $n \mid \binom{n}{k}$, $k = 1, 2, \cdots, n-1$. Hence, by inspection of (b), it follows that $Q(n)$ is an integer whenever $n$ is prime.

II. *Solution of* (1) *and* (2) *by M. J. Pelling, University of Benin, Nigeria.* For any prime $q < n$ let $L(n, q)$ denote the power of $q$ which occurs in $Q(n)$. The contribution to $L(n, q)$ from $k = mq^s$ where $q \nmid m$ is

$$2smq^s - s(n+1) = \sum_{i=1}^{s} (2(mq^{s-i})q^i - (n+1)).$$

If we now define for real $x > 0$, $[x]$ = the largest integer *strictly* less than $x$, and for integral $b$, $0 < b < n$, set

$$F(n, b) = \sum_{t=1}^{[n/b]} (2tb - (n+1)),$$

it follows that $L(n, q) = F(n, q) + F(n, q^2) + \cdots + F(n, q^k)$ where $q^k$ is the largest power of $q$ strictly less than $n$. Computing $F(n, b)$ we get

$$F(n, b) = [n/b](b([n/b] + 1) - n - 1).$$

If $b \mid n$ then $[n/b] = n/b - 1$ so that $F(n, b) = -[n/b] = 1 - n/b$. If $b \nmid n$ let $n = rb + m$, $0 < m < b$, so that $[n/b] = r$ and

$$F(n, b) = (b - m - 1)[n/b] \geqq 0.$$

Hence if $q \nmid n$ we certainly have $L(n, q) \geqq 0$, and if $n$ itself is prime it follows that all $L(n, q) \geqq 0$, and $Q(n)$ is an integer. This proves (1).

To investigate (2) we need only study primes $q$ which divide $n$. Suppose $n = Kq^m$, where $q \nmid K$, $m \geqq 1$ and the case $K = m = 1$ is excluded since $n$ is composite. Then $F(n, q^i) = 1 - Kq^{m-i}$ for $i = 1, 2, \cdots, m$. Let $K \equiv m_s \pmod{q^s}$ where $0 < m_s < q^s$ for $s = 1, 2, \cdots, k$, $q^k$ being the largest power of $q$ less than $K$. Then

$$F(n, q^{m+s}) = (q^{m+s} - m_s q^m - 1)[K/q^s] \text{ for } s = 1, 2, \cdots, k.$$

Adding up, we have

$$L(n, q) = m - K\frac{(q^m - 1)}{(q-1)} + \sum_{s=1}^{k} (q^m(q^s - m_s) - 1)[K/q^s].$$

LEMMA. *In the case $n = Kq^m$ under consideration, $L(n, q) < 0$ if and only if either $K < q$ or $K$ is of the form $aq^k - 1$ where $1 < a \leqq q$ and $k \geqq 1$.*

*Proof.* If $K < q$ then $L(n, q) = m - K(q^m - 1)/(q-1) < 0$. If $K = aq^k - 1$, then

$$L(n, q) = m - (aq^k - 1)\cdot\frac{(q^m - 1)}{(q-1)} + (q^m - 1)\cdot\sum_{s=1}^{k} (aq^{k-s} - 1)$$

$$= m - \frac{(q^m - 1)}{(q-1)}(aq^k - 1 + k(q-1) - aq^k + a)$$

$$= m - (a + k(a-1) - 1)\frac{(q^m - 1)}{(q-1)} < 0.$$

If $K$ is not of the form $aq^k - 1$ nor is less than $q$ then we can set $K = aq^r + m_r$ where $a \geqq 1$ and $q - m_1 = q^2 - m_2 = \cdots = q^{r-1} - m_{r-1} = 1$ but $q^r - m_r \neq 1$. Then $K \equiv -1 \pmod{q^{r-1}}$ so that $m_r = tq^{r-1} - 1$, $1 \leqq t \leqq q-1$, and $q^r - m \geqq q^{r-1} + 1$.

The expression $(q^m(q^s - m_s) - 1)[K/q^s]$ is then $\geqq (q^m - 1)aq^{r-s}$ for $s = 1, 2, \cdots, r-1$ and for

$s=r$ is $\geqq a(q^m(q^{r-1}+1)$. For $r<s\leqq k$ it is positive. Hence

$$\begin{aligned}L(n, q)&\geqq m-(aq^r+m_r)\frac{(q^m-1)}{(q-1)}+(q^m-1)(aq^{r-1}+aq^{r-2}+\cdots+aq)+a(q^m(q^{r-1}+1)-1)\\&=m-(aq^r+m_r)\frac{(q^m-1)}{(q-1)}+a(q^r-1)\frac{(q^m-1)}{(q-1)}+aq^m\cdot q^{r-1}\\&=m+aq^{r-1}+((a-1)(q^r-q^{r-1}-1)+(q^r-m_r)-(q^{r-1}+1))\cdot\frac{(q^m-1)}{(q-1)}\\&>0.\end{aligned}$$

This concludes the proof of the lemma.

The following theorem, which answers (2) completely, is immediate from the preceding lemma:

THEOREM. *$Q(n)$ is an integer if and only if,*

(i) *$n$ is prime, or*

(ii) *If $n$ is composite then for every prime divisor $p$ of $n$ if $p^m \mid n$ but $p^{m+1} \nmid n$ then $n > p^{m+1}$ and also $n$ is not of the form $(ap^k-1)p^m$ with $1<a\leqq p$ and $k\geqq 1$.*

Thus for example $Q(36)$ is an integer and 36 is the smallest composite case.

Both parts were also solved by David Hertzig. Solutions to (1) and partial solutions to (2) were given by M. T. Bird, P. Erdös (Hungary), Irving Gerst, Carl Hurd, Dennis Jespersen, Jeff Lagarias, Albert Leisinger, L.E. Mattics, M.R. Modak (India), Ram Murty and Kumar Murty (Canada), J. D. O'Neill, and E. Zaslawsky. Bruckman also gives a partial solution to (2).

Correct solutions to (1) only were submitted by Ezra Brown, Ron Evans, Alex Ferrer (Mexico), Daniel Finkel, Rev. J. E. Fischer, M. G. Greening (Australia), L. Kuipers, Harry Lass, Vojtech László · (Czechoslovakia), K. R. P. Singh (India), D. P. Sumner, Charles Wexler, and the proposer.

*Editor's comment.* Both Erdös and Pelling remark that the density of $n$ such that $Q(n)$ is an integer is $1-\log 2$.

### Intersecting Triangles and Their Circumcircles

E 2512 [1975, 73]. *Proposed by Eugene A. Herman, Brandeis University*

Let $T_1$ and $T_2$ be two triangles with circumcircles $C_1$ and $C_2$ respectively. Show that if $T_1$ meets $T_2$ then some vertex of $T_1$ lies in (or on) $C_2$ or vice versa. Generalize.

*Solution by Harold W. Schneider, Roosevelt University.* Suppose $T_1$ and $T_2$ are triangles for which this is false. Then certainly the circumcircle of one cannot be contained in that of the other, nor can $C_1$ and $C_2$ be tangent. Thus $C_1$ and $C_2$ intersect in two points. Call the line through these points $L$.

Because of the hypothesis, the vertices of $T_1$ must all lie on the same side of $L$, and the vertices of $T_2$ must all lie on the other side. Thus the vertices of $T_1$ and $T_2$ lie on opposite sides of $L$, and hence $T_1$ and $T_2$ do not meet, contrary to the hypothesis.

This argument also shows that the assertion is true for any two polygons inscribed in circles.

Also solved by M. G. Greening (Australia), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), and the proposer.

*Editor's comment.* Lossers points out that the above proof applies equally well to simplexes and circumspheres in $\mathbb{R}^n$. Combining these two generalizations we obtain that suggested by Victor Klee: If $B_1$, $B_2$ are closed balls in $\mathbb{R}^n$ and $S_1 \subseteq B_1$, $S_2 \subseteq B_2$ are such that their convex hulls meet, then either $S_1$ meets $B_2$ or $S_2$ meets $B_1$.

### A View of an Edge

E 2513 [1975, 73]. *Proposed by Neal Felsinger, Branford, Connecticut*

Let $P$ be a simple (non-self-intersecting) planar polygon. If $A$ is a point in the plane, and if $E$ is an

edge of $P$, then $E$ is *viewable from* $A$ if for every point $x$ of $E$, the line segment joining $A$ to $x$ contains no point of $P$ other than $x$. (I.e., one's view of $E$ from $A$ is not "blocked" by any part of $P$.)

(a) Let $A$ and $P$ be arbitrary. Must some edge of $P$ be viewable from $A$? Examine the cases of $A$ exterior to $P$ and interior to $P$ separately.

(b) Find sufficient conditions on $A$ in order that some edge of $P$ is viewable from $A$.

*Solution by Norman Bauman, Nanuet, New York.* (a) The answer is no, as is shown by the accompanying two figures.

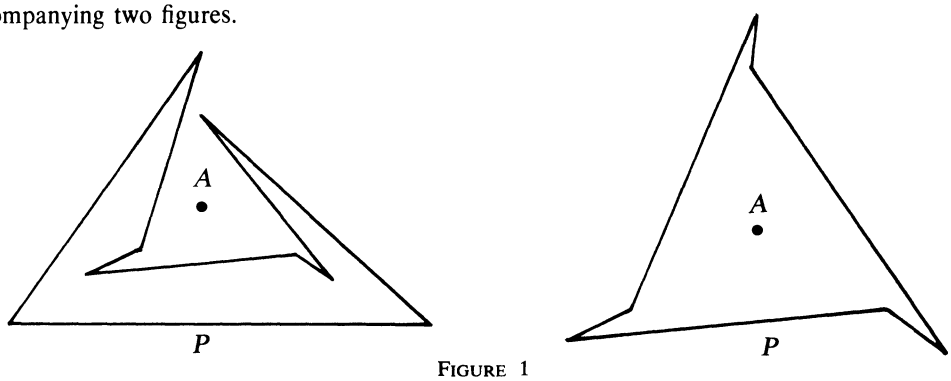



FIGURE 1

(b) If there is a ray from $A$ which does not intersect $P$, some side of $P$ is viewable from $A$. For, rotating a ray $R$ about $A$ which initially does not intersect $P$, one successively encounters the viewable vertices of $P$, $V = V_1, \cdots, V_k$ say. Label $V_i$ with $+$ if the ray $R$ strikes one of the edges of $P$ at $V_i$ *before* it passes through $V_i$, and label $V_i$ with $-$ if $R$ strikes one of the edges of $P$ at $V_i$ *after* it passes through $V_i$. Then each $V_i$ carries at least one label, $V_1$ is labelled $-$ and $V_k$ is labelled $+$. Hence there must be a pair $V_i$, $V_{i+1}$ labelled $-$, $+$ respectively, and $V_i V_{i+1}$ will then be an edge of $P$ viewable from $A$.

Also solved by Ron Evans and by the proposer.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before May 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6072*. *Proposed by Wayne Lawton, Institute for Advanced Study*

Let $a_1, \cdots, a_n$ be $n$ distinct complex numbers such that $0 < |a_k| < 1$ for $1 \leqq k \leqq n$. Let $B = (b_{ij})$ be the $n \times n$ Hermitian matrix defined by $b_{ij} = a_i\bar{a}_j/(1 - a_i\bar{a}_j)$ for $1 \leqq i, j \leqq n$. Then $B$ is positive definite and the following equality is valid

$$\max_{x_i \in \mathbf{C}} \left\{ |x_1 + \cdots + x_n|^2 : \sum_{1 \leqq i,j \leqq n} b_{ij} x_i \bar{x}_j = 1 \right\} = \prod_{k=1}^{n} |a_k|^{-2} - 1.$$

6073. *Proposed by George Crofts, Virginia Polytechnic Institute and State University*

Let $f$ be an increasing real-valued function from $[a, b]$ onto $[c, d]$ and let $m$ denote Lebesgue measure. If there is a set $E \subset [a, b]$, with $m(E) = 0$, for which $m(f(E)) = d - c$, must $f$ be singular (i.e., $f' = 0$ almost everywhere)?

6074*. *Proposed by H. L. Montgomery, University of Michigan*

Let $f$ be a weakly increasing continuous function defined on $[0, 1]$, with $f(0) = 0$, $f(1) = 1$ and let $l$ denote the arc length of the curve $(x, f(x)), 0 \leqq x \leqq 1$. Prove that $l \leqq 2$, with equality if and only if $f'(x) = 0$ almost everywhere. (Compare Problem 6007 [1975, 84] which gives a partial result.)

6075. *Proposed by H. L. Montgomery, University of Michigan*

If $f(x) \in L^1(-\infty, \infty)$, $f(x) \geqq 0$, $\hat{f}(t) \geqq 0$ for all $x$, $t$ where $\hat{f}$ is the Fourier transform, then for any integer $k \geqq 1$,

$$\int_{-k}^{k} f(x)dx \leqq (2k+1)\int_{-1}^{1} f(x)dx.$$

6076*. *Proposed by Robert L. Anderson, Spartanburg, South Carolina*

Given $n$ real numbers $p_1, p_2, \cdots, p_n$, find a continuous function $x(t)$ with piece-wise continuous derivative $x'(t)$ on $[0, n]$ such that $x(t)$ minimizes

$$L(x) = \int_0^n \sqrt{1+[x'(t)]^2}dt$$

subject to the $n$ constraints

$$\int_{i-1}^{i} x(t)dt = p_i, \qquad i = 1,2,\cdots,n.$$

Is the solution unique?

6077*. *Proposed by H. L. Montgomery, University of Michigan*

Let $s(n)$ denote the sum of the base 10 digits of $(1974)^n$. Show that $s(n) \to \infty$ as $n \to \infty$. (I can also do 1975, 1976, but not 1977.)

## SOLUTIONS OF ADVANCED PROBLEMS

### Interchange of Limits

5589 [1968, 415]. *Proposed by Richard Bumby and Erik Ellentuck, Rutgers University*

Let $f$ be any real valued function defined on the natural numbers. Define $f^{(m)}(n) = (f(n)+f(n+1)+\cdots+f(n+m-1))/m$. Prove that $\liminf_m \operatorname{glb}_n f^{(m)}(n) = \operatorname{lub}_m \liminf_n f^{(m)}(n)$.

*Solution by Dan Marcus, Yale University.* The statement is false, as the following counterexample shows: Let $f(n)$ be the $n$th term of the sequence

$$1, 0, 2, -1, 2, -1, 3, -2, 3, -2, 3, -2, \ldots,$$

where the pair $r$, $1-r$ is repeated $r$ times. By considering the $2r-1$ terms following the first occurrence of $r$, one sees that $\liminf_m \operatorname{glb}_n f^{(m)}(n) \leqq 0$, while on the other hand $f^{(2)}(n) \geqq \frac{1}{2}$ shows that $\operatorname{lub}_m \liminf_n f^{(m)}(n) \geqq \frac{1}{2}$.

If we include the condition that $f$ be bounded from below, then the statement is true. Assume, without loss, that $f \geqq 0$. Then we have the following estimates for $\mu$, $\nu \leqq m$:

$$\frac{m}{\mu} f^{(m)}(n) \geqq f^{(\mu)}(n). \tag{1}$$

$$f^{(m)}(n) \geqq \left(1-\frac{\nu}{m}\right) f^{(m-\nu)}(n+\nu). \tag{2}$$

If $k \leqq m$ and $q = [m/k]$, then from (1) we obtain

$$\left(1+\frac{1}{q}\right) f^{(m)}(n) > f^{(qk)}(n). \tag{3}$$

Now we can begin the proof. The inequality in one direction is immediate. For the other, we show that if $\operatorname{lub}_m \liminf_n f^{(m)}(n) > \alpha > \beta > \gamma$, then $\liminf_m \operatorname{glb}_n f^{(m)}(n) \geqq \gamma$. By our hypothesis, there exist

integers $k$ and $\nu$ such that $f^{(k)}(n) > \alpha$ for $n > \nu$. Then $f^{(qk)}(n) > \alpha$ for $n > \nu$ and $q \geq 1$. Using (3) we obtain $f^{(m)}(n) > \beta$ for $n > \nu$ and all sufficiently large $m$. Then (2) shows that $f^{(m)}(n) > \gamma$ for all $n$ and all sufficiently large $m$. This completes the proof.

Also solved by L. E. Clarke (England), Charles Riley, Harold Shapiro, and the proposers.

**The Strip**

5973 [1974, 525]. *Proposed by G. Tsintsifas, Thessaloniki, Greece*

Let $G = \{A_1, A_2, \cdots A_n\}$ be a bounded set of points in the plane. If any three of these points can be covered by a strip of breadth $d$, show that $G$ can be covered by a strip of breadth $2d$.

Find also the minimum real number $k$, so that any point set $G$ with the given property can be covered by a strip of breadth $kd$.

I. *Partial solution by the proposer.* Let $D$ be the diameter of $G$ and $A_\kappa$, $A_\lambda$ be two points with $A_\kappa A_\lambda = D$. For $A_i \in G$, let $h_i$ be the distance of $A_i$ from the line $A_\kappa A_\lambda$. The shortest altitude of the triangle $A_\kappa A_\lambda A_i$ is $h_i$, and therefore $h_i \leq d$. This means that the point $A_i$ lies in a strip of breadth $2d$, with boundary two lines parallel to the line $A_\kappa A_\lambda$, one on each side of it and at a distance $d$ from it.

II. *Comments by Michael Goldberg, Washington, D.C.* Let us say that a set has a strip-width of $d$ if it can be covered by a strip of width $d$. Consider the sets of strip-width $kd$, as shown in the figure. The

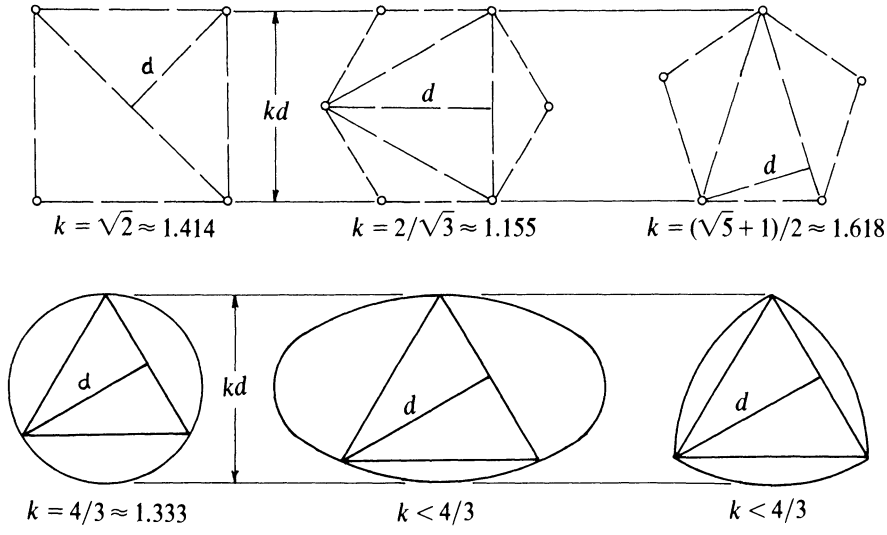



FIG. 1

first three are the vertices of regular $n$-gons. The second three are a circle, an arbitrary oval containing the circle, and an oval of constant width which does not contain the circle. Then the strip-width of three vertices of the square is $d$, from which $k = \sqrt{2} \approx 1.414$. The strip-width of three alternate vertices of the hexagon is $d$, from which $k = 2/\sqrt{3} \approx 1.155$. The strip-width of three vertices of the pentagon is $d$, from which $k = (\sqrt{5}+1)/2 \approx 1.618$. As $n$ increases, the set approaches a circle. The value of $k$ decreases to $4/3 \approx 1.333$.

For the arbitrary oval containing the circle, the maximum strip-width of an inscribed triplet is greater than $3kd/4$, and similarly for the oval of constant width. For each of these two cases, the value of $k$ is less than 4/3.

Hence, for all the sets considered, a value of $k = (\sqrt{5}+1)/2 \approx 1.618$ will suffice in permitting a strip

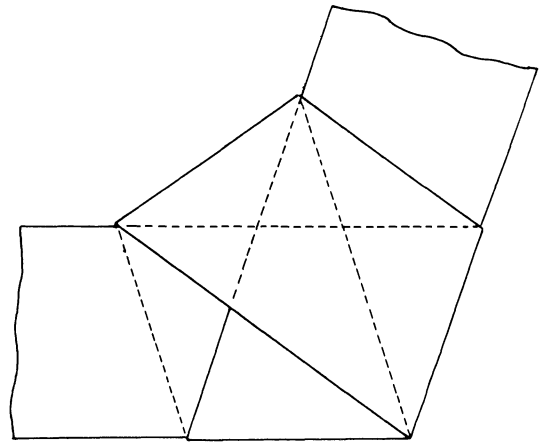


FIG. 2

of width $kd$ to cover the set. It is conjectured that this is the sought minimum. Also, it is interesting to note that a simple knot tied in the strip of width $d$ will generate the regular pentagon of strip-width $kd$.

**An $n$-tuple Integral Formula**

5974 [1974, 525]. *Proposed by M. S. Klamkin, University of Waterloo, Ontario*

Prove that aside from a polynomial of integration of degree $2n-1$,

$$x^{2n-1}\int\frac{dx}{x^2}\int\frac{dx}{x^2}\int\cdots\int\frac{dx}{x^2}\int x^{2n-1}F(x)dx=\int\int\cdots\int F(x)(dx)^{2n}$$

where there are $2n$ integrals on each side.

*Solution by A. B. Farnell, Colorado State University.* Consider

$$y_n = x^{n-1}\int\frac{dx}{x^2}\int\frac{dx}{x^2}\int\cdots\int\frac{dx}{x^2}\int G(x)dx,$$

where there are $n$ integrals involved. We propose to show by induction that

$$y_n^{(n)} = x^{1-n}G(x).$$

This expression is readily verified for $n=1,2,3$. Thus we assume it valid for $y_{n-1}^{(n-1)}$. Then

$$y_n' = (n-1)x^{n-2}\int\cdots\int G(x)dx + x^{n-3}\int\cdots\int G(x)dx,$$
$$xy_n' = (n-1)y_n + y_{n-1}.$$

Differentiating $(n-1)$ times, we obtain

$$xy_n^{(n)} + (n-1)y_n^{(n-1)} = (n-1)y_n^{(n-1)} + x^{2-n}G(x),$$

or

$$y_n^{(n)} = x^{1-n}G(x).$$

This shows that

$$x^{n-1}\int\frac{dx}{x^2}\cdots\int x^{n-1}F(x)dx=\int\cdots\int F(x)(dx)^n$$

modulo a polynomial of degree $(n-1)$ for all $n$.

Also solved by K. S. Anant (India), Günter Bach (Germany), Robert Breusch (New Zealand), P. S. Bruckman, L. Carlitz, Paul Chauveheid (Belgium), L. E. Clarke (England), P. G. Comba, L. J. Dickson (Australia), A. A. Jagers (Netherlands), J. P. Matelski, L. E. Mattics, M. R. Modak (India), V. T. Norton, Jr., B. S. Popov (Yugoslavia), O. G. Ruehr, St. Olaf College Students, Allen Stenger, Robert Weinstock, and the proposer.

**Ordinal Types Thick and Thin**

5975 [1974, 525]. *Proposed by Richard Rado, University of Reading, England, and Albert Wilansky, Lehigh University*

An ordinal type $A$ is called *thick* if there exists a family $(S_\alpha : \alpha \in A)$ of sets $S_\alpha \subset S_\beta$ properly for $\alpha < \beta$ and card $\cup\{S_\alpha : \alpha \in A\} <$ card $A$. Show that $R$ (the reals) is thick and that any well-ordered type (i.e., ordinal number) is thin (= not thick).

Determine other thick and thin ordinal types.

*Solution by Paul R. Meyer, Lehman College.* We give a necessary and sufficient condition that an ordinal type $A$ be thick. The two special cases mentioned in the problem clearly follow. The condition is stated in terms of two cardinal invariants for $A$: let $dA$ denote the least cardinality of a subset dense in $A$ (in the topological sense), and let $rA$ denote the cardinality of the subset of $A$ consisting of those elements $\alpha$ which have an immediate successor $\alpha^+$.

THEOREM. *An ordinal type $A$ is thick if and only if $dA + rA <$ card $A$.*

*Proof.* If $dA + rA <$ card $A$, there is a subset $B$ of $A$ such that $B$ is dense in $A$, $B$ contains all elements which have successors, and card $B <$ card $A$. For $\alpha \in A$, define $S_\alpha$ to be the set $\{\lambda \in B : \lambda < \alpha\}$. To show that $S_\alpha \subset S_\beta$ properly for $\alpha < \beta$ there are two cases: $\beta = \alpha^+$ and $\beta \neq \alpha^+$, in each of which the proper inclusion follows. In the case of the reals, one can let $B$ be the rationals.

To show the converse, we assume that $dA + rA =$ card $A$ and show that $A$ is thin. Assume we have $\{S_\alpha\}$ satisfying the conditions of the problem, put $S = \cup\{S_\alpha : \alpha \in A\}$, and show card $S \geqq$ card $A$. We dispose of the easier case first. If $\alpha$ has a successor, there exists $x \in S_{\alpha+} - S_\alpha$; thus the result follows if $rA =$ card $A$. This clearly includes the case in which $A$ is well ordered.

For the remaining case assume $dA =$ card $A$. We construct a function $g : S \to A^*$ (= Dedekind completion of $A$) such that $B = g(S)$ is dense in $A^*$. The desired inequality follows from

$$\text{card } S \geqq \text{card } B \geqq d(A^*) \geqq dA = \text{card } A.$$

(Density character does not increase in passing to subspaces of ordered spaces, although this is not true for topological spaces in general.) Since each $x$ in $S$ determines a cut in $A$, we can define $g$ by $g(x) = \inf\{\beta : x \in S_\beta\}$. The routine verification that $B$ is dense in $A^*$ completes the proof.

REMARK. The two parts of the condition are independent. An example in which $dA < rA =$ card $A$ is obtained by letting $A$ consist of the top and bottom edges of the lexicographic ordered square. The rationals, of course, satisfy the opposite inequality.

Also solved by Neal Felsinger, H. L. Hiller, A. A. Jagers (Netherlands), I. G. Kastanas (Greece), R. L. Poss, Oto Strauch (Czechoslovakia), and the proposers.

*Editor's comment.* Felsinger's condition for thickness of $A$ is that card $B <$ card $A$ whenever $B$ is dense in $A$; density being defined: if for all $x, y \in A$, $x < y$, $\exists z \in B$ such that $x \leqq z < y$. Hiller gives a collection of examples with his solution including

(i) Every ordinal number is thin.
(ii) Every countable order type is thin.
(iii) There exist arbitrarily large thick order types.
(iv) If $K$ is strongly inaccessible, then every order type of size $K$ is thin.

**Trivial Centralizer Groups**

5976 [1974, 670]. *Proposed by Desmond MacHale, University College, Cork, Ireland*

A group element $x$ is said to have trivial centralizer if it commutes only with its powers, i.e., $C_0(x) = \langle x \rangle$, the cyclic group generated by $x$.

Characterize all finite groups in which every non-identity element has trivial centralizer. *Question*: Do there exist infinite trivial centralizer groups?

*Solution by C. Y. Tang, University of Waterloo, Canada.* If $G$ is a finite group with trivial centralizer property then every Sylow-$p$-subgroup must be cyclic of prime order. It is well known that such groups must be metacyclic [W. R. Scott, *Group Theory*, Theorem 12.6.17]. Hence there are exactly two types of such groups: (1) $G$ is trivial or cyclic of prime order; (2) $G$ is a nonabelian metacyclic group of order $pq$ where $p, q$ are distinct primes. To see case (2), we note that (by the above-cited theorem) $G = \langle x, y; x^m = y^n = 1, y^{-1}xy = x^r \rangle$, where $(m, n) = 1 = (m, r-1)$ and $r^n \equiv 1 \bmod m$. Now by the trivial centralizer property $m, n$ must be distinct primes, say, $m = p$, $n = q$. Since $\langle x \rangle \Delta G$ it follows that $|G| = pq$.

Recent works of S. I. Adjan [Proc. Steklov Inst. Math., vol. 112 (1971)] showed that Burnside groups $B(n, p)$ where $n$ is the number of generators and $p$ is any prime greater than 665 (previously 4381) are infinite, whose abelian subgroups are either trivial or finite of order $p$. Thus these groups satisfy the trivial centralizer property.

Also solved by W. H. Gustafson, A. A. Jagers (Netherlands), A. Krishnamurthy (India), G. A. Losey (Canada), R. C. Lyndon, D. M. Rocke, Kenneth Schilling, R. C. Williams, and the proposer.

NOTE. Joseph Altinger points out that the *rim* of a group $G$ was defined by Norman Levine (*The anticenter of a group*, this MONTHLY, Jan. 1960, p. 61) as $R(G) = \{a \mid ab = ba$ implies $\exists\, c \in G$ such that $a = c^i$, $b = c^j$ for some integers $i$ and $j\}$. The *anticenter* $AC(G)$ is the set of products of elements of the rim. Thus a trivial centralizer group $G$ is one with $G = AC(G)$. Jim Dombek (*Anticenters of several classes of groups*, this MONTHLY, Oct. 1962, p. 738) characterized anticenters for many groups.

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## REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.*

*Calculus and Analytic Geometry.* By Sherman K. Stein. McGraw-Hill, New York, 1973. xv + 1062 pp. $14.95. (Telegraphic Review, November 1973.)

This book develops the calculus by solving compelling real-world problems. It eschews contrived or irrelevent applications and it is capable of stimulating a broad spectrum of students. It is considerably more than a revision of the author's *Calculus in the First Three Dimensions.*

Having used *Calculus and Analytic Geometry* in 1973–75, we find Stein's overall writing style lucid and readable. It effectively appeals to students' geometrical and physical intuition. The author motivates each new concept with good examples before presenting the theory. Nearly all the definitions and theorems included are within the grasp of the average students. The absolute rigor is

carefully consigned to the appendices; thus, those items that might strain one's good will if inserted in the main body of the text are still available to those who delight at this level of discourse. Indeed, this is a preferred mode of presentation.

Each chapter has a summary that students regularly describe as an excellent study aid. Also included are review exercises and guide quizzes that are extremely helpful to students conscientious enough to use them. There is a sufficient number of well-chosen exercises from routine to difficult which illustrate and reinforce the concepts and issues addressed. In addition to the standard applications, the author includes numerous illustrations from various fields. For instance, there are examples and exercises on the present value of future income, minimization and maximization in economics, and the application of calculus to probability.

In using this text for a sequence of four semester-courses (meeting 3 hours a week) during 1973–75, we covered chapters 1–6 on the computation and application of the derivative in the first semester. The second semester was mainly spent on definite integrals over intervals (chapters 7–10). The third course was devoted to chapters 11–14 and chapters 19–22, dealing with partial derivatives and their applications, definite integrals over plane and solid regions as well as vector functions. Finally, topics including series, high derivatives and line integrals (chapters 15–17 and chapters 24–26) were treated in the last course of the sequence. Chapter 18 (Traffic and the Poisson Distribution) was left as optional material for the fourth semester. We found the treatment of series in chapter 15 to be somewhat deficient; and we agreed to supplement it by including, for example, the discussion of the comparison test and the ratio test.

The students' reaction to this text was favorable as evidenced by their performance and the course evaluation forms. Most students urged the instructor to adopt the text again. In fact, we plan to use the book in 1975–76.

Of course, the text, while superior, is not absolutely so. The section in the first chapter that is structured to motivate the concepts of the integral tends to alarm the non-mathematicians, or anyone weak in algebra. The theme returns in still more detail in chapter 7 and again seems to set many students on self-destruct. However, the summary at the end of chapter 7 is excellent. The first chapter, titled "The two main problems of calculus" is clearly misnamed. One looks in vain for the two "problems" and finds instead the author should have said "concepts," for he evidently meant the derivative and the integral. Chapters 5 and 6 are very long and each might be divided in half. The definition of "critical number" in chapter 6 is incomplete and should be corrected and amplified with more examples.

In proving the fundamental theorems of calculus in chapter 8, the author mentioned that he would assume the existence of the integral of a continuous function over $[a, b]$. While this is quite reasonable at this level, the reviewers think the author should single out this crucial assumption by listing it as an important theorem without proof. This assumption should be especially emphasized prior to the corollary on p. 293 which says a continuous function on $[a, b]$ has an antiderivative.

Chapter 9 on computing antiderivatives is not clear in several places. The section on integration by partial fractions does not have enough examples to illustrate the various techniques. The remark on p. 338 that certain examples are included for those who prefer to use integral tables tends to dissuade the students from further study of methods of integration. In chapter 10, the sections on computing cross-sectional lengths and areas are poorly carried out and students find them confusing. Also this is the first chapter to skip the review exercises. They reappear subsequently, but only in chapters 12 and 16. We note that deeper into the text the absolute number of exercises decreases.

The foregoing selection of comments simply demonstrates the need for a sensitive instructor. All in all, we applaud Professor Stein for bringing forth this excellent effort. This text is indeed a very attractive addition to the literature of elementary calculus.

CHARLES PHILLIPS and JEAN CHAN STANEK, California State College, Sonoma

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = approximate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(13). *The Metrics Are Coming! The Metrics Are Coming!* Richard L. Cardwell. Dorrance & Company, 1975, x + 91 pp, $6.95. Brief, elementary introduction to the metric system for the non-mathematician. Clear presentation with many interesting cartoons. RSK

GENERAL, T(13-16: 1). *The Math Book.* Nancy Myers. Hafner Pr, 1975, viii + 406 pp, $9.95. Intended for liberal arts students and elementary education majors. Includes chapters on the history of the real numbers; set theory, logic, algebraic structures, groups, probability, finite geometries; Euclidean and non-Euclidean geometries. SG

BASIC, T**(13: 1), S**. *Basic Mathematical Skills, A Guided Approach.* Robert A. Carman, Marilyn J. Carman. Wiley, 1975, xii + 528 pp, $9.95 (P). A guided workbook which gives a strong course in arithmetic and measurement along with an introduction to algebra. A lot of thought is behind this book. An outstanding text. CEC

BASIC, T(13: 1), S, *A First Course in College Mathematics, Module 4: The Integers and Their Operations; Equations and Inequalities; Squares, Square Roots, and Similar Triangles.* Margaret F. Willerding. Prindle, 1975, xii + 124 pp, $3.75 (P). Designed to be suitable for a self-paced or a traditional course. FLW

PRECALCULUS, T(13: 1), *Precalculus Mathematics: Algebra, Trigonometry, Analytic Geometry.* Harvey Braverman. Williams & Wilkins, 1975, xi + 533 pp, $11.95. After an "algebra review", the author covers functions and graphs, inequalities, polynomials, logarithms, trigonometry, polar coordinates, the straight line, and conic sections in that order. SG

FOUNDATIONS, P. *Minimal Degrees of Unsolvability and the Full Approximation Construction.* Richard L. Epstein. Memoirs No. 162. AMS, 1975, viii + 136 pp, $7.30 (P). A lively examination of the full approximation method for constructing minimal degrees of unsolvability, written for the recursion theorist. Each chapter begins with a long section entitled "Motivation" in which, with the aid of diagrams, the nature of the problem and the ideas necessary to the solution are presented in detail.LCL

FOUNDATIONS, S(16-17). *Cours de Logique Mathématique: Tome 3: Récursivité et Constructibilité.* Roland Fraissé. Gauthier-Villars (U.S. Dist.: SMPF), 1975, 134 pp, 58F (P). An exposition of recursiveness, both absolute and relative, Gödel's incompleteness theorem, the independence of the axiom of constructibility. Very few exercises. SG

COMBINATORICS, S(15-17), L, *Combinatorial Algorithms.* Albert Nijenhuis, Herbert S. Wilf. Comp. Sci. and Appl. Math. Acad Pr, 1975, xiv + 253 pp, $19.50. A collection of algorithms from combinatorial analysis. Each algorithm is presented with a flow chart, subroutine specifications, code for the subroutine version in FORTRAN, and sample output. Bibliography and notes. Index. RJA

ALGEBRA, T(18: 1), S, P. *Simple Noetherian Rings.* John Cozzens, Carl Faith. Tracts in Math., No. 69. Cambridge U Pr, 1975, xvii + 135 pp, $12.95. A survey, including all the known structure theorems, and many proofs. Includes a section on open problems, and appendices presenting the needed background in homological algebra and category theory. Extensive bibliography. No exercises. JG

ALGEBRA, T(18), P, *Rings of Quotients, An Introduction to Methods of Ring Theory.* Bo Stenström. Grund. math. Wissenschaften, B. 217. Springer-Verlag, 1975, viii + 309 pp, $39.60. Intended as an introduction to ring theory and to the theory of rings of fractions, this book concentrates on the general theory of rings of fractions which leads to the consideration of Grothendieck categories. Conversely, a representation theorem for such categories is established. Includes many illustrative examples and a good range of exercises. SG

ALGEBRA, T*(16-17), S*, L, *n-gons.* Friedrich Bachmann, Eckart Schmidt, Cyril W.L. Garner. U of Toronto Pr, 1975, xii + 191 pp, $15. A very interesting exposition of a very pretty subject. A probe into the geometry of n-gons which leads to topics in linear algebra and modern algebra, including vector spaces, Boolean algebra, cyclotomic polynomials, and Galois correspondence. Well-written with a fair number of exercises. Ideal for a senior-level seminar. SG

CALCULUS, S(13), *Problems in Higher Mathematics.* V.P. Minorsky. Trans: Yuri Ermolyev. MIR, 1975, 395 pp. A collection of 2570 problems in analytic geometry and calculus through multiple and line integrals. The selection approximates in scope and difficulty those found in most texts. TAV

CALCULUS, T(13: 2), *Basic Calculus.* Darel W. Hardy. Brooks/Cole, 1975, x + 459 pp, $12.95. Two semester text intended to be compatible with computer-oriented courses as well as those for students of biology and social sciences. Intuitive presentation; minimum of precalculus preliminaries in the text; begins with integration. SG

DIFFERENTIAL EQUATIONS, P. *Linear Differential Equations with Periodic Coefficients.* V.A. Yakubovich, V.M. Starzhinskii. Trans: D. Louvish. Halsted Pr, 1975. *V. 1:* xii + 386 pp; *V. 2:* xii + 452 pp, $49.50 set. An exposition of recent advances in the theory of systems of linear differential

equations with several degrees of freedoms. Written for the "mathematically mature" engineer. Topics covered include: linear algebra prerequisites and general theory of systems of linear differential equations, Hamiltonian systems, perturbation theory, parametric resonance, Lyapunov's method, Hill's equation. SG

DIFFERENTIAL EQUATIONS, P. *Global Theory of a Second Order Linear Ordinary Differential Equation with a Polynomial Coefficient.* Yasutaka Sibuya. Math. Stud., V. 18. North-Holland, 1975, xv + 290 pp, $18 (P). The author focuses on the equation $y'' + P(x)y = 0$ where $P$ is a polynomial. He develops methods for finding asymptotic solutions in case integral representations of solutions are not available. SG

DIFFERENTIAL EQUATIONS, T(16-17). *Partial Differential Equations.* E.T. Copson. Cambridge U Pr, 1975, vii + 280 pp, $27.50, $9.95 (P). Written for seniors and first year graduate students. Covers equations of first and second order, boundary and initial value problems, equations of hyperbolic and elliptic type, wave and heat equations, potential theory in the plane. Some, but not many, exercises included. SG

FUNCTIONAL ANALYSIS, T(18: 2), S, P*. *Banach Lattices and Positive Operators.* Helmut H. Schaefer. Grund. math. Wissenschaften, B. 215. Springer-Verlag, 1974, xi + 376 pp, $40.20. Contains fair amount of author's and his students' research work (done between 1963 and 1973) which the author feels might have decisively narrowed the gap that existed previously between functional analysis and vector lattice theory. A book well-written for a good cause. Makes the theory of Banach lattices and positive linear operators inseparable from the main stream of operator theory. I-CH

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-104: Eigenfunction Branches of Nonlinear Operators, and their Bifurcations.* George H. Pimbley, Jr. Springer-Verlag, 1969, 128 pp, $7.40 (P).

OPTIMIZATION, S(16). *Problems and Exercises in the Calculus of Variations.* M.L. Krasnov, G.I. Makarenko, A.I. Kiselev. Trans: George Yankovsky. MIR, 1975, 222 pp, R 1.50 (P). Major concepts such as invariance of Euler's equation, conditional extremum, moving boundary, Hamilton-Jacobi equation, etc., all are well-illustrated through many solved examples. Virtually with no proofs, yet covers enough of skills and techniques to attract readers to theories. A good useful little book, worth owning. I-CH

ANALYSIS, T(17: 1), P. *Lecture Notes in Mathematics-458: Ergodic Theory--Introductory Lectures.* Peter Walters. Springer-Verlag, 1975, vi + 198 pp, $9.90 (P). Intended to bring the novice to the point where he can read the research papers in the field. After an introduction to the basics, the author discusses entropy, topological dynamics and the modern treatment of topological entropy. As an introduction, the definition-theorem-corollary style is somewhat ponderous. TAV

ANALYSIS, P. *Theory and Application of Special Functions.* Ed: Richard A. Askey. Acad Pr, 1975, xi + 560 pp, $20. Proceedings of a 1975 MRC seminar in Madison, Wisc. "Reports of the death of special functions were premature." LAS

GEOMETRY, S(14), L. *Inversions.* I. Ya. Bakel'man. Trans: Joan W. Teller, Susan Williams. U of Chicago Pr, 1974, viii + 70 pp, $2.95 (P). From the Popular Lectures in Mathematics Series, this easily readable volume contains the necessary mathematical prerequisites for its discussion of inversions, their applications, their relation to linear fractional transformations and the foundation of geometry via group theory. JNC

GEOMETRY, T(14: 1). *Geometry.* Joseph Verdina. Merrill, 1975, xiv + 394 pp, $14.95. Extensive use of set symbolism in a Birkhoff-type treatment of Euclidean geometry for prospective secondary teachers. Includes a chapter on projective geometry and a brief discussion of non-Euclidean geometry. No transformation theory. JNC

TOPOLOGY, T(18: 1), S, P. *Cohomology Theory of Topological Transformation Groups.* Wu Yi Hsiang. Ergebnisse der Math., B. 85. Springer-Verlag, 1975, x + 163 pp, $25. Presentation of the cohomology theory of topological transformation groups developed along the same lines as the classical linear representation theory of compact connected Lie groups. Content is excellent, but the text abounds with grammatical mistakes and typos. References. RJA

TOPOLOGY, P. *Topological Transformation Groups 1: A Categorical Approach.* J. de Vries. Math. Centre Tracts, No. 65. Math Centrum, 1975, v + 251 pp, Dfl. 27 (P). Topological transformation groups in various categories of topological spaces, and their completeness properties as categories in their own right. PJM

PROBABILITY, P. *Decomposition of Multivariate Probabilities.* Roger Cuppens. Prob. and Math. Stat., V. 29. Acad Pr, 1975, xv + 244 pp, $26.50. Devoted to the analysis of multivariate characteristic functions, it contains both classical results (pre-1940) and recent work, most of the latter having been originally published in Russian or French journals. Theorem-proof format. Good bibliography.RSK

PROBABILITY, P. *Markov Chains.* D. Revuz. Math. Lib., V. 11. North-Holland, 1975, x + 336 pp, $35.50. Presumes substantial background in the abstract theory of homogeneous Markov chains with a general state space, Martingales and topological groups. Potential theory, the classification of chains by asymptotic behavior and the Chacon-Ornstein theorem are used to treat random walks on locally compact Abelian groups. Formidable reading. TAV

PROBABILITY, P. *Lecture Notes in Mathematics-440: Markov Processes: Ray Processes and Right Processes.* Ronald K. Getoor. Springer-Verlag, 1975, v + 118 pp, $7.80 (P). A highly-technical treatise extending the results of J.B. Walsh and P.A. Meyer on Ray processes to state spaces of greater generality. Results of D.B. Ray and F. Knight for Ray processes are extended to the more general class of "right" processes. TAV

PROBABILITY, T(18: 1), P. *Topics in Stochastic Processes.* Robert B. Ash, Melvin F. Gardner. Prob. and Math. Stat., No. 27. Acad Pr, 1975, viii + 321 pp, $34.50. Written for readers well versed in the area, it includes $L^2$ stochastic processes, spectral theory and prediction, ergodic theory, the Ito integral, stochastic differential equations. Many exercises with solutions. TAV

STATISTICS, T(13: 1). *Introduction to Applied Statistics.* Marvin Lentner. Prindle, 1975, xi + 388 pp, $13.95. Covers the standard topics, including regression and correlation, non-parametric techniques, analysis of multinomial data, and analysis of variance, in a nontheoretical manner. Contains many applied problems and examples. RSK

STATISTICS, T(17), P. *Applied Multivariate Analysis and Experimental Designs.* N. Krishnan Namboodiri, Lewis F. Carter, Hubert M. Blalock, Jr. McGraw, 1975, xvi + 688 pp, $18.50. Written for sociologists and political scientists as a bridge between elementary texts and more advanced specialized texts. Presumes some knowledge of analysis of variance and multiple regression. Contains four main parts: the general linear model, experimental designs, simultaneous-equation models, and models involving measurement errors. Includes an introduction to Fortran and an appendix on matrices. RSK

STATISTICS, P. *Testing Research Hypotheses Using Multiple Linear Regression.* Keith A. McNeil, Francis J. Kelly, Judy T. McNeil. So Ill U Pr, 1975, xi + 587 pp, $7.95 (P); $12.50. Goal of book is to present one powerful technique, multiple linear regression, in a language that is understandable to the non-mathematically trained behavioral scientist. Particularly geared to a multiple regression computer program described in an appendix, and for which a separate manual is available. RSK

STATISTICS, S(15-17), P, L. *The Design of Experiments to Find Optimal Conditions: A Programmed Introduction to the Design of Experiments.* Yu P. Adler, E.V. Markova, Yu V. Granvosky. Trans: G. Leib. MIR, 1975, 286 pp. A programmed introduction to the (Box-Wilson) method of steepest ascent.FLW

STATISTICS, S(15-17), P, L. *Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics.* Kali S. Banerjee. Statistics, V. 12. Dekker, 1975, xiii + 141 pp, $12.75. Recent results on Hotelling's problem of how most efficiently to weigh several objects using a spring or chemical balance. FLW

STATISTICS, S*(15-16), L. *The Monte Carlo Method.* I.M. Sobol'. Trans: Robert Messer, John Stone, Peter Fortini. U of Chicago Pr, 1974, viii + 65 pp, $2.95 (P). A compact introduction with a brief review of random variables and several examples of applications of the method. FLW

STATISTICS, P. *Lecture Notes in Economics and Mathematical Systems-103: Optimal Subset Selection, Multiple Regression, Interdependence and Optimal Network Algorithms.* D.E. Boyce, A. Farhi, R. Weischedel. Springer-Verlag, 1974, xiii + 187 pp, $8.20 (P).

STATISTICS, S, P, L. *Reliability and Fault Tree Analysis: Theoretical and Applied Aspects of System Reliability and Safety Assessment.* Ed: Richard E. Barlow, Jerry B. Fussell, Nozer D. Singpurwalla. SIAM, 1975, xxxix + 927 pp, $29.50.

STATISTICS, T(15-17: 1, 2), S, P, L. *Sequential Statistical Procedures.* Z. Govindarajulu. Prob. and Math. Stat., V. 26. Acad Pr, 1975, xvi + 565 pp, $39.50. Sequential probability ratio tests, sequential tests for composite hypotheses, and sequential estimation. Appendices contain some mathematical details. Extensive bibliography. FLW

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-32: Mathematical Foundations of Computer Science 1975.* Ed: J. Bečvář. Springer-Verlag, 1975, x + 476 pp, $16.80 (P). Proceedings of a September 1975 conference in Mariánské Láznè, Czechoslovakia. LAS

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-29: Interval Mathematics.* Ed: K. Nickel. Springer-Verlag, 1975, vi + 331 pp, $12.90 (P). Proceedings of the International Symposium at Karlsruhe in May 1975. JAS

COMPUTER SCIENCE, T(14-15: 1), S, L. *Structured Programming and Problem Solving with ALGOL W.* Richard B. Kieburtz. P-H, 1975, xiv + 365 pp, $10.95. A good introductory text. Worth reading even if one does not have access to Algol W. Introduces Algol W and concepts of structured programming and problem solving such as top-down methods, step-wise refinement, algorithm correctness, recursion and readability. Includes chapters on simulation and computer organization, a glossary, and a formal (BNF) description of Algol W. Good choice of examples and exercises. Typescript print. RWN

COMPUTER SCIENCE, T(15: 1), S, L*. *Software Systems Principles, A Survey.* Peter Freeman. SRA, 1975, xxi + 663 pp, $15.95. A broad-based introduction including hardware, operating systems, information management, and languages. Makes excellent use of survey papers including several from *Computing Surveys.* Although it's very readable, the application of the book would be enhanced by a fair amount of practical exposure. Good bibliography. RNW

COMPUTER SCIENCE, S(17), P. *Algol 60 Compilation and Assessment.* B.A. Wichmann. APIC Stud. in Data Proc., No. 10. Acad Pr, 1973, vii + 320 pp, $18.25. A novel, thorough study of 20 compilers with respect to the implementation of the various constructs, execution speed and other measures of performance, "trouble-spots", and documentation. Includes many general recommendations for the design of compilers and languages. RWN

COMPUTER SCIENCE, T(13: 1). *FORTRAN Fundamentals, A Short Course.* Jack Steingraber. Hayden, 1975, vi + 90 pp, $4.45 (P). A brief, relatively expensive, summary. Intended to be used with a reference manual. Examples and problems. RWN

COMPUTER SCIENCE, T(13-14), S, L. *Programming Language/One With Structured Programming, Third Edition.* Frank Bates, Mary L. Douglas. P-H, 1975, xii + 336 pp, $12.50, $9.95 (P). This popular text has been updated to include PL/C, structured programming concepts and a wider range of programming exercises. RWN

COMPUTER SCIENCE, S(16), P. *Distribution Sampling for Computer Simulation.* T.G. Lewis. Heath, 1975, xvi + 150 pp, $17. Part I concerning random number generation from uniform distributions discusses the generalized feedback shift register method which is equivalent to a linear congruential method and tests for randomness, especially n-space behavior. Methods in Part II on sampling from nonuniform distributions include inversion, rejection, and the author's conditional bit method which is a bit-wise generator using conditional probabilities. Includes several program listings for generating and testing. The study is generally empirical and computer-oriented. RWN

SYSTEMS THEORY, T(14-15; 1), S, L. *Computer Simulation of Continuous Systems.* R.J. Ord-Smith, J. Stephenson. Cambridge U Pr, 1975, vi + 327 pp, $9.95 (P). Describes the use of digital, analog and hybrid computers to simulate problems in engineering and science. Includes scaling, function generation, analog memory, and sequential vs. simultaneous computation. Appendices on analog and hybrid hardware and on preparation and checking the models. Numerous practical examples, including feedback control, are quite understandable. Good exercises. Bibliography. Prerequisite: differential equations. RWN

APPLICATIONS (BIOLOGY), P. *Sex and Evolution.* George C. Williams. Monographs in Pop. Bio., V. 8. Princeton U Pr, 1975, x + 200 pp, $13.50. The maladaptive character of sexual reproduction in higher plants and animals is inconsistent with its observed frequency. This monograph documents that thesis in a relatively non-quantitative manner, and proposes appropriate minor changes in current evolutionary theory. LAS

APPLICATIONS (BIOLOGY), S(16-18), P, L. *Group Selection in Predator-Prey Communities.* Michael E. Gilpin. Princeton U Pr, 1975, xiii + 108 pp, $10.50. Group selection (as contrasted to Mendelian individual selection) is a controversial new paradigm of evolution bearing on social dynamics, psychology and theology: it "explains" the evolution of social and religious characteristics that control population levels. This monograph employs differential equation and computer models to test out certain special cases of the group selection hypothesis. LAS

APPLICATIONS (CONTROL THEORY), T(16-17; 1), S. *Introduction to Mathematical Control Theory.* Stephen Barnett. Oxford U Pr, 1975, viii + 264 pp, $16. A concise, readable book on the basics of control theory. Concentrates on so-called state space methods; avoids undue abstraction; lacks control systems with uncertainty. Contents accessible to students with background in calculus and matrix theory. Contains good problems, challenging and inspiring. I-CH

APPLICATIONS (CONTROL THEORY), T(16-17; 2), S. *Applied Optimal Control: Optimization, Estimation, and Control, Revised Printing.* Arthur E. Bryson, Jr., Yu-Chi Ho. Halsted Pr, 1975, 481 pp, $18.50. A text that can be taught in either the deterministic-stochastic or the introductory-advanced sequence. Major part of the book investigates the structure and parameters of dynamic systems and precise measurements for feedback controls; the remaining part proceeds to the design of the "best" control system. Some exercises constitute a semi-research problem. I-CH

APPLICATIONS (ECONOMICS), S(16-17), P. *To Stop or Not to Stop: Some Elementary Optimal Stopping Problems with Economic Interpretations.* Björn Leonardz. Halsted Pr, 1973, 178 pp, $11.95. Written from an economic viewpoint, this monograph provides a clear treatment of some elementary optimal stopping problems, special cases of sequential decision-making under uncertainty as well as arguments that make plausible in economic terms the results derived by mathematics and probability theory. I-CH

APPLICATIONS (ECONOMICS), P. *Adaptive Economic Models.* Ed: Richard H. Day, Theodore Groves. Acad Pr, 1975, ix + 581 pp, $24.50. Papers from an October 1974 symposium at the Mathematics Research Center, Madison, by economists, biologists and engineers exploring economic behavior from an adaptive point of view. LAS

APPLICATIONS (ELECTRONICS), P. *Algorithmic Analysis of Electronic Circuits.* Anatole V. Petrenko, Vitaly P. Sigorsky. Trans: Carroll O. Wilde. Western Periodicals, 1975, v + 593 pp, $30 (P). Computer assisted analysis of electronic circuits: how to do it. Methods used here apply to other areas as well. PJM

APPLICATIONS (INFORMATION THEORY), S(15-18), P, L. *Information-Lossless Automata of Finite Order.* A.A. Kurmit. Trans: D. Louvish. Halsted Pr, 1974, x + 186 pp, $25. Construction of inverse automata to reproduce input words given output and either initial or final state. Original research using graphs whose vertices are graphs. LH

APPLICATIONS (MECHANICS), P. *Mathematical Theory of Creep and Creep Rupture, Second Edition.* Folke K.G. Odqvist. Oxford U Pr, 1974, ix + 200 pp, $25. A book useful to the designers of high-temperature machines, jet engines. Builds up a mathematical theory of creep in metals, analogous to the theory of elasticity, comprising theories of beams, membranes, plates, shells, thick-walled tubes, etc. Devotes special attention to finite deformation, stability, creep rupture, and relaxation. I-CH

APPLICATIONS (SOCIAL SCIENCE), T(17-18), P. *Réseaux sociaux et classifications sociales.* Francois Lorrain. Hermann (U.S. Dist.: SMPF), 1975, 290 pp, 86F (P). Networks and categories of relations between individuals. Subtitle: Essay on the algebra and geometry of social structures. PJM

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Jennifer Galovich, St. Olaf; Steven Galovich, Carleton; Ih-Ching Hsu, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

*Rensselaer Polytechnic Institute*: Assistant Professor H. W. McLaughlin will be on leave for the 1975–76 academic year at the Applied Mathematics Division of the National Bureau of Standards in Gaithersburg, Maryland; Assistant Professor L. A. Rubenfeld will be on leave at Tel-Aviv University and the Weizmann Institute of Science, Israel, for the 1975–76 academic year.

## SYMPOSIUM IN HONOR OF PROFESSOR DUREN

Professor W. L. Duren, Jr., will retire from his position of University Professor on the faculty of the University of Virginia in May 1976. For this occasion the University has arranged a symposium on May 6 and 7 in honor of Professor Duren.

The theme of this symposium will be *Education in Mathematics and Computing.* The program will be held in the Applied Mathematics building in the Engineering School, Room 200 and is scheduled as follows:

| | | |
|---|---|---|
| May 6 | 1:00 P.M. | Professor A. W. Tucker, Princeton University, *Combinatorial Mathematics Looks Ahead.* |
| | 2:30 P.M. | Coffee. |
| | 3:00 P.M. | Professor E. E. Moise, Queens College, *Present Failures in the Teaching of Mathematics and Some Possible Remedies.* |
| May 7 | 9:00 A.M. | Dr. H. O. Pollak, Bell Telephone Laboratories, *How Should Applications Influence Mathematics Education?* |
| | 10:30 A.M. | Coffee. |
| | 11:00 A.M. | Professor E. J. McShane, University of Virginia, *Contributions of the Chicago School of Calculus of Variations to Control Theory and Other Applications, and Implications for the Future.* |

In addition, there will be a social function in the evening of May 6.

Colleagues, friends, and students of Bill Duren are cordially invited to attend. It will be immensely helpful if those who plan to attend will notify Robert H. Owens, Department of Applied Math and Computer Science, Thornton Hall, University of Virginia, Charlottesville, VA 22901.

## IX INTERNATIONAL SYMPOSIUM ON MATHEMATICAL PROGRAMMING

The Mathematical Programming Society announces that the IX International Symposium on Mathematical Programming will be held in Budapest, in the premises of the Hungarian Academy of Sciences, August 23–27, 1976. The Bolyai János Mathematical Society will be responsible for the organization.

Contributed papers on all theoretical, computational, and applicational aspects of mathematical programming are welcome. Abstracts should be sent before March 1, 1976, to the Chairman of the Program Committee, Professor András Prékopa.

All letters should be directed to the following address: IX International Symposium on Mathematical Programming, Bolyai János Mathematical Society, H 1061 Budapest, VI. Anker köz 1, Hungary.

## SYMPOSIUM ON SYMBOLIC MATHEMATICAL COMPUTATION AT AAAS ANNUAL MEETING

A symposium on the theme "Symbolic Mathematical Computation: Its Potential for Science and Education" has been arranged for the AAAS Annual Meeting in Boston. It will be held on February 22, 1976, in Room 204 of the John B. Hynes Auditorium at the Prudential Center from 3:00 to 6:00 P.M. The symposium is being cosponsored by the AAAS Section on Mathematics (Section A), the Association for Computing Machinery, and the Society for Industrial and Applied Mathematics. The program will consist of the following talks:

*Symbolic Mathematical Computation — A Survey*, by James H. Griesmer, IBM Thomas J. Watson Research Center.

*Implications of Symbolic Computation for the Teaching of Mathematics*, by Paul S. Wang, Department of Mathematics, M. I. T.

*Scientific Applications of Symbolic Systems*, by Wayne Fullerton, Los Alamos Scientific Laboratory.

## MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### CALENDAR OF FUTURE MEETINGS

Fifty-sixth Summer Meeting, University of Toronto, August 26–28, 1976.

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A&M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Franklin College, Franklin, May 1, 1976.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Biloxi, Mississippi, February 13–14, 1976.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Bowie State College, Bowie, Maryland, April 24, 1976.

METROPOLITAN NEW YORK, Hofstra University, Hempstead, N.Y., April 25, 1976.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY, Montclair State College, Upper Montclair, May 8, 1976.

NORTH CENTRAL, St. Cloud State University, St. Cloud, Minnesota, April 30–May 1, 1976.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, University of California, Davis, February 21, 1976.

OHIO, Youngstown State University, Youngstown, May 7–8, 1976.

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, April 30–May 1, 1976.

SEAWAY, College of St. Rose, Albany, April 30–May 1, 1976.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, N. Carolina, March 26–27, 1976.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, April 1976.

TEXAS, Texas A & M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), April or May 1976.

### FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Boston, February 18–24, 1976.

AMERICAN MATHEMATICAL SOCIETY, University of Toronto, August 24–27, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, University of Nevada, Reno, April 23–24, 1976.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, March 31–April 2, 1976.

PI MU EPSILON

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SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

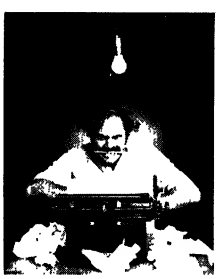
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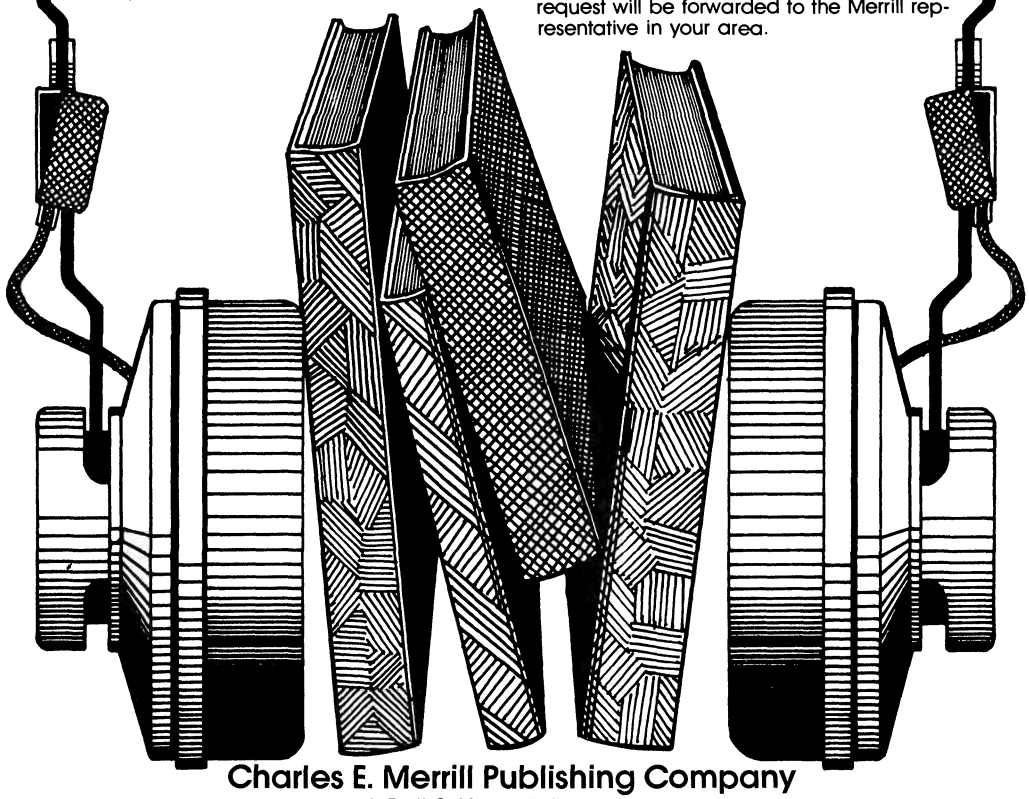
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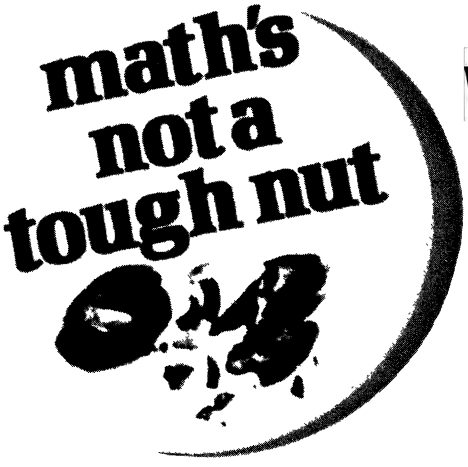
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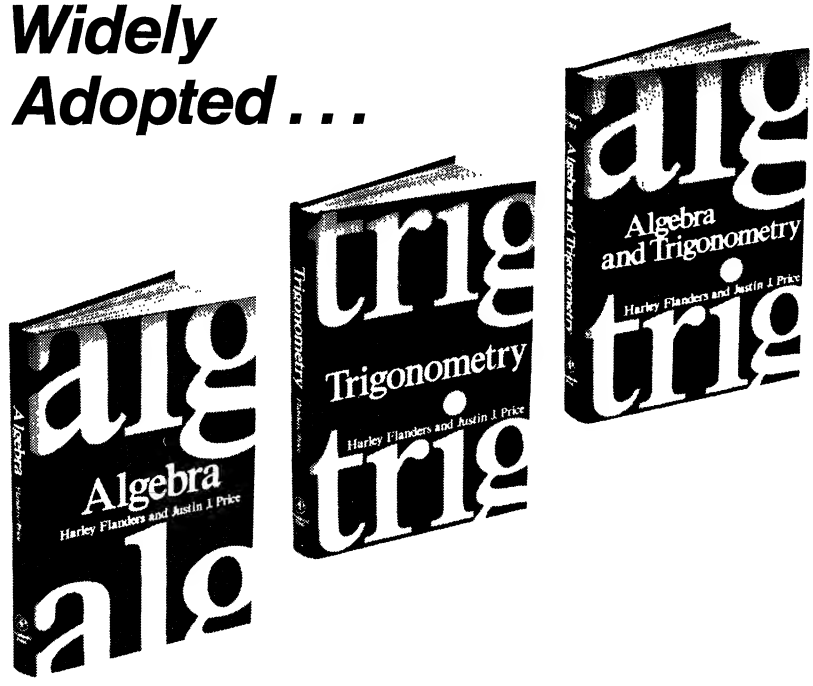
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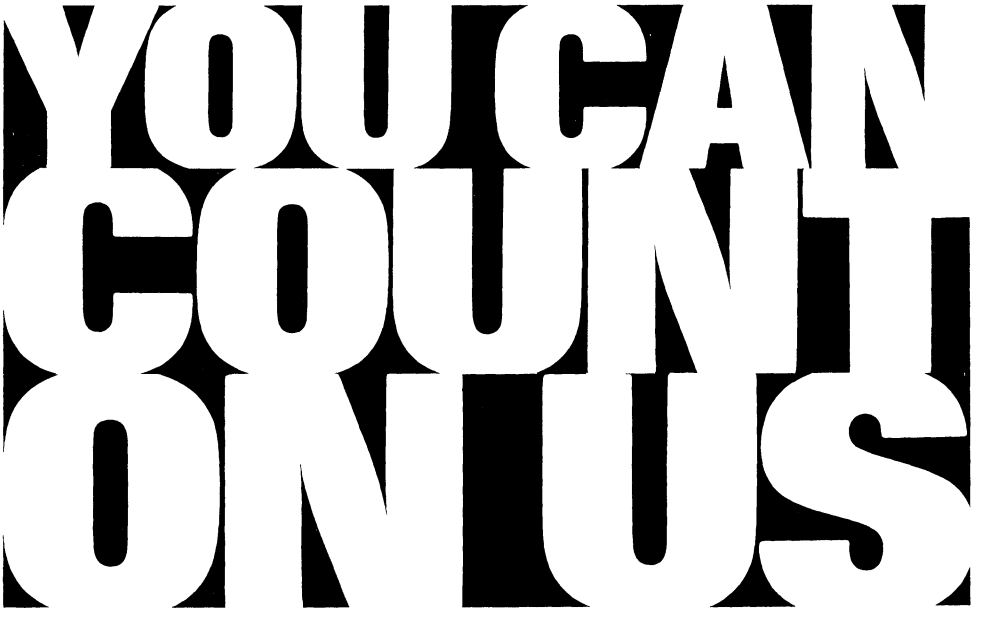
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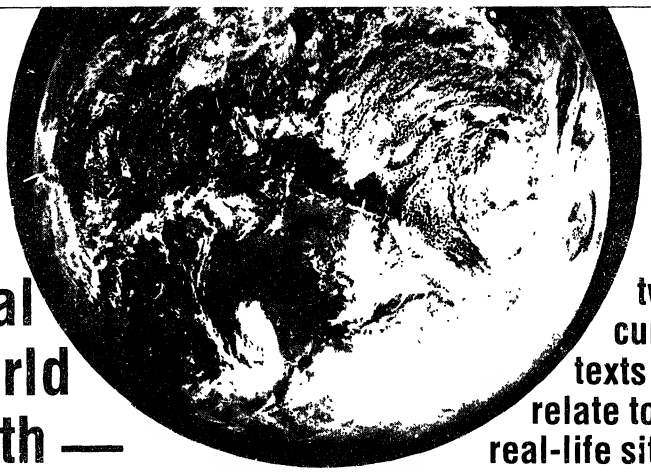
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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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## NOTICE

Because of the lack of interest displayed by the readership and also the increasing age and ill health of its Editor, Professor Alfonse Caracalla Zitronenbaum, the QUERIES DEPARTMENT is being discontinued with this issue. If a substantial fraction of the readership writes in to protest, it is possible that this Department may be reinstituted at some future date.

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## DIRECTIONS IN MATHEMATICAL PSYCHOLOGY, II

ANATOL RAPOPORT

**Pseudo-Probabilistic and Stochastic Models.** We have mentioned some problems of measurement and scaling that arise when the psychologist supposes that some subjective quantities underlie the relative frequencies of observed events, for example in paired comparisons. Such problems are by-passed if the frequencies themselves are the quantities of interest, or if they are viewed simply as estimates of the probabilities of the corresponding events. This is because probabilities, being dimensionless, are already given on an absolute scale, which admits only the identity transformation. Hence, models of behavior in which probabilities of observed acts are at the center of interest (stochastic models) are not subject to restrictions of the sort that govern models constructed on weaker scales.

Although the earliest mathematical models of conditioning processes were based on some probabilistic considerations, they were not true stochastic models. The probabilities were translated at the start into derivatives, so that the point of departure was typically a differential equation, whose solution was supposed to predict deterministically some aspect of the process observed. For this reason, we call them pseudo-probabilistic. H. D. Landahl's learning model [**28**] is an example.

Landahl assumed that in a two-choice learning situation, the "correct" response occurred whenever the level of excitation $\varepsilon_c$ in the neural channel leading to it exceeded the level of excitation $\varepsilon_w$ in the neural channel leading to the "wrong" response. Learning occurred because each correct response added a constant increment $b$ to the excitation at the corresponding channel (the effect of "reward") and each wrong response subtracted a constant quantity $\beta$ from the excitation at the channel leading to it (the effect of "punishment"). At a particular instant, however, the difference $(\varepsilon_c - \varepsilon_w)$ was assumed to be perturbed by a random variable $X$. Thus, after $n$ trials (stimulus presentations), the difference is given by

$$\Delta(n) = \varepsilon_c(0) - \varepsilon_w(0) + bc + \beta w + X, \tag{53}$$

where $c$ is the cumulated number of correct responses, $w$ of wrong responses, $b$ and $\beta$ constants. Since $c + w = n$, we have for the probability of wrong response

$$p_w(n) = \Pr[\Delta(n) < 0] = \Pr[X < \varepsilon_w(0) - \varepsilon_c(0) - bn + (b - \beta)w]. \tag{54}$$

If $n$ is approximated by a continuous variable, $p_w(n)$ can be represented by $dw/dn$. If a specific probability distribution is assumed for $X$, (54) reduces to a differential equation to be solved for $w(n)$, the number of cumulated wrong responses, as a function of the number of trials.

Landahl assumed a Laplacian density for $X$:

$$f(x) = -\frac{k}{2}e^{-k|x|}, \quad k \text{ constant}. \tag{55}$$

Assuming no initial bias in excitation levels, i.e., $\varepsilon_w(0) = \varepsilon_c(0)$, and initial condition $w(0) = 0$, he

obtains the following solution in terms of the "psychological parameters," $b$ and $\beta$:

$$w(n)=\frac{1}{k(b-\beta)}\log_e\frac{2b}{2b-(b-\beta)(1-e^{-kbn})}. \tag{56}$$

The curve gives a good fit to some data, for example those produced by rats in two choice situations.

Some critical comments are in order. The assumption of a Laplacian frequency for $X$ certainly seems fortuitous and was admittedly suggested by mathematical convenience. The "more natural" normal frequency leads to a differential equation that cannot be solved in closed form in terms of elementary functions. Yet, the assumption may not be as opportunistic as it seems when we recall that $X$ represents fluctuations of the *difference* of two random variables, namely the excitations affecting the respective channels to the two responses. If the latter were exponentially distributed, their difference would be Laplacian. Exponential distributions are not hard to take.

A more serious objection relates to the test of the model by cumulative data, which tend to mask discrepancies. As we shall see, genuine stochastic models of learning or conditioning can be subjected to much more severe tests.

Nevertheless, the very simplicity of the deterministic model brings out certain features that may be lost in the more refined approaches. We note, for example, that if we set $\beta = 0$ in (22) and examine $p_c = dc/dn$, it turns out that this variable satisfies the differential equation

$$dp_c/dn = b'p_c(1-p_c), \tag{57}$$

where $b' = kb$ in (56). This equation is suggestive of a contagion process, where $p_c$ represents the fraction of the infected in a population in which "infection" results from random equiprobable contacts between the infected and the uninfected. If we think of $p_c$ as the fraction of conditioned neuroelements, it is intriguing to picture learning as a progressive spread of conditioning (by random contacts) in the population of neuroelements.

It is also interesting to examine Landahl's formula (56) with $b = \beta$, that is, when the effect of rewarding the correct response is identical with that of punishing the wrong response. In that case, the right side of (56) becomes indeterminate, but its limit, as $\beta$ approaches $b$, evaluated by Hôpital's rule is

$$w(n)=K(1-e^{-kbn}),\ K \text{ constant}. \tag{58}$$

Exactly the same formula is derived from the simplest stochastic model of conditioning, where the response is assumed to depend on the state of a single "stimulus element," which either is or is not conditioned to the correct response [2]. If it is conditioned, the correct response occurs; otherwise the wrong response. Further, once the stimulus element becomes conditioned to the correct response, it remains conditioned; that is, the correct response will occur on every succeeding trial. If the element is not conditioned on a particular trial, it will become conditioned following the trial with probability $\theta$, independent of $n$. This model can be represented by a two-state Markov chain with one absorbing state. Denoting the correct (conditioned) state by 1 and the wrong state by 0, the Markov chain is represented by the following matrix of transition probabilities:

$$\begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1-\theta & \theta \\ 0 & 1 \end{pmatrix} \end{array}.$$

As has been said, stochastic models can be subjected to more severe tests than pseudo-probabilistic one, since the former predict not only the expected values of a random variable but *all* the statistics of the protocols. For instance, the above very simplified stochastic model of learning

predicts that every protocol is a sequence (possibly empty) of wrong responses followed by a sequence of correct responses. It predicts not only the mean number of cumulated errors in a population of subjects but the entire distribution of this random variable. In stochastic models, many other random variables can be singled out and their distributions determined, for instance, lengths of runs of responses, alternation frequencies, number of trials to first, second, third correct responses, to the last wrong response, etc., etc.

Clearly, the two state Markov chain with one absorbing state is a model of "all-or-none" learning in a two choice situation, inasmuch as the probability of the correct response does not increase as long as the subject is in state 0, and becomes 1 as soon as the subject passes to state 1. The question arises whether there are learning situations of this sort. It turns out that there are. For instance, in paired associate learning experiments, where a human subject has to learn to associate each of a set of stimuli with one of two responses, the all-or-none learning model predicts the statistics of protocols with good results.

Another situation to which the finite Markov chain with an absorbing state seems appropriate is that of concept identification. In a typical experiment, a number of stimuli differing on several dimensions are classified by the experimenter according to a particular dimension. For instance, if the stimuli are objects differing in size (large, medium, small), color (red, white, and blue) and shape (triangular, circular, or square), they may be classified as square and non-square. The correct response is then "yes" for square and "no" for non-square. The subject is shown one object at a time and responds with "yes" or "no." He then gets a feedback, "right" or "wrong" and is shown the next object. Clearly, the subject solves the problem (classifies correctly) once he identifies shape as the relevant dimension and square as the correct value on that dimension.

The so-called hypothesis-testing model of the learning process assumes that the subject has a repertoire of possible hypotheses (classification rules), that he samples one or more of them on each presentation, and eliminates them in accordance with the information received in the feedback. A great many assumptions can be made concerning the sampling process (for instance, how many hypotheses are sampled at a time), the elimination process (for instance, whether or under what conditions eliminated hypotheses can be sampled again), and the way the hypothesis to be tested is selected among the sampled ones. Each set of assumptions leads to a specific model, i.e., a Markov chain with specifically defined states and transition probabilities. The method will be illustrated by one example [**40**]. The model is based on the following assumptions.

1. The repertoire of hypotheses consists of one hypothesis for each value on each dimension, for instance, "Red is yes," "White is yes," "Blue is no," "Large is yes," "Square is no," etc.

2. After each trial, exactly one hypothesis is sampled and tested on the next trial, being eliminated if the response is wrong and kept otherwise for the successive trial.

3. An eliminated hypothesis is returned to the repertoire, i.e., may be sampled again on successive trials (nothing is learned from errors).

This model is represented by a Markov chain with three states. $L$ represents the state in which the subject has sampled the relevant hypothesis (an absorbing state). $C$ represents the state when an irrelevant hypothesis has been sampled, which is nevertheless consistent with the feedback on the current trial. $E$ represents the state when an irrelevant hypothesis, inconsistent with the current feedback, has been sampled; i.e., following an error. The matrix of transition probabilities becomes

$$
\begin{array}{cc} & \text{Trial } n+1 \\ & \begin{array}{ccc} L & C & E \end{array} \\ \text{Trial } n \begin{array}{c} L \\ C \\ E \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pi & 1-\pi \\ \theta & \pi(1-\theta) & (1-\pi)(1-\theta) \end{bmatrix} \end{array}
$$

where $\pi$ is the probability of correct response on the basis of an irrelevant hypothesis, and $\theta$ is the probability of sampling the relevant hypothesis. Assuming that initially, the irrelevant hypothesis was not sampled, the initial probability vector is $V_1 = [\gamma\theta, \gamma(1-\theta), 1-\gamma]$ where $\gamma$ is the probability that the first response is a correct guess. Thereby the stochastic process is determined.

It is interesting to note that the predictions of the model can be strengthened so as to differentiate between different orders of stimulus presentations. Suppose, for example, the correct classification is "square" and that on trial $n$ a large, red square was presented and on trial $n+1$ a small white square. Then, if the subject was in state $C$ on trial $n$, and if his response was correct ("yes"), it must have been based on either "Large is yes," or "Red is yes" (not on "Square is yes," for in that case the subject would have been in state $L$). Consequently, keeping one of these hypotheses on trial $n+1$, the subject will give the wrong response with probability 1. Similarly, if the current response were made on the basis of an irrelevant hypothesis, and the stimulus were repeated on the next trial, the response on that trial would also have to be correct. If now probabilities of *sampling* the several hypotheses are assigned as parameters, the Markov chain can be formulated as a non-stationary one with the successive matrices determined by the order of stimulus presentations. Specifically, for each pair of trials, $n$ and $n+1$, the set of hypotheses, excluding the relevant one, can be partitioned into two subsets, $H_c$, those that are consistent with the classification on trials $n$ and $n+1$, and $H_e$, those that are inconsistent with the classification on the two trials. If $\Gamma_n$ represents the partition on trial $n$ produced by the stimuli on trials $n$ and $n+1$, $\theta_i$, the probability of selecting each of the irrelevant hypotheses $h_i$ $(i = 1, \cdots, N-1)$, and $\theta_\rho$ the probability of selecting the relevant hypothesis $h$, the matrix of transition probabilities for trial $n$ will be given by

$$
\text{Trial } n \qquad T_{\Gamma_n} = \begin{array}{c} \\ L \\ C \\ E \end{array} \begin{array}{c} \begin{array}{ccc} \text{Trial } n+1 & & \\ L & C & E \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & (1-\theta_\rho)^{-1} \sum_{h_i \in H_c} \theta_i & (1-\theta_\rho)^{-1} \sum_{h_i \in H_e} \theta_i \\ \theta_\rho & \sum_{h_i \in H_c} \theta_i & \sum_{h_i \in H_e} \theta_i \end{bmatrix} \end{array}
$$

The testing of this model requires the estimation of $N$ parameters. However, the model generates a generally different set of probability vectors for each schedule of presentations and so has stronger predictive leverage. For instance, it permits inference about the probabilities with which the subjects select the different hypotheses to be tested.

One of the most commonly used stochastic models is the so-called linear model [7, 14]. It results from a generalization of the above single stimulus model to one with a large set of stimuli, from which a "sample" is conditioned at each trial. Thus, the proportion of conditioned stimuli keeps changing during the learning process. Here, assuming two possible responses, the system can be in any of a continuum of states, each state presenting the *probability* of making one of two responses. The transition equation for this probability $p(n)$ is given by

$$
p(n+1) = \alpha p(n) + \lambda(1-\alpha) \quad (0 \leqq \alpha, \lambda \leqq 1). \tag{59}
$$

The parameters $\alpha$ and $\lambda$ depend, in general, on the experimental event associated with the previous trial, this event being, in turn, dependent on both the response and on the associated outcome.

Thus, in a two-choice, two-outcome situation, where each outcome can be paired with each response, four pairs of experimental events, hence four pairs of parameters are involved. It is, however, usual to combine experimental events into equivalence classes, which reduces the number of

parameters. In some experiments, for example, it seems that only the outcome has a bearing on the conditioning event; in others only the response; in still others outcomes are *determined* by the responses alone, etc.

If the probability of response together with the event following the trial is taken as the state, then the linear model is a Markov process, since the transition probability to each possible next state depends only on the state. However, the state space now contains an infinity of "points." Moreover, the states now include probabilities of responses instead of just responses, as in the case of the one-stimulus model. The stochastic model now leads to predictions of "probabilities of the second order," so to say, that is, the probabilities that the probability of a response will have certain values after $n$ trials. In the general case, even the asymptotic values of these probabilities are not expressible in closed form. Special cases, when they can be so expressed, are singled out for attention and influence the choice of experiments. Also some approximation methods have been worked out. The development of such methods and the problem of estimating parameters from the data involve a respectable mathematical apparatus. On the whole, the technical difficulties are considerable already in the two choice cases. They multiply when more than two choices are involved.

Extensive treatments of applications of Markov processes to learning theory are given in [**1**] and in [**43**].

Comparing the older pseudo-probabilistic models of conditioning with the more recent stochastic models, we see that each provides a different source of leverage to mathematical psychology. The pseudo-probabilistic models with their several free parameters and few variables (often a single one) are usually easy to corroborate. Therefore they do not pinpoint the structure of the mechanism supposedly underlying conditioning, because several models can be fitted by the same data. Such models can, however, serve as points of departure in investigations where the interpretation and manipulation of the parameters is of principal interest. Of course, the parameters of genuinely stochastic models can also be interpreted and manipulated. However, the large amount of work involved in the estimation of parameters in stochastic models tends to focus attention on the fine details of the experimental protocols rather than on constructing psychological theories based on the interpretation of the parameters. The emphasis of the stochastic approach is on getting reasonable assurance that a particular proposed model is adequate rather than on using the model as a springboard for developing a psychological theory.

It should also be kept in mind that the earlier mathematical psychologists faced the task of demonstrating that mathematical methods can be used *at all* in investigations of interest to a psychologist. Thus, many of the early models should be viewed as demonstrations or "études" rather than the building blocks of an adequately supported theory. The stochastic approach reflects a rigorous and demanding attitude toward a mathematical model. Its goal is to derive a more or less compelling (instead of merely adequate) set of hypotheses concerning processes underlying some aspect of behavior.

In more recent applications stochastic learning models have also been used to explain the effects of experimental manipulations that introduce discontinuities into the learning process while supposedly leaving the parameters intact. Examples of these interventions are those where the learning task is changed, for instance where the erstwhile "correct" response becomes the "wrong" response and vice versa, or where the relevant dimension of the stimulus to be attended to is changed.

Consider the rat faced with the choice of jumping toward a window covered with a black card or one covered with a white card. The black window is "correct" in the sense that it can be opened to obtain food behind; the white window is "wrong" in the sense that it is locked. If the black window is always on the left and if the rat eventually learns to jump only toward it, we do not know whether the rat has learned that "black" was correct or that "left" was correct. For this reason, if we want to know whether the rat can learn that "black" is correct, we must randomize the position of the black window. But then, it is possible that in learning the correct response, the rat must learn at least *two* things, (1) to attend to brightness rather than position; (2) to choose "black" rather than "white."

The situation can be generalized. The learning task is conceived as involving the singling out of a relevant dimension among several and the correct value on that dimension. In this connection, it is interesting to compare the results of changing the learning task in two ways: (1) by changing the "correct" value of the same dimension of the stimulus; (intradimensional reversal); (2) by changing the relevant dimension (extra-dimensional reversal). Of interest is the relative difficulty of relearning in each case. Also of interest are so-called overlearning effects. If training on one task continues *beyond* some "criterion" assumed to be an indication that the task has been learned, is it easier or harder to re-learn when the task is changed? On common sense grounds, one would expect that it is harder to relearn under these conditons, because presumably the erstwhile correct response has been rooted more firmly by overlearning. Yet counter-examples are known. Combining probabilities of "attending" to the relevant dimension with those of responding to a particular value on that dimension, stochastic learning models provide explanations for many of the observed effects [**44**].

Let $v = \Pr[br]$, the probability that the subject attends to brightness in the abovementioned experiment; $y = \Pr[B \mid br]$, the probability that he attends to black, given that he attends to brightness. Then the "state of learning" of the subject on any given trial is the pair $x = (v, y)$. The basic assumption of the model is

$$\Pr[B \mid x] = yv + \tfrac{1}{2}(1 - v), \tag{60}$$

where the second term says that if brightness is *not* attended to, black and white are equiprobable choices. The linear model (cf., p. 156) is now specified as follows:

$$y(n+1) = y(n) + \theta_1[1 - y(n)] = y(n)(1 - \theta_1) \tag{61}$$

if the subject attends to brightness, chooses black and is rewarded;

$$y(n+1) = y(n) - \theta_2 y(n) = y(n)(1 - \theta_2) \tag{62}$$

if the subject attends to brightness, chooses black and is not rewarded, and so forth, including all the combinations of directed attention, choice, and outcome. This linear model contains, in addition to the learning parameters discussed earlier, also learning parameters related to attending to the relevant dimension.

In the simplest case, consider two dimensions of the stimulus, say, color and shape, with two values on each dimension, say red and green for color, circle and triangle for shape. These values were used in an experiment on children. In phase 1 of the experiment, all subjects learned by trial and error that color was relevant and that red was correct (always rewarded) and that green was wrong (never rewarded).

Let $\phi$ and $\theta$ be the learning parameters associated respectively with attending to the relevant dimension and responding correctly. (Note that these parameters correspond to $1 - \alpha$ in the earlier notation, hence are positively related to the rate of learning.) Let " # green" be the random variable representing the total number of responses "green." Then the stochastic model predicts

$$E[\,\#\,\text{green}] = \phi^{-1}(1 - v_0) + 2\theta^{-1}(1 - y_0), \tag{63}$$

where $v_0$ and $y_0$ respectively are the expected values of the initial probabilities of attending to color and of responding "red."

In phase 2, the learning task is changed in one of three ways:

Reversal ($R$) : the stimuli remain the same as in phase 1 but now green is correct and red is wrong. Presumably, then, at the start of phase 2, $v_0 = 1$, $y_0 = 0$.

Intradimensional shift ($I$) : color is still relevant; however, in phase 1 the stimuli were blue or yellow and cross or square. Presumably, in this case, the initial expected values of $v$ and $y$ are 1 and $\frac{1}{2}$ respectively, assuming no color generalization and no initial bias.

Extradimensional shift ($E$) : form instead of color is relevant in phase two. Now $v_0 = 0$, $y_0 = \frac{1}{2}$.

Substituting these values into (63), we obtain the predicted values of $E[\#\text{ green}]$ in the three types of shift, namely $2\theta^{-1}$ for $R$, $\theta^{-1}$ for $I$, $\phi^{-1}+\theta^{-1}$ for $E$. Thus, learning the new task is always easier under $I$ than under $E$. The relative difficulty under $E$ and $R$, however, depends on the relative magnitudes of $\phi$ and $\theta$. If $\phi$ is smaller than $\theta$, $R$ is easier; otherwise $E$ is easier.

In deriving theoretically the effect of over-learning on reversal, it is necessary to postulate that learning did not go to completion in phase 1; otherwise, in terms of the stochastic model, over-learning could not affect the initial conditions of phase 2.

In terms of the experiment on rats, let $V_n$ and $Y_n$ be random variables representing respectively the probabilities of attending to brightness and choosing black on trial $n$ of phase 1 ($V_0<1$; $Y_0<1$). Let $T_n$ be the expected total number of errors after reversal on trial $n$, given $V_n$ and $Y_n$. The stochastic model predicts

$$T_n = \phi^{-1}(1-V_n)+2\theta^{-1}Y_n, \tag{64}$$

from which it can be shown that if $\phi<\theta$, $T_n$ decreases for $n$ sufficiently large and if $\phi>\theta$, $T_n$ increases for $n$ sufficiently large. Referring to the discussion of reversal and extradimensional shift, we see that the condition for observing the "paradoxical" over-learning effect is precisely the same as the condition that makes re-learning under reversal easier than under extradimensional shift. The experimental prediction, therefore, is that a shift from position to brightness as the relevant dimension in the experiment with rats will result in a task more difficult than black–white reversal when and only when over-learning facilitates relearning in the latter condition. Perhaps it is somewhat premature to test this prediction in view of the drastic simplification of considering only two learning parameters. Still a summary of findings on conditions involving shifts and the overlearning effect are consistent with the model. Generalizations involving probabilistic reinforcement, more than two dimensions of stimuli, etc. are straightforward and lead to testable quantitative predictions.

**Descriptive models.** A mathematical model has a descriptive and a predictive aspect. The demarcation between the two is not sharp. In fact, the difference can be viewed as a quantitative one: the predictive aspect is the stronger the longer and the more sophisticated is the chain of mathematical deduction induced by the model. For instance, the predictive aspect of the Newtonian model of the solar system is very strong — witness the long chains of mathematical deductions leading to the accurate predictions of eclipses, the discovery of Neptune, etc. In contrast, an empirically suggested formula fitted to a set of observations linking two variables is essentially a descriptive model. To be sure, this model is not entirely devoid of predictive content, for it predicts where other, not yet observed points will fall on the curve. But these predictions, if corroborated, do little more than increase our confidence in the descriptive accuracy of the formula. They do not shed *further* light on the process that generates the data.

To take another example of a purely descriptive model, consider a sociogram, constructed by determining a binary relation on a set of individuals, for instance, who talks to whom. The relation can be represented by a directed graph, or in the symmetric case, by an ordinary linear graph. Being a mathematical object, the graph can be viewed as a mathematical model of an aspect of the social profile of the set of individuals. It can be equivalently represented by a matrix with entries 0 and 1. From the powers of this matrix we can deduce the relation $R^k$, where $AR^kB$ $(k=1,2,\cdots)$ means that a "communication chain" of at most $k$ links connects $A$ and $B$. In spite of these opportunities for deduction, the graph is essentially a descriptive rather than a predictive model of the group, since the deduced relations are no more than a more detailed account of the structure of the graph. The model would be predictive if it implied some behavioral predictions deduced from the structure.

The principal value of descriptive models is in that they facilitate conceptualization. For instance, from the diagram of the graph one may readily detect a "critical node," an individual whose removal would make communication between members of subsets impossible.

Descriptive models play a special role in mathematical psychology by helping the investigator form a relatively simple picture of the situation investigated.

Factor analysis is a classical example of a method that leads to a descriptive model facilitating conceptualization. Because these models are essentially descriptive, there is some question of whether factor analysis should be properly subsumed under the methods of mathematical psychology rather than those of descriptive quantitative psychology or psychometrics. However, as pointed out and as we shall see in what follows, the line of demarcation cannot be drawn sharply.

The fundamental problem of factor analysis can be stated as follows. Suppose we obtain the scores of a large number of subjects on $n$ different tests. Given any two tests, $i$ and $j$, the product moment correlation coefficient $r_{ij}$ of the scores over the subjects can be computed. We ask whether given such a correlation matrix $(r_{ij})(i, j = 1, 2, \cdots, n)$, we can construct a Euclidean space of $m \ll n$ dimensions such that if each of the $n$ tests is represented by a vector $x$ in that space, the cosine of the angle between $x_i$ and $x_j$ will be $r_{ij}$ $(i, j = 1, 2, \cdots, n)$.

In the extreme case, suppose $r_{ij} = 1$ for all $i$, $j$. Then all tests (although they may be of varying difficulty) "measure the same thing." The space would then be one-dimensional, and each subject's score would be simply his coordinate on the line. If the $r_{ij}$ can be accommodated in two dimensions, the tests can be assumed to measure two independent abilities, each test being related in a particular degree to each, as reflected by the direction cosines of the corresponding vector. From each subject's scores, his "factor scores," that is, the measures of his two abilities can be calculated. And so on for any number of dimensions.

Clearly, a substantial reduction of the number of dimensions is possible only if the rank of the correlation matrix is substantially smaller than $n$. In practice, considerable leeway is allowed to leave a portion of the total variance of the scores unaccounted for by the factors singled out. In psychological testing, the purpose of the procedure is (1) to see whether most of the variance can be accounted for by relatively few factors, permitting a compact description of the data; and (2) whether the interpretation of the factors provides some insight into the underlying abilities. For instance, the factors of I.Q. scores may be revealed to be "mathematical ability," "verbal facility," "memory," "visual imagination," "logical reasoning," etc.

An interesting application of factor-analytic techniques is an instrument called the *semantic differential* [**45**]. The set of tests consists of "target words," which usually evoke some qualitative connotations, such as big, good, rough, hot, gentle, timid, etc. Each of a large population of subjects is asked to rank each target word on a number of scales, whose extreme points are qualitative opposites, such as good-bad, rough-smooth, etc. These test scores are then factor-analyzed. It turns out that three dimensions account for most of the variance. These are interpreted as (1) the evaluative factor, typically represented by the good-bad scale, (2) a potency factor, typically represented by the big-little scale, and (3) an activity factor, e.g., fast-slow, active-passive or the like. For each subject, then, a target word can be roughly represented as a point in the three-dimensional "connotation space." The configuration of positions of a set of target words in this space can be viewed as a representation of a certain aspect of the subject's personality profile, to be compared with profiles of other subjects. The test has been used in research involving inter-cultural comparisons and in clinical studies.

Factor-analytic techniques are an outgrowth of statistical methods and are based on assumptions most frequently made in the applications of those methods, for instance the normality of certain random variables. In investigations where standard statistical assumptions are not easy to justify, other techniques have been developed not dependent on assumptions concerning underlying distributions. The common feature in all of these techniques is the search for a geometric "space," in which subjects or objects or both can be imbedded in such a way that the data appear as consequences of the positions assigned to them.

In the simplest case, consider two subjects $A$ and $B$ and three objects, $X$, $Y$, and $Z$. The subjects are asked to rank the objects in order of preference. If the objects are labeled so that $A$'s preference order is $X > Y > Z$, there are six possibilities for $B$'s preference order. In all cases, the data can be represented in a single dimension. That is, the two subjects and the three objects can be placed on a line equally spaced in such manner that the distances from each subject to each of the objects will

represent the preferences, the smallest distance corresponding to the strongest preference. This is illustrated in Table 4.

TABLE 4

| $B$'s preference order | Order of subjects and objects | with |
|---|---|---|
| $X\ Y\ Z$ | $A\ B\ X\ Y\ Z$ | |
| $Y\ Z\ X$ | $A\ X\ Y\ B\ Z$ | $B$ nearer to $Y$ |
| $Z\ X\ Y$ | $B\ Z\ X\ A\ Y$ | $A$ nearer to $X$ |
| $Z\ Y\ X$ | $A\ X\ Y\ Z\ B$ | |
| $Y\ X\ Z$ | $B\ Y\ A\ X\ Z$ | $A$ nearer to $X$ |
| $X\ Z\ Y$ | $Z\ B\ X\ A\ Y$ | $A$ and $B$ nearer to $X$ |

On the other hand, with three subjects and three objects, some configurations cannot be represented in one dimension, for instance if $A$'s preference order is $X > Y > Z$, $B$'s $Z > Y > X$ and $C$'s $X > Z > Y$. In two dimensions, however, the configuration can be represented, for assigning the following coordinates to the subjects and objects. $X$: $(0,0)$, $Y$: $(1,0)$; $Z$: $(0,3)$; $A$: $(0,-1)$; $B$: $(1,3)$; $C$: $(0,2)$. Note that in order to accommodate some of the configurations, restrictions must be placed on some of the distances (cf. last column of Table 4). This means that the descriptive model generates not only structural information relating to the dimensionality of the space required to represent the data but also some metric information. In general, the more subjects and objects there are the more metric information will be generated by the model. The process has been called *unfolding* [**10, 11**]. If the subjects and objects are all on one line and if the line is folded about a subject's position, the objects will arrange themselves in his order of preference. Consequently the total picture is obtained by "unfolding."

In the data leading to the unfolding model, only the orders of distances between each subject and each object are observed. In other situations, such distances are observed between every two "points" in the set. Once again consider the sociogram represented by a linear graph. It is natural to define the "social distance" between two individuals as the smallest number of edges in the paths connecting them. We wish to assign coordinates to each individual in a space of the smallest number of dimensions so that the orders of distances shall be preserved. We need not, however, confine ourselves to a space with a Euclidean metric, where the distance between $n$-dimensional vectors $x$ and $y$ is defined as

$$d(x,y)=\left[\sum_{i=1}^{n}|x_i-y_i|^2\right]^{1/2}. \tag{65}$$

We may wish to consider the entire class of Minkowski metrics with

$$d(x,y)=\left[\sum_{i=1}^{n}|x_i-y_i|^r\right]^{1/r}\quad r\geqq 1 \tag{66}$$

to find one that will enable us to reduce the number of dimensions to a minimum [**26, 50**]. In that case, the solution of the problem will provide us not only with the number of dimensions and with some information about the coordinates of each of the persons in the sociogram but also with the "appropriate" metric that achieves the compression. This information can be interpreted psychologically and so the model can serve as a generator of concepts for a psychological theory. The dimensions may suggest relevant aspects of position in "social space" (wealth? status? political complexion? some personality trait?). Some coordinates (like wealth) may be reasonably inferred from observations; others will be inferred from the solution and so play the part of the free parameters. Finally, the "most appropriate metric" may shed some light on what kind of a thing "social distance" is. Note, in particular, that the parameter $r$ of the metric reflects the relative weight of the sizes of the components of a vector in determining its absolute magnitude.

The common task in the models discussed is that of finding a geometric space in which the data can be embedded as points, so that the relations among these points will reflect the relations observed in the data. For example, the fundamental idea of factor analysis is that of finding a suitable space in which to imbed a set of vectors such that the cosines of angles between any two reflect correlations obtained from a set of data. Since its formulation, the method was extended to a great variety of contexts; and, as frequently happens, came to be applied routinely in much the same way as statistical tests of significance are often applied without much regard for the conditions on which the justification of the model rests (the normality of the underlying distributions). Thereby the theoretical significance of the method was adumbrated, since any set of data could be subjected to the routine, grinding out an "answer" in the form of so many dimensions of the embedding space accounting for so much variance in the data.

Multidimensional analysis was at first also developed as a purely computational method, which could always yield an "answer": the accommodation of a set of "distances" in a space of so many dimensions with a residual error, which, like the level of significance of a statistical test, reflected the "tolerance" of the investigator.

Unlike factor analysis, multidimensional analysis (sometimes called a non-parametric version of factor analysis) does not depend on specific distributions of random variables. Moreover, the space in which multidimensional analysis imbeds the "distances" can be more general than the usual Euclidean space. Consequently, the method raises a number of interesting mathematical questions and points to a direction of theoretical development. One set of questions relates to the requirements to be fulfilled if the metric in terms of which the distances are to be defined is to have certain properties.

To fix ideas, suppose the raw data are subjective estimates of degrees of "dissimilarity" between pairs of stimuli. Thus, given quadruples of stimuli $x$, $y$, $z$, and $w$, the subject responds whether $x$ and $y$ are at least as "dissimilar" as $w$ and $z$. The responses induce an ordering of dissimilarities, e.g., $xy \gtrsim zw$. This ordering can be represented by a distance function $d(x, y)$, provided the data are consistent with the usual three axioms for a metric:

$M_1$ : $d(x, x) \geqq 0$; $d(x, y) > 0$ *for* $x \neq y$
$M_2$ : $d(x, y) = d(y, x)$
$M_3$ : $d(x, y) + d(y, z) \geqq d(x, z)$.

Note that such a metric does not guarantee the existence of $x$, $y$, $z$, such that $d(x, y) + d(y, z) = d(x, z)$, that is, the existence of "collinear points." As an example, consider a set of points on the circumference of a circle, where the distance between every pair is defined as the length of the joining cord. This distance satisfies $M_1 - M_3$, but collinear points do not exist. We are thus led to inquire about conditions that must be satisfied to define a space with "segmental additivity," where any two points can be joined by a segment along which distances are additive. Such a definition is embodied in

$M_4$ : *For any distinct* $x$, $z$, *there exists a set of points* $Y$, *which can be put into one-to-one correspondence with an interval of real numbers, so that* $x$ *and* $z$ *correspond to the end points and* $d(y_1, y_2) = |t_1 - t_2|$, *where* $t_1$ *and* $t_2$ *are the real numbers corresponding to* $y_1$ *and* $y_2$ *respectively,* [4].

The problem now is to formulate a set of properties which, if satisfied by the data, permit a representation of the dissimilarity ordering by a scale satisfying $M_1 - M_4$. Of these there are six. The first two reflect $M_1$ and $M_2$; the next three are continuity properties, insuring sufficient "richness" of the set of stimuli (cf. $A_2$ of conjoint measurement). The last is the "key" property with concrete empirical content: It concerns the "isosimilarity contours" about any stimulus $x$, that is, the set of points equidistant from $x$ in the subject's perception:

For any two distinct contours around $x$ and any point $z$ on the outer contour, there is a $y$ on the inner contour, which is co-segmental with $x$ and $z$.

This is essentially a requirement of spatial homogeneity: the psychological distance between two

concentric contours is independent of the direction taken from the center. To test this property experimentally, we choose pairs $x_1z_1$ and $x_2z_2$. Among all the $y$ for which $x_1$ is constant, find $y_1$, such that $y_1z_1$ is minimal. Find $y_2$ in the same relation to $x_2$ and $z_2$. If the property in question holds, the psychological distance between $y_1$ and $z_1$ should be the same as between $y_2$ and $z_2$.

The theorem on the existence of a scale satisfying $M_1 - M_4$ (given the above six properties of the data) was proved by Beals and Krantz [3].

As has been said, multidimensional scaling was developed as a technique of imbedding a set of data (typically an ordered set of psychological distances between pairs of objects) into some $n$-dimensional space with an appropriate metric. The family of metrics predominantly used is that of so called power metrics (Minkowski $r$-metrics), where for $n$-dimensional vectors $x$, $y$

$$d(x, y) = \left[\sum_{i=1}^{n} | x_i - y_i |^r\right]^{1/r}.$$

The two fundamental properties of these metrics are interdimensional additivity (the distance between two points is an additive combination of contributions of their components to the distance) and intradimensional subtractability (the distance is a function of the absolute differences of corresponding coordinates). Generalizing these properties we can express interdimensional additivity thus:

$$d(x, y) = F\left[\sum_{i=1}^{n} \phi_i(x_i, y_i)\right], \tag{67}$$

where $F$ is an increasing function of its argument. Intra-dimensional subtractability is expressed by

$$d(x, y) = F(| X_i - Y_i |, \cdots, | X_n - Y_n |), \tag{68}$$

where $F$ is an increasing function of each of its $n$ arguments and $X_i = f_i(x_i)$ is the psychological (subjective) scale value of $x_i$, here assumed to be some "objective" (e.g., physical) magnitude. Both properties are satisfied if

$$d(x, y) = F\left[\sum_{i=1}^{n} \varphi_i (| X_i - Y_i |)\right], \tag{69}$$

where now $F$ and each $\varphi_i$ are increasing functions of their respective arguments.

The foregoing analysis relates to the problem of determining the conditions under which a subjective "distance" associated with a pair of stimuli has certain properties expected of a metric.

Suppose now that the dimensionality of a perceptive space is known (or assumed). The problem now is to assign an appropriate metric to this space in order to represent the stimuli as points in it. In the case of a one-dimensional space, this is simply a scaling problem; hence the term multi-dimensional scaling. For instance, the geometry of visual space can be assumed to be three-dimensional, or, if we confine our attention to the horizontal plane through the two eyes, two-dimensional. The investigations of R. K. Luneburg [35] led to the conclusion that a hyperbolic rather than a Euclidean metric is appropriate to this space. Again, a set of visual stimuli, differing in hue, brightness, and saturation, can be assumed to be representable by a three-dimensional space, but finding an appropriate metric for it is a scaling problem.

Let us assume, then, that the space of stimuli is given, where each stimulus is specified by a point $x = (x_1, x_2, \cdots, x_n)$ in $n$-dimensional space, where the $x_i$ are the "objective" coordinates of the point (e.g., reflecting physical magnitudes). Also given is a dissimilarity ordering $M(x, y)$, and the data satisfy the following two properties:

$A_1$ : $M(x, x) = M(y, y) < M(x, y) = M(y, x)$ *for* $x \neq y$. (Again this property reflects the first two metric axioms),

$A_2$ : *Given any three concentric isosimilarity contours, if two stimuli on the inner and outer contour respectively coincide on k dimensions* $(0 \leqq k < n)$, *there exists a stimulus on the middle contour that coincides with them on those same dimensions.*

To see the significance of this property, consider the extreme case $k = n - 1$, so that the stimuli on the inner and outer contours differ on only one dimension. Then we should be able to pass through all the intermediate contours by varying only the coordinate of that dimension.

Beals, Krantz, and Tversky [4] derived necessary and sufficient conditions on the data to justify metrical representations with interdimensional additivity, intradimensional subtractability, or both.

Clearly for sets of data that satisfy both the assumptions of interdimensional additivity and intradimensional subtractability, the set of metrics is quite restricted. In fact, if the data satisfy one additional property, namely that the midpoint between $x$ and $z$ is the closest point to $z$ on the isosimilarity contour around $x$, then there are only two known examples of metric appropriate to such data, one of which is the power metric.

**Mathematically Suggested Conceptual Schemes.** It appears from the foregoing that mathematical models in psychology are, as a rule, tailored to specific, narrowly circumscribed situations, small islands of research, and that there is little prospect of an emergent synthesis of the sort that united several formerly isolated branches of physical science. Still, some comparatively recent mathematical developments have aroused hopes that such synthesizing principles might, after all, emerge. I have in mind information theory, game theory, the theory of automata, and mathematical linguistics. These developments are characterized by considerable generality, a pre-requisite for a synthesizing role. At the same time, they derive not from problems arising in the study of "matter" as such but from abstractions of activities of human beings, for example, communication, contingent planning, and strategic conflict, and so seem to be suitable for the level of analysis proper for psychology.

As is well known, information theory arose in consequence of the analysis of the communication process completely abstracted from the physical nature of communication systems. The amount of information in a sequence of signals remains the same whether the signals are voltages, sound waves or smoke rings. At the same time, "information" is what is stored in memory, retrieved in recognition, and exchanged in social intercourse. Shortly after information theory was established as a theoretical tool of communication engineering, it seemed to some psychologists that its mathematical apparatus might serve as a unifying principle for a variety of psychological phenomena.

The theory of automata revealed the isomorphism between logical calculi and the structure of devices (in hardware) that can realize logical formulae of arbitrary complexity, in other words, a physical realization of the reasoning process.

Mathematical linguistics offers analytical tools for the study of language structure. Since the capacity to communicate via *grammatical* language seems to be a unique characteristic of the human organism, this tool of analysis appeared as a means of entry into human psychology.

Game theory offers mathematical models of *strategic* conflict, where calculation of consequences of alternative courses of action, taking into account the alternatives open to other actors with different goals, is the fundamental underlying idea. Again it appears that a uniquely human characteristic has been tapped.

It can be argued that all four conceptualizations actually carry abstraction beyond the scope of psychology, since the psychology of the individual nowhere enters the analysis. For this very reason, however, these formulations seemed, at least to some, full of promise as synthesizing principles, providing general foundations for the construction of psychological theories. We shall briefly examine the actual or potential impact of these developments on psychology.

Applications of information theory to psychology suggested themselves immediately in two study areas, namely, choice reaction times and memory span. In fact, some of the earliest psychological experiments have been on reaction times [12]. It was noted that if the subject is asked to respond differentially to a set of different stimuli, the reaction time is a monotone increasing function of the

number of choices. Specifically, the function appears to be a logarithmic one, which immediately suggests a relation to the "average amount of information" carried by each stimulus, since the latter is proportional to the logarithm of the number of (equi-probable) stimuli.

With the advent of information theory, the "average amount of information per stimulus" could be related not only to the number of stimuli in the set but also to their relative frequencies of presentation and to the sequential constraints on the presentations. If it could be shown that the average reaction time in a choice reaction time experiment was determined by this "average amount of information per stimulus," then "amount of information" could indeed play a central role in a whole variety of experiments, analogous to the role played by "amount of energy" in physical events. For instance, the human organism could be viewed as a link in a communication channel and characterized by a certain capacity $C$ (units: bits per second). A fundamental theorem of information theory states that given a source (a repertoire of signals), characterized by an "entropy" $H$, i.e., the average number of bits per signal, it is possible to transmit over the channel up to the rate of $C/H$ signals per second with an arbitrarily small probability of error [**49**]. In this conception, average reaction time can be interpreted as the average time necessary to process the information contained in a signal. Hence, assuming a constant capacity, reaction time ought to be proportional to the entropy of the source.

This was the fundamental idea underlying the experiments of R. Hyman [**20**]. In the simplest case, consider a source with a repertoire of binary signals, 0 and 1. The entropy of this source can be controlled by varying the relative frequencies, $p(i)$, $i = 0, 1$, of randomly presented signals; for in that case, the entropy is $H = -p(0)\log_2 p(0) - p(1)\log_2 p(1)$ bits per signal. $H$ can also be controlled by varying sequential probabilities of the signals presented. For instance, if the presentation is a Markov chain with transition probabilities $p_{ij}$ $(i, j = 0, 1)$, its entropy is calculated as

$$H = -\frac{p_{10}}{1 - p_{00} + p_{10}}(p_{00}\log_2 p_{00} + p_{01}\log_2 p_{01}) - \frac{1 - p_{00}}{1 - p_{00} + p_{10}}(p_{10}\log p_{10} + p_{11}\log p_{11}). \tag{70}$$

Hyman performed a number of experiments, varying the source as indicated, and obtained very high correlations between average reaction times and the entropy of the source.

The suggested conclusion that the subject plays the role of a channel with a fixed capacity has been questioned on two grounds. First, the concept of entropy derives from averaging the information associated with each signal. For instance, if the signals are independent events, the information associated with each is $-\log_2 p$, where $p$ is the probability of its occurrence. If reaction times are determined by the corresponding "amounts of information," then they ought to be proportional to the negative logarithm of the frequencies. Hyman's experiments did not bear out this conclusion. The average reaction time to the less probable signal was consistently shorter than expected and that to the more probable one longer. Second, the redundancies in the source introduced by unequal frequencies of signals and their sequential dependence can be "utilized" only by a proper coding of *sequences* of signals. But it is difficult to imagine how this coding can be accomplished if the subject is required to respond to one signal at a time, as he was in the experiments cited.

Subsequently, S. Kornblum [**23**], showed that average reaction times can be widely different corresponding to two sources with the *same* entropy but with different arrangements of repetitions and alternations of signals, thus casting a long shadow of doubt on the usefulness of the information concept in reaction time experiments.

Another quasi-neurological model proposed by L. S. Christie and R. D. Luce [**9**] led to the conclusion that the logarithmic relation between reaction time and the number of equiprobable stimuli can be derived on other than information-theoretic grounds.

A detailed discussion of the theory of reaction times based on stochastic latency mechanisms was given by William J. McGill [**39**].

Experiments with memory span also revealed some difficulties for the information-theoretic approach. It turns out that the short-term memory span for strings of binary digits is not much longer than that for decimal digits, although a decimal digit carries over three times the amount of information as a binary digit. If subjects are *trained* to identify triples of binary digits with octal digits, the memory span increases accordingly. Thus the problem of relating memory span to the entropy of the source is really a problem of determining a coding scheme used by the subject (if that is the case).

This problem becomes especially acute if one tries to explain the prodigious feats of memory performed by chess masters who are able to play several games (as many as forty in some cases) simultaneously without seeing the boards. First, it is necessary to assign some reasonable amount of information contained in a typical chess position. It certainly will not do to assume that the pieces are randomly placed, for in that case each set of positions would carry an amount of information, exceeding by several orders of magnitude the memory spans observed in experiments. We must assume that the chess master perceives the set of positions as a recognizable entity, a "word" endowed with a "meaning," and that at each round he keeps one such "word" in mind. Similar assumptions must be made about Mozart, who is said to have been able to write out the complete score of an orchestral composition after a first hearing, about the actor who has memorized the leading parts in a dozen of Shakespeare's plays, etc.

Thus, attempts to quantity information processing in human beings in terms of "amount of information" stored (as formally defined) very soon lead to the realization of the inadequacy of the "amount of information" as the central concept, analogous to the amount of energy in physics. Nevertheless, the conceptual value of information-theoretical constructs need not be impaired on that account. It is only necessary to realize that the value of the constructs resides not in their immediate and straightforward applicability to testable mathematical models in psychology (as, for example, concepts of energy and entropy are in physics) but rather as generators of newly defined problems, for example, the problem of coding, which is inherent in any model of information processing.

Quantitative studies in linguistics have been almost exclusively statistical, concerned with the empirical determination of frequency distributions of linguistic units, say letters or words in large sample of verbal output. In fact, A. A. Markov, after whom the Markov process is named, studied the frequencies of letters conditional upon preceding letters in Pushkin's verse novel *Eugene Onegin*, [**37**].

In the 1940's, G. K. Zipf [**61**], attempted to account for some regularities in the frequency distributions of words by what he called the Principle of Least Effort, which he mistakenly thought to be an analogue of the Principle of Least Action in physics. Zipf's attempts to formulate the Principle of Least Effort rigorously was hampered by lack of mathematical competence.

In 1955, B. Mandelbrot [**36**] offered a mathematical model which specified clearly what Zipf was saying vaguely. If we consider words as strings of letters (or phonemes), then the amount of information that can be carried by a word clearly increases with word length. On the other hand, the cost (effort) of producing a word can also be supposed to increase with length. Suppose we fix the mean cost of a word and derive the frequency distribution that maximizes the mean information per word; or alternately fix the mean information per word, and derive the frequency distribution that minimizes cost. Mandelbrot showed that such optimizing frequencies follow a somewhat generalized version of "Zipf's Law." Following Zipf's extensive empirical studies, the parameters of frequency distributions in verbal outputs of different populations (authors, politicians, children, mental patients, etc.) were compared and some interesting systematic differences were discovered. Statistical linguistics has also been used in content analysis and, at times, applied to problems of disputed authorship [**17**].

Mathematical analysis applied to the study of syntactic structures is conceived in an entirely different framework, far removed from the statistical or stochastic. The term "mathematical linguistics" is now predominantly used to refer to the structural rather than the statistical approach.

The central problem is to specify with mathematical rigor a set of rules, i.e., a "grammar," so that, given a set of symbols representing a vocabulary of linguistic units (phonemes, morphemes or the like), the rules will generate all possible grammatical sentences in the language governed by that grammar and only those. The number of possible sentences can be considered to be infinite but this creates no difficulty, because the rules can be recursive.

A category of such rules constitutes a category of language structures. For instance, one such category comprises languages that can be adequately described by a set of rules governing the behavior of a finite automaton. The state of the automaton is determined by a string of fixed length just produced. The rules specify the subset of the vocabulary from which the next unit (including " # ", which denotes the end of a sentence) can be selected when the automaton is in a given state. N. Chomsky [**8**] showed that, in view of the possibility of unlimited imbedding of strings inside other strings in, say, English, the finite automaton model is inadequate for that type of language. He went on to consider so-called phrase structure models of grammar, where the reduction of a sentence to a string of elementary units is accomplished by a recursive branching process (which permits unlimited imbedding). This model, too, was rejected as inadequate. So-called transformation grammars, on the next level of complexity, are now intensively studied by linguists.

An example of how this taxonomy could be related to a psychological context is provided by E. M. Gold's study on the theoretical "learnability" of languages in various categories [**15**].

An impetus to the use of automaton or computer models as a theoretical framework in psychology was provided by the widely cited paper of W. S. McCulloch and W. Pitts [**38**]. These authors showed that given a statement involving an arbitrarily complex concatenation of conditions relating a set of inputs (stimuli) to a set of outputs (responses), including sequences of both, it is possible to construct an automaton that will realize these conditions. The elements of such automata are simple input-output devices, each receiving a number of pulse inputs and producing a single pulse output. These, being discrete events, are analogous to the "all-or-none" firing of a neuron. The inputs impinge on such a "neuron" via terminal bulbs of axones that lead from other neurons. These axones transmit the outputs of neurons. For instance, an element with two impinging terminal bulbs that fire if and only if both terminal bulbs fire within a specified time interval represents an "and" element. One that fires if either of the terminal bulbs fire represents an "or" element. Some terminal bulbs inhibit the output of a neuron on which they terminate. Neurons where these originate therefore represent the logical constant "not." More generally, an element may have a "threshold," that is, will fire if and only if the number of firing "excitatory" bulbs impinging on it exceeds the number of firing "inhibitory" bulbs by a given integer, which is the threshold.

The structure of such an automaton, that is, the connections among the elements, determines the mapping of stimulus patterns on response patterns. It is, however, important to note that this model is by no means a "telephone switchboard" representation of the nervous system, that is, a one-to-one mapping of a stimulus space on a response space. Rather the output of a McCulloch-Pitts automaton depends both on imputs impinging from the outside and on its internal state. Hence the automaton can have memory banks, enabling it to refer to past "experiences" in modifying its behavior, in short to "learn," "form concepts," etc. It is these opportunities for simulating "intelligence," widely publicized with the development of computer technology, that attracted psychologists to such automaton models as possible representations of living nervous systems governing behavior and cognitive processes. A class of such models is described in [**41**].

The impact of game theory on psychology is best appreciated with reference to generation of new concepts and problems related to conflicts of interest. As is well known, the mathematical theory of games rests on a fundamental theorem proved by J. von Neumann [**59**], to the effect that in every two person zero-sum game representable in matrix form, there is available to each player an optimal strategy or an optimal probability distribution over the finite set of available strategies. Optimality is to be understood in terms of maximizing expected payoff under the constraint that the opponent is attempting to do the same. In this context, the theory is essentially normative, for one can conceive its

task as that of prescribing optimal play to "rational" players. The impact of game theory upon psychology, however, was via the non-zero-sum game, where prescription of optimal strategies is beset with ambiguities.

The most widely used matrix game in psychological experiments has been the so-called Prisoner's Dilemma game, represented by Matrix 1.

| | | $A_2$ | $B_2$ |
|---|---|---|---|
| $A_1$ | | 1 | 2 |
| | 1 | | −2 |
| $B_1$ | | −2 | −1 |
| | 2 | | −1 |

MATRIX 1.

Player 1 chooses a horizontal row; Player 2 a vertical column. The lower left entry in the resulting outcome is payoff to Player 1; upper right to Player 2.

The dilemma arises from the circumstance that if each player chooses his dominating strategy (which is best against either of the other's two strategies), outcome $B_1B_2$ results, to which *both* players would prefer $A_1A_2$. This discrepancy between the individually rational choice and the collectively rational choice raises interesting psychological questions, related to the actual choices made by subjects.

In most experiments with Prisoner's Dilemma, sequences of plays are examined, where the outcome of each play is announced to the subjects. The object is to see to what extent the players will establish "tacit cooperation," even in the absence of opportunities to communicate and to coordinate their choices. Thus the game has come to be looked upon as a tool in experiments on "conflict and cooperation" whereby the frequencies of the corresponding choices serve as natural measures of tendencies to "cooperate" or to "act in accordance with self interest."

Experiments of this sort, suggesting quantitative behavioral measures of degrees of motivation and of "social interaction" (the behavior of paired players being strongly interdependent) are understandably attractive to empirically oriented social psychologists. By the nature of their discipline, however, the attention of experimenting social psychologists has been directed primarily not toward constructing mathematical models of decision-making but toward uncovering relations between dependent and independent variables of special interest to them, for example, between personalities of the players or experimental conditions simulating social situations and the relative frequencies of choices. Clearly, game theory has nothing to say about these matters. The mathematical theory of the non-cooperative game is directed toward singling out equilibria, distinguishing between dominated and undominated, interchangeable and non-interchangeable sets of equilibria, etc. Of these matters social psychologists know little and care less.

Yet an amalgam of game-theoretic and social psychological considerations is quite possible, via relating psychological pressures to the strategic structure of the game. Consider the set of possible structures of $2 \times 2$ games (involving two players each with two available strategies), where the payoffs are given only on an ordinal scale. Assuming strict inequality among each player's four payoffs, the eight payoffs can be placed in the cells of the game matrix in $(4!)(4!) = 576$ different ways. However, since interchanges of rows or columns of the matrix or re-labeling the players does not change the strategic structure of a game, this number is easily seen to be reduced to 78 [**47**]. Among the 78 structurally distinct $2 \times 2$ games, some are of considerable psychological interest.

Consider the game represented by Matrix 2.

| | $A_2$ | $B_2$ |
|---|---|---|
| $A_1$ | 2<br>0 | 0<br>2 |
| $B_1$ | −1<br>−2 | −2<br>−1 |

MATRIX 2

As in Prisoner's Dilemma, both players have dominating strategies, and there is a single equilibrium at $A_1A_2$. Nevertheless, the outcome may seem "unfair" to Player 1. "Cooperation" may ensue if Player 2 alternates between $A_2$ and $B_2$, while Player 1 keeps choosing $A_1$. In fact, the theory of the cooperative game [**42**] provides the "equitable" distribution between outcomes $A_1A_2$ and $A_1B_2$, namely, in the ratio 3 : 1. Leaving aside, for the moment, the question to what extent this "solution" will be discovered by experimental subjects playing this game, note that "cooperation" in this game entails something quite different from what it entails in Prisoner's Dilemma. There, cooperation means sticking to the "cooperative" outcome $A_1A_2$. In the present game, Player 1's "cooperation" involves sticking to his maximin strategy, while Player 2's cooperation does not. Moreover, in Prisoner's Dilemma, if one player does not "cooperate" the other can improve his own payoff by retaliating. In the present game, on the other hand, retaliation punishes the retaliator as well as the offender.

As a final example, consider Matrix 3.

| | $A_2$ | $B_2$ |
|---|---|---|
| $A_1$ | 2<br>2 | 4<br>3 |
| $B1$ | 3<br>4 | 0<br>0 |

MATRIX 3

Here outcomes $B_1A_2$ and $A_1B_2$ are both equilibria, one of which favors Player 1 and the other Player 2. "Cooperation" in this game would be effected by coordinated alternation between the two equilibrium outcomes. Each player, however, may be tempted to "pre-empt" his strategy $B$ in the expectation that retaliation, being costly, will not be undertaken. Clearly, attempts to "pre-empt" by both leads to disaster. An interesting question is whether a cooperative pattern, once established in this game would be more "stable" than in Prisoner's Dilemma. In favor of the affirmative conjecture is the supposition that the very difficulty of establishing the coordinated pattern (in the absence of communication) may serve to stabilize it. That is, the players may be reluctant to jeopardize a successfully achieved "rhythmic coordination."

Thus, we see that a variety of respectable psychological questions are suggested by the considerable variety of strategic structures of even the simplest games. It is in this respect that the potential impact of game theory on psychology should be evaluated.

Clearly, the opportunities of linking game-theoretic ideas to psychology are greatly broadened in larger contexts. Experiments with $n$-person cooperative games have been inspired by game-theoretic concepts much more directly than those with two-person non-cooperative games. In particular, the "relevance" of various solution concepts has been examined, such as the von Neumann-Morgenstern solution, the Shapley Value, the Bargaining Set, the Kernel, the Competitive Bargaining Set, and the Core under restricted changes of coalition structure [**6, 19, 21, 27**].

The importance of all these theoretical constructs as "mathematical models of behavior" is questionable for several reasons. First, it is very difficult to ascertain to what extent the monetary payoffs used in game experiments actually represent "utilities" to the respective players, or, for that matter, whether consistent utility scales can be at all established. Second, some of the solution concepts, notably the von Neumann-Morgenstern solution, single out large sets of outcomes as consistent with a solution and make very weak predictions. Third, there is much overlap among the outcomes singled out by the various solution concepts, so that in many experiments the results are consistent with more than one theory. In my opinion, the proposed "solutions" of $n$-person cooperative games should be viewed not as predictive mathematical models but rather as organizers of conceptualization and generators of hypotheses, the latter being more often inspired by discrepancies between results implied by the solution and the results observed. It is at this juncture that psychology, which is ignored in formal game theory, can come into its own. Mathematical psychology rooted in the theory of games, then, becomes not so much a collection of mathematical models of conflict behavior as a body of ideas, conjectures, and hypotheses generated or suggested by a preliminary formal-mathematical analysis of conflict situations.

**Concluding Remarks.** The three "arms" of science (to borrow military terminology) are observation, induction, and deduction. In the "exact" physical sciences, the most significant input of mathematics has been to deduction in the shape of the enormously productive algorithms of reasoning. Outside the realm of exact science, the role of mathematics in strengthening the observational and inductive "arms" has been proportionately greater than in deduction, because problems of observation and induction are more complex in the inexact sciences. Problems of observation in the physical sciences are effectively attacked by refined techniques of instrumentation, that is, by applying known physical laws (optics, electro-magnetics, etc.) to extend the range of the senses. Induction is relatively straightforward, because confidence in the generality of physical laws, hence in the legitimacy of inductive leaps, is almost always justified. The situation beyond the physical sciences, particularly in psychology, is different. Many of the observed regularities are statistical. An experiment is often conceived as an observation on a random sample from a population in which a variable of interest is defined by a distribution. Consequently, the degree of confidence in the inductive leap must be quantified. Mathematical statistics is rigorized inductive inference.

Statistical techniques serve also as instruments of observation, for example, by revealing correlations "not visible to the naked eye" or by bringing into focus the principal components of a complex source of variance (as in factor analysis).

Observation, induction, and deduction are synthesized by conceptualization, that is, the singling out of fundamental categories *about* which scientific assertions are made. Here mathematics has also played a crucial role in the physical sciences. The clarification of the distinctions among force, momentum, and energy, once only dimly perceived, was accomplished by their mathematical definitions. However, mathematical definitions are not confined to the task of turning intuitively grasped notions into operationally precise concepts. Mathematical definitions often generate entirely new concepts, not rooted in any intuited notions, for example, "magnetic flux," "white noise," "space-time curvature," etc.

In psychology, mathematics plays a similar dual role in conceptualization. On the one hand, the development of mathematical psychology requires the rigorization of intuitively grasped notions; hence mathematical-operational definitions of "threshold," "utility," "habit strength," etc. How far this quantification can go and remain fruitful is an open question. Success in quantifying "intelligence," "personality profile," and the like has been spotty. The other conceptualizing .role of mathematics, that of generating new concepts *in the process of mathematicizing* (rather than making familiar notions operational by quantifying them) is perhaps the more important one in psychology. Witness the plethora of concepts crystalized in information theory, automaton theory, mathematical linguistics, and game theory. These new concepts, in turn, generate new questions, hypotheses, and

conjectures, possibly of great importance for the future development of psychology. Without the influx of mathematical ideas, they would hardly have occurred to psychologists, because, like the new concepts generated by mathematicized physical science, they are, for the most part, not rooted in unaided intuition, which feeds on every day experience. The full import of mathematical psychology can be appreciated only if its role in "stretching conceptual horizons" is clearly seen.

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# WHAT IS A GROUP RING?

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**1. Introduction.** Let $K$ be a field. Suppose we are given some three element set $\{\alpha, \beta, \gamma\}$ and we are asked to form a $K$-vector space $V$ with this set as a basis. Then certainly we merely take $V$ to be the collection of all formal sums $a \cdot \alpha + b \cdot \beta + c \cdot \gamma$ with $a, b, c \in K$. In the same way if we were originally given four, five or six element sets we would again have no difficulty in performing this construction. After awhile, however, as the sets get larger, the plus sign becomes tedious and at that point we would introduce the $\Sigma$ notation. In general, if we are given a finite set $S$, then the $K$-vector space $V$ with basis $S$ consists of all formal sums $\Sigma_{\alpha \in S} a_\alpha \cdot \alpha$ with coefficients $a_\alpha \in K$. Finally, there is no real difficulty in letting $S$ become infinite. We merely restrict the sums $\Sigma a_\alpha \cdot \alpha$ to be finite, by which we mean that only finitely many nonzero coefficients $a_\alpha$ can occur.

Of course, addition in $V$ is given by

$$\left(\sum a_\alpha \cdot \alpha\right) + \left(\sum b_\alpha \cdot \alpha\right) = \sum (a_\alpha + b_\alpha) \cdot \alpha$$

and scalar multiplication is just

$$b\left(\sum a_\alpha \cdot \alpha\right) = \sum (ba_\alpha) \cdot \alpha.$$

Moreover, by identifying $\beta \in S$ with the element $\beta' = \Sigma b_\alpha \cdot \alpha \in V$, where $b_\beta = 1$ and $b_\alpha = 0$ for $\alpha \neq \beta$ we see that $V$ does indeed have this copy of $S$ as a basis and our original problem is solved.

Now, how do we multiply elements of $V$? Certainly the coefficient-by-coefficient multiplication

$$\left(\sum a_\alpha \cdot \alpha\right) \circ \left(\sum b_\alpha \cdot \alpha\right) = \sum (a_\alpha b_\alpha) \cdot \alpha$$

is exceedingly uninteresting and other than that no natural choice seems to exist. So we are stuck. But suppose finally that we are told that $S$ is not just any set, but rather that $S$ is in fact a multiplicative group. We would then have a natural multiplication for the basis elements and by way of the distributive law this could then be extended to all of $V$.

Let us now start again. Let $K$ be a field and let $G$ be a multiplicative group, not necessarily finite. Then the **group ring** $K[G]$ is a $K$-vector space with basis $G$ and with multiplication defined distributively using the given multiplication of $G$. In other words, for the latter we have

$$\left(\sum_{x \in G} a_x \cdot x\right)\left(\sum_{y \in G} b_y \cdot y\right) = \sum_{x,y \in G} (a_x b_y) \cdot (xy) = \sum_{z \in G} c_z \cdot z,$$

where

$$c_z = \sum_{xy = z} a_x b_y = \sum_{x \in G} a_x b_{x^{-1}z}.$$

Certainly the associative law in $G$ guarantees the associativity of multiplication in $K[G]$ so $K[G]$ is a ring and in fact a $K$-algebra. It is clear that these easily defined group rings offer rather attractive objects of study. Furthermore, as the name implies, this study is a meeting place for two essentially different disciplines and indeed the results are frequently a rather nice blending of group theory and ring theory.

Suppose for a moment that $G$ is finite so that $K[G]$ is a finite dimensional $K$-algebra. Since the study of finite dimensional $K$-algebras (especially semisimple ones over algebraically closed fields) is in far better shape than the study of finite groups, the group ring $K[G]$ has historically been used as a tool of group theory. This is of course what the ordinary and modular character theory is all about (see [2] for example). On the other hand, if $G$ is infinite then neither the group theory nor the ring theory is particularly advanced and what becomes interesting here is the interplay between the two. Our main concern in this article will be with infinite groups and in the following we shall discuss and prove a few

selected results. These results, however, are by no means representative of the entire scope of the subject, but rather were chosen because their proofs are both elegant and elementary. Few references will be given here, but the interested reader is invited to consult the book [5] or the more recent surveys, [4] and [6].

**2. Zero divisors.** Before we go any further, two observations are in order. First, once we identify the elements of $G$ with a basis of $K[G]$, then the formal sums and products in $\sum a_x \cdot x$ are actually ordinary sums and products. In particular, we shall drop the dot in this notation. Second, if $H$ is a subgroup of $G$ then, since $H$ is a subset of the basis $G$, its $K$-linear span is clearly just $K[H]$. Thus $K[H]$ is embedded naturally in $K[G]$.

Now let $x$ be a nonidentity element of $G$ so that $G$ contains the cyclic group $\langle x \rangle$. Then $K[G]$ contains $K[\langle x \rangle]$ and we briefly consider the latter group ring. Suppose first that $x$ has finite order $n > 1$. Then $1, x, \cdots, x^{n-1}$ are distinct powers of $x$ and the equation

$$(1-x)(1+x+\cdots+x^{n-1}) = 1-x^n = 0$$

shows that $K[\langle x \rangle]$, and hence $K[G]$, has a proper divisor of zero. On the other hand, if $x$ has infinite order then all powers of $x$ are distinct and $K[\langle x \rangle]$ consists of all finite sums of the form $\Sigma_{-\infty}^{\infty} a_i x^i$. Thus this group ring looks something like the polynomial ring $K[x]$ and indeed every element of $K[\langle x \rangle]$ is just a polynomial in $x$ divided by some sufficiently high power of $x$. Thus $K[\langle x \rangle]$ is contained in the rational function field $K(x)$ and is therefore an integral domain.

Now what we have shown above is the following. If $G$ has a nonidentity element of finite order, a torsion element, then $K[G]$ has a nontrivial divisor of zero, but if $G$ has no nonidentity element of finite order, then there are at least no obvious zero divisors. Because of this, and with frankly very little additional supporting evidence, it was conjectured that $G$ is torsion free if and only if $K[G]$ has no zero divisors. Remarkably this conjecture has held up for over twenty-five years.

We still know very little about this problem. In fact all we know is that the conjecture is true for some rather simple classes of groups as, for example, free groups, or abelian groups. In the latter case the proof is even quite easy. Thus, suppose that $G$ is torsion free abelian, and let $\alpha$, $\beta$ be elements of $K[G]$ with $\alpha\beta = 0$. Then, clearly, $\alpha$ and $\beta$ belong to $K[H]$ for some finitely generated subgroup $H \subseteq G$ and, by the fundamental theorem of abelian groups, $H$ is just the direct product of the infinite cyclic groups $\langle x_1 \rangle$, $\langle x_2 \rangle, \cdots$ and $\langle x_t \rangle$. It is then quite easy to see that $K[H]$ is contained naturally between the polynomial ring $K[x_1, x_2, \cdots, x_t]$ and the rational function field $K(x_1, x_2, \cdots, x_t)$. Indeed, $K[H]$ is just the set of all elements in $K(x_1, x_2, \cdots, x_t)$ which can be written as a polynomial in $x_1, x_2, \cdots, x_t$ divided by some sufficiently high power of $(x_1 x_2 \cdots x_t)$. Thus $K[H]$ is certainly an integral domain and hence, clearly, either $\alpha = 0$ or $\beta = 0$. Actually the best result to date on this conjecture concerns supersolvable groups. Here for the first time nontrivial ring theory comes into play but the proof is unfortunately too complicated to give here.

On a more positive note, there is a variant of the zero divisor problem which we can handle effectively. A ring $R$ is said to be **prime** if $\alpha R \beta = 0$ for $\alpha, \beta \in R$ implies that $\alpha = 0$ or $\beta = 0$. Clearly this agrees with the usual definition in the commutative case. In addition, this concept makes more sense and is more important from a ring-theoretic point of view than the very much more stringent condition of no zero divisors. For example, the matrix ring $K_n$ is always prime, even though it does have zero divisors for $n \geqq 2$. The main result of interest here is as follows.

THEOREM I. *The group ring $K[G]$ is prime if and only if $G$ has no nonidentity finite normal subgroup.*

The proof in one direction is quite easy. Suppose $H$ is a nonidentity finite normal subgroup of $G$ and set $\alpha = \Sigma_{h \in H} h$, the sum of the finitely many elements of $H$. If $h \in H$ then $hH = H$ so $h\alpha = \alpha$ and thus

$$\alpha^2 = \sum_{h \in H} h\alpha = |H|\alpha.$$

In particular, if $\beta = |H|1 - \alpha$ then $\alpha\beta = 0$. Furthermore, since $H$ is normal in $G$, we have for all $x \in G$, $Hx = xH$, and hence $\alpha x = x\alpha$. Therefore $\alpha$ commutes with a basis of $K[G]$, so it is central and we have

$$\alpha K[G]\beta = K[G]\alpha\beta = 0.$$

Finally, since $\alpha$ and $\beta$ are clearly not zero we see that $K[G]$ is not prime.

The converse direction is much more difficult and as a starter we must ask ourselves how we are going to find a finite normal subgroup in $G$. Observe that if $H$ is such a subgroup and if $h \in H$, then certainly $h$ has finite order, and furthermore, for all $x \in G$ we have

$$h^x = x^{-1}hx \in x^{-1}Hx = H.$$

Thus all $G$-conjugates of $h$ are contained in $H$ and so certainly there are only finitely many distinct ones. This, therefore, leads us to define two interesting subsets of $G$, namely,

$$\Delta(G) = \{x \in G \mid x \text{ has only finitely many conjugates in } G\}$$

and

$$\Delta^+(G) = \{x \in G \mid x \text{ has only finitely many conjugates in } G \text{ and } x \text{ has finite order}\}.$$

At this point a certain amount of group theory obviously comes into play. While the proofs are by no means difficult it does seem inappropriate to offer them here. Therefore we just tabulate the necessary facts below. See [**5**, Lemma 19.3] for details.

LEMMA 1. *Let $G$ be a group and let $\Delta(G)$ and $\Delta^+(G)$ be defined as above. Then*
(i) *$\Delta(G)$ and $\Delta^+(G)$ are normal subgroups of $G$.*
(ii) *$\Delta(G) \supseteq \Delta^+(G)$ and the quotient group $\Delta(G)/\Delta^+(G)$ is torsion free abelian.*
(iii) *$\Delta^+(G) \neq \langle 1\rangle$ if and only if $G$ has a nonidentity finite normal subgroup.*

Part (iii) above certainly seems to answer our first question. Now observe that if $x$ is any element of $G$, then the $G$-conjugates of $x$ are in a natural one-to-one correspondence with the right cosets of $\mathbb{C}_G(x)$, the centralizer of $x$. Thus $x \in \Delta(G)$ if and only if $[G : \mathbb{C}_G(x)] < \infty$ and we shall need the following two basic properties of subgroups of finite index ([**5**, Lemmas 1.1 and 1.2]).

LEMMA 2. *Let $G$ be a group and let $H_1, H_2, \cdots, H_n$ be a finite collection of subgroups of $G$.*
(i) *If $[G : H_i]$ is finite for all $i$, then $[G : \cap H_i]$ is finite.*
(ii) *If $G$ is the set theoretic union $G = \bigcup_{ij} H_i g_{ij}$ of finitely many right cosets of the subgroups $H_i$, then for some $i$ we have $[G : H_i]$ finite.*

We can now proceed with the remainder of the proof of Theorem I. If $\alpha = \Sigma a_x x \in K[G]$, let us define the **support of $\alpha$, Supp $\alpha$,** to be the set of all group elements which occur with nonzero coefficient in this expression for $\alpha$. Thus Supp $\alpha$ is a finite subset of $G$ which is empty precisely when $\alpha = 0$. Now suppose that $K[G]$ is not prime and choose nonzero group ring elements $\alpha$ and $\beta$ with $\alpha K[G]\beta = 0$. If $x \in \text{Supp } \alpha$ and $y \in \text{Supp } \beta$ then certainly $(x^{-1}\alpha)K[G](\beta y^{-1}) = 0$, and furthermore, $1 \in \text{Supp } x^{-1}\alpha$, $1 \in \text{Supp } \beta y^{-1}$. Thus without loss of generality, we may assume that $\alpha$ and $\beta$ have 1 in their supports.

Let us now write $\alpha = \alpha_0 + \alpha_1$ and $\beta = \beta_0 + \beta_1$ where Supp $\alpha_0$, Supp $\beta_0 \subseteq \Delta(G)$ and Supp $\alpha_1$, Supp $\beta_1 \subseteq G - \Delta(G)$. In other words, we have split $\alpha$ and $\beta$ into two partial sums, the first one containing the group elements in $\Delta(G)$ and the second one containing the rest. Since 1 is contained in the supports of $\alpha$ and $\beta$ we see that $\alpha_0$ and $\beta_0$ are nonzero elements of $K[\Delta(G)]$. Our goal here is to prove that $\alpha_0\beta_0 = 0$. Once this is done the theorem will follow easily. Let us suppose by way of contradiction that $\alpha_0\beta_0 \neq 0$. Then clearly $\alpha_0\beta = \alpha_0\beta_0 + \alpha_0\beta_1$ is not zero since Supp $\alpha_0\beta_0 \subseteq \Delta(G)$ and

Supp $\alpha_0\beta_1 \subseteq G - \Delta(G)$ so there can be no cancellation between these two summands. Thus we can choose $z \in G$ to be a fixed element in Supp $\alpha_0\beta$.

We observe now that if $u \in \text{Supp}\,\alpha_0$ then $[G : \mathbb{C}_G(u)]$ is finite and thus, by Lemma 2 (i), $H = \bigcap_{u \in \text{Supp}\,\alpha_0} \mathbb{C}_G(u)$ has finite index in $G$. Moreover, if $h \in H$, then $h$ centralizes all elements in Supp $\alpha_0$ and hence $h$ centralizes $\alpha_0$. Let us now write Supp $\alpha_1 = \{x_1, x_2, \cdots, x_r\}$ and Supp $\beta = \{y_1, y_2, \cdots, y_s\}$. Furthermore, if $x_i$ is conjugate to $zy_j^{-1}$ in $G$ for some $i, j$, then we choose some fixed $g_{ij} \in G$ with $g_{ij}^{-1} x_i g_{ij} = zy_j^{-1}$.

Now let $h \in H$ and recall that $\alpha K[G]\beta = 0$. Then since $\alpha h\beta = 0$ and $h$ centralizes $\alpha_0$, we have

$$0 = h^{-1}\alpha h\beta = h^{-1}(\alpha_0 + \alpha_1)h \cdot \beta = \alpha_0\beta + h^{-1}\alpha_1 h\beta.$$

Thus $z$ occurs in the support of $h^{-1}\alpha_1 h\beta = -\alpha_0\beta$ so there must exist $i, j$ with $z = h^{-1}x_i h y_j$ or in other words $h^{-1}x_i h = zy_j^{-1}$. But then this says that $x_i$ and $zy_j^{-1}$ are conjugate in $G$, so by definition of $g_{ij}$ we have

$$h^{-1}x_i h = zy_j^{-1} = g_{ij}^{-1} x_i g_{ij},$$

and thus $hg_{ij}^{-1}$ centralizes $x_i$. We have therefore shown that for each $h \in H$ there exists an appropriate $i, j$ with $h \in \mathbb{C}_G(x_i)g_{ij}$ and hence

$$H \subseteq \bigcup_{i,j} \mathbb{C}_G(x_i)g_{ij}.$$

Now $[G : H]$ is finite, so $G = \bigcup_k Hw_k$ is a finite union of right cosets of $H$ and we conclude from the above that

$$G = \bigcup_{i,j,k} \mathbb{C}_G(x_i)g_{ij}w_k.$$

We have therefore shown that $G$ can be written as a finite set theoretic union of cosets of the subgroups $\mathbb{C}_G(x_i)$ and hence we deduce from Lemma 2 (ii) that for some $i$, $[G : \mathbb{C}_G(x_i)]$ is finite. But this says that $x_i \in \Delta(G)$ and this is the required contradiction, since by definition $x_i \in \text{Supp}\,\alpha_1 \subseteq G - \Delta(G)$. Thus $\alpha_0\beta_0 = 0$.

We can now complete the proof quite quickly and easily. Since $\alpha_0$ and $\beta_0$ are non zero elements of $K[\Delta(G)]$ with $\alpha_0\beta_0 = 0$ we see that $K[\Delta(G)]$ has nontrivial zero divisors. Hence $\Delta(G)$ cannot possibly be torsion free abelian. On the other hand, according to Lemma 1 (ii), $\Delta(G)/\Delta^+(G)$ is torsion free abelian so we must have $\Delta^+(G) \neq \langle 1 \rangle$. Thus finally we deduce from Lemma 1 (iii) that $G$ has a nonidentity finite normal subgroup and the theorem is proved.

In conclusion, we remark that Theorem I has an amusing application to the zero divisor problem. Namely, we can show that if $G$ is a torsion free group then $K[G]$ has nontrivial zero divisors if and only if it has nonzero elements of square zero. Of course if $\alpha \in K[G]$, $\alpha \neq 0$ with $\alpha^2 = 0$, then certainly $K[G]$ has zero divisors, so this direction is really trivial. In the other direction let $\alpha$ and $\beta$ be nonzero elements of $K[G]$ with $\alpha\beta = 0$. Since $G$ is torsion free, Theorem I implies immediately that $K[G]$ is prime so we have $\beta K[G]\alpha \neq 0$. But

$$\beta K[G]\alpha \cdot \beta K[G]\alpha = \beta K[G](\alpha\beta)K[G]\alpha = 0$$

since $\alpha\beta = 0$ and hence we see that every element of $\beta K[G]\alpha$ has square zero.

**3. Idempotents.** Let us return again to the group ring of a finite group and consider its regular representation. That is, we view $V = K[G]$ as a $K$-vector space on which $K[G]$ acts as linear transformations by right multiplication. In particular, since $V$ is finite dimensional here, each choice of a basis for $V$ gives rise to a certain matrix representation for $K[G]$. More precisely, for each such

basis we obtain an appropriate faithful homomorphism $\tau : K[G] \to K_n$ from $K[G]$ into $K_n$, the ring of $n \times n$ matrices over $K$. Clearly $n = \dim V = |G|$.

Suppose first that $\tau$ corresponds to the natural basis for $V$, namely $G$ itself. Then for each $x \in G$ right multiplication by $x$ merely permutes the basis elements. Thus $\tau(x)$ is a permutation matrix, that is a matrix of 0's and 1's having precisely one 1 in each row and column. Moreover, if $x \neq 1$, then $gx \neq g$ for all $g \in G$ so $\tau(x)$ has only 0 entries on its main diagonal. Hence clearly trace $\tau(x) = 0$. On the other hand, $\tau(1)$ is the identity matrix, so trace $\tau(1) = n = |G|$. Now matrix trace maps are $K$-linear so for $\alpha = \Sigma a_x x \in K[G]$ we obtain finally

$$\text{trace } \tau(\alpha) = \sum_x a_x \cdot \text{trace } \tau(x) = a_1 |G|.$$

In other words, the trace of $\tau(\alpha)$ is just a fixed scalar multiple of $a_1$, the identity coefficient of $\alpha$.

We are therefore led, for arbitrary groups $G$, to define a map $\text{tr} : K[G] \to K$, called **the trace**, by

$$\text{tr}\left(\sum_x a_x \cdot x\right) = a_1.$$

Moreover, it seems reasonable to expect that tr should have certain trace-like behavior and that it should prove to be an interesting object for study. Indeed, we observe immediately that tr is a $K$-linear functional on $K[G]$, and furthermore, that for $\alpha = \Sigma a_x x$, $\beta = \Sigma b_y y$ we have

$$\text{tr } \alpha\beta = \sum_{xy=1} a_x b_y$$

and this is symmetric in $\alpha$ and $\beta$ since $xy = 1$ if and only if $yx = 1$. Thus $\text{tr } \alpha\beta = \text{tr } \beta\alpha$.

Now one of the most interesting questions about tr concerns idempotents $e \in K[G]$ and the values taken on by tr $e$. To see what we might expect these values to be, let us again assume that $G$ is finite. Since the trace of a linear transformation is independent of the choice of basis, we compute

$$|G| \text{ tr } e = \text{trace } \tau(e)$$

by taking a basis more appropriate for $e$. Namely we write $V$ as the vector space direct sum $V = Ve + V(1-e)$ and then we choose as a basis for $V$ the union of ones for $Ve$ and for $V(1-e)$. Since $e$ acts like one on the first set and like zero on the second, we conclude therefore that

$$|G| \text{ tr } e = \text{trace } \tau(e) = \dim Ve.$$

Thus if we are able to divide by $|G|$ in K (that is, if the characteristic of $K$ does not divide $|G|$) then

$$\text{tr } e = (\dim Ve)/|G|.$$

In other words, we see that tr $e$ is contained in the prime subfield of $K$, i.e., the rationals $Q$ if char $K = 0$ or the Galois field GF($p$) if char $K = p$. Furthermore, in characteristic 0 since $0 \leqq \dim Ve \leqq |G|$ we have $0 \leqq \text{tr } e \leqq 1$.

Now it turns out that these two properties of tr $e$ are indeed true in general, but we should observe that there is a basic distinction between them. The fact that tr $e$ is contained in the prime subfield is clearly an algebraic property while the inequality $0 \leqq \text{tr } e \leqq 1$ is in some sense analytic in nature. With this basic dichotomy in mind, we now proceed to consider some proofs.

THEOREM II. *Let $e \in K[G]$ be an idempotent. Then* tr $e$ *is contained in the prime subfield of $K$.*

We first consider fields $K$ of characteristic $p > 0$, and as it turns out, the property of $e$ we use here is $e^p = e$. The reason for this is that in algebras over fields of characteristic $p$ the $p$th power map is fairly well behaved. Indeed the identity $(a+b)^p = a^p + b^p$ always holds in commutative $K$-algebras and for noncommutative ones an appropriate generalization exists. To be more precise let $A$ be a

$K$-algebra and define $[A, A]$, the commutator subspace of $A$, to be the subspace generated by all Lie products $[a, b] = ab - ba$ with $a, b \in A$. Then the result we are alluding to is the following and a proof can be found in [**5**, Lemma 3.4].

LEMMA 3. *Let $A$ be an algebra over a field $K$ of characteristic $p > 0$. If $a_1, a_2, \cdots, a_m \in A$ and if $n > 0$ is a given integer then there exists an elements $b \in [A, A]$ with*

$$(a_1 + a_2 + \cdots + a_m)^{p^n} = a_1^{p^n} + a_2^{p^n} + \cdots + a_m^{p^n} + b.$$

With this fact we can now prove the first part of Theorem II. Let $e = \Sigma a_x x$ be an idempotent in $K[G]$ and let $S$ denote the subset of Supp $e$ consisting of all those elements of order a power of $p$, the characteristic of $K$. Then since $S$ is finite, there exists an appropriate $p$th power, say $p^s$, with $x^{p^s} = 1$ for all $x \in S$. Now let $n$ be any integer with $n \geqq s$ and we apply Lemma 3 to $e^{p^n} = e$. Thus there exists an element $\gamma$ in the commutator subspace of $K[G]$ with

$$e = e^{p^n} = \sum (a_x)^{p^n} x^{p^n} + \gamma,$$

and we proceed to compute the traces of both sides of this equation. Observe first that $\operatorname{tr} \gamma = 0$ since for any $\alpha, \beta \in K[G]$ we have $\operatorname{tr} \alpha\beta = \operatorname{tr} \beta\alpha$ and hence $\operatorname{tr} [\alpha, \beta] = 0$. Also since $n \geqq s$ it follows that for any $x \in \operatorname{Supp} e$ we have $x^{p^n} = 1$ if and only if $x \in S$. Thus clearly

$$\operatorname{tr} e = \sum_{x \in S} (a_x)^{p^n} = \left( \sum_{x \in S} a_x \right)^{p^n}.$$

Now this equation holds for all integers $n \geqq s$ and in particular if we take $n = s$ and $s + 1$ we obtain

$$(\operatorname{tr} e)^p = \left( \sum_{x \in S} a_x \right)^{p^s p} = \left( \sum_{x \in S} a_x \right)^{p^{s+1}} = \operatorname{tr} e.$$

Therefore $\operatorname{tr} e$ is an element $k \in K$ which satisfies $k^p = k$ and since all such elements $k$ are contained in $\mathrm{GF}(p)$, the theorem is proved in characteristic $p > 0$.

We now proceed to consider the characteristic 0 case and here we will need the following (see [**7**, Assertions IV, V, VI]).

LEMMA 4. *Let $A = Z[a_1, a_2, \cdots, a_r]$ be an integral domain in characteristic 0 which is finitely generated as a ring over the integers $Z$ and let $b \in A$ be an element not contained in the rational numbers $Q$. Then there exists a maximal ideal $M$ of $A$ such that $F = A/M$ is a field of characteristic $p > 0$ for some prime $p$ and such that the image of $b$ in $F$ is not contained in* $\mathrm{GF}(p)$.

We remark that this fact is an easy consequence of the Extension Theorem for Places if $b$ is transcendental over $Q$. But if $b$ is assumed algebraic, then its possible images in the fields $A/M$ are greatly restricted. Fortunately in this case we can apply the Frobenius Density Theorem, a result from algebraic number theory, to prove the lemma.

Now let $K$ have characteristic 0 and let $e = \Sigma a_x x$ be an idempotent in $K[G]$. If $A = Z[a_x \mid x \in \operatorname{Supp} e]$, then $A$ is clearly an integral domain in characteristic 0 and $A$ is finitely generated as a ring over the integers $Z$. Furthermore, $e$ is an idempotent in $A[G]$, where the latter is the subring of $K[G]$ consisting of all elements with coefficients in $A$. Now let $M$ be any maximal ideal of $A$ with $F = A/M$ a field of characteristic $p > 0$. Then under the natural homomorphism

$$A[G] \to (A/M)[G] = F[G]$$

the image of $e$ is an idempotent and hence by the first part of Theorem II the image of $\operatorname{tr} e$ is contained in $\mathrm{GF}(p)$. The second part of Theorem II now follows immediately from Lemma 4 with $b = \operatorname{tr} e$.

Thus we have proved the first conjectured property of $\operatorname{tr} e$, namely that it is always contained in the prime subfield of $K$ and it remains to consider the second property, namely that $0 \leqq \operatorname{tr} e \leqq 1$ if $K$ has

characteristic 0. Here the proof is much more technical in nature, it is analytic rather than algebraic, and we shall present some basic reductions and then indicate why the proof should work.

Let $K$ be a field of characteristic 0 and let $e = \Sigma a_x x$ be an idempotent in $K[G]$. If $F = Q(a_x \mid x \in \operatorname{Supp} e)$ then $F$ is a finitely generated field extension of $Q$ and $e$ is an idempotent in $F[G]$. Furthermore, $F$ can be embedded in the complex numbers $C$ and hence again $e$ is an idempotent in $C[G]$. Thus, as a first reduction, we may clearly assume that $K = C$ is the field of complex numbers. A second minor observation is that we need only show that $\operatorname{tr} e \geqq 0$. The reason for this is that if $e$ is an idempotent, then so is $1 - e$ and then $1 - \operatorname{tr} e = \operatorname{tr}(1 - e) \geqq 0$ yields $\operatorname{tr} e \leqq 1$.

Let us now consider the complex group ring $C[G]$. As is to be expected when one combines the structure of group rings with the richness of the complex numbers, many nice additional properties emerge. For example, if $\alpha = \Sigma a_x x$ and $\beta = \Sigma b_x x$ are elements of $C[G]$ we set

$$(\alpha, \beta) = \sum_x a_x \bar{b}_x$$

and

$$\| \alpha \| = (\alpha, \alpha)^{1/2} = \left( \sum_x | a_x |^2 \right)^{1/2}$$

where $\bar{a}$ is the complex conjugate of $a$ and $| a |$ is its absolute value. Then clearly $( \, , \, )$ defines a Hermitian inner product on $C[G]$ with the group elements as an orthonormal basis and with $\| \ \|$ the usual associated norm. Furthermore let $\alpha^*$ be given by

$$\alpha^* = \sum_x \bar{a}_x x^{-1}.$$

Then the map $^*$ is easily seen to satisfy

$$(\alpha + \beta)^* = \alpha^* + \beta^*, (\alpha\beta)^* = \beta^* \alpha^*, \ \alpha^{**} = \alpha,$$

so that $^*$ is a ring antiautomorphism of $C[G]$ of order 2, that is, an involution. Now observe that

$$(\alpha, \beta) = \operatorname{tr} \alpha\beta^* = \operatorname{tr} \beta^* \alpha$$

and hence we deduce easily that if $\gamma$ is a third element of $C[G]$, then

$$(\alpha, \beta\gamma) = (\alpha\gamma^*, \beta) = (\beta^* \alpha, \gamma).$$

In other words, $^*$ is the adjoint map with respect to this inner product for both right and left multiplication.

We can now use the above machinery to obtain an alternate proof of the assertion $\operatorname{tr} e \geqq 0$ at least when $G$ is finite. Let $I = eC[G]$ be the right ideal of $C[G]$ generated by the idempotent $e$ and let $I^\perp$ be its orthogonal complement. Then since $C[G]$ is a finite dimensional vector space, we know that $I + I^\perp = C[G]$ is a direct sum decomposition. But $I^\perp$ is not just a subspace of $C[G]$, it is in fact also a right ideal. To see this let $\alpha \in I$, $\beta \in I^\perp$ and $\gamma \in C[G]$. Then since $I$ is a right ideal $\alpha\gamma^* \in I$ and hence

$$(\alpha, \beta\gamma) = (\alpha\gamma^*, \beta) = 0.$$

Thus $\beta\gamma$ is orthogonal to all $\alpha \in I$ and we conclude that $\beta\gamma \in I^\perp$.

We have therefore shown that $I + I^\perp = C[G]$ is a decomposition of $C[G]$ as a direct sum of two right ideals and we let $f + f^\perp = 1$ be the corresponding decomposition of 1. As is well known, $f$ and $f^\perp$ are then both idempotents with $I = fC[G]$ and $I^\perp = f^\perp C[G]$. Now $f$ is orthogonal to $f^\perp C[G] = (1 - f)C[G]$ so for all $\alpha \in C[G]$ we have

$$0 = (f, (1 - f)\alpha) = ((1 - f)^* f, \alpha)$$

and hence $(1-f)^*f \in C[G]^\perp = 0$. Thus $f = f^*f$ so

$$f^* = (f^*f)^* = f^*f = f$$

and we see that $f$ is a self-adjoint idempotent, a projection. It is now an easy matter to complete the proof. Since $eC[G] = I = fC[G]$ both $e$ and $f$ are left identities for this ideal and hence we have

$$\operatorname{tr} e = \operatorname{tr} fe = \operatorname{tr} ef = \operatorname{tr} f.$$

Furthermore $f = f^*f$ so

$$\operatorname{tr} e = \operatorname{tr} f = \operatorname{tr} f^*f = \|f\|^2 \geqq 0$$

and the result follows.

It is amusing to observe that this proof yields only $\operatorname{tr} e \geqq 0$ and not $\operatorname{tr} e \in Q$, but it does have the redeeming virtue of being extendable to infinite groups. Of course the finiteness of $G$ was used crucially here. In fact it was used to deduce that $I + I^\perp = C[G]$ since such decompositions are not in general true for infinite dimensional inner product spaces. But the proof really rests upon the auxiliary element $f$ and there is an alternate characterization of this element which we can use. Let $\alpha \in I$ and consider the distance between $\alpha$ and $1 \in C[G]$. Then by definition

$$d(\alpha, 1)^2 = \|\alpha - 1\|^2 = (\alpha - 1, \alpha - 1)$$

and since $1 = f + f^\perp$ and $(\alpha - f, f^\perp) = 0$ we have

$$d(\alpha, 1)^2 = (\alpha - f - f^\perp, \alpha - f - f^\perp) = \|\alpha - f\|^2 + \|f^\perp\|^2.$$

Thus $d(\alpha, 1) \geqq \|f^\perp\|$ and equality occurs if and only if $\alpha = f$. In other words, $f$ is the unique element of $I$ which is closest to 1.

Now let $G$ be an arbitrary group, let $e$ be an idempotent in $C[G]$ and set $I = eC[G]$. Then we define the distance from $I$ to 1 to be

$$d = \inf_{\alpha \in I} d(\alpha, 1) = \inf_{\alpha \in I} \|\alpha - 1\|.$$

Since $C[G]$ is no longer complete if $G$ is infinite, there is no reason to believe that some element of $I$ exists which is closest to 1. But there certainly exists a sequence of elements of $I$ whose corresponding distances approach $d$. As it turns out, it is convenient to choose a sequence $f_1, f_2, \cdots, f_n, \cdots$ of elements of $I$ with

$$d^2 \leqq \|f_n - 1\|^2 < d^2 + 1/n^4.$$

Then this sequence plays the role of the auxiliary element $f$ of the finite case and indeed the final conclusion mirrors the original formula $\operatorname{tr} e = \|f\|^2$ obtained above. Namely after a certain amount of work with inequalities and approximations, which we will not give here (see [5, Section 22] for details), we deduce finally that

$$\operatorname{tr} e = \lim_{n \to \infty} \|f_n\|^2 \geqq 0.$$

This completes the proof.

**4. Semisimplicity.** We close with one final problem, the semisimplicity problem. The amusing thing about this one is that while it has been worked on for over twenty-five years, it is only recently that a viable conjecture has been formulated. In this section we shall discuss the conjecture along with some of the early work in characteristic 0.

We first offer some definitions. If $R$ is a ring, then an $R$-module $V$ is an additive abelian group which admits right multiplication by the elements of $R$. More precisely, we are given a ring

homomorphism $R \to \text{End } V$ from $R$ to the endomorphism ring of $V$ and by way of this map there is a natural action of $R$ on $V$ which we denote by right multiplication. Thus in essence, $V$ is an $R$-vector space. Furthermore we say that $V$ is an irreducible $R$-module if there are no $R$-submodules of $V$ other than 0 or $V$ itself. For example, if $R = F$ is a field then the irreducible $F$-modules are precisely the one-dimensional $F$-vector spaces.

Now what we would like to do is study $R$ in terms of its irreducible modules but there is a natural obstruction here. It is quite possible that we cannot even tell the elements of $R$ apart in this manner, that is, there could exist two distinct elements $r, s \in R$ such that for all irreducible $R$-modules $V$ and for all $v \in V$ we have $vr = vs$. Of course if the above occurs then $v(r-s) = 0$ and hence we cannot distinguish the element $r - s$ from 0. Thus what is of interest here is

$$JR = \{r \in R \mid Vr = 0 \text{ for all irreducible } R\text{-modules } V\},$$

the Jacobson radical of $R$. This is easily seen to be a two sided ideal of $R$ which can be alternately characterized as follows (see [3, Theorem 1. 2. 3]).

LEMMA 5. *Let $R$ be a ring with 1. Then*

$$JR = \{r \in R \mid 1 - rs \text{ is invertible for all } s \in R\}.$$

Now a ring is said to be **semisimple** if its Jacobson radical is zero and this then leads to two problems in group rings. The first is to characterize those fields $K$ and groups $G$ with $K[G]$ semisimple and then the second, more ambitious problem, is to characterize in general the Jacobson radical $JK[G]$. The initial work here concerned fields of characteristic 0 since for $G$ finite it was a classical fact that $K[G]$ is semisimple. Presumably $K[G]$ is always semisimple in characteristic 0 and the first result here on infinite groups concerned the field $C$ of complex numbers.

THEOREM III. *For all groups $G$, $C[G]$ is semisimple.*

For our proof, which closely mirrors the original, we revert to the notation of the previous section and we furthermore introduce an auxiliary norm on $C[G]$ by defining $|\alpha| = \Sigma |a_x|$ if $\alpha = \Sigma a_x x$. Clearly $|\alpha + \beta| \leqq |\alpha| + |\beta|$ and $|\alpha\beta| \leqq |\alpha| \cdot |\beta|$. Now let $\alpha$ be a fixed element of $JC[G]$. Then for all complex numbers $\zeta$, $1 - \zeta\alpha$ is invertible by Lemma 5 and we consider the map

$$f(\zeta) = \text{tr}\,(1 - \zeta\alpha)^{-1}.$$

This is a complex function of the complex variable $\zeta$ and we shall show that $f$ is in fact an entire function and we will find its Taylor series about the origin.

For convenience we set $g(\zeta) = (1 - \zeta\alpha)^{-1}$ so that $f(\zeta) = \text{tr}\, g(\zeta)$. Furthermore, it is clear that all $g(\zeta) \in C[G]$ commute and hence we have the basic identity

$$\begin{aligned} g(\zeta) - g(\eta) &= (1 - \zeta\alpha)^{-1} - (1 - \eta\alpha)^{-1} \\ &= [(1 - \eta\alpha) - (1 - \zeta\alpha)](1 - \zeta\alpha)^{-1}(1 - \eta\alpha)^{-1} \\ &= (\zeta - \eta)\alpha\, g(\zeta) g(\eta). \end{aligned}$$

We first show that $|g(\eta)|$ is bounded in a neighborhood of $\zeta$. Now by the above $g(\eta) = g(\zeta) - (\zeta - \eta)\alpha g(\zeta) g(\eta)$ so $|g(\eta)| \leqq |g(\zeta)| + |\zeta - \eta|\,|\alpha g(\zeta)|\,|g(\eta)|$ and hence

$$|g(\eta)| \cdot \{1 - |\zeta - \eta|\,|\alpha g(\zeta)|\} \leqq |g(\zeta)|.$$

In particular, if we choose $\eta$ sufficiently close to $\zeta$ then we can make the factor $\{\cdots\}$ larger than 1/2 and thus we deduce that $\eta \sim \zeta$ implies $|g(\eta)| \leqq 2|g(\zeta)|$.

Next we show that $f(\zeta)$ is an entire function. To do this we first plug the above formula for $g(\eta)$ into the right hand side of the basic identity to obtain

$$g(\zeta) - g(\eta) = (\zeta - \eta)\alpha g(\zeta)\{g(\zeta) - (\zeta - \eta)\alpha g(\zeta) g(\eta)\}.$$

Then we divide this equation by $\zeta - \eta$ and take traces to obtain

$$\frac{f(\zeta)-f(\eta)}{\zeta-\eta} - \operatorname{tr}\alpha g(\zeta)^2 = -(\zeta-\eta)\operatorname{tr}\alpha^2 g(\zeta)^2 g(\eta).$$

Finally since $|\operatorname{tr}\gamma| \leqq |\gamma|$ we conclude from the boundedness of $|g(\eta)|$ in a neighborhood of $\zeta$ that

$$\lim_{\eta\to\zeta} \frac{f(\zeta)-f(\eta)}{\zeta-\eta} = \operatorname{tr}\alpha g(\zeta)^2.$$

Hence $f(\zeta)$ is an entire function with $f'(\zeta)$ given by $f'(\zeta) = \operatorname{tr}\alpha g(\zeta)^2$.

Now we compute the Taylor series for $f$ about the origin. Since $f(\zeta) = \operatorname{tr}(1-\zeta\alpha)^{-1}$ what we clearly expect here is that for $\zeta$ small we can write $(1-\zeta\alpha)^{-1}$ as the sum of an appropriate geometric series and then obtain $f$ by taking traces. To be more precise and rigorous set

$$s_n(\zeta) = \sum_{i=0}^{n} \zeta^i \operatorname{tr}\alpha^i.$$

Then

$$\begin{aligned} f(\zeta) - s_n(\zeta) &= \operatorname{tr}\left\{ g(\zeta) - \sum_{i=0}^{n} \zeta^i\alpha^i \right\} \\ &= \operatorname{tr} g(\zeta)\left\{ 1-(1-\zeta\alpha)\sum_{i=1}^{n} \zeta^i\alpha^i \right\} \\ &= \operatorname{tr} g(\zeta)\zeta^{n+1}\alpha^{n+1} \end{aligned}$$

and thus

$$|f(\zeta) - s_n(\zeta)| \leqq |\zeta|^{n+1}|\alpha|^{n+1}|g(\zeta)|.$$

Now by our previous remarks $|g(\zeta)|$ is bounded in a neighborhood of zero and hence for $\zeta$ sufficiently small we deduce that

$$\lim_{n\to\infty} s_n(\zeta) = f(\zeta).$$

We have therefore shown that

$$f(\zeta) = \sum_{i=0}^{\infty} \zeta^i \operatorname{tr}\alpha^i$$

is the Taylor series expansion for $f(\zeta)$ in a neighborhood of the origin. Furthermore, $f$ is an entire function and hence we can invoke a well-known theorem from complex analysis ([**1**, Theorem 3, pg. 142]) to deduce that the above series describes $f(\zeta)$ and converges for all $\zeta$. In particular we have

$$\lim_{n\to\infty} \operatorname{tr}\alpha^n = 0$$

and this holds for all $\alpha \in JC[G]$.

We conclude the proof by showing that if $JC[G] \neq 0$ then there exists an element $\alpha \in JC[G]$ which does not satisfy the above. Indeed, suppose $\beta$ is a nonzero element of $JC[G]$ and set $\alpha = \beta\beta^*/\|\beta\|^2$. Then $\alpha \in JC[G]$ since the Jacobson radical is an ideal. Furthermore we have $\alpha = \alpha^*$ and

$$\operatorname{tr}\alpha = \|\beta\|^{-2}\operatorname{tr}\beta\beta^* = \|\beta\|^{-2}\|\beta\|^2 = 1.$$

Now the powers of $\alpha$ are also symmetric under $^*$ so for all $m \geqq 0$ we have

$$\operatorname{tr}\alpha^{2m+1} = \operatorname{tr}\alpha^{2m}(\alpha^{2m})^* = \|\alpha^{2m}\|^2 \geqq (\operatorname{tr}\alpha^{2m})^2.$$

Hence by induction $\operatorname{tr}\alpha^{2^m} \geqq 1$ for all $m \geqq 0$ and this contradicts the fact that $\operatorname{tr}\alpha^n \to 0$. Thus $JC[G] = 0$ and the result follows.

Needless-to-say this analytic proof both excited and annoyed the algebraists who rightly viewed group rings are an algebraic subject. Later algebraic proofs were indeed given and the best result to date in characteristic 0 is as follows. Let $K$ be a field of characteristic 0 which is not algebraic over the rationals $Q$. Then for all groups $G$, $K[G]$ is semisimple. Furthermore, for all the remaining fields the semisimplicity question is equivalent to that of the rational group ring (see[**5**], Theorem 18.3).

We should mention at this point that there is an amusing ploy to try to extend the above argument to $Q[G]$. Namely, we again consider $f(\zeta) = \operatorname{tr}(1-\zeta\alpha)^{-1}$, this time as a map from $Q$ to $Q$, and we observe as before that for $|\zeta|$ small

$$f(\zeta) = \sum_0^\infty \zeta^i \operatorname{tr}\alpha^i.$$

Now the right hand side here describes an analytic function in a neighborhood of the origin and this function has the rather strange property that it takes rationals to rationals. The question then is: must such a function necessarily be a polynomial? Unfortunately this is not the case and a simple counterexample is as follows. Let $r_1, r_2, \cdots, r_n, \cdots$ be an enumeration of the nonnegative rationals and define

$$h(\zeta) = \sum_{n=1}^\infty \frac{1}{n!} \frac{(r_1^2-\zeta^2)(r_2^2-\zeta^2)\cdots(r_n^2-\zeta^2)}{(r_1^2+1)(r_2^2+1)\cdots(r_n^2+1)}.$$

Then $h$ is easily seen to be an entire function which takes $Q$ to $Q$ and which is not a polynomial.

Actually, it now seems that the semisimplicity problem for $Q[G]$ will be settled by purely ring theoretic means since the result will follow from an appropriate noncommutative analog of the Hilbert *Nullstellensatz*. To be more precise, it has been conjectured that the Jacobson radical of a finitely generated algebra is always a nil ideal. This conjecture, along with the known fact ([**5**] Theorem 18.5) that $Q[G]$ has no nonzero nil ideals, would then easily yield $JQ[G] = 0$. Thus what is really of interest is the case of fields of characteristic $p > 0$ and here we are just beginning to guess at what the answer might be. Suppose first that $G$ is finite. Then $K[G]$ is semisimple here if and only if $p \nmid |G|$. Of course this latter condition does not make sense for infinite groups, but the equivalent condition, namely that $G$ has no elements of order $p$, does. None-the-less this is not the right answer. It is not true that $JK[G] \neq 0$ if and only if $G$ has an element of order $p$. What does seem to be true is that elements of order $p$ do play a role, provided that they are suitably well placed in $G$. To give this ambiguous idea some more meaning we start by quoting a simple fact ([**5**] Lemmas 16.9 and 17.6).

LEMMA 6. *Let $\alpha$ be an element of the group ring $K[G]$. Then $\alpha \in JK[G]$ if and only if $\alpha \in JK[H]$ for all finitely generated subgroups $H$ of $G$ with $H \supseteq \operatorname{Supp}\alpha$.*

This of course says that whatever "well placed" means exactly, it must surely depend upon how the particular element sits in each finitely generated subgroup of $G$. Now we get more specific. Let $H$ be a normal subgroup of $G$. Then we say that $H$ carries the radical of $G$ if

$$JK[G] = JK[H] \cdot K[G].$$

Our goal here is to find an appropriate carrier subgroup $H$ of $G$ such that the structure of $JK[H]$ is reasonably well understood. Admittedly this is a somewhat vague statement but we would certainly insist that $JK[H]$ be so simple in nature that we can at least decide easily whether or not it is zero.

The first candidate for $H$ is based upon the $\Delta^+$ subgroup and Lemma 6 above. We define

$$\Lambda^+(G) = \{x \in G \mid x \in \Delta^+(L) \text{ for all finitely generated subgroups } L \text{ of } G \text{ containing } x\}.$$

Then $\Lambda^+(G)$ is a characteristic subgroup of $G$ and in all of the examples computed so far $\Lambda^+(G)$ does

indeed carry the radical. Furthermore, the ideal

$$I = JK[\Lambda^+(G)] \cdot K[G]$$

has been studied in general and it has been found to possess many of the properties expected of the Jacobson radical. Thus it now appears that $\Lambda^+(G)$ carries the radical but we are unfortunately a long way from a proof of this fact. Finally we remark that this conjecture has a nice ring theoretic interpretation. Namely it is equivalent to the assertion that if $G$ is a finitely generated group, then $JK[G]$ is a union of nilpotent ideals.

Now given the above conjecture, the next natural step is to study $\Lambda^+(G)$ and $JK[\Lambda^+(G)]$ and it soon becomes apparent that these objects are not as nice as we had hoped for. Indeed $\Lambda^+(G)$ turns out to be just any locally finite group and so the problem of determining $JK[\Lambda^+(G)]$ is decidedly nontrivial. We are therefore faced with the problem of studying locally finite groups in general and here a new ingredient comes into play.

Let $G$ be a locally finite group so that by definition all finitely generated subgroups of $G$ are finite. Now if $A$ is such a finite subgroup of $G$ we say that $A$ is locally subnormal in $G$ if $A$ is subnormal in all finite groups $H$ with $A \subseteq H \subseteq G$. In other words, we demand that each such $H$ has a chain of subgroups.

$$A = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = H$$

for some $n$ with $H_i$ normal in $H_{i+1}$. Then using these locally subnormal subgroups as building blocks in a certain technical manner we can define a new and interesting characteristic subgroup of $G$ denoted by $\mathcal{S}(G)$. Again it turns out that in all computed examples $\mathcal{S}(G)$ carries the radical of $G$ and that furthermore the ideal

$$I = JK[\mathcal{S}(G)] \cdot K[G]$$

possesses in general many properties expected of the Jacobson radical. Thus it now appears that if $G$ is locally finite then $\mathcal{S}(G)$ carries the radical of $G$.

The semisimplicity problem for groups in general has therefore been split into two pieces, namely the cases of finitely generated groups and of locally finite groups. Moreover, if we combine the corresponding conjectures for each of these cases, then what we expect, or at least hope, is that the group $\mathcal{S}(\Lambda^+(G))$ carries the radical of $G$. While this group may appear to be somewhat complicated in nature, there are in fact some nice structure theorems for it. Furthermore, and most important, we have $JK[\mathcal{S}(\Lambda^+(G))] \neq 0$ if and only if $\mathcal{S}(\Lambda^+(G))$ contains an element of order $p$.

*Added in Proof.* Hopefully reference [4] will appear soon in translation in the Journal of Soviet Mathematics. There is a new monograph in Russian by A. A. Bovdi entitled *Group Rings*. A second volume is anticipated. This author is presently working on an expanded and updated version of [5] which will be entitled *The Algebraic Structure of Group Rings*. It will be published in two volumes by Marcel Dekker, Inc. There has been some exciting progress recently on the zero divisor problem, in particular in the case of polycyclic-by-finite groups. This can be found in the papers by K. A. Brown, *On zero divisors in group rings*, to appear in the Journal of the London Math Society, and by D. R. Farkas and R. L. Snider, $K_0$ *and Noetherian group rings*, to appear in the Journal of Algebra.

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# MATHEMATICAL NOTES

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## A CONNECTED, LOCALLY CONNECTED, COUNTABLE HAUSDORFF SPACE

GERHARD X. RITTER

In this MONTHLY, 76 (1969) 169–171, A. M. Kirch gave an example of a connected, locally connected countable Hausdorff space. We give another, geometrically much simpler, example. The example we provide will be a modified version of a connected, but not locally connected, countable Hausdorff space constructed by R. H. Bing in [1].

Let $X = \{(a, b) \in Q \times Q \mid b > 0\}$, where $Q$ denotes the set of rational numbers, and let $\theta$ be a fixed irrational number. The points of our space will be the elements of $X$. With each $(a, b) \in X$ and each $\varepsilon > 0$ we associate two sets $L_\varepsilon(a, b)$ and $R_\varepsilon(a, b)$ which consist of all points of $X$ in the triangles $ABC$ and $A'B'C'$, respectively, where $A = (a - b/\theta, 0)$, $B = (a - b/\theta + \varepsilon, 0)$, $C = (a - b/\theta + \varepsilon/2, \theta\varepsilon/2)$, $A' = (a + b/\theta - \varepsilon, 0)$, $B' = (a + b/\theta, 0)$, and $C' = (a + b/\theta - \varepsilon/2, \theta\varepsilon/2)$. We now define a basic $\varepsilon$-neighborhood of $(a, b) \in X$ by $N_\varepsilon(a, b) = \{(a, b)\} \cup L_\varepsilon(a, b) \cup R_\varepsilon(a, b)$.

Since $\theta$ is irrational, no two points of $X$ can lie on a line with slope $\theta$ or $-\theta$. Thus, for any two distinct points of $X$, we may choose a sufficiently small $\varepsilon > 0$ such that the points lie in disjoint $\varepsilon$-neighborhoods. Therefore, $X$ is Hausdorff and, obviously, a countable space.

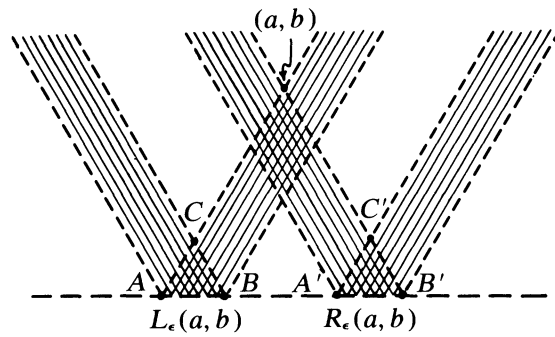



FIG. 1

Figure 1 shows that the closure of each basic $\varepsilon$-neighborhood $N_\varepsilon(a, b)$ consists of the union of four infinite strips with slope $\pm\theta$ emanating from the bases of the triangles $R_\varepsilon(a, b)$ and $L_\varepsilon(a, b)$. Hence the closures of each pair of basic neighborhoods intersect. Therefore, $X$ is connected but not regular and, hence, not metrizable.

For analogous reasons we have that for each $(a, b) \in X$ and $\varepsilon > 0$ the subspace $N_\varepsilon(a, b)$ is connected. For if $(a', b') \in N_\varepsilon(a, b)$, then the closure of any relatively open set containing $(a', b')$ must intersect the closure of any relatively open set containing $(a, b)$. Thus, $X$ is locally connected.

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## A REMARK ON ROUCHÉ'S THEOREM

I. GLICKSBERG

The purpose of this note is to point out a slightly strengthened and symmetric form of the classical Rouché result [3, p. 157] (as well as its "extension" to Banach algebras due to Holladay [2, p. 145]) which does not seem to have been noted, and has an almost equally simple proof. Assume $f$ and $g$ are analytic within and on a rectifiable simple closed curve $\partial$, and that

$$|f+g|<|f|+|g| \text{ on } \partial. \tag{1}$$

Then $f$ and $g$ have the same number of zeroes within $\partial$.

Trivially (1) is equivalent to the fact that, on $\partial$, $f/g$ is defined and never assumes a value in $\mathbf{R}_+$, the nonnegative reals. Thus we can form the integral

$$I_t = \frac{1}{2\pi i}\int_\partial \left[\left(\frac{f}{g}(z)-t\right)'\Big/\left(\frac{f}{g}(z)-t\right)\right]dz \qquad (t \geqq 0)$$

yielding a continuous integer-valued function of $t$ on $\mathbf{R}_+$[3, p. 191]. But $I_0$ is the number of zeroes of $f$ within $\partial$ less the corresponding number for $g$, while clearly $I_t \to 0$ as $t \to \infty$, which proves the result. (Note that no extension involving three functions is possible: as $|z^{2n}+z-1|<3$ for $|z|=1$ shows, the triangle inequality on $\partial$ for three functions can be strict without any consequent relation between the number of zeroes. Also note that (1) is not the most general hypothesis for our argument to work; it is simply a convenient expression implying that $f/g(\partial)$ does not separate 0 and $\infty$ in $\mathbf{C}$, which is what is really needed.)

Now let $A$ be a commutative semisimple Banach algebra with identity, having spectrum $M$ and Šilov boundary $\partial$, considered as an algebra of continuous functions on $M$ (via the Gelfand representation of course). Our direct extension of Holladay's result then asserts that if $f, g \in A$ satisfy (1) then both are invertible, or both are non-invertible. In fact if $f$ is invertible (1) implies

$$|1+f^{-1}g|<1+|f^{-1}g| \text{ on } \partial,$$

hence that $f^{-1}g(\partial)$, the range of $f^{-1}g$ on $\partial$, misses $\mathbf{R}_+$. But [2, p. 143], $f^{-1}g(\partial)$ contains the boundary of $\mathrm{sp}_A(f^{-1}g)$, the spectrum of $f^{-1}g \in A$, so $\mathbf{R}_+$ misses $\mathrm{sp}_A(f^{-1}g)$ as well, which implies $f^{-1}g$ is invertible, so $g$ is invertible.

(The same sort of argument, using an abstract Schwarz lemma [1, 4.8], provides more information when $f$ and $g$ are non-invertible and (1) holds: then unless $f/g$ and $g/f$ are both unbounded we have $f^{-1}(0) = g^{-1}(0)$ and (1) holds on all of $M\backslash g^{-1}(0) = M\backslash f^{-1}(0)$; in fact $f = ge^h$ there, where $h$ lies in the closed subalgebra of $C(M\backslash g^{-1}(0))$ generated by $A$ and $f/g$.)

Work supported in part by the National Science Foundation.

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## A NOTE ON GROUPS WITH UNIQUE EXTRACTION OF ROOTS

COLIN D. FOX

In a recent article, Hirshon [2] proved a theorem for the class of finitely generated groups in which extraction of roots is unique and whose automorphism group is periodic. The purpose of this note is to show that this class is precisely the class of infinite cyclic groups.

A group, $G$, in which extraction of roots is unique (that is $x^n = y^n$ implies $x = y$ for all $x$ and $y$ in $G$ and non-zero integers, $n$) is called an *$R$-group*. Kurosh ([3], volume 2, p. 244) gives the following basic property of $R$-groups

$$(*) \qquad x^{-m}yx^m = y \text{ implies } x^{-1}yx = y.$$

It follows easily from this that any $R$-group with periodic automorphism group is abelian. For suppose $x$ and $y$ are any elements of such a group, $G$. Then, in particular, the inner automorphism $g \to x^{-1}gx$ of $G$ is periodic. That is, there is a positive integer, $n$, such that $g = x^{-n}gx^n$ for all $g$ in $G$. So $x^{-n}yx^n = y$ and, by (*), $x^{-1}yx = y$. Hence $yx = xy$ for all $x$ and $y$ in $G$.

Now suppose, in addition, that $G$ is finitely generated. Then $G$ is the direct product of $k$ infinite cyclic groups, $A_i = \langle a_i \rangle$, $i = 1, 2, \cdots, k$ (see [3], volume 1, p. 147 and volume 2, p. 242). If $k \geqq 2$, a non-periodic automorphism of $G$ is given by

$$(a_1^{m_1}, a_2^{m_2}, \cdots, a_k^{m_k}) \to (a_1^{m_1+m_2}, a_2^{m_2}, \cdots, a_k^{m_k}).$$

If $k = 1$, that is $G = \langle a_1 \rangle$, then the only automorphisms of $G$ are $a_1^m \to a_1^m$ and $a_1^m \to a_1^{-m}$, both of which are periodic. Since an infinite cyclic group is an $R$-group, we have proven the following theorem.

THEOREM. *Let $G$ be a finitely generated $R$-group. Then the automorphism group of $G$ is periodic if, and only if, $G$ is an infinite cycle.*

The existence of non-finitely generated, torsion-free abelian groups whose automorphism groups are periodic (Fuchs [1], p. 271), and of finitely generated groups which are not $R$-groups and whose automorphism groups are periodic (a 2-cycle, for example), shows that the conditions of the theorem cannot be relaxed.

As a corollary to this theorem and Theorem 3 of [2], we have

COROLLARY. *Let $G$ be a finitely generated $R$-group. If $G$ has a subgroup of finite index whose automorphism group is periodic, then $G$ is an infinite cyclic group.*

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## RESEARCH PROBLEMS

EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, 16 Mill Lane, Cambridge, England CB2 1SB.)*

### THE SOFA PROBLEM

NEAL R. WAGNER

There is an old problem in combinatorial geometry which has to do with moving furniture around corners.

PROBLEM. *What is the region of largest area which can be moved around a right-angled corner in a corridor of width one?*

Note that the problem is a two-dimensional one. The region is assumed to be rigid, and only combinations of rotations and translations are allowed.

The problem has an interesting history. It was first formally stated in print by L. Moser [**6**] in 1966, although it had been discussed informally by many people prior to 1966. The problem was also mentioned by H. T. Croft [**1**] in 1967. Croft included a rough sketch called the "Shephard piano," but the word "sofa" has now become firmly linked to the problem. In 1968, Hammersley [**3,** p. 84] gave what is currently the best-known lower bound for the maximum area: $(\pi/2)+(2/\pi)$, or approximately 2.2074. In a private communication to the author, Hammersley stated: "I had always guessed (but never been able to prove) that the lower bound above was also the exact value of [the maximum area]." Hammersley's solution (see Figure 1) can be obtained using simple calculus by considering only outer boundaries made up of two 90° arcs from a unit circle plus a variable length line segment. (The inner boundary must then be two unit segments plus a half circle.) Another lower bound was published in [**2**].

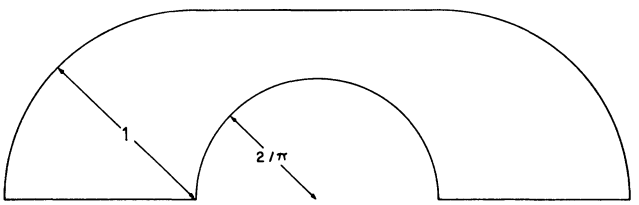



FIGURE 1.

The best known upper bound for the maximum area, $2\sqrt{2}$, was also given by Hammersley in 1968, with a proof appearing as "Editorial commentary" in 1969. (See [**3**].) Westwell [**3**] and Sebastian [**7**] claimed to have established the above upper bound, but both erroneously assumed that a bounding rectangle for the region must be rotated through a 90° angle as the corner is negotiated.

There have been several attempts to find approximate solutions to the problem. Howden [**4**] considers a general problem-solving algorithm which, to quote the referee, "comes to grief when confronted by an actual problem." Maruyama [**5**] has a much more interesting algorithm which, for a right-angled corner, produced a region of area 1.98 fairly closely resembling the solution by Hammersley. However, Maruyama does not give an error estimate or any other proof that his area is close to the maximum area.

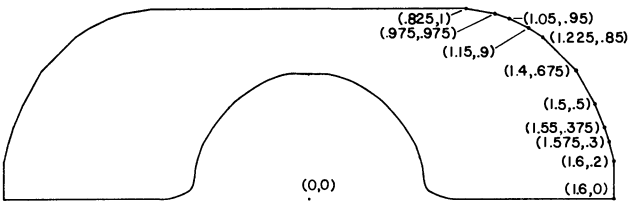



FIGURE 2.

The author independently constructed a type of Monte Carlo simulation which produced the region shown in Figure 2. This simulation might be compared to the evolution of a rigid organism which must traverse right-angled corners. All variation occurs in the outer boundary (the inner boundary is determined by turning the corner), while selection is for larger area.

Starting with widely different initial regions, the author's simulation evolved in each case to a region in close agreement with Hammersley's. The largest such region is a union of 3450 squares 1/40 units on a side, and thus with a total area of about 2.15. Figure 2 uses rectangular coordinates to specify the convex hull of the outer boundary of this region, while the inner boundary as shown is simply what would be cut out while going around the corner. The area of this figure is certainly very close to the area obtained by Hammersley. (The author has not been able to obtain an exact value for the area of Figure 2.)

It is clear that small adjustments to the outer boundary in Figure 2 will produce small improvements in the area, but the close agreement of regions obtained by entirely different methods leads one to conjecture that the areas in Figures 1 and 2 are very near to the actual maximum.

For generalizations of the sofa problem, see Maruyama [5], who considered general kinds of two-dimensional corridors and included shapes of various regions produced by his algorithm. Croft [1] included a rough sketch of a region called the "Conway Car" for reversing in a $T$-junction. (See also [5, Figure 13].) In the original problem statement [6], the editors noted: "For the three-dimensional version, in which the hallway has a fixed height $h$, it seems reasonable to suppose the answer would be a cylinder of height $h$ whose cross-section is given by the solution of the [two-dimensional sofa problem]." Hammersley [3, p. 75] suggests what is perhaps a more legitimate three-dimensional version: "$\cdots$, where the passage has three straight sections each at right-angles to the other two," as in a corner in a corridor plus an elevator shaft. Concerning convex solutions to the sofa problem, the author can only remark that the maximum area is surely larger than the value of $\pi/2$ given by a semicircle.

**Acknowledgements.** The author would like to thank the University of Texas at El Paso for a research grant and the use of its computer. For a great deal of help with references, the author is also indebted to R. K. Guy, J. Wetzel, S. Greibach, J. M. Hammersley, and especially the two (not necessarily distinct) referees.

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## CLASSROOM NOTES

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### PAPPUS IMPLIES DESARGUES

A. SEIDENBERG

In a projective plane, an easy consequence of the Fundamental Theorem is that a projectivity $\sigma$ from a line $m$ to a line $m' \neq m$ that leaves the intersection $m \cdot m'$ fixed is a perspectivity.[1] The special case of this in which $\sigma$ is a product of two perspectivities we will call Theorem B. Well-known proofs that the Fundamental Theorem implies Pappus and Desargues really only use Theorem B. Thus in Pappus there are three distinct points $A, B, C$ on a line $n$, three distinct points $A', B', C'$ on a line $n' \neq n$, all six points distinct from $n \cdot n'$, and it is asserted that $F = AB' \cdot A'B$, $E = AC' \cdot A'C$, $D = BC' \cdot B'C$ are collinear. For the proof, consider the perspectivity from $BA'(= m)$ to $n'$ with center $A$ (notation: $(BA')(A/\Lambda)n'$), the perspectivity from $n'$ to $BC'(= m')$ with center $C$, and their product

$$\sigma\colon (BA')\frac{A}{\Lambda}n'\frac{C}{\Lambda}(BC').$$

Then $BA' \cdot BC' = B$ is invariant under $\sigma$ (since $A, B, C$ are collinear); and $AC'$, $A'C$, and $FD$ join corresponding points of $\sigma$ (none $= B$) so $F, D$, and $E = AC' \cdot A'C$ are collinear. Desargues' Theorem follows by a similar proof. Conversely, Pappus and Desargues, with an equally simple proof, imply Theorem B.[2] In Theorem B if we write $\sigma\colon m \,\overline{\wedge}\, m^* \,\overline{\wedge}\, m'$, there are two cases: (a) $m$, $m^*$, $m'$ are not concurrent, and (b) $m$, $m^*$, $m'$ are concurrent. Then Pappus is equivalent to (a) and Desargues to (b). Thus Pappus and Desargues appear on the same level, as special cases of Theorem B.[3]

Now to prove Desargues from Pappus. Let $m_1, m_2, m_3$ be three distinct lines meeting at $E$; let $E, A_i, B_i$ be distinct points on $m_i$, $i = 1, 2, 3$; let $A_1A_2A_3$, $B_1B_2B_3$ be triangles; and let $A_1A_2 \cdot B_1B_2 = R_3$, $A_2A_3 \cdot B_2B_3 = R_1$, $A_3A_1 \cdot B_3B_1 = R_2$. Desargues says that $R_1, R_2, R_3$ are collinear. Consider the projectivity

$$\sigma\colon m_1\frac{R_3}{\Lambda}m_2\frac{R_1}{\Lambda}m_3.$$

We wish to prove that $\sigma$ is a perspectivity. Let $R_1R_3$ meeť $m_1$ in $G$; note that $R_1 \neq R_3$. Assume for a moment that $R_1R_3$ does not pass through $E(G \neq E)$ and that $R_1$ is not on $m_1(R_1 \neq G)$. Take a line $m^*$ through $G$, $m^* \neq m_1$, and $m^*$ not on $R_1$ (which we could not do if $R_1$ were on $m_1$). Then write $\sigma$ in the form

$$\sigma\colon m_1\frac{R_3}{\Lambda}m_2\frac{R_1}{\Lambda}m^*\frac{R_1}{\Lambda}m_3.$$

Now apply Pappus to remove $m_2$:

$$\sigma\colon m_1 \,\overline{\wedge}\, m^*\frac{R_1}{\Lambda}m_3,$$

and apply it once more to remove $m^*$ and get that $\sigma$ is a perspectivity. $R_2$ is its center and $R_1R_3$ lies on corresponding points (neither $= E$), so $R_1, R_2, R_3$ are collinear.

There are many ways to complete the proof, but we shall use a general technique, and, moreover, one corresponding to the intuitive notion that the special cases are "limits" of non-special ones.

Let the notation be as before and let $R_1R_3 = m'$ with $m'$ not on $E$. Think of $A_1, A_2, B_1, B_2, m_3$, and $m'$ as fixed and $R_1$ as variable on $m'$ (so $E, m_1, m_2, m_3$ are fixed, $R_1, A_3, B_3, R_2$ variable). Since $R_1$ is not on $A_1A_2$ or $B_1B_2$, $m'$ is not on $A_1, A_2, B_1$, or $B_2$. Consider the projectivities

$$\lambda\colon m' \frac{A_2}{\Lambda} m_3 \frac{A_1}{\Lambda} m' \quad \text{and} \quad \mu\colon m' \frac{B_2}{\Lambda} m_3 \frac{B_1}{\Lambda} m',$$

so our assertion takes the form: $\lambda(R_1) = \mu(R_1)(= R_2)$. Note that every point $R_1$ of $m'$, except for $R_3$ and $m' \cdot m_2$ and $m' \cdot m_3$, arises as a value of $R_1$ from a Desargues configuration (namely for $A_3 = R_1A_2 \cdot m_3$ and $B_3 = R_1B_2 \cdot m_3$). One may check separately (and trivially) that $\lambda$ and $\mu$ agree at these exceptional points. Hence $\lambda, \mu$ agree, except possibly at one point, namely, $R_1 = m' \cdot m_1$. Now two one-to-one correspondences (i.e., bijections) can obviously not differ at exactly one point. Hence $\lambda = \mu$.[4]

The remaining case, that some two of $R_1, R_2, R_3$, e.g., $R_1, R_3$, are collinear with $E$, the so-called Little Theorem of Desargues, can also be looked at as a "limit," but may now be taken care of more quickly as follows. If some two of $R_1, R_2, R_3$ are not collinear with $E$, then we know already that $R_1, R_2, R_3$ are collinear; and if each two of $R_1, R_2, R_3$ are collinear with $E$, then obviously $R_1, R_2, R_3$ are collinear. Thus in every case $R_1, R_2, R_3$ are collinear.

**Postnote.** Hessenberg [**4**] originally gave two proofs of his theorem: the first (§4) is wrong; and the second, he says, is essentially the same as the first. Desargues' Theorem as usually stated has a symmetry in it. Hessenberg places $R_1$ and $R_3$ at infinity, whereupon the symmetry is lost, but then falsely uses the symmetry. Hilbert [**5**; p. 111] essentially gives Hessenberg's first proof. Though he is aware that the proof is incomplete, he does not explicitly explain how to meet the issue. In Hartshorne's proof [**3**; p. 84], if $C'$ lies on $AB$, then "Step 1" is, as Hartshorne says, trivial, as is "Step 2", but "Step 3" does not apply at all. Cronheim [**2**], citing Hessenberg (1905 and 1930), Reidemeister, de B. Robinson, and Coxeter [**1**] (who in turn cites Pasch and Dehn, Baker, and Hodge and Pedoe), says that "all" proofs of Hessenberg's Theorem merely consider the so-called general case; and wishes to complete the proof by an orderly, though minute, examination of the special configurations. Pedoe [**6**] criticizes the proof (unjustly) and proposes to fix *it*. (Pedoe's second proof (p. 70) merely begs the question by using Desargues' Theorem in the course of proving it.) Hessenberg expressed dissatisfaction with his first proof, as disclosing no leading idea; and the same holds for all the proofs that follow up this part of Hessenberg's paper.

The idea of the present note was given in [**9**; pp. 93–94], but the possibility that $R_1$ is on $m_1$ was overlooked. Pickert [**7**; p. 145], after reproducing Hessenberg's second proof (with an unnecessary emendation), suggests a simplification in the case $R_1R_3$ is not on $E$ ($C \notin \gamma$, in his notation), but here makes the same slip.

G. de B. Robinson's proof [**8**; p. 125] fails when he says (in our notation) that $A_1A_2 \cdot B_2B_3$ cannot lie on $A_1B_3$. However, the proof can be fixed as follows. Let

$$\sigma_{12}\colon m_1 \frac{R_3}{\Lambda} m_2, \quad \sigma_{23}\colon m_2 \frac{R_1}{\Lambda} m_3.$$

To prove that $\sigma_{12}\sigma_{23} = \sigma_{13}$ is a perspectivity. By [**4**; §6], except in the trivial case of three points per line or that $\sigma_{13}$ is already a perspectivity, one can find a line $m_0$ cutting $m_1, m_2, m_3$ in $S_1, T_2, U_3$ such that

$$\text{(a)}: \ (S_1)\sigma_{12} \neq T_2, \qquad \text{(b)}: \ (T_2)\sigma_{23} \neq U_3; \quad \text{and} \quad \text{(c)}: \ (S_1)\sigma_{13} \neq U_3.[5]$$

By (a) $m_0 \neq R_3S_1$; by (b) $m_0 \neq R_1U_3$ (so $R_3S_1 \neq R_1U_3$ and $R_3S_1 \cdot R_1U_3 = R_0$ is not on $m_0$); and by (c) $R_0$ is not on $m_2$. Let $\sigma_{20}\colon m_2(R_0/\Lambda)m_0$, $\sigma_{02} = \sigma_{20}^{-1}$. Then $\sigma_{12}\sigma_{23} = \sigma_{12}\sigma_{20}\sigma_{02}\sigma_{23}$. By Pappus, $\sigma_{12}\sigma_{20}$ is a perspectivity $\sigma_{10}$; $\sigma_{02}\sigma_{23}$ is a perspectivity $\sigma_{03}$; and $\sigma_{10}\sigma_{03} = \sigma_{12}\sigma_{23}$ is a perspectivity.

**Notes:**

1. The statement of the Fundamental Theorem and the implication indicated are of no logical consequence for the following, but are merely introductory to the formulation of Theorem B (below). The Fundamental Theorem says that any three distinct collinear points can be sent into any other such three, respectively, by one and only one projectivity.

2. The details are left to the reader.

3. Of the various implications so far mentioned, the only one we use is that Pappus implies case (a) of Theorem B; for a proof of this see [9; p. 92f]. The others are mentioned for expository purposes.

4. *Proof.* If $\lambda$ and $\mu$ agree except at $P$, then $\{\lambda(P)\} = \operatorname{comp}\{\lambda(Q) \mid Q \neq P\} = \operatorname{comp}\{\mu(Q) \mid Q \neq P\} = \{\mu(P)\}$, so $\lambda(P) = \mu(P)$, q.e.d. The analogy with "limits" shows itself here: if $\lambda, \mu$ are "continuous" and agree, except at $P$, then $\lambda(P) = \mu(P)$ by continuity; if $\lambda, \mu$ are bijections (into the same set) and agree except at $P$, then again $\lambda(P) = \mu(P)$ because $\lambda(P)$, $\mu(P)$ are determined (as in the case of continuity) by the values of $\lambda, \mu$ at the points $Q \neq P$.

5. On $R_1R_3$ take a point $H, H \neq R_1, H \neq R_3$, and $H \neq E$ (if $R_1R_3$ is on $E$). Let $m_0$ be a line on $H$ cutting $m_1, m_2, m_3$ in $S_1, T_2, U_3$. Unless $\sigma_{13}$ is a perspectivity with center $H$, $m_0$ can be so taken that $(S_1)\sigma_{13} \neq U_3$. It will then also be the case that $(S_1)\sigma_{12} \neq T_2$ and $(T_2)\sigma_{23} \neq U_3$.

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## AN ENTIRE FUNCTION BOUNDED IN EVERY DIRECTION

D. J. NEWMAN

We produce a non-constant entire function, $f(z)$, such that, for every fixed (real) $\theta$, $f(re^{i\theta})$ remains bounded as $r \to \infty$. This is exactly what is needed to confuse students who have just struggled to comprehend the meaning of Liouville's theorem.

We achieve our aim by showing that the entire function, $g(z)$, given by $\int_0^\infty (e^{zt}/t')dt$ is uniformly bounded outside some strip around the real axis. This done we may simply shift the origin, i.e., choose $f(z) = g(z + iA)$ for a sufficiently large $A$. For then every ray from the origin will spend only finitely much length in the strip of unboundedeness.

What we show precisely is

$$\text{For Im } z = \frac{\pi}{2} + c, \qquad c > 0, \qquad g(z) \ll \frac{1}{c}, \tag{1}$$

(where here as later we employ the notation $\alpha \ll \beta$ to mean $|\alpha| \leqq |\beta|$). By symmetry it will then follow that the same bound holds for Im $z = -\frac{1}{2}\pi - c$ and so it will suffice to pick any $A$ above $\pi/2$.

The strategy is to use Cauchy's theorem to replace the integral in question by the integral along the imaginary $t$-axis. Justifying this shift and estimating the new integral becomes the "job of work" of this note.

To begin with we introduce polar coordinates $t = re^{i\theta}$, $-\pi < \theta \leqq \pi$, $0 \leqq r$, and then observe that the function

$$t^t = e^{t \log t} = \exp[r(\log r \cos\theta - \theta \sin\theta) + ir(\log r \sin\theta + \theta \cos\theta)]$$

is analytic on the whole "slit plane," $r \neq 0$ and $\theta \neq \pi$. Thus Cauchy's theorem applies with the integrand $e^{zt}/t^t$ and the path made up of

A: the positive axis from $\varepsilon$ to $R$

B: the (large) quarter circle from $R$ to $iR$

C: the positive imaginary axis from $iR$ to $i\varepsilon$

D: the (small) quarter circle from $i\varepsilon$ to $\varepsilon$.

We have, in short,

$$\int_A + \int_B + \int_C + \int_D = 0, \tag{2}$$

and we now estimate the latter three integrals. So if we write $z = x + i(\frac{1}{2}\pi + c)$ we shall have

$$\left|\frac{e^{zt}}{t^t}\right| = \exp[r\cos\theta(x - \log r) + r\sin\theta(\theta - \tfrac{1}{2}\pi - c)]. \tag{3}$$

$\int_B$: Here $r = R$, $0 \leqq \theta \leqq \pi/2$. Thus if we insist that $\log R > x + c$ and use the fact that $\theta \leqq \pi/2$, then by (3) the integrand is bounded by $e^{-cR(\cos\theta + \sin\theta)} \leqq e^{-cR}$, since $\cos\theta + \sin\theta \geqq 1$. But $dz$ is bounded by $R\,d\theta$ and so all in all we obtain

$$\int_B \ll \tfrac{1}{2}\pi R e^{-cR} \quad \text{for} \quad R > e^{x+c}. \tag{4}$$

$\int_C$: Here $\theta = \frac{1}{2}\pi$, $\varepsilon \leqq r \leqq R$. The integrand, by (3), is $e^{-cr}$ while $dz = -idr$. We therefore obtain the bound $\int_C \ll \int_\varepsilon^R e^{-cr}\,dr < \int_0^\infty e^{-cr}\,dr = 1/c$, i.e.,

$$\int_C \ll 1/c. \tag{5}$$

$\int_D$: Here $r = \varepsilon$, $0 \leqq \theta \leqq \pi/2$. Thus the integrand, by 3, is bounded by $\exp[\varepsilon \cdot 1(x - \log\varepsilon) + \varepsilon \cdot 0]$. Since $-\varepsilon \log\varepsilon$ takes its maximum at $\varepsilon = 1/e$ this is, furthermore, bounded by $e^{ex+1/e}$. Since $dz$ is bounded by $\varepsilon d\theta$, finally, we obtain

$$\int_D \ll \frac{\pi}{2}\varepsilon e^{ex+1/e}. \tag{6}$$

By (2) through (6) and letting $\varepsilon \to 0$, $R \to \infty$, we obtain (1) and the construction is complete.

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## MINIMAL ELEMENTS IN AN INTEGER REPRESENTING ALGORITHM

GRAHAM LORD

Let $A$ be an infinite sequence, $\{a(i)\}_{i=1,2,\cdots}$, of strictly increasing natural numbers with $a(1)$ equal to 1. Every natural number $n$ can be represented as the sum of terms from this sequence:

$$n = a(i_1) + a(i_2) + \cdots + a(i_j),$$

where $a(i_1)$ is the largest term of $A$ not exceeding $n$ and $a(i_k)$ for $2 \leqq k \leqq j$ is the largest term of $A$

which does not exceed

$$n - [a(i_1) + a(i_2) + \cdots + a(i_{k-1})].$$

Such a representation of $n$ is unique. Denote its length $j$ by $\alpha(n)$. Then $\alpha(n)$ equals 1 if and only if $n$ is an element of $A$.

I. Kátai, **[2], [3]** and **[4],** has considered the distribution of the values $\alpha(n)$ and the asymptotic behaviour of the sums $\Sigma_{n \leq x} \alpha^k(n)$, especially for sequences $A$ in which the differences of consecutive terms have a limiting distribution. This note will consider the construction of the minimal element of a fixed length.

For any natural number $m$, consider the set of positive integers $n$ whose representation is of length $m$ and define, if this set is not empty, $Y(m)$ to be the minimum such $n$, i.e., $Y(m) = \min_n \{n \mid \alpha(n) = m\}$. Say that $Y(m)$ does not exist if the preceding set is empty. Hence $Y(1) = 1$ and if $Y(m)$ exists, so does $Y(j)$ for $j < m$. Also, $Y(1) < Y(2) < Y(3) < \cdots < Y(m)$.

THEOREM. *Let $Y(m-1)$ be given, $m > 1$. $Y(m)$ exists if and only if there exists an $i$ such that $a(i) - a(i-1) > Y(m-1)$. Furthermore, if $Y(m)$ does exist, then it is equal to $Y(m-1) + a(j-1)$ where $j$ is $\min_i \{i \mid a(i) - a(i-1) > Y(m-1)\}$.*

*Proof.* If there is an $i$ satisfying $a(i) - a(i-1) > Y(m-1)$, then there is a smallest such $i$, say $j$. Let $z = Y(m-1) + a(j-1)$. Thus $a(j) > z > a(j-1)$, that is, $a(j-1)$ is the largest term in $A$ less than or equal to $z$. Consequently, as $\alpha(z - a(j-1)) = m - 1$, then $\alpha(z) = m$: $Y(m)$ exists and is less than or equal to $z$.

Now consider any positive integer $n$ of length $m$. Let its representation be $a(i_1) + a(i_2) + \cdots + a(i_m)$. Then $\alpha(n - a(i_1))$ equalling $m - 1$ implies

$$Y(m-1) \leqq n - a(i_1) < a(i_1 + 1) - a(i_1).$$

So $i_1 + 1 \geqq j$ and $n \geqq Y(m-1) + a(i_1) \geqq Y(m-1) + a(j-1) = z$. In all $Y(m) = z$.

Conversely, if $Y(m)$ exists and $a(i-1)$ is the largest term in $A$ less than or equal to $Y(m)$, then $Y(m) - a(i-1) \geqq Y(m-1)$. The required inequality follows from this and $a(i) > Y(m)$.

For some sequences the minimal elements $Y(m)$ can be determined explicitly. With $a(1) = 1$, $a(2) = 2$, and $a(n+1) = F_{n+2} = F_n + F_{n+1}$, the Fibonacci sequence, $Y(m)$ equals $F_{2m+1} - 1$. Or if $a(1) = 1$, $a(2) = 3$, $a(n+2) = L_{n+2} = L_{n+1} + L_n$, the Lucas sequence, $Y(m)$ is equal to $L_{2(m-1)} - 1$. For the sequence of triangular numbers, $a(n) = n(n+1)/2$, the minimal elements satisfy the following equation:

$$2^{m-1} Y(m) = [Y(1) + 3][Y(2) + 3] \cdots [Y(m-1) + 3].$$

This latter result was established by E. Lemoine at the turn of the century. (See Dickson **[1]**, vol. 2, p. 32.)

A consequence of the theorem is that $Y(m)$ exists for all natural numbers $m$ if and only if $\limsup\{a(n) - a(n-1)\} = +\infty$. In particular, such minimal elements of all lengths exist if $A$ is taken to be the primes (including 1), the initial values being $Y(1) = 1$, $Y(2) = 4$, $Y(3) = 27$, $Y(4) = 1354$. Note that this is not in contradiction to the result of I. M. Vinogradov **[5],** that every sufficiently large odd positive integer can be represented as a sum of three primes, in this case there being no restriction as to the form of the representation.

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## MATHEMATICAL EDUCATION

EDITED BY PAUL T. MIELKE AND SHIRLEY HILL

*Material for this Department should be sent to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933, or to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110.*

We have recently rejected several articles that provide personal advice on how to improve one's teaching. Given our current backlog, we will continue to reject such articles if they do not add substantially to the spirit and content of the CUPM pamphlet "Suggestions on the Teaching of College Mathematics," which is still available free of charge from MAA Special Projects Office, Department of Mathematics, California State University, Hayward, California 94542. — *The Editors.*

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### COLLEGE CREDIT FOR ALGEBRA AND GEOMETRY: A VIEWPOINT

ROBERT W. HUNT

(The Transfer Credit Review Board of the California Community Colleges and the California State University and Colleges, at a meeting on June 3, 1974, considered a challenge regarding transferability of two mathematics courses. This commentary is written in partial response to that issue.)

The basic question at issue is whether or not courses in elementary algebra (introductory algebra, beginning algebra) and plane geometry should carry elective college credit towards the baccalaureate degree when transferred from a California Community College to a California State University or College. A related question is whether or not the latter institutions should offer such courses for credit.

First, it should be noted that there is a significant demand for a basic algebra course at most of the two and four year campuses in the CCC and CSUC systems. The need for plane geometry is not nearly as great, tending to be either for cultural reasons or as preparation for teaching, engineering, or additional mathematical course work.

The population for the algebra course consists of students who need both the accompanying levels of understanding and the skills in order to pursue other mathematical studies, including statistics and computer programming, a variety of courses in the physical and behavioral sciences and business, and for a miscellany of other motivations. These students either failed to obtain this background in high school (whether or not they actually took an algebra course) or they have retained virtually none of what was once encountered (whether through lack of understanding or the erosion of time). If these students are to be served, then, is it reasonable to grant elective baccalaureate credit for their time and effort?

Careful consideration of this matter tends to make it more elusive than it might first seem to be. Concerns with reducing the stature of the degree or diluting academic standards are difficult to solidify

in view of the fact that the credit is elective (not aligned with any major or minor programs nor with upper division course requirements) and since such credit is often in excess of the minimum number of required units at the time of graduation. Furthermore, the assignment of course credit is useful for student motivation, for calculation of veterans' benefits, and for granting of teaching credit. Any arguments against credit merit that are based upon duplication of course content with traditional high school offerings must weigh carefully the results of similar comparisons for literally dozens of other baccalaureate level courses to which credit is routinely assigned (including foreign language courses, physical science courses, other "precalculus" mathematics courses, and many areas of the humanities). While it is true that these courses, when translated to the college level, are often expanded and accelerated, the same extensions can certainly be incorporated in introductory algebra and plane geometry. Furthermore, a variety of current and relevant applications can be used to enhance such courses significantly. Textbooks and other materials that accomplish these expansions and that are designed for college use are available in increasing supply.

While there are many related matters to consider in addressing this issue (such as enrollment eligibility and mastery levels), it is my opinion that, under certain conditions and specifications (such as appropriate elaboration of content and applicability), it is neither inappropriate nor academically irresponsible to grant baccalaureate level credit for courses in elementary algebra and plane geometry.

(According to the By-Laws of the Transfer Credit Review Board, eight supporting votes were needed for sustaining the challenge and thus disallowing transfer credit for elementary algebra and plane geometry. There were only six votes to sustain and hence the view expressed here supports and reflects the Board decision.)

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## REPLY TO J. M. SCANDURA ON MATHEMATICAL PROBLEM SOLVING

J. DICKINSON

Scandura [1] pointed out that individuals differ in their capacity to solve new mathematical problems and suggested two possible explanations for this. Those who solve new problems "...either know something which the others do not or they are in some way inherently superior individuals." The remainder of Scandura's paper implicitly assumes that the former is the case and he demonstrated that training in the development of higher order rules enables subjects to solve those problems to which the higher order rule applies. However, two points concerning the nature of Scandura's conclusions need to be made. Firstly, any subjects who were capable of generating higher order rules without training were removed from further experimentation. Thus, evidence for inherent superiority versus training was never examined. Only those who failed to generate higher order rules for themselves participated in experimentation. The initial question of problem solving ability as an innate or acquired characteristic cannot be answered by inspection of data derived from a homogeneous group of non-problem solvers.

Secondly, Scandura found that after training in the use of higher order rules only those problems to which the acquired rules applied could be solved. There was no evidence of training producing any capacity in subjects enabling them to generate higher order rules more effectively than before. In fact, subjects were unable to solve a problem for which they had not been taught the appropriate rule.

In essence, therefore, all Scandura was able to demonstrate was that if people are trained in the use of a specific higher order rule they will be able to use that rule. To use one of Scandura's examples:

training a child in the knowledge that there are 12 inches in a foot and three feet in a yard and training in multiplication enables him to determine how many inches there are in a given number of yards which is hardly a novel observation.

Rather than the optimism apparent in Scandura's paper, his results in fact present a pessimistic view of the value of problem solving education. Subjects do not learn how to seek and generate new higher order rules after training in the use of this kind of rule. Only an examination of differences between those who failed to develop new higher order rules and the superior problem solvers rejected by Scandura can determine whether this ability is acquired, the nature of the acquisition process and the contribution of genetic factors.

**Reference**

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## PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

### ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before June 30, 1976.*

E 2581. *Proposed by Clark Kimberling, University of Evansville*

Let $\phi$ denote Euler's totient function and let $\{F_n\}$ be the sequence of Fibonacci numbers: $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$. Show that $\phi(F_n)$ is divisible by 4 if $n \geq 5$.

E 2582. *Proposed by Ioan Tomescu, University of Bucharest, Rumania*

Let $\{A_i; 1 \leq i \leq n\}$, $\{B_i; 1 \leq i \leq n\}$, and $\{C_i; 1 \leq i \leq n\}$ be three partitions of a finite set $M$. If for every $i$, $j$, $k$ we have

$$|A_i \cap B_j| + |A_i \cap C_k| + |B_j \cap C_k| \geq n,$$

prove that $|M| \geq n^3/3$, and that this inequality cannot be improved when $n$ is divisible by 3.

E 2583. *Proposed by C. L. Mallows, Bell Laboratories, Murray Hill, New Jersey*

Find all continuous $g: \mathbb{R} \to \mathbb{R}$ such that, for some continuous $f: \mathbb{R}^2 \to \mathbb{R}$, we have $g(xy) = f(x, g(y))$ for all $x, y$ in $\mathbb{R}$.

E 2584. *Proposed by H. S. M. Coxeter, University of Toronto*

Describe an infinite complex of congruent isosceles triangles, extending systematically throughout three-dimensional Euclidean space in such a way that each side of every triangle belongs to just two other triangles.

2585. *Proposed by Jan Mycielski, University of Colorado*

Prove that, for every triangulation of a 2-dimensional closed surface, the average number of edges meeting at a vertex approaches 6 in the limit as the number of triangles used approaches infinity.

E 2586. *Proposed by Walter Egerland, U.S. Army Ballistic Research, Aberdeen, Maryland*

Evaluate $\det A$ where $A = (a_{ij})$ is the $(n+1) \times (n+1)$ matrix defined by

$$a_{ij} = 0 \quad \text{if} \quad i - j \neq 0, 2, -2,$$

$$a_{ii} = \lambda_i + \lambda_{i-1} \quad (\text{where } \lambda_0 = \lambda_{n+1} = 0),$$

$$a_{i+2,i} = 1, \qquad a_{i,i+2} = \lambda_i \lambda_{i+1}.$$

(The scalars $\lambda_i$ may belong to any commutative ring.)

## SOLUTIONS OF ELEMENTARY PROBLEMS

### Sequence of Polygons

E 2401 [1973, 203]. *Proposed by Vladimir F. Ivanoff, San Carlos, California*

The exterior angle bisectors of a convex polygon $P_0$ form a polygon $P_1$, whose exterior angle bisectors form a polygon $P_2$, and so on. Prove that $P_n$ approaches a regular polygon as $n \to \infty$.

*Solution adapted from that of Michael Golomb, Purdue University.* Let $m$ be the number of vertices of $P_0$. Then each $P_n$ has $m$ vertices which we denote by $A_n(i)$, $i = 0, 1, \cdots, m-1$. The variable $i$ should always be reduced mod $m$. The labelling is chosen so that $A_n(i)$ and $A_n(i+1)$ are the endpoints of a side of $P_n$, which we denote by $a_n(i)$. Moreover, we require that $A_{n+1}(i)$ is the intersection of the exterior angle bisectors of $P_n$ at the vertices $A_n(i)$ and $A_n(i+1)$. Let $\alpha_n(i)$ be the interior angle of $P_n$ at the vertex $A_n(i)$. Then we have

$$\alpha_{n+1}(i) = \tfrac{1}{2}(\alpha_n(i) + \alpha_n(i+1)),$$

$$a_{n+1}(i) = a_n(i) \frac{\cos \frac{1}{2}\alpha_n(i)}{\sin \alpha_{n+1}(i)} + a_n(i+1) \frac{\cos \frac{1}{2}\alpha_n(i+2)}{\sin \alpha_{n+1}(i+1)}.$$

In matrix form, we have

$$\alpha_{n+1} = \alpha_n S, \qquad a_{n+1} = a_n T_n,$$

where $\alpha_n$ (resp. $a_n$) is the row vector whose coordinates are $\alpha_n(i)$ (resp. $a_n(i)$). The matrix $S$ is cyclic, doubly stochastic, and has distinct eigenvalues

$$\tfrac{1}{2}(1 + \omega^i), \qquad i = 0, 1, \cdots, m,$$

where $\omega$ is a primitive $m$th root of 1. Hence $\lim S^n = P$ (all limits are as $n \to \infty$) exists and $P$ is a doubly stochastic matrix of rank 1. This implies that all entries of $P$ are equal to $1/m$. Therefore

$$\lim \alpha_n = \lim (\alpha_0 S^n) = \alpha_0 P$$

and if we set $\alpha(i) = \lim \alpha_n(i)$, then $\alpha(0) = \alpha(1) = \cdots = \alpha(m-1) = (m-2)\pi/m$.

Now put $p = \cos(\pi/m)$, $H_n = pT_n$, $b_n(i) = p^n a_n(i)$ and $b_n = p^n a_n$. Then we have $\lim H_n = S$ and $b_{n+1} = b_n H_n$. Thus

$$b_n = b_0 H_1 H_2 \cdots H_n.$$

Let $G_n = H_1 H_2 \cdots H_n$ and let $g_{ij}(n)$ be the entries of $G_n$.

We can use the Strong Ergodicity Theorem as stated and proved in E. Seneta, *Non-Negative Matrices*, Wiley, New York 1973, pp. 73–76. It is easy to verify that our matrices $H_n$ satisfy the assumptions $A'$ and $B'$ on p. 73 of this book. Instead of this theorem we shall use its corollary on p. 76. Since the Perron-Frobenius row-eigenvector of $S$ has all coordinates equal to $1/m$ we deduce that

$$\lim_{n\to\infty} \frac{g_{ij}(n)}{g_{ik}(n)} = 1$$

for all $i$, $j$, $k$. This implies that $\lim_{n\to\infty}(b_n(i)/b_n(j)) = 1$ and consequently that

$$\lim_{n\to\infty} \frac{a_n(i)}{a_n(j)} = 1.$$

Hence $P_n$ approaches a regular polygon as $n \to \infty$ in the following sense: each interior angle of $P_n$ approaches $(m-2)\pi/m$ and the ratio of any two sides of $P_n$ approaches 1.

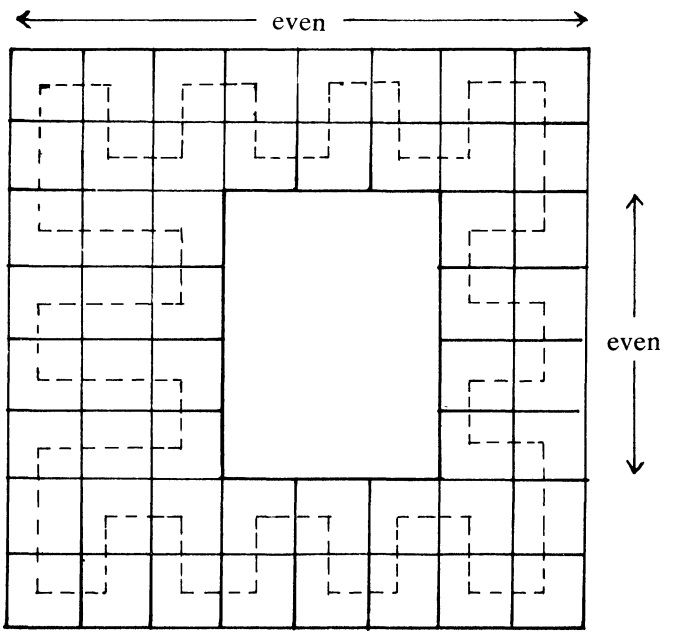
Partially solved by Daniel Finkel and by Kenneth Rosen.

*Editor's comment.* Golomb also proves that the limits $\lim_{n\to\infty} b_n(i)$ exist and are non-zero.

**Tiling a Checkerboard with Dominoes**

E 2508 [1974, 1111]. *Proposed by Sidney Penner, Bronx Community College*

Let four squares, exactly two of each color, be removed from a $2n \times 2n$ checkerboard. Show that if at least one of the deleted black squares and at least one of the deleted white squares are interior squares, then the remaining portion of the board can be tiled with $2 \times 1$ dominoes.



*Solution by Chip Ashlar, State University of New York at Binghamton.* Call the unit squares which compose the checkerboard "cells." There will be two interior deleted cells of opposite color which form diagonally opposite corners of a rectangle $R$ so that $R$ does not contain the other two deleted cells. The sides of $R$ have lengths of opposite parity; suppose the row length of $R$ is odd. Thus the rows of $R$ containing the two deleted cells in $R$ may be tiled and then so may be the untiled portions of the columns of $R$. The cells in the checkerboard (including the two deleted cells not in $R$) which remain when $R$ is deleted may be traversed by a Hamiltonian circuit in many ways (see figure for one). Since the two cells deleted from this circuit are of opposite colors, the remaining parts of the circuit have even lengths, and so they may be tiled.

Also solved by Tim Cling, Michael Goldberg, Carl Hurd, Carolyn MacDonald, the St. Olaf Problem Group, and the proposer.

**Area of a Convex Polygon**

E 2514 [1975, 73]. *Proposed by G. A. Tsintsifas, Thessaloniki, Greece*

Let $P$ be a convex polygon and let $K$ be the polygon whose vertices are the midpoints of the sides of $P$. A polygon $M$ is formed by dividing the sides of $P$ (cyclically directed) in a fixed ratio $p:q$ where $p+q=1$. Show that

$$|M| = (p-q)^2|P| + 4pq|K|,$$

where $|M|$ denotes the area of $M$, etc.

I. *Solution by M. Ram Murty and V. Kumar Murty, Undergraduates, Carleton University, Canada.* Consider a triangle $ABC$ with $D$ between $A$ and $B$ and with $E$ between $A$ and $C$ such that $CE:EA = p:q$ and $AD:DB = p:q$. Then $|ADE| = p|ABE| = pq|ABC|$.

Now let $X_0X_1\cdots X_{n-1}$ be the convex polygon $P$ and let $Y_0Y_1\cdots Y_{n-1}$ be the polygon $M$ such that $Y_i$ is between $X_i$ and $X_{i+1}$ (subscripts taken modulo $n$), and $X_iY_i: Y_iX_{i+1} = p:q$. With the above observation, we get

$$|Y_iX_{i+1}Y_{i+1}| = pq|X_iX_{i+1}X_{i+2}|$$

and hence

$$\sum_{i=1}^{n} |Y_iX_{i+1}Y_{i+1}| = pq\sum_{i=1}^{n} |X_iX_{i+1}X_{i+2}| = |P| - |M|.$$

Taking $p = q = \frac{1}{2}$ gives $\frac{1}{4}\sum_{i=1}^{n} |X_iX_{i+1}X_{i+2}| = |P| - |K|$. Thus

$$4pq(|P| - |K|) = |P| - |M|.$$

Since $p+q=1$, we have $1-4pq = (p-q)^2$ and so obtain

$$|M| = (1-4pq)|P| + 4pq|K| = (p-q)^2|P| + 4pq|K|.$$

II. *Comment by M. S. Klamkin, University of Waterloo.* It follows as an easy consequence that $\min|M| = |K|$ occurs for $p = q = \frac{1}{2}$.

Also solved by Brother Alfred Brousseau, Paul Bruckman, Leon Gerber, F. Gerrish (England), M. G. Greening (Australia), P. L. Hon (Hong Kong), Kao Hwa Sze, M. S. Klamkin (Canada), L. Kuipers (Switzerland), Harry Lass, Graham Lord (Canada), O. P. Lossers (Netherlands), Carolyn MacDonald, A. McD. Mercer (Canada), R. D. Nelson (England), Daniel Pedoe, Harry Ruderman, Michael Somos, Charles Wexler, and the proposer.

### The Careless File Clerk

E 2515 [1975, 74]. *Proposed by C. L. Mallows, Bell Telephone Laboratories, Murray Hill, N.J.*

A careless file clerk has documents $D_1, \cdots, D_d$ that should go respectively into files $F_1, \cdots, F_d$; instead he places them independently, at random, into a total of $f$ files ($f \geqq d \geqq 1$) so that each of the $f^d$ possible arrangements is equally likely. Show that the event that some non-empty subset $S$ of the files $F_1, \cdots, F_d$ can be made to have the correct contents by redistributing within $S$ the union of their contents has probability $d/f$.

*Solution by O. P. Lossers, Technological University, Eindhoven, The Netherlands.* Denote the required probability by $p(f, d)$, and let $N_j(f, d)$ be the number of documents winding up in the $j$th file. Proceeding by induction with respect to $f$, we then have

$$\begin{aligned} p(f+1, d) &= \sum_{n=0}^{d} p[N_{f+1}(f+1, d) = n]p(f, d-n) \\ &= \sum_{n=0}^{d} \binom{d}{n}\left(\frac{1}{f+1}\right)^n\left(\frac{f}{f+1}\right)^{d-n}\frac{d-n}{f} \\ &= \frac{1}{f+1}\sum_{n=0}^{d}\binom{d}{n}\left(\frac{1}{f+1}\right)^n(d-n)\left(\frac{f}{f+1}\right)^{d-n-1} = \frac{d}{f+1}. \end{aligned}$$

The last identity is established by writing $x = 1/(f+1)$, $y = f/(f+1)$ and noting that $d(x+y)^{d-1}$ is equal to the sum of the series obtained by differentiating each term of the binomial expansion of $(x+y)^d$ with respect to $y$.

REMARK. If all documents are placed in different files and if all $f!/(f-d)!$ such arrangements are equally likely, the answer remains the same. This can be proved by induction along similar lines.

Also solved by David Bloom, E. T. Dixon, Milton Eisner, Daniel Gallin, Ellen Hertz, Carl Hurd, Marc Kilgour (Canada), Murray Klamkin (Canada), Harry Lass, Joel Levy, Ram Murty & Kumar Murty (Canada), Michael Skalsky, Michael Somos, and the proposer.

### Permutation Equivalence

E 2516 [1975, 168]. *Proposed by Morris Newman and Charles Johnson, National Bureau of Standards*

Two matrices $A$ and $B$ are *permutation equivalent* if $B$ can be obtained from $A$ by first permuting the rows of $A$ and then permuting the columns of the resulting matrix.

Call an $n \times n$ matrix of zeros and ones a $k$-$k$ matrix if there are precisely $k$ ones in each row and in each column. Show that if $n \leqq 5$, then every $k$-$k$ matrix is permutation equivalent to its transpose, but that this is no longer true if $n \geqq 6$.

I. *Solution for $n \leqq 5$ by Detlef Laugwitz, Technische Hochschule, Darmstadt, Germany.* A sufficient condition for $A$ and its transpose $A^t$ to be permutation equivalent is that there exist permutation matrices (i.e., 1-1 matrices) $P_1$, $P_2$ such that $S = P_1AP_2$ will be a symmetric matrix. This follows from $S^t = P_2^tA^tP_1^t$ and

$$A^t = (P_2^t)^{-1}P_1AP_2(P_1^t)^{-1} = PAQ,$$

where $P = (P_2^t)^{-1}P_1$ permutes the rows of $A$, and $Q = P_2(P_1^t)^{-1}$ permutes the columns of $PA$. Since by interchanging 1's and 0's an $(n-k)$-$(n-k)$ matrix will become a $k$-$k$ matrix, we only need discuss $k \leqq n/2$. All cases $n \leqq 5$ are covered by the following lemma.

LEMMA. *For $k = 0, 1, 2$, (and $k = n, n-1, n-2$) all $k$-$k$ matrices may be transformed into symmetric matrices $S = P_1AP_2$, $P_1$ and $P_2$ being permutation matrices; in particular, such matrices are permutation equivalent to their transposes.*

*Proof.* For $k = 0$, $A$ is already symmetric, and for $k = 1$ it is a permutation matrix and can be transformed into the identity $I$ by row permutations. For $k = 2$ the proof goes by induction over $n$, and $S$ will moreover have 1 in the upper left corner. The case $n = 2$ is obvious. If $n > 2$ then by permutations of rows and permutations of columns we obtain either

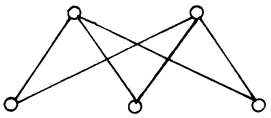
$$\begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & & & \\ 0 & 0 & & \tilde{A} & \\ \cdots & & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ \cdots & & & \tilde{A} & \\ & & & & \end{bmatrix}$$

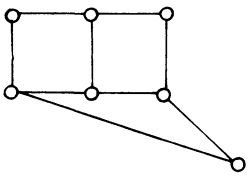
In the first case $\tilde{A}$ is a 2-2 matrix of order $n-2$, and hence, by permutations of rows (columns) which leave the first two rows (columns) of $A$ fixed, $A$ will become a symmetric matrix with 1 in the upper left corner. In the second case we consider the matrix $\tilde{A}$ of order $n-1$ which we obtain by deleting the first row and the first column of $A$, and we replace the 0 in the upper left corner of $\tilde{A}$ by 1. For the new matrix the assumptions hold, and we can transform it into a symmetric matrix. After this we insert 0 again instead of the 1 in the corner, thus completing the proof.

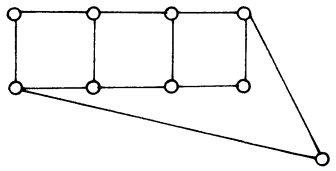
II. *Composite solution for $n \leqq 5$ by Allen J. Schwenk, Michigan State University, and Aage Bondesen. Espergaerde, Denmark.* Associate with every $n \times n$ zero-one matrix $A$ a bipartite graph $G$ consisting of two sets of points each labelled from 1 to $n$, with an edge between the $i$th point in the first set and the $j$th point in the second set if and only if $a_{ij} = 1$. No edge connects any two points in the same set. Thus each $k$-$k$ matrix is represented by a $k$-regular bipartite graph. Note that an $(n-k)$-$(n-k)$ matrix is obtained from a $k$-$k$ matrix by interchanging ones and zeros; thus it suffices to consider only values of $k \leqq [n/2]$. Now a matrix is permutation equivalent to its transpose if and only if the bipartite graph has an automorphism interchanging the two sets of points. For a 0-0 matrix $G$ is totally disconnected and any 1-1 mapping will suffice. If $k = 1$, the matrix corresponds to a 1-regular bipartite graph which is the union of independent edges. The required automorphism is obtained by mapping every point onto its only neighbor. In the case where $k = 2$, $G$ will be composed of one or more disjoint even circuits. Orient each circuit arbitrarily and map each point onto its immediate successor. For any $n \times n$ $k$-$k$ matrix with $n \leqq 5$ we have $k \leqq [5/2] = 2$, so that the matrix is permutation equivalent to its transpose.

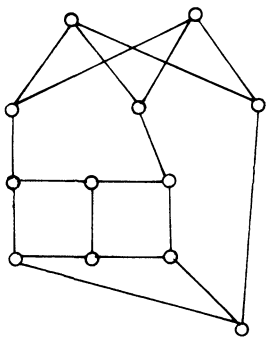
III. *Solution for $n \geqq 6$ by Allen J. Schwenk.* For all $n \geqq 6$ and $k = 3$, we construct a family of graphs $G_n$ having no automorphism of the above type. Let $H_n = K_2 \times P_{n-3}$ be the ladder graph. This bigraph has $2n-6$ vertices of which 4 have degree 2. Form $J_n$ by adding a new point to $H_n$ joined to a pair of degree 2 vertices which are at an even distance. Bigraph $J_n$ now has 3 vertices of degree 2. The 3-regular bigraph $G_n$ is now formed by adding three independent edges joining the vertices of degree 2 in $J_n$ to the vertices of degree 2 in $K_{3,2}$. The resulting bigraph has no automorphism switching the sets because one set has a pair of vertices with all three neighbors in common, whereas the other set has no such pair of vertices.

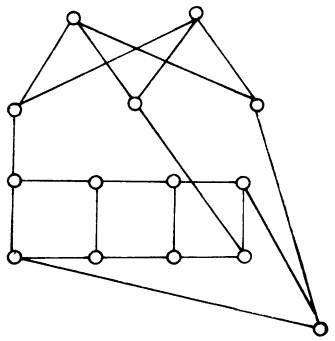
This construction is illustrated below for $n = 6$ and 7. The unique pair of vertices with three common neighbors are at the top of $G_n$.

$K_{3,2} =$ 

$J_6 =$ 

$J_7 =$ 

$G_6 =$ 

$G_7 =$ 

IV. *Solution for $n \geq 6$ by Aage Bondesen.* For every $n > 5$, and every $k$, $3 \leq k \leq n-3$, imagine an $n \times n$ $k$-$k$ matrix constructed in the following way. For every $i, j, 1 \leq i, j \leq n$, put

$$\tilde{a}_{ij} = \begin{cases} 1 & \text{if } j - 1 \equiv 0, 1, \cdots, k-1 \pmod{n} \\ 0 & \text{otherwise.} \end{cases}$$

Then in the matrix $\tilde{A} = (\tilde{a}_{ij})$ all the columns are different and so are the rows. Example, with $n = 9$ and $k = 4$:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1. |

Now change $\tilde{a}_{11}$ (which is one) into zero, and $\tilde{a}_{31}$ (which is zero) into one, and at the same time change $\tilde{a}_{1,k+1}$ (from zero) into one, and $\tilde{a}_{3,k+1}$ (from one) into zero. The resulting matrix $A$ is an $n \times n$ $k$-$k$ matrix in which the first and second rows are equal, and all the columns are different, for the following

reason: only two columns have been changed, and they are (after the change) quite clearly mutually different, and different from all the other columns in that each of them has *in more than two places* a zero adjacent to a one. In our example, $A$ looks like this:

$$\begin{matrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1. \end{matrix}$$

No amount of shuffling the columns and rows of $A$ can produce a matrix with two equal columns, so $A$ is *not* permutation equivalent to $A^t$.

Also solved by Robert Brandon, Carl Hurd, O. P. Lossers (Netherlands), John H. Smith, Richard Stevens, and the proposers. Laugwitz also presents a correct solution for $n \geqq 6$.

### Another Triangle Inequality

E 2517 [1975, 168]. *Proposed by A. G. Ferrer, Ensenada, Mexico*

Let $P$ denote a point interior to the triangle $ABC$, and let $r_1$, $r_2$, $r_3$ denote the distances from $P$ to the sides of the triangle. If $p$ denotes the perimeter of the pedal triangle, show that

$$\sum (r_1 + r_2)\cos\tfrac{1}{2}C \leqq p.$$

When does equality occur?

*Solution by Larry Walker and Cornelio Abungu, Jarvis Christian College, Texas.* Let $p_3$ denote the length of the side of the pedal triangle which faces $C$. Then we have

$$\begin{aligned} p_3^2 &= r_1^2 + r_2^2 + 2r_1r_2\cos C \\ &= (r_1 - r_2)^2 + 4r_1r_2\cos^2\tfrac{1}{2}C \\ &\geqq [(r_1 - r_2)^2 + 4r_1r_2]\cos^2\tfrac{1}{2}C \\ &= (r_1 + r_2)^2\cos^2\tfrac{1}{2}C. \end{aligned}$$

Thus $p_3 \geqq (r_1 + r_2)\cos\frac{1}{2}C$ and equality holds if and only if $r_1 = r_2$. By adding we get

$$p = \sum p_3 \geqq \sum (r_1 + r_2)\cos\tfrac{1}{2}C$$

and it is clear that equality holds if and only if $r_1 = r_2 = r_3$.

Also solved by Aage Bondesen (Denmark), Erhard Braune (Austria), Leonard Carlitz, Jane Evans, Michael Goldberg, Leonard Goldstone, M. G. Greening (Australia), M. S. Klamkin (Canada), L. Kuipers (Switzerland), A. Oppenheim (Nigeria), and the proposer.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before June 30, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6078. *Proposed by Albert Wilansky, Lehigh University*

Is it possible for a continuous linear functional on a normed space to map every bounded closed set onto a closed set of scalars? (Exclude trivial cases.)

6079. *Proposed by D. J. Kleitman, Massachusetts Institute of Technology*

Given a bipartite graph connecting $n$ vertices with $n$ others. If the symmetry group of the graph is transitive on both parts of the graph, must it be transitive on the whole graph? (Due to Bohdan Zelinka and announced at the Kesthely Conference, June 1973.)

6080*. *Proposed by R. N. Hevener, Jr., University of South Carolina*

A theorem of Abel states that if $\Sigma_{n=0}^{\infty} a_n z^n$ converges on the closed interval $A$, then (i) convergence is uniform on $A$, whence (ii) it determines a continuous function on $A$. Is either part of this theorem true if $A$ denotes a closed disk instead of an interval? If we impose the additional hypothesis, trivially satisfied in Abel's theorem, that the function be continuous on the boundary of $A$, is either part true?

6081*. *Proposed by T. Šalát, University of J. A. Komensky, Czechoslovakia*

Let $(X, d)$ be a metric space. We call $f: X \to R$ quasicontinuous at $x_0$ if for each positive $\varepsilon$ and $\delta$ there exists an open sphere $S(x_1, \delta_1) = \{x: d(x, x_1) < \delta_1)\} \subset S(x_0, \delta)$, such that $f[S(x_1, \delta_1)] \subset (f(x_0 - \varepsilon), f(x_0 + \varepsilon))$. Does there exist a metric space of first category and with no isolated points which allows a quasicontinuous function which is nowhere continuous?

REMARK. The set of points of discontinuity of a quasicontinuous function on $X$ must be of first category in $X$. (S. Marcus, Colloq. Math. 8 (1961), 45–53.)

6082. *Proposed by Thomas C. Craven, University of Hawaii*

Let $K(t)$ be the rational function field in one variable over a field $K$ of arbitrary characteristic. Does the equation $x^n - y^2 = 1$ have a nonconstant solution in $K(t)$ when $n > 2$?

6083. *Proposed by Emil Grosswald, Temple University*

Prove that, for real $p > 0$, the following identity holds:

$$\sum_{r=1}^{\infty} \frac{p}{r(p+r)} = \sum_{r=1}^{\infty} (-1)^{r-1} \binom{p}{r} \frac{1}{r}.$$

What is the function represented by both sides of this identity?

## SOLUTIONS OF ADVANCED PROBLEMS

### Topological Groups

5977 [1974, 670]. *Proposed by J. F. Chew, University of Akron*

Let $(G, \cdot, d)$ be a triple such that $(G, \cdot)$ is a group, $(G, d)$ is a metric space and the function $(x, y) \to x \cdot y$ is a continuous map from $G \times G$ to $G$. If $d$ is left invariant ($d(a \cdot x, a \cdot y) = d(x, y)$ for all $a, x, y \in G$), then $(G, \cdot, d)$ is a topological group. Can this statement be made without the assumption that $d$ is left invariant?

I. *Solution* (A) *by Lewis Lum, University of Tennessee, Knoxville.* Assume the metric $d$ is left invariant. To show $(G, \cdot, d)$ is a topological group it suffices to show the function $x \to x^{-1}$ is a continuous map from $G$ to $G$, where $x^{-1}$ denotes the inverse of $x$ in $G$. To that end, let $x_n$ be a sequence in $G$ converging to $x$. Observe that the continuity of the function $(x, y) \to x \cdot y$ implies the sequence $x_n \cdot x^{-1}$ converges to $x \cdot x^{-1} = e$, the identity element in $G$. Now from

$$d(x_n^{-1}, x^{-1}) \leqq d(x_n \cdot x_n^{-1}, x_n \cdot x^{-1}) = d(e, x_n \cdot x^{-1})$$

we see that the sequence $x_n^{-1}$ converges to $x^{-1}$. That is, the function $x \to x^{-1}$ is continuous and $(G, \cdot, d)$ is a topological group.

Note that $d(x, y) \leqq d(a \cdot x, a \cdot y)$ for all $a, x, y \in G$ actually implies that $d$ is left invariant since

$$\begin{aligned} d(a \cdot x, a \cdot y) &\leqq d(a^{-1} \cdot (a \cdot x), a^{-1} \cdot (a \cdot y)) \\ &= d((a^{-1} \cdot a) \cdot x, (a^{-1} \cdot a) \cdot y) \\ &= d(e \cdot x, e \cdot y) = d(x, y). \end{aligned}$$

II. *Solution* (B) *by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands.* To show that the answer to the final question is "No," we construct an appropriate metric $d$ on $(\mathbf{Q}, +)$, the additive group of the rationals. Let $n \to q_n$, $n \in \mathbb{N}$, be an enumeration of $\mathbf{Q}$. Let $a_1 + a_2 + a_3 + \cdots$ be a convergent series with only positive terms. Define an injective mapping $f$ from $\mathbf{Q}$ into $\mathbb{R}$ by $f(x) = x + \Sigma_{q_n \leqq x} a_n$, and for $x, y \in \mathbf{Q}$, put $d(x, y) = |f(x) - f(y)|$. Clearly $d$ is a metric on $\mathbf{Q}$. Moreover for $x < y$ we have $d(x, y) = y - x + \Sigma_{x < q_n \leqq y} a_n$. In particular, if $y = q_m$ and $x$ increases to $y$, then $d(x, y)$ decreases to $a_m$. On the other hand, if $y$ decreases to $x$, then $d(x, y)$ decreases to 0. It follows that $x \to -x$ is nowhere continuous on $\mathbf{Q}$ with respect to $d$. Continuity of addition is easily verified.

Also solved by L. Emery, Lee Erlebach, and by Lewis Lum (part B also).

**One-One Continuous Mapping $\mathbb{R} \to \mathbb{R}$**

5978 [1974, 670]. *Proposed by F. D. Hammer, Berkeley, California*

An exercise in Wilansky, *Topology for Analysis*, (Ginn 1970, p. 59, #110) asks for a continuous function $f: \mathbb{R} \to \mathbb{R}$ which is $1-1$ on the rationals, but not $1-1$ everywhere. Can "rational" be replaced by "irrational"?

*Solution by Brian White, student, Davidson (N.C.) College.* Suppose there is such an $f$ with $a < b$ rationals and $f(a) = f(b) = h$. The set $A = \{x \in [a, b] \mid f(x) = h\}$ is not dense in $[a, b]$, since otherwise $f$ would be constant on $[a, b]$. Thus there are two points $a' < b'$ in $A$ such that no point between them lies in $A$. By the Intermediate Value Theorem, the image of the open interval $(a', b')$ is either greater than $h$ or less than $h$. Suppose the former (a similar argument for the latter) so that:

$$f([a', b']) = [h, f(m)] \quad \text{with} \quad h < f(m) \quad \text{and} \quad m \in (a', b').$$

Then

$$f([a', m]) = f([m, b']) = [h, f(m)].$$

Since $f$ is one-one on the irrationals, $f$ must map the irrationals of $(a', m)$ to images of rationals of $[m, b']$. But this gives a one-one mapping of an uncountable set into a countable set, which is impossible.

Also solved by G. A. Aeppli, K. F. Anderson (Canada), Donald Cook, T. C. Craven, A. J. D'Aristotle, R. J. Evans, T. H. Foregger, C. H. Franke, David Grinstein, I. G. Kastanas (Greece), P. W. Lindstrom, O. P. Lossers (Netherlands), Stephen Noltie, Mary Powderly, Arthur Solomon, Art Steger, A. J. Vince, A. Vogt, and the proposer.

**Schlicht Cubics on $|z|<1$**

5979 [1974, 670]. *Proposed by Benjamin Volk, Far Rockaway, N. Y.*

Let the function $z + a_2z^2 + a_3z^3$ be schlicht in the open unit disk, $|z|<1$. Then show that $|a_2| \leqq 2\sqrt{2}/3$.

*Solution by A. C. Hindmarsh, Lawrence Livermore Laboratory, University of California.* The following stronger results hold:

(1) $|a_2| \leqq 2\sqrt{2}/3$ and $|a_3| \leqq 1/3$;

(2) Any given smaller bound on $|a_3|$ implies a smaller bound on $|a_2|$, given below; and

(3) All of these bounds are sharp.

We define first the difference quotient of the given function, $f$, as

$$D(z, w) = \frac{f(z) - f(w)}{z - w} = 1 + a_2(z + w) + a_3(z^2 + zw + w^2).$$

By schlichtness, $D(z, w) \neq 0$ for all $z$ and $w$ in the unit disk $U$. For a fixed $\rho$ with $|\rho| \leqq 1$, $\rho \neq 1$, define also a mapping from $U$ into the extended plane by

$$g_\rho(z) = \frac{-1 - a_3z^2(1 + \rho + \rho^2)}{(1 + \rho)z}.$$

This is constructed so that $a_2 = g_\rho(z) \Leftrightarrow D(z, \rho z) = 0$. Thus the image $G_\rho = g_\rho(U)$ does *not* contain $a_2$. Conversely, if $G$ is the union of all $G_\rho$ with $|\rho| \leqq 1$, $\rho \neq -1$, for a fixed value of $a_3$ with $|a_3| < 1$, then $a_2 \notin G$ implies that $f$ is schlicht. (The case $\rho = -1$ is handled by noting that $D(z, -z) = 1 + a_3z^2$ cannot vanish.) Hence sharp bounds on $a_2$ are given precisely by the complement of the region $G$.

The region $G_\rho$ can be described by reference to the special slit mapping $s(z) = z + 1/z$. Recall that $s$ maps $U$ onto the exterior of the slit $-2 \leqq x \leqq 2$, and maps $\partial U$ onto that slit. Also, the image under $s$ of the disk $|z| \leqq r$, $r < 1$, is the exterior of an ellipse whose major radius is $r + 1/r$. Letting $a_3(1 + \rho + \rho^2) = A^2e^{2i\alpha}$, $A > 0$ and $\alpha$ real, we have

$$g_\rho(z) = -(1 + \rho)^{-1}Ae^{i\alpha}(\zeta + 1/\zeta), \quad \zeta = Ae^{i\alpha}z.$$

If $A > 1$, $\zeta$ covers the closed disk $\bar{U}$, and $G_\rho$ is the entire extended plane. This contradicts $a_2 \notin G_\rho$ and we conclude $A \leqq 1$, or $|a_3| \leqq |1 + \rho + \rho^2|^{-1}$. The choice $\rho = 1$ gives $|a_3| \leqq 1/3$.

We observe further that with $A \leqq 1$, $G_\rho$ is the exterior of an ellipse (or a slit if $A = 1$) whose major radius is

$$B(\rho) = |1 + \rho|^{-1}A(A + 1/A) = (1 + |a_3||1 + \rho + \rho^2|)/|1 + \rho|.$$

The condition $a_2 \notin G_\rho$ thus implies $|a_2| \leqq B(\rho)$. A bound on $|a_2|$ will be obtained by computing the minimum of $B(\rho)$ with $|\rho| = 1$.

Letting $\rho = e^{i\theta}$ and $r_3 = |a_3|$, we have

$$B^2(\rho) = (1 + r_3|1 + 2\cos\theta|)^2/(2 + 2\cos\theta).$$

We let $x = 2\cos\theta$ and write

$$B^2(\rho) = b(x) = (1 + r_3|1 + x|)^2/(2 + x), \quad -2 < x \leqq 2.$$

On $-2 < x \leqq -1$, we observe that $b(x)$ decreases monotonically from $b(-2+) = +\infty$ to $b(-1) = 1$. If $x > -1$, we find that $b'(x)$ has the two roots $1/r_3 - 3$ and $-1/r_3 - 1$. The latter is outside the interval $|x| \leqq 2$, and the former is inside if $r_3 \geqq 1/5$, but outside otherwise. From the sign of $b'(x)$, we deduce that the minimum of $b(x)$ in $|x| \leqq 2$ occurs at $x = 2$ if $r_3 < 1/5$ and at $x = 1/r_3 - 3$ otherwise.

Evaluating these we find that $|a_2|$ is bounded by

$$\lim_{|\rho|=1} B(\rho) = \begin{cases} (1+3r_3)/2 & \text{if } r_3 < 1/5 \\ 2\sqrt{r_3(1-r_3)} & \text{if } r_3 \geqq 1/5 \end{cases} \equiv A_2(r_3).$$

The inequality $r_3 \leqq 1/3$ yields the bound $|a_2| \leqq 2\sqrt{2}/3$.

To prove sharpness, we let $a_3 = r_3$, $0 \leqq r_3 \leqq 1/3$, and $a_2 = A_2(r_3)$. The ellipse whose exterior is $G_\rho$ intersects the positive real axis at a point $R$ such that $g_\rho(e^{i\varphi}) = R > 0$ for some real $\varphi$, or

$$-(1+\bar{\rho})[e^{-i\varphi} + r_3(1+\rho+\rho^2)e^{i\varphi}] = |1+\rho|^2 R.$$

Writing the above as $ae^{-i\varphi} + be^{i\varphi} = c > 0$, we have also $\bar{b}e^{-i\varphi} + \bar{a}e^{i\varphi} = c$. Solving this system for the two unknowns $e^{i\varphi}$ and $e^{-i\varphi}$, we obtain

$$e^{i\phi} = c(a-\bar{b})/(|a|^2 - |b|^2), \quad c = \left| \frac{|a|^2 - |b|^2}{\bar{a} - b} \right|,$$

$$R = R(\rho) = \frac{1 - r_3^2|1+\rho+\rho^2|^2}{|1+\rho - r_3(1+\bar{\rho})(1+\rho+\rho^2)|}.$$

To determine the admissible real values of $a_2$, we must compute $\min R(\rho)$ over $\rho$ such that $|\rho| \leqq 1$, $\rho \neq -1$. $R$ diverges to $\infty$ as $\rho \to -1$, and is smooth elsewhere in $|\rho| \leqq 1$. We begin by computing $R(\rho)$ on $\partial U$:

$$R(e^{i\theta}) = \frac{1 - r_3^2|1+e^{i\theta}+e^{2i\theta}|^2}{|1+e^{i\theta}||1 - r_3 e^{-i\theta}(1+e^{i\theta}+e^{2i\theta})|}$$

$$= \frac{1 - r_3^2(1+2\cos\theta)^2}{|1+e^{i\theta}|(1 - r_3 - 2r_3\cos\theta)} = \frac{1 + r_3 + 2r_3\cos\theta}{\sqrt{2+2\cos\theta}}.$$

This function (or rather its square) was studied earlier, when examining $B(e^{i\theta})$ for $2\cos\theta > -1$, and found to have a minimum of $A_2(r_3)$.

The study of $R(\rho)$ for $|\rho| < 1$ appears to necessitate a rather tedious calculation. In terms of Cartesian coordinates, $\rho = x + iy$, $R^2(\rho)$ is a rational function of $x$ and y,

$$R^2(\rho) = P^2(x, y)/Q(x, y),$$

where $P$ is the numerator of $R$. Since $R(\bar{\rho}) = R(\rho)$, this is an even function of $y$, and thus a rational function of $x$ and $y^2$. Consider the vertical partial derivative,

$$\frac{\partial R^2}{\partial(y^2)} = Q^{-2}PH, \quad H \equiv (2QP_y - PQ_y)/2y.$$

For the moment, assume $0 < r_3 < 1/3$, and set $\varepsilon = 1 - 3r_3$. Then the polynomial $H(x, y)$ can be computed in terms of the new coordinates $\xi = (1-x)r_3/\varepsilon$ and $\eta = (1 - x^2 - y^2)r_3/\varepsilon$, and is found to be

$$H = \varepsilon^3 \sum_{j=0}^{4} H_j(\xi)\eta^j,$$

$$H_0(\xi) = -(1 - r_3)(8\xi^3 + 12\xi^2 + 6\xi + 1)$$

$$H_1(\xi) = -16\varepsilon\xi^3 - 28\varepsilon\xi^2 - 2(8\varepsilon + 1)\xi - (3\varepsilon + 1)$$

$$H_2(\xi) = -2[6\varepsilon\xi^2 + (7\varepsilon + 2)\xi + (2\varepsilon + 1)]$$

$$H_3(\xi) = -(18\varepsilon\xi + 11\varepsilon + 1)/3$$

$$H_4(\xi) = -\varepsilon.$$

(This calculation was done by the algebraic manipulation program REDUCE.) By inspection, the coefficients in each of the $H_j(\xi)$ are all negative for $\varepsilon > 0$, and thus $H \leqq 0$ for $\xi \geqq 0$, $\eta \geqq 0$.

We conclude that $\partial R^2/\partial(y^2) \leqq 0$ in $\bar{U}$, and by continuity this must be true for $0 \leqq r_3 \leqq 1/3$. It follows that $R^2$ assumes its minimum on $\partial U$, where $\min R = A_2(r_3) = a_2$. Thus, the point $a_2$ lies in the complement of $G$, and $f$ is schlicht. In the case $a_3 = 1/3$, $G_1$, and therefore $G$, is the complement of a horizontal slit, and so the slit $|a_2| \leqq 2\sqrt{2}/3$, $\operatorname{Im}(a_2) = 0$, is precisely the set of values of $a_2$ for which $f$ is schlicht.

Also solved by O. G. Ruehr.

### Separating Spheres in $\mathbb{R}^n$

5981 [1974, 671]. *Proposed by R. A. McCoy, Virginia Polytechnic Institute and State University*

Let $i$, $j$, and $n$ be positive integers. Prove or disprove: $i + j \leqq n$ if and only if for every two sets $\{B_1, \cdots, B_i\}$ and $\{D_1, \cdots, D_j\}$ of balls in $\mathbb{R}^n$,

$$\left(\bigcap_{k=1}^{j} D_k\right) \setminus \left(\bigcap_{k=1}^{i} B_k\right)$$

is connected.

*Solution by F. B. Strauss, University of Texas at El Paso.* We offer a counterexample for $n \geqq 3$. Let $D = \{x \in \mathbb{R}^n : \|x\| \leqq 2\}$, choose $x_0 \in \mathbb{R}^n$ with $\|x_0\| = 4$, and let $B_1 = \{x \in \mathbb{R}^n : \|x + x_0\| \leqq 5\}$ and $B_2 = \{x \in \mathbb{R}^n : \|x - x_0\| \leqq 5\}$. Then $\frac{1}{2}(-1)^k x_0 \in D \setminus B_k$, $k = 1, 2$. If $x \in D \setminus B_1$, $\|x\| \leqq 2$ and $\|x + x_0\| > 5$; then $25 < \|x + x_0\|^2 = \|x\|^2 + 2(x, x_0) + \|x_0\|^2 \leqq 20 + 2(x, x_0)$, and hence $(x, x_0) > 0$. Similarly, $(x, x_0) < 0$ for $x \in D \setminus B_2$. Thus $D \setminus B_1$, $D \setminus B_2$ lie on different sides of the hyperplane $H = \{x \in \mathbb{R}^n : (x, x_0) = 0\}$; and since neither is empty, $(D \setminus B_1) \cup (D \setminus B_2) = D \setminus (B_1 \cap B_2)$ is not connected. Here $i = 2$, $j = 1$, $i + j = 3 \leqq n$.

Also solved by Robert Schlafly, Peter Ungar, and B. R. Wenner.

### Algebra Generators

5982 [1974, 784]. *Proposed by R. L. Bishop, University of Illinois*

Let $A$ be a (not necessarily associative) finite dimensional algebra over a field $K$. If $x_1, \cdots, x_p$ are indeterminates and $w$ is a nonassociative word in them, then for each $a_1, \cdots, a_p$ in $A$ we get $w(a_1, \cdots, a_p) \in A$ by substituting $x_i = a_i$. Prove that there are words $w_1, \cdots, w_q$ such that for every $a_1, \cdots, a_p$ the algebra generated (using products and linear combinations over $K$) by $a_1, \cdots, a_p$ is the linear span of $w_i(a_1, \cdots, a_p)$, $i = 1, \cdots, q$.

*Solution by D. J. Samuelson, Pennsylvania State University at McKeesport.* Let $p$ be a fixed positive integer. A *monomial* is any word constructed from the indeterminates $x_1, \cdots, x_p$ using multiplication only. We denote by $W_i$ the set of all monomials of length $i$. Thus, for $p = 2$, $W_1 = \{x_1, x_2\}$, $W_2 = \{x_1x_1, x_1x_2, x_2x_1, x_2x_2\}$, $W_3 = \{x_1(x_1x_1), (x_1x_1)x_1, x_1(x_1x_2), (x_1x_1)x_2, \cdots\}$, etc.

THEOREM. *The words $w_1, \cdots, w_q$ which suffice to establish the proposition above can be taken to be $W_1 \cup W_2 \cup \cdots \cup W_{2^{n-1}}$, the monomials of length at most $2^{n-1}$, where $n$ = dimension $(A)$.*

This theorem follows immediately from Lemma 2 below.

Let $a = (a_1, \cdots, a_p)$ be a sequence of elements of $A$ and put

$$M_i = \{w(a_1, \cdots, a_p) \mid w \in W_i\}.$$

Whenever $S \subseteq A$, let $L(S)$ denote the linear span of $S$ in $A$.

LEMMA 1. *Suppose that* $M_1 \cup \cdots \cup M_{2i} \subseteq L(M_1 \cup \cdots \cup M_i)$ *for some fixed* $i$. *Then* $M_j \subseteq L(M_1 \cup \cdots \cup M_i)$ *whenever* $j \geq 2i+1$.

*Proof.* Consider first the case when $j = 2i+1$. If $m \in M_{2i+1}$, then $m = m_1 m_2$ where $m_1, m_2 \in M_1 \cup \cdots \cup M_{2i}$. By hypothesis we have that $m_1, m_2 \in L(M_1 \cup \cdots \cup M_i)$ and thus that $m (= m_1 m_2) \in L(M_1 \cup \cdots \cup M_{2i})$. Since $m$ is arbitrary and $L(M_1 \cup \cdots \cup M_{2i}) \subseteq L(M_1 \cup \cdots \cup M_i)$, it follows that $M_{2i+1} \subseteq L(M_1 \cup \cdots \cup M_i)$. The lemma can now be completed by induction on $j$ using arguments similar to those above.

Let $[a]$ denote the subalgebra of $A$ (using products and linear combinations over $K$) generated by the sequence $a = (a_1, \cdots, a_p)$.

LEMMA 2. *If dimension* $(A) = n$, *then* $[a] = L(M_1 \cup \cdots \cup M_{2^n-1})$.

*Proof.* Define inductively a sequence $B_1, B_2, B_3, \cdots$ of sets as follows:

$B_1$ is a maximal linearly independent subset of $M_1$;
$B_i$ is a maximal linearly independent subset of $M_1 \cup \cdots \cup M_i$ such that $B_i \supseteq B_{i-1}$ $(i = 2, 3, \cdots)$.

Since $A$ is a finite dimensional algebra, there is a smallest positive integer $r$ for which $B_r = B_{r+1} = B_{r+2} = \cdots$ Set $B = B_r$. From Lemma 1 we conclude:

(*) If $B - (M_1 \cup \cdots \cup M_{2^{i-1}})$ is nonempty,
then $B \cap (M_{2^{i-1}+1} \cup \cdots \cup M_{2^i})$ is nonempty,

for $i = 1, 2, 3, \cdots$. Since dimension $(A) = n$, $B$ contains at most $n$ elements and from (*) it follows that $B \subseteq M_1 \cup \cdots \cup M_{2^n-1}$. Since every element of $[a]$ is expressible as a linear combination of monomials in the elements $a_1, \cdots, a_p$ we deduce that

$$[a] = L(B) = L(M_1 \cup \cdots \cup M_{2^n-1}).$$

Also solved by Joseph Gruendler and by the proposer.

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## REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

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*Multivariable Mathematics: Linear Algebra, Differential Equations, Calculus.* By Richard E. Williamson, Hale F. Trotter. Prentice-Hall, Englewood Cliffs, N. J., 1974. ix + 630 pp. $16.50. (Telegraphic Review, October 1974.)

As indicated by the title of this *portmanteau* of a book, the purpose of its eighteen chapters is to present the material commonly taken up following a first course in calculus. (In passing, we note that there is no discussion of infinite series, a topic often not covered in elementary calculus and saved for a second course.) A number of different one- and two-semester courses (outlined by the authors) can be taught from this flexible text. Linear algebra provides the framework on which the book is

constructed; but adequate as it is for most applications, this frame would have to be strengthened to support a complete course in linear algebra.

The reviewers used this text for a one-semester course in ordinary differential equations with linear algebra given to students in engineering, biology, and chemistry. Slightly more than half the book is devoted to a careful coordination of these topics, one which avoids the pitfall of using too much algebra at the expense of analysis and its applications. The linear differential operator is used in solving first-order equations and equations with constant coefficients, while the algebraic tools needed to study linear systems of o.d.e.'s are brought in via examples, so that the student does not have to wait too long before seeing a system of o.d.e.'s solved. The chapter on first-order equations is an attractive combination of technique, theory, numerical methods (Euler's method), and applications such as bacterial population growth, satellite motion, and the psychological study of stimulus and response. The treatment of equations with constant coefficients includes a nicely done section on oscillations and sections covering the Laplace transform and convolutions. There is a chapter on existence and uniqueness theory which includes the Picard iteration method applied to nonlinear equations.

The material on multivariable calculus is based in part on an earlier text, *Calculus of Vector Functions*, written by the authors in collaboration with R. H. Crowell (Telegraphic Review, January 1973), but has been rewritten "to remove ... an overemphasis, at this level, of the theoretical and abstract point of view." The authors have succeeded admirably, while preserving the high quality of exposition found in their earlier text.

The treatment of differential calculus proceeds systematically through the study of functions from $R$ to $R^n$, from $R^n$ to $R^n$, and from $R^n$ to $R^m$. The presentation of the derivative in terms of a local affine approximation is well-done and gives the student an exciting glimpse of the blending of algebra and analysis.

The chapter dealing with multiple integration contains an unusually well-motivated section on the change-of-variable technique. The authors first illustrate the special cases of polar, spherical, and cylindrical coordinate changes and then show that these can all be viewed in a uniform way.

The important integral results of vector field theory—Green's Theorem, Stokes' Theorem, and the Divergence theorem—are presented in a clear way that emphasizes physical interpretations.

The only real criticism on the part of students using this book is that, although there is an adequate supply of exercises (both computational and theoretical), it seems as if less than half the computational problems have their answers given in the text.

In conclusion, we recommend this attractive text strongly. It is comprehensive and well-written, geometrical and physical in its flavor, with many examples and figures to help the reader learn the material.

HENRY J. RICARDO and CHARLES H. STOLZE, Manhattan College

*Recursive Function Theory & Logic.* By Ann Yasuhara. Academic Press, New York, 1971. xv + 338 pp. $21.50. (Telegraphic Review, April 1972.)

*Introduction to the Theory of Computation.* By Erwin Engeler. Academic Press, New York, 1973. viii + 231 pp. $13.95. (Telegraphic Review, October 1974.)

*Theory of Computation.* By Walter S. Brainerd and Lawrence H. Landweber. Wiley, New York, 1974. xxi + 336 pp. $19.95. (Telegraphic Review, October 1974.)

Once in each of the four years 1972–1975 I taught a one-quarter five credit hour course entitled "Computability and Unsolvability." This course is listed with identical description under both the department of computer and information sciences and the department of mathematics. During the four years I used the three books indicated above for textbooks. Altogether thirty-nine students took

the course: they were computer science or mathematics majors at the advanced undergraduate or beginning graduate level. I found teaching this course very enjoyable: the class was small and this not being a required course the students who took it were good students, interested in the subject matter, although not necessarily sophisticated in mathematics.

During the first two years, in 1972 and 1973, I used Yasuhara's text. This is an ambitious book and covers a great deal of material on mathematical logic and recursive function theory. The part on recursive function theory comes before the part on mathematical logic. This review pertains only to Part I of the book. (The first time I taught the course I covered chapters I, III, IV, V, and VII; the second time I covered chapters I–V.)

Chapter I contains basic results about Turing machines: a construction is sketched for a universal Turing machine, the unsolvability of the halting problem is proved, and some decision problems are explained. Unfortunately at the beginning of the chapter the reader runs into a problem which seems to recur in various portions of the book; namely, the author has a tendency to disregard her own conventions. For example, on page 16 in the definition of a Turing machine the following sentence appears, "For every pair $q_i s_j$, there is to be a quintuple beginning with that pair." However, on page 17 in Example 2 this rule is not followed. Also the author sometimes states problems in a vague or incorrect manner. For example on pages 19–21 a copy machine is outlined. This machine changes a tape inscription $\cdots{**}s_1 s_2 \cdots s_k X{**}\cdots$ to $\cdots{**}X s_1 s_2 \cdots s_k{**}\cdots$. The author constructs a fairly complicated Turing machine to solve this problem. But the problem as stated can be solved simply by moving $X$ only. On page 32 she states and proves (Theorem 1.1) the unsolvability of the halting problem. The halting problem is stated for arbitrary Turing machines, but the proof given works only for a restricted class of Turing machines. Additional comments are needed to carry the proof over to arbitrary Turing machines.

Chapter II deals with the Semi-Thue and Thue systems. Then follows a short chapter on enumerations and Gödel numberings. Chapter IV introduces recursive functions. In chapter V the author proves the equivalence of recursive and Turing-computable functions. Chapter VI covers topics in computational complexity. Several students tried to use this chapter for a project but could not understand the material. Chapter VII deals with recursive and recursively enumerable sets. In chapter VIII the author proves Kleene's normal form theorem, describes the arithmetic hierarchy, and proves the Friedberg-Muchnik theorem.

There are enough good exercises in the book. Unfortunately the arbitrarily changing conventions are confusing to the students. Also the writing is uneven: some sections are much better than others. Nevertheless this is a good book; its level is appropriate; and I liked teaching from it both times that I used it.

In 1974 I used Engeler's text. Unlike Yasuhara, it contains material for a one-quarter or one-semester course. There are three chapters and I covered the first two. In chapter 1 the author introduces regular languages and finite automata, and gives a construction for a minimal finite transducer. Chapter 3 contains a short introduction to formal grammars with an emphasis on context-free grammars.

Chapter 2 is the main chapter. An abstract universal calculator is introduced for which one writes programs. A function is called computable if there is a program which computes it. The author then introduces recursive functions, and by analyzing the structure of programs proves the equivalence between partial recursive and computable functions. Afterwards a Gödel numbering is given for programs. Several portions of the construction of a universal program are slightly incorrect. Then Kleene's normal form theorem is proved and the halting problem is shown to be unsolvable. The latter proof is not entirely right but is easily corrected. At this point Turing machines are introduced, but the presentation is sketchy. The chapter ends with the construction of random-access stored program machines.

The students enjoyed reading this book, and so did I. The author writes informally but carefully. The book proceeds relatively slowly in chapter 1 so as to allow the beginning student to digest the

material. The pace quickens in chapters 2 and 3. The exercises are clearly stated. Because of the many good features of this book I originally planned to use it again the following year. However, when the next book appeared I decided to try it.

So in 1975 I used the text by Brainerd and Landweber. This book is clearly computer science oriented. Also, like Yasuhara, it contains much more material than can be covered in a one-quarter or one-semester course. I went through the first six chapters. Chapter 1 is a high-powered introductory chapter on sets, relations, functions, mathematical induction, operations on functions, and cardinality. The authors try to be as general as possible, so, for example, functions go from $r$-tuples to $s$-tuples. Given the background of the students in the class, it seemed that I could spend much of the quarter on chapter 1 alone. Instead of doing that I went into chapter 2. Here the language PL is defined. PL is a simple algorithmic language and is used to introduce the notion of PL-computable function. Chapter 3 covers recursive functions. The operations of exponentiation and repetition are used instead of primitive recursion and minimization respectively. It is shown that the primitive recursive functions are the ones computable in PL without GOTO's. At the end of the chapter a Gödel numbering is given.

In chapter 4 several abstract models are used to study the notion of computability, namely the unlimited-register machine, the single-register machine, and the Turing machine. In chapter 5 the authors define labeled Markov algorithms and discuss connections with the language SNOBOL. Also in this chapter they complete the proof of the equivalence of various notions of computability. Then they show the existence of a universal function and prove the undecidability of the halting problem. Chapter 6 is about recursively enumerable sets. Rice's theorem is proved and used for the study of undecidable problems. Chapters 7–11 contain further material. The chapter headings are as follows: Formal Languages, Reducibility, Complexity of Computation, Subrecursive Hierarchies, Introduction to Combinatory Logic (by George W. Petznick).

In contrast to my experiences with Yasuhara and Engeler I did not find any errors and even very few misprints. This is a carefully written book. I enjoyed the emphasis on the connections between the theory of computation and programming language concepts. There are many excellent exercises. However, the mathematical sophistication was way over the student's heads. This text is probably more suitable for a graduate course.

In the last two years several additional textbooks have appeared on the theory of computation. Most of these books were given telegraphic reviews in the MONTHLY. I list here only the authors and dates of review: Davis, October 1974; Jones, November 1974; Boolos and Jeffrey, March 1975; Manna, May 1975; Hamlet. Portions of Davis, Boolos and Jeffrey, and Manna were read by students for their projects. They had good comments about the readability of the first two. Boolos and Jeffrey give the solutions to the exercises right after the problems; the presentation in Davis is very clear but there are no exercises. A major portion of Manna is on flowchart programs, flowchart schemes, and the fixedpoint theory of programs. Jones and Hamlet also contains nice material for a course. It is a pleasure to conclude that there are a number of good textbooks available on the theory of computation.

JOHN GRANT, University of Florida

*Discrete Mathematical Structures and Their Applications.* By Harold S. Stone. Science Research Associates, Inc., Palo Alto, California, 1973. vii + 402 pp. $15.50. (Telegraphic Review, February 1974.)

Aimed at course B3 of the 1968 ACM curriculum, this text differs from others on discrete structures in containing "much less graph theory and substantially more material on abstract algebra." The author, a computer scientist, has produced a book which offers a new direction for undergraduate courses in abstract algebra. A discussion of many computer-related topics serves to illustrate and apply the development of algebraic structures; in turn, the "abstract" material is presented in a very

concrete manner. Most of my students found both the content and the style of the exposition quite appealing. There are excellent chapter introductions, section summaries, and ample references to articles and other books (but not, for example, to the text by Birkhoff and Bartee). In addition, there is an abundance of illustrative tables and diagrams. This is, all in all, a most interesting book.

After greeting this volume with much enthusiasm, I began to discover a great need for caution in using it. As with many attractive textbooks, this one is pitched at the better students (the "guinea pigs" were seniors and graduate students at Stanford). A more typical class may require some extra material and exercises. Worse yet, there are many errors – of almost every possible description – guaranteed to mislead all but the exceptionally wary. My best student actually enjoyed the abundance of mistakes, but the lack of care taken throughout the book is a serious drawback for almost anyone attempting to learn from it.

The first chapter, an ample discussion of preliminaries, is followed by a good chapter on groups, made very concrete and clear by the extensive use of generators and group graphs. Unfortunately, however, the reader must unscramble Figures 2.2.1 and 2.2.2. before the clarity becomes apparent. Chapter 3, the Polya Theory of Enumeration, is a clear presentation of an often cumbersome topic. Here again, the reader needs to be careful. The example introduced on p. 119 uses Burnside's Theorem erroneously. Also, the role played by the degree of the underlying permutation group could be brought out more clearly. Finally, a forward reference to section 9.4, where there is an enumeration of equivalence classes of switching functions, would be helpful to readers intrigued but left hanging in Chapter 3.

Chapters 4 and 5 contain applications of group theory and residue arithmetic. The Hamming codes provide a good illustration of subgroups and cosets, and the section on dynamic memories contains some groups which students can enjoy playing with. As "applications", however, these chapters are somewhat disappointing. The topics are interesting, but an understanding of group theory does not shed much light on them.

Chapters 6 and 7 are devoted to semigroups and to finite-state machines, respectively. The later chapter is a bit long for inclusion in an algebra course (and a bit short for a full discussion of finite-state machines). If the important ideas could be illustrated for a single group of examples, considerable economy would result.

Chapter 8, rings and fields, contains the standard results on rings, ideals, polynomials, and field extensions, plus a large amount of material on finite fields. Students are asked to perform extensive computations with concrete representations of finite fields, in a way which illustrates most of the algebraic ideas very sharply. Much of this material is used in Chapter 9, linear finite-state machines, perhaps the most appealing chapter in the entire book. Here one finds shift-registers, Polya theory for switching functions, maximum-length sequences, and encoding/decoding techniques for Hamming codes. With a little thought, and a lot of pencil and paper, the reader can pull together most of the ideas of the book, and see them working very well together in these applications.

In the final chapter, Stone takes up lattices, Boolean algebras, and the disjunctive normal form for $n$-variable switching functions.

I would use this book again, preferably in a corrected form. Student reaction was largely favorable, though there was a great deal of frustration with the many errors, and the lack of clear introductory examples in several places. The problem sets are ample and challenging (including the "fun" of deciding which exercises are correct), and in fact provide a strong motivation for exploring the theory in detail.

This book, despite its shortcomings, is a most welcome offering. It should influence future courses for general mathematics majors, and for prospective secondary teachers. (A similar direction has already been suggested, for the latter group, in the British ATM handbook, *Some Lessons in Mathematics*, Cambridge University Press, 1964.) I hope that future variations of this theme will retain the freshness of the original.

PHILIP A. LEONARD, Arizona State University

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S*(13), L*. *Riddles in Mathematics, A Book of Paradoxes.* Eugene P. Northrop. Krieger, 1975, viii + 262 pp, $7.95. Reprint of original 1944 edition. Over 200 examples of the unexpected in mathematics (arithmetic, geometry, logic, probability, infinity, calculus)--things that appear to be false, but are actually true; or appear to be true, but are actually false, or simply self-contradictory. LCL

GENERAL, T(13-15: 2), L. *Introduction to Mathematics for Life Scientists, Second Edition.* Edward Batschelet. Biomathematics, V. 2. Springer-Verlag, 1975, xv + 643 pp, $27.60. About 100 new pages, primarily of problems, illustrations, appendices and additional material on integration and matrices. See extended review of first edition, February 1974. LAS

GENERAL, T(13-15: 1), S*, L*. *Symmetry Discovered: Concepts and Applications in Nature and Science.* Joe Rosen. Cambridge U Pr, 1975, xi + 138 pp, $11.95. A lively attempt at communicating the "symmetry disease" via illustrations from geometry, nature and science. Introduces group theory in an optional role. Contains an extensive bibliography arranged by topic and carefully referenced in the text. A good choice for a freshman seminar or an interim (short-course) text. LAS

GENERAL, S(9-13). *The Impossible in Mathematics.* Irving Adler. NCTM, 1975, 25 pp, $1.10 (P). A reprint of one chapter of Adler's 1957 book *Monkey Business.* Elementary treatment of classical construction problems, 15 puzzles, etc.. LAS

GENERAL, S(15-18). *Essai sur la Psychologie de l'Invention dans le Domaine Mathématique.* Jacques Hadamard. Gauthier-Villars, 1975, 134 pp, 26F (P). A reissue of the French translation (Blanchard, 1959) of the English original (Princeton, 1945; Dover, 1954; Oxford U Press, 1955). JAS

GENERAL, S(13), L. *The Application of Mechanics to Geometry.* B. Yu. Kogan. Trans: David J. Sookne, Robert A. Hummel. U of Chicago Pr, 1974, vi + 57 pp, $2.95 (P). An entry in the "Popular Lectures in Mathematics" series. Uses elementary laws of mechanics to prove some familiar theorems of plane geometry and to compute volumes and surface areas of certain solids. Illustrated. Very readable. DFA

BASIC, T(13: 1, 2). *X-Rated Algebra for Mature Beginning Students, 6 Volumes.* Patricia Fernandez, Richard R. Miller. Brooks/Cole, 1975. *V. I: Basics,* 152 pp, $3.95 (P); *V. II: Linear and Polynomial Functions,* ix + 102 pp, $3.95 (P); *V. III: Exponents and Logarithms,* ix + 110 pp, $3.95 (P); *V. IV: Quadratic and Rational Functions,* ix + 116 pp, $3.95 (P); *V. V: Linear Systems, Linear Programming, and Curve Fitting,* ix + 126 pp, $3.95 (P); *V. VI: Permutations and Combinations,* vii + 46 pp, $2.95 (P). This series consists of twelve units in beginning and intermediate algebra, including a unit on linear programming. The text is well written and the examples are helpful. The price is steep. CEC

BASIC, T(13: 1). *Introduction to Algebra: A Personalized Approach.* George L. Henderson, Mary I. Van Beck, Walter W. Leffin. Prindle, 1975, xi + 387 pp, $9.95 (P). This text is equally suitable for self-study as well as for regularly taught classes. It consists primarily of brief expositions, examples, and long lists of problems covering the material usually found in beginning algebra courses up to quadratic equations and systems of linear equations. CEC

PRECALCULUS, T(13: 1). *Introductory Algebra, A College Approach, Third Edition.* Milton D. Eulenberg, Theodore S. Sunko, Howard A. James. Wiley, 1975, xii + 360 pp, $10.95. Adult treatment with stress on mastery of content. Includes logarithms; no trigonometry. LCL

EDUCATION, T(13-14: 1), S. *Mathématiques Buissonnières.* André Deledicq. CEDIC, 1975, 287 pp, (P). Written to give (European) secondary school teachers ideas for enrichment of teaching. This book contains snapshots of games, graph theory, vector spaces, distances (pre-topology), functions. The approach is informal and fresh. Could well have been a text (outline) for a liberal arts course written by N. Bourbaki themselves. JAS

HISTORY, P, L*. *Historia Mathematica, Vol. 2, pp. 425-624, November 1975.* Ed: Garrett Birkhoff. U Toronto Pr. Proceedings of an August 1974 American Academy Workshop on the evolution of modern mathematics. A unique record of a dialogue between active mathematicians (e.g., E. Bishop, J. Dieudonné, S. Abhyankar) and historians of mathematics (e.g., J. Grabiner, K.O. May, I. Grattan-Guinness, T. Hawkins). LAS

HISTORY, L. *The Mathematicall Praeface to the Elements of Geometrie of Euclid of Megara (1570).* John Dee. Sci Hist Pub, 1975, $8.95. John Dee (1527-1608), mathematician, alchemist, astronomer and mystic, wrote a wide-ranging introduction to the first English translation of Euclid. This preface--reprinted here for the first time since 1661--had a profound influence on sixteenth century science, for it contained an unusually detailed classification of the "mathematical arts." This edition is introduced by extensive commentary by Allen Debus. LAS

HISTORY, P. *Historical Studies in the Physical Sciences, V. 6.* Ed: Russell McCormmach. Princeton U Pr, 1975, xiv + 550 pp, $30. Nine papers on the history of physics and chemistry, including a lengthy essay on the emergence of physics in Japan. LAS

HISTORY, P, L. *Einar Hille: Classical Analysis and Functional Analysis, Selected Papers.* Ed: Robert R. Kallman. MIT Pr, 1975, xlvii + 708 pp, $35. Hille's own selection of his important papers that document the transition from classical to functional analysis. Eleventh in the MIT series Mathematicians of Our Time. LAS

HISTORY, P. *The Emergence of Science in Western Europe.* Ed: Maurice Crosland. Science History Pub, 1976, 201 pp, $18. Edited texts of papers from a 1974 conference at Leeds University focused on scientific evolution in specific national contexts, in particular, Italy, Holland, England, Scotland, France and Germany. LAS

HISTORY, P. *Mathematische Werke.* Gotthold Eisenstein. Chelsea, 1975. *Band I,* xiii + 502 pp; *Band II,* xiii + 426 pp, $60 set. The first complete collection of Eisenstein's works together with his letters to Hermite, Richelot, Stern and Gauss (published for the first time). Also includes a youthful autobiography and a biographical article by Kurt-R. Biermann. JAS

HISTORY, P. *Helmut Hasse Mathematische Abhandlungen.* Heinrich Wolfgang Leopoldt, Peter Roquette. Walter de Gruyter, 1975, 474 DM. *Band 1:* xv + 535 pp; *Band 2:* xv + 525 pp; *Band 3:* x + 532 pp. A selection of important papers up to 1961. The third volume contains a complete bibliography (to 1975), including books. JAS

FOUNDATIONS, P*. *Entailment, The Logic of Relevance and Necessity, V. I.* Alan Ross Anderson, Nuel D. Belnap, Jr. Princeton U Pr, 1975, xxvii + 542 pp, $21. Entailment, a strong version of the "if... then..." implication requiring that the premises be relevant to the conclusion, was a central feature in the canon of logical thought until its elimination in the formalization of logic in this century. In 1956 W. Ackermann proposed a formalization of entailment. Anderson and Belnap now offer the first half of an unusually lucid (even witty) encyclopedic survey of the research generated by Ackermann's paper. LAS

FOUNDATIONS, P, L*. *Mathematics, Matter and Method, Philosophical Papers, Volume I.* Hilary Putnam. Cambridge U Pr, 1975, xiv + 328 pp, $25. 19 papers on the philosophy of mathematics and science and on the nature of scientific enquiry. A valuable gathering of important yet dispersed sources of contemporary reflection on the nature of mathematical truth. LAS

FOUNDATIONS, P. *The Logic of the Plausible and Some of its Applications.* René Leclercq. Plenum Pr, 1974, vii + 80 pp, $11.75. A terse formalization of analogy and generalization, preceded by a sparse outline of the history of systematic thinking. LAS

COMBINATORICS, S(17-18), P*, L. *Finite Operator Calculus.* Gian-Carlo Rota. Acad Pr, 1975, x + 159 pp, $9.75. A reprinting of five papers dealing with recent unifying theories in combinatorics, many awaiting further development. The approach is reflected by the author's feeling that for him, "understanding a problem in combinatorics means reducing it to a problem in linear operators or at least to modules over a commutative ring." No index. LCL

COMBINATORICS, T(17-18: 1), P. *Lecture Notes in Economics and Mathematical Systems-109: Introduction to the Theory of Matroids.* Rabe von Randow. Springer-Verlag, 1975, ix + 102 pp, $7.80 (P). Compact, formal study of all the equivalent axiomatic definitions of matroids, together with their most commonly used properties. Two additional chapters on the greedy algorithm, and exchange properties of matroid bases. LCL

COMBINATORICS, P. *Lecture Notes in Mathematics-387: Eléments de la Théorie des Matroïdes.* Claude P. Bruter. Springer-Verlag, 1974, 141 pp, $7.40 (P). Thorough introduction to matroids with explicit axiomatic development of basics and chapters on geometric and algebraic representations of matroids. Applications to more "classical" combinatorial topics, including systems of distinct representatives (transversal theory), graphs and optimization problems. SS

NUMBER THEORY, T(17-18), P. *Les nombres p-adiques.* Yvette Amice. Pr U France (U.S. Dist: SMPF), 1975, 189 pp, (P). A well-written introduction to p-adic analysis. Discusses p-adic numbers, non-archimedean fields and Banach spaces, analytic functions, and rationality theorems. Includes a sufficient number of exercises. SG

NUMBER THEORY, T(18: 1, 2), S, P. *Les nombres premiers.* William John Ellison, Michel Mendès France. Hermann (U.S. Distr: SMPF), 1975, xiv + 442 pp, 118 F (P). A splendid introduction to the theory of prime numbers. Includes proofs of the prime number theorem, discussions of the error term, and a proof of Dirichlet's theorem. Many recent developments are discussed in appendices at the end of each chapter. Lots of good exercises and an extensive bibliography. CEC

NUMBER THEORY, T**(14-15: 1), S*, P, L*. *Elementary Number Theory.* David M. Burton. Allyn, 1976, ix + 358 pp, $14.95. A superbly written introduction to number theory. The blend between the mathematics and its history is excellent. Includes a good selection of problems, suggestions for further reading, and some useful tables. CEC

NUMBER THEORY, P. *Lecture Notes in Mathematics-475: Répartition Modulo 1.* Ed: G. Rauzy. Springer-Verlag, 1975, 258 pp, $10.30 (P). Proceedings of the colloquium at Marseille-Luminy in June 1974. JAS

LINEAR ALGEBRA, T(17-18: 1), S, P. *Finite Dimensional Multilinear Algebra, Part II.* Marvin Marcus. Pure and Appl. Math., V. 23. Dekker, 1975, xvi + 718 pp, $36.50. Final volume. Contains chapters on Grassmann algebras, Clifford algebras, representation theory, invariants of tensor products, and special topics now undergoing active development. Problems, many with hints. Extensive bibliography. Expensive for reproduction of typescript. JD-B

LINEAR ALGEBRA, T(14-15: 1), *Introduction to Linear Algebra.* Irene and Miklos Farkas. Adam Hilger, 1975, 205 pp. An introduction to the theory of finite-dimensional real and complex vector spaces. Generally conventional in topics and treatment, but has chapter on complex numbers, and almost nothing on canonical forms. JD-B

ALGEBRA, P, *Simplicial Methods and the Interpretation of "Triple" Cohomology.* J. Duskin. Memoirs No. 163. AMS, 1975, v + 135 pp, $7.30 (P). Introduces $K(\pi,n)$-torsors which are analogs of higher dimensional locally trivial fiber bundles. These semisimplicial objects over a category yield a Yoneda-like interpretation for triple cohomologies. JAS

ALGEBRA, P, *Lecture Notes in Mathematics-488: Representations of Algebras.* Ed: V. Dlab, P. Gabriel. Springer-Verlag, 1975, xii + 377 pp, $13.20 (P). Proceedings of the International Conference at Carleton University, Ottawa in September 1974. JAS

ALGEBRA, T(16-18: 1, 2), S*, *Algebra für Naturwissenschaftler und Ingenieure.* Rudolf Lidl. Walter de Gruyter, 1975, 329 pp, 19,80 DM (P). A substantial exposition of the basic definitions and results from modern algebra with exercises and an introduction to many applications. Among the relatively special topics covered are: lattices, automata, character tables, coding theory, and switching algebras. JAS

FINITE MATHEMATICS, T(13: 1, 2), *Applied Mathematics: An Introduction, Mathematical Analysis for Management, Third Edition.* Chris A. Theodore. Irwin, 1975, xv + 611 pp, $14.95. Almost all appended sections in the second edition have been dropped. Coverage has been reduced and simplified (particularly in the calculus) to allow for a wider range of topics (e.g., reintroduction and substantial expansion of fifth part on probability and statistics). LCL

CALCULUS, T*(14: 1), *Vector Calculus.* Jerrold E. Marsden, Anthony J. Tromba. Freeman, 1976, xiv + 449 pp, $14.95. In order of appearance: Vectors in 3- and n-dimensional Euclidean space, differentiation of functions $R^n \to R^m$, vector fields, maxima and minima of functions $R^n \to R$, double and triple integrals, line and surface integrals, theorems of Green, Stokes, Gauss. To follow one-variable calculus. Attractive, with many appealing features. Should be examined carefully for use in the calculus sequence. DFA

REAL ANALYSIS, T(15-16: 1, 2), *The Elements of Real Analysis, Second Edition.* Robert G. Bartle. Wiley, 1976, xv + 480 pp, $16.95. Major changes from the well-known 1964 edition: simplification of introductory material, expansion of calculus in $R^p$. Features extensive "projects" (related series of advanced exercises) and a detailed flow chart permitting flexible use for a variety of course objectives. LAS

COMPLEX ANALYSIS, P, *Lecture Notes in Mathematics-476: Modular Functions of One Variable IV.* Ed: B.J. Birch, W. Kuyk. Springer-Verlag, 1975, 150 pp, $8.60 (P). Continues the proceedings published in Lecture Notes 320, 349, 350 (TR, November 1974) of the conference at Antwerp in July and August 1972. JAS

COMPLEX ANALYSIS, P, *Lecture Notes in Mathematics-482: Fonctions de Plusieurs Variables Complexes II.* Ed: Francois Norguet. Springer-Verlag, 1975, ix + 367 pp, $13.20 (P). A partial text of the presentations between January 1974 and June 1975 in the Séminaire Francois Norguet. JAS

DIFFERENTIAL EQUATIONS, T(18: 1), *The Heat Equation.* D.V. Widder. Pure and Appl. Math., V. 67. Acad Pr, 1975, xiv + 267 pp, $22.50. Concerns chiefly the two-dimensional equation, and assumes no previous acquaintance with the theory of partial differential equations. Develops the integral transform theory needed, and includes many of the author's results on expansion of solutions into infinite series. A challenging text which omits many details. DFA

DIFFERENTIAL EQUATIONS, P, *Lecture Notes in Mathematics-459: Fourier Integral Operators and Partial Differential Equations.* Ed: J. Chazarain. Springer-Verlag, 1975, 372 pp, $13.80 (P). Papers from the colloquium at Nice, France in May 1974. JAS

DIFFERENTIAL EQUATIONS, P, *Lecture Notes in Mathematics-464: Hypoellipticity and Eigenvalue Asymptotics.* Charles Rockland. Springer-Verlag, 1975, 171 pp, $8.60 (P). Examines local solvability and hypoellipticity properties of certain pseudo-differential operators with multiple characteristics, from the standpoint of certain geometric invariants of the operators. DFA

NUMERICAL ANALYSIS, P, *A Bibliography for Finite Elements.* J.R. Whiteman. Acad Pr, 1975, $9.25 (P). Over 2000 items, first arranged alphabetically by author, then with titles arranged by keyword. Computer printout. LAS

NUMERICAL ANALYSIS, S(17-18), P, *Finite Elemente und Differenzenverfahren.* Eds: J. Albrecht, L. Collatz. Int. Ser. Num. Math., V. 28. Birkhauser, 1975, 186 pp, $12. Contains chiefly papers presented at a conference on the methods of finite elements and finite differences, held in 1974 at the Technical University of Clausthal. JD-B

NUMERICAL ANALYSIS, S(17-18), P, *Numerische Methoden bei graphentheoretischen und kombinatorischen Problemen.* Eds: L. Collatz, *et al.* Int. Ser. Num. Math., V. 29. Birkhauser, 1975, 159 pp, $10.50. Texts of some of the lectures given at a conference on numerical methods in graph theory and combinatorics, held at Oberwolfach in December 1974. JD-B

NUMERICAL ANALYSIS, P, *Lecture Notes in Computer Science-6: Matrix Eigensystem Routines--EISPACK Guide.* B.T. Smith, *et al.* Springer-Verlag, 1974, x + 387 pp, $11.50 (P). EISPACK is a highly-developed, widely-available package for obtaining matrix eigenvalues and eigenvectors. This guide includes recommended and possible usages, examples, execution times, validation and documentation. RWN

PROBABILITY, T(18: 2), P. *Discrete-Parameter Martingales.* J. Neveu. Trans: T.P. Speed. Math. Lib., V. 10. North-Holland, 1975, viii + 236 pp, $26.95. The author presumes a knowledge of measure theory and probability theory at about the level of Feller's Volume II. The treatment is terse and abstract with little motivational material. Covers recent developments through Banach spaces of martingales and their associated quadratic variations and maximal functions. Contains an extensive bibliography. TAV

PROBABILITY, T*(17: 1, 2). *Queueing Systems, Volume I: Theory.* Leonard Kleinrock. Wiley, 1975, xvii + 417 pp, $19.95. Unlike many other treatments which are too elementary and limited or far too elegant and precise, the author "treads the boundary inbetween." An unusually lucid treatment for the beginner, the book is a fine choice for a first course. Appendices on transform theory and elementary probability make the book self contained. Coming: Volume II on computer applications. TAV

PROBABILITY, T(16-17: 1), P. *Lecture Notes on Queueing Systems.* Brian Conolly. Wiley, 1975, 176 pp, $14.95 (P). "Although mostly theoretical in content, the frank objective of the course is a practical one--the analysis and evaluation of methods available for the reduction of congestion...". A clearly written introduction to queueing theory consistent with its objectives. The mathematical tools needed are primarily differential equations, Laplace transforms, some complex analysis. Several examples and substantial bibliography. TAV

PROBABILITY, P. *Lecture Notes in Mathematics-480: Ecole d'Eté de Probabilités de Saint-Flour IV-1974.* X.M. Fernique, J.P. Conze, J. Gani. Springer-Verlag, 1975, xi + 293 pp, $11.50 (P).

PROBABILITY, P. *Lecture Notes in Mathematics-472: Probability-Winter School.* Ed: Z. Ciesielski, K. Urbanik, W.A. Woyczyński. Springer-Verlag, 1975, vi + 283 pp, $12.10 (P). Proceedings of the fourth winter school at Karpacz, Poland in January 1975. JAS

PROBABILITY, P. *Limit Theorems of Probability Theory.* Ed: P. Révész. North-Holland, 1975, 420 pp, $39.75. Proceedings of the colloquium held at Keszthely, Hungary in June 1974. JAS

PROBABILITY, T(16-18: 1), P. *Branching Processes with Biological Applications.* P. Jagers. Wiley, 1975, xiii + 268 pp, $28.50. Stresses "the general branching process whose individuals may give birth at random points during life." Avoids multi-type processes. FLW

STATISTICS, T?(13: 1). *A Programmed Text in Statistics.* J. Hine, G.B. Wetherill. Halsted Pr, 1975. *Book One: Summarizing Data,* viii + 95 pp, $5.50 (P); *Book Two: Basic Theory,* x + 109 pp, $6.50 (P); *Book Three: The t-test and $\chi^2$ Goodness of Fit,* viii + 53 pp, $4.50 (P); *Book Four: Tests on Variance and Regression,* viii + 69 pp, $5 (P). In typical programmed "frame" format. Easily understood. Intended for precalculus "service" course. Contains fewer topics than standard texts, e.g., no analysis of variance, no nonparametric test. Unfortunately, the number of exercises seems meager and the total price excessive. TAV

STATISTICS, T(18: 1), P*. *Basic Concepts in Information Theory and Statistics, Axiomatic Foundations and Applications.* A.M. Mathai, P.N. Rathie. Halsted Pr, 1975, x + 137 pp, $9.95. Research results in a theorem-proof format. Includes chapters on the concepts of uncertainty (Shannon's entropy), directed divergence and inaccuracy, related statistical concepts, and other analogous measures. Each chapter concludes with applications and open problems. Good set of references. RSK

STATISTICS, P**. *Multivariate Statistical Methods.* Ed: William R. Atchley, Edwin H. Bryant. Benchmark Papers in Systematic and Evolutionary Biology. Dowden, Hutchinson & Ross, 1975. V. 1, *Among-Groups Covariation,* xv + 464 pp, $27; V. 2, *Within-Groups Covariation,* xv + 436 pp, $26. These volumes contain classical papers integrated with papers illustrating recent applications from biology, particularly systematics and ecology. Sectioned by topic; each section contains an explanatory editor's introduction and a short bibliography. Topics in the first volume are multivariate analysis of variance, discriminant analysis, cluster analysis, and nonmetric scaling. Papers in the second volume deal with principal-component analysis, factor analysis, and multivariate regression and correlation. RSK

STATISTICS, T(16-17: 1, 2). *Introduction to Probability and Statistics, Part II: Statistics.* Narayan C. Giri. Statistics, V. 7. Dekker, 1975, viii + 314 pp, $14.75. Theoretical, classical approach, assuming background of *Part I: Probability* (TR, November 1975). Includes chapters on order statistics, estimation, hypothesis testing, sequential analysis, nonparametric methods, and the general linear hypothesis and analysis of variance, but nothing on correlation or regression analysis. RSK

STATISTICS, T(17-18: 1), P*. *Statistical Prediction Analysis.* J. Aitchison, I.R. Dunsmore. Cambridge U Pr, 1975, xi + 273 pp, $24.50. Concerned with theory and problems in which the past (informative) experiment is independent of the future experiment. First half introduces the concept of the predictive density function and includes decisive, informative, mean coverage tolerance and guaranteed coverage tolerance prediction. Last half describes specific areas of application, including sampling inspection, regulation and optimization, calibration, diagnosis, and treatment allocation. Uses both frequentist and Bayesian approaches. RSK

STATISTICS, T?(1), S. *Statistical Methods for the Earth Scientist, An Introduction.* Roger Till. Halsted Pr, 1974, xiii + 154 pp, $11.75. Elementary treatment, containing many interesting examples of applications of statistics to geology, and many good references to uses of particular statistical techniques by others in the field. However the presentation of material is sparse, confusing and sometimes inaccurate, and the book includes no exercises. It would require much effort on the part of the instructor to make it useable as a text. RSK

STATISTICS, S(15-18), P. *Multivariate Distributions.* Kenneth S Miller. Krieger, 1975, viii + 167 pp, $16.50. Consists of the author's *Multidimensional Gaussian Distributions* (1964) plus a new chapter on distributions associated with the complex normal density function. FLW

STATISTICS, T(15-17: 1-3), S. *Statistics in Research, Third Edition.* Bernard Ostle, Richard W. Mensing. Iowa St U Pr, 1975, xiv + 596 pp, $18.95. "Expanded material on probability and more coverage on the use of computers" in this edition of this rather comprehensive treatment of statistical methods. FLW

COMPUTER SCIENCE, T(15: 1, 2), L. *Applied Algebra for the Computer Sciences.* Arthur Gill. P-H, 1976, xv + 432 pp, $16.50. A mathematically precise introduction to the techniques and applications of modern algebra which are of particular use to the beginning student in computer science. Includes a good selection of problems and lots of good diagrams. A knowledge of higher-level programming is needed for some of the problems. CEC

COMPUTER SCIENCE, T(14-15: 1), S, L. *Data Structures.* Mark Elson. SRA, 1975, xii + 307 pp, $13.95. Data structures: vectors, arrays, stacks, queues, linked lists, trees, graphs, PL/1-like structures, plexes, strings, etc. Application to expression evaluation, storage management, sorting and searching. Includes the two chapters on LISP and SNOBOL from the author's *Concepts of Programming Languages.* A good base for a course with few prerequisites. RWN

COMPUTER SCIENCE, T(13-14: 1), S, L. *Computing with Mini Computers.* Fred Gruenberger, David Babcock. Wiley, 1973, xv + 288 pp, $13.25. Introductory text on computing centered around minicomputers and, in particular, the Varian 620/L computer. Begins with machine language, followed by assembly language, and then FORTRAN and BASIC. Discusses flowcharts, loops, addressing, subroutines, etc. Contains information specific to the minis throughout. Exercises and selected answers. Index. RJA

COMPUTER SCIENCE, T(15-17: 1), S, L. *The Structure and Design of Programming Languages.* John E. Nicholls. A-W, 1975, xx + 572 pp, $17.95. Text on design of high-level procedural programming languages for applications and systems programmers. Presents features common to most high-level languages: implementation, program structure and representation, data structures, expressions and assignment, control structures, subroutines, etc. Chapter bibliographies. Exercises. Appendix on languages used in text. Index. RJA

COMPUTER SCIENCE, T(16-18: 1), S, P, L. *Sorting and Sort Systems.* Harold Lorin. A-W, 1975, xv + 459 pp, $16.95. Discusses internal and external sorts. For the former, the discussion proceeds from example to properties and variations of the method, attempting to provide a great amount of detail. For the latter, a more general approach is used. Several appendices. Bibliography. Index. RJA

COMPUTER SCIENCE, T(16-18: 1), S, P, L. *An Introduction to Database Systems.* C.J. Date. A-W, 1975, xvii + 366 pp, $15.95. Discusses three approaches to design of a database system: relational, hierarchical, and network. The bias is for the first, although implementations exist only for the second and third methods. Exercises and selected answers. Chapter bibliographies. Index. RJA

COMPUTER SCIENCE, T(15-17: 1), S, L. *The Program Development Process, Part 1: The Individual Programmer.* J.D. Aron. A-W, 1974, xxii + 264 pp, $12.95. A text on the individual programmer's role in systems programming. Covers procedural aspects of programming: problem analysis, planning, program design, coding, debugging, documentation, delivery. Index. RJA

COMPUTER SCIENCE, T(16-18: 1), S, P, L. *Recursive Programming Techniques.* William H. Burge. A-W, 1975, xv + 277 pp, $15.95. Develops and uses a language based on the lambda calculus, much the same way as LISP. Main thesis of book is that expressions denoting end results, as opposed to instruction formats, simplify and improve the task of programming. Includes material on program structure, data structures, parsing, and sorting. Chapter bibliographies. Index. RJA

COMPUTER SCIENCE, T(15-17: 1), L. *Artificial Intelligence.* Earl B. Hunt. Acad Pr, 1975, xii + 468 pp, $29. A systematic survey of issues from statistics, logic, mathematics and psychology that bear on artificial intelligence. A "very conservative" book, it focuses on major unsolved conceptual problems of machine comprehension and perception. LAS

COMPUTER SCIENCE, L. *Computer Graphics: 118 Computer-Generated Designs.* Melvin L. Prueitt. Dover, 1975, viii + 69 pp, $3 (P). Sample plots from a versatile program PICTURE that draws in perspective, removes hidden lines, rotates, zooms and even provides color. LAS

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science-34: GI-5. Jahrestagung.* J. Mühlbacher. Springer-Verlag, 1975, x + 755 pp, $24.20 (P). Proceedings of the fifth annual meeting of the Gesellschaft für Informatik at Dortmund in October 1975. JAS

COMPUTER SCIENCE, T(14-15: 1), S. *Computer Concepts and Assembler Programming: 360/370 Systems.* Richard H. Stark, Donald W. Dearholt. Acad Pr, 1975, xii + 397 pp, $11.95. Discusses, somewhat broadly, such concepts as computer evolution, machine arithmetic, character manipulation, assembly and execution. Uses a simplified 360, named BAREBONZ, to introduce assembly languages. A good combination of general concepts and necessary details. RWN

COMPUTER SCIENCE, T(16-17), P. *Pattern Recognition Principles.* Julius T. Tou, Rafael C. Gonzalez. Appl. Math. and Comp., No. 7. A-W, 1974, xxii + 377 pp, $19.50. Intended as a broad introduction and general reference. Includes deterministic, statistical and syntactic methods; distance and likelihood functions; trainable classifiers; pattern preprocessing and feature extraction. Exercises. References. Prerequisites: programming, statistics, linear algebra. RWN

APPLICATIONS (ECONOMICS), T(16-17), S, L. *General Equilibrium.* Michael Allingham. Halsted Pr, 1975, ix + 113 pp, $19.95. A concise exposition of "the essence of economic science" based on interpretable axioms, expounded in propositions expressed in their "simplest meaningful versions." Includes competitive and cooperative equilibrium, stability, and an appendix on social choice. An excellent primer on mathematical economics. LAS

APPLICATIONS (ECONOMICS), P. *Lecture Notes in Economics and Mathematical Systems-111: Variable Structure Systems with Application to Economics and Biology.* Ed: A. Ruberti, R.R. Mohler. Springer-Verlag, 1975, vi + 321 pp, $12.90 (P). Proceedings of the second US-Italy seminar in May 1974. JAS

APPLICATIONS (ENGINEERING), T(18: 1), P. *Perturbation Methods in Fluid Mechanics.* Milton Van Dyke. Parabolic Pr (P.O. Box 3032, Stanford, CA 94305), 1975, xiv + 271 pp, $7. Concerns chiefly singular perturbation problems and the method of matched asymptotic expansions and of strained coordinates. Updates 1964 edition by adding 34 pages of notes and 150 recent references. Published by the author in order to retain low 1964 price. DFA

APPLICATIONS (MECHANICS), T(18: 2), P. *Mathematical Theory of Dislocations and Fracture.* R.W. Lardner. U of Toronto Pr, 1974, xi + 363 pp, $20. Contains brief historical review. Discusses physical significance of dislocations. Studies those aspects of dislocation theory which are closely related to elasticity theory, to modern continuum mechanics, and to the theory of cracks and fracture. Uses some advanced mathematical tools, such as tensor calculus, contour integration, Fourier transforms, and Green's functions. I-CH

APPLICATIONS (MECHANICS), P. *Lecture Notes in Mathematics-461: Computational Mechanics.* Ed: J.T. Oden. Springer-Verlag, 1975, vii + 328 pp, $12.90 (P). Invited lectures from the International Conference on Computational Methods in Nonlinear Mechanics at Austin, Texas in 1974. JAS

APPLICATIONS (OPERATIONS RESEARCH), T(15-17: 2), S*, L. *Computer Applications in Operations Analysis.* Bennet P. Lientz. P-H, 1975, xiv + 289 pp, $12.95. Provides valuable discussions of major fields of operations research (mathematical programming, network analysis, graph theory, simulation, queueing theory) which utilize computers. Supplies information on the existence and sources of running program codes. Emphasizes the how and the methods rather than the why or the proofs of results. An introductory book with unique features. I-CH

APPLICATIONS (PHYSICS), P. *Rheometry.* K. Walters. Chapman & Hall, 1975, x + 278 pp, $32. Theoretical and experimental rheology for persons involved in measurements. By a mathematician, but assumes little mathematical background. Cites over 300 recent research papers. DFA

APPLICATIONS (PHYSICS), T(17-18: 1). *Statistical Mechanics.* Ryuzo Abe. Trans: Yasushi Takahashi. U of Tokyo Pr (U.S. Distr: Intern. Scholarly Book Serv.), 1975, 178 pp, $10.50.

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-32: Particles, Quantum Fields and Statistical Mechanics.* Ed: M. Alexanian, A. Zepeda. Springer-Verlag, 1975, 132 pp, $7.80 (P). Proceedings of the 1973 summer institute in theoretical physics held at the Centro de Investigacion y de Estudios Avanzados del IPN in Mexico City. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-33: Classical and Quantum Mechanical Aspects of Heavy Ion Collisions.* Ed: H.L. Harney, *et al.* Springer-Verlag, 1975, vii + 311 pp, $12.10 (P). Symposium papers from the Max-Planck-Institut, October 1974. LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Mathematics-449: Hyperfunctions and Theoretical Physics.* Ed: F. Pham. Springer-Verlag, 1975, 218 pp, $9.90 (P). Proceedings of a colloquium at Nice in 1972-73 organized on the occasion of the visit by M. Sato and two of his students, T. Kawai and M. Kashiwara. The theory of hyperfunctions is used to illuminate certain aspects of field theory in physics. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-35: Proceedings of the Fourth International Conference on Numerical Methods in Fluid Dynamics.* Ed: Robert D. Richtmyer. Springer-Verlag, 1975, v + 457 pp, $16 (P). Proceedings of the conference held at the University of Colorado in June 1974.JAS

APPLICATIONS (PHYSICS), P. *Lectures on Turbulence Theory.* Alexandre Joel Chorin. Publish or Perish, 1975, iii + 159 pp, $7.50 (P). Derives the moving average representation of a random field as a special case of a general representation theorem of which the more usual spectral representation is another special case. Also describes attempts to construct a dynamical theory. Probabilistic background provided. DFA

APPLICATIONS (PHYSICS), T(17: 1), P. *Multiple Scattering Processes, Inverse and Direct.* Harriet H. Kagiwada, Robert Kalaba, Sueo Ueno. Appl. Math. and Comp., No. 8. A-W, 1975, xvi + 336 pp, $13.50 (P); $21.50. Formulates multiple scattering problems as initial value problems. Enables one to make thorough parameter studies with numerical computations which are accurate and stable, as shown by numerous tables and graphs. Derives the fundamental equations both physically and analytically. Methods and results useful in radiative transfer and transport theory. I-CH

APPLICATIONS (PHYSICS), P. *Interdisciplinary Mathematics, V. VII-IX.* Robert Hermann (18 Gibbs St, Brookline, MA 02146), 1974. *V. VII: Spinors, Clifford and Cayley Algebras,* viii + 276 pp, $15 (P); *V. VIII: Linear Systems Theory and Introductory Algebraic Geometry,* ix + 282 pp, $18 (P); *V. IX; Geometric Structures of Systems--Control Theory and Physics,* xi + 442 pp, $24 (P).

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Ih-Ching Hsu, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; R.W. Nau, Carleton; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

*Belmont College*: Associate Professors W. D. Driskill, O. B. McCowan, and R. H. Medley, Jr., have been promoted to Professors.

*Case Western Reserve University*: Dr. D. A. Singer, Cornell University, has been appointed Assistant Professor; Dr. T. R. Butts, Michigan State University, has been appointed Executive Officer and Senior Instructor; Drs. D. A. Smith, Duke University, and N. L. Max, Carnegie-Mellon University, hold visiting appointments for the year 1975–76.

*Cleveland State University*: Dr. B. M. Scott, University of Wisconsin, Madison, has been appointed Assistant Professor; Associate Professor A. J. Silberger, University of Massachusetts, has been appointed Associate Professor.

*College of Wooster*: Associate Professor D. G. Beane has been promoted to Professor; Assistant Professor P. M. Brown, Gonzaga University, has been appointed Assistant Professor.

*Fort Lewis College*: Associate Professor L. S. Johnson, Chairman of the Department of Mathematics, has been appointed Director of the School of Arts and Sciences; Assistant Professor G. W. Grefsrud has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics; Assistant Professor R. A. Gibbs has been promoted to Associate Professor.

*Georgia Institute of Technology*: Dr. R. G. Bartle, University of Illinois, has been appointed Professor; Dr. M. J. Christensen, Wayne State University, has been appointed Assistant Professor; Mr. B. R. Caine, Emory University, has been appointed Instructor; Dr. R. A. Duke has been promoted to Associate Professor.

*Oakland University*: Dr. B. R. Sands, Oregon State University, has been appointed Assistant Professor; Professor L. R. Bragg has been appointed Chairman of the Department of Mathematical Sciences; Assistant Professor L. J. Nachman has been promoted to Associate Professor.

*The Citadel*: Assistant Professor I. S. Metts, Jr., has been promoted to Associate Professor and appointed Acting Head of the Department of Mathematics; Professor George E. Reves, formerly Head of the Department of Mathematics, retired on June 30, 1975, with the title of Professor Emeritus.

*University of South Carolina, Aiken*: Dr. David Koster, University of Wisconsin, Madison, has been appointed Assistant Professor; Professor Ralph Surasky retired in July 1975 with the title of Professor Emeritus.

*University of Toledo*: Dr. Carl Looney, Fort Hays Kansas State College, has been appointed Assistant Professor; Assistant Professors Junior Stein and Stuart Steinberg have been promoted to Associate Professors; Associate Professors H. L. Bentley and Martin Kummer have been promoted to Professors; Associate Professor Violet Davis has retired with the title of Emeritus Associate Professor.

*Washington and Lee University*: Assistant Professor R. L. Wilson, Jr., University of Wisconsin, Madison, has been appointed Associate Professor; Associate Professor R. S. Johnson has been promoted to Professor.

Assistant Professor L. J. Alex, State University College at Oneonta, has been promoted to Associate Professor.

Professor Arthur Brown, Queens College, retired on August 31, 1975, with the title of Professor Emeritus.

Dr. James Caristi, University of Iowa, has been appointed Assistant Professor at Texas Lutheran College.

Dr. W. F. Denny, II, University of Oklahoma, has been appointed Assistant Professor of Mathematics and Computing Science at McNeese State University.

Dr. R. A. Dobyns has been appointed Academic Dean at Carson-Newman College.

Reverend J. C. Friedell, Loras College, has been promoted from Associate Professor to Professor.

Professor Michael Golomb, Purdue University, retired after thirty-three years on its faculty; he is spending the first semester of 1975–76 as a Visiting Professor of Applied Mathematics at Brown University.

Assistant Professor R. M. Gravina, University of Lowell, has been promoted to Associate Professor.

Professor Alton C. Grimes, Mississippi State University, retired on August 3, 1975, with the title of Professor Emeritus.

Professor John P. Hoyt, Indiana University of Pennsylvania, retired on May 23, 1975, after more than forty-seven (teaching) years with the title of Professor Emeritus; he was also given the Outstanding Teacher Award at the University for 1974–75. In 1967, Professor Hoyt retired from the U. S. Naval Academy.

Ms. R. Ibele, University of North Carolina at Chapel Hill, has been appointed Visiting Instructor at Virginia Wesleyan College.

Assistant Professor T. R. Lucas, University of North Carolina at Charlotte, has been promoted to Associate Professor.

Dr. Robert McGuigan has been appointed Chairman of the Mathematics Department at Westfield State College.

Professor Lyle E. Mehlenbacher, University of Detroit, retired on July 1, 1975, from his positions of Professor of Mathematics, Associate Dean of the Graduate School, and Director of Grants Administration, with the title of Professor Emeritus of Mathematics.

Professor Thomas Mickewich, Lake Superior State College, received the Distinguished Teacher Award for 1974–75.

Professor J. Winfield Poole, St. Thomas University, has been appointed Acting Vice-President (Academic).

Dr. Michael Poole, Auburn University, has been appointed Assistant Professor at Macon Junior College.

Associate Professor Dave Robison, DeVry Institute of Technology, has been promoted to Academic Dean.

Dr. Marcus Saegrove, University of Guam, has been appointed Assistant Professor at Viterbo College.

Professor Arthur Schlissel has been appointed Chairman of the Mathematics Department at John Jay College of Criminal Justice, CUNY.

Mr. Gary Seldomridge, West Virginia University, has been appointed Instructor at Potomac State College of West Virginia University.

Instructor John Shaw, University of Wisconsin, Fond du Lac, has been promoted to Assistant Professor.

Mr. Mark Smith, University of Illinois, has been appointed Assistant Professor at Lake Forest College.

Assistant Professor Alexia Sontag, Mt. Holyoke College, has been appointed Assistant Professor at Wellesley College.

Associate Professor M. A. Spikell, Lesley College, has been appointed Visiting Associate Professor at Boston University.

Professor Rothwell Stephens, Knox College, has retired after forty-four years with the title of Distinguished Service Professor Emeritus.

Associate Professor L. R. Tanner, Jamestown College, has been promoted to Professor.

Dr. Ellen Torrance, Kansas State University, has accepted a position as actuarial student and assistant to the President at Cologne Life Reinsurance Company in Stamford, Connecticut.

Dr. S. A. Wiitala, Dartmouth College, has been appointed Assistant Professor at Nebraska Wesleyan University.

## FELLOWSHIP AND RESEARCH OPPORTUNITIES IN THE MATHEMATICAL SCIENCES

In its annual brochure on *Fellowship and Research Opportunities in the Mathematical Sciences*, the Office of Mathematical Sciences of the Assembly of Mathematical and Physical Sciences, National Research Council, calls attention to a number of fellowships and other kinds of support for research in the mathematical sciences at both the predoctoral and postdoctoral levels to be awarded during the year 1975–76. Copies of this brochure are available from: Office of Mathematical Sciences/AMPS, National Research Council, 2101 Constitution Avenue, N. W., Washington, D. C. 20418.

## HOUSTON JOURNAL OF MATHEMATICS

The *Houston Journal of Mathematics* is now accepting papers for publication. All manuscripts will be refereed and may be submitted in duplicate to any of the following members of the Editors Board: R H Bing, David G. Bourgin, Gordon G. Johnson (Managing Editor), Bjarni Jonsson, Andrzej Lelek, John S. Mac Nerney, J. W. Neuberger, B. H. Neumann, and Jurgen Schmidt.

The Journal is being supported in part by the American Mathematical Society.

For further information write to Gordon G. Johnson, Houston Journal of Mathematics, Department of Mathematics, University of Houston, Houston, Texas 77004.

## GROUP TRAVEL TO INTERNATIONAL MATH ED CONGRESS

A group travel plan at reduced rates is being arranged for MAA members who wish to attend the Third International Congress on Mathematics Education, to be held August 16–21 in Karlsruhe, West Germany. The travel arrangements will be sponsored jointly by the MAA and the National Council of Teachers of Mathematics. The Pan American flight for 180 passengers will cost approximately $380 a person for the round trip, leaving Kennedy International Airport July 31, 1976, for Frankfurt and returning August 22 from Frankfurt to Kennedy. This plan allows for a week at Karlsruhe and two weeks for travel prior to the Congress. Further details of a tour option and other travel and housing arrangements are being developed. For information write to the MAA Washington Office.

## SYMPOSIUM ON APPLICATIONS OF STATISTICS

A Symposium on Applications of Statistics is planned to be held at Dayton, Ohio, during the period June 14–18, 1976. This symposium is sponsored by the Air Force Flight Dynamics Laboratory. Several distinguished scientists who have made significant contributions to the applications of statistics are expected to present invited papers on a very broad spectrum of topics. The proceedings of the invited paper sessions are expected to be published. Attendance at the symposium is open to anyone interested. Persons interested in presenting contributed papers should send abstracts of their papers (not exceeding 200 words) to P. R. Krishnaiah as soon as possible and not later than March 31, 1976. Further details regarding the symposium may be obtained from P. R. Krishnaiah, Air Force Flight Dynamics Laboratory, AFFDL/FYS, Bldg. 125, Wright-Patterson AFB, Ohio 45433.

## WORKSHOPS SPONSORED BY THE MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA SECTION

The MD–DC–VA section of the MAA will sponsor two five-day workshops, June 7–11 and 14–18, 1976, at Salisbury State College, Maryland. The topic of the first will be "Models: Deterministic and Stochastic," and the topic of the second will be "Calculus, Computing, and Constructive Mathematics." The first is designed for teachers of statistics and finite mathematics, and will be conducted by Dr. D. C. Cathcart. He worked with Professors Maki and Thompson and will base the workshop on their text "Mathematical Models and Applications." The second is designed for teachers of calculus, and will be conducted by Dr. Harry Cheng. He has worked and published with Errett Bishop, the founder of Constructive Mathematics. There will be Section-sponsored lectures on topics of special interest to teachers of undergraduates. The total cost (including room and board) for each workshop will be $115.00. For further information contact Dr. B. A. Fusaro, S.S.C., Salisbury, MD 21801.

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## CALENDAR OF FUTURE MEETINGS

Fifty-sixth Summer Meeting, University of Toronto, August 26–28, 1976.

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, Florida A&M University, Tallahassee, March 5–6, 1976.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Franklin College, Franklin, May 1, 1976.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, Fort Hays Kansas State College, Hays, probably March 26–27, 1976.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 mths. bef. mtg.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Bowie State College, Bowie, Maryland, April 24, 1976.

METROPOLITAN NEW YORK, Hofstra University, Hempstead, April 25, 1976.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY, Montclair State College, Upper Montclair, May 8, 1976.

NORTH CENTRAL, St.Cloud State University, St.Cloud, Minnesota, April 30–May 1, 1976.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Youngstown State University, Youngstown, May 7–8, 1976.

OKLAHOMA-ARKANSAS, Hendrix College, Conway, Arkansas, March 26–27, 1976.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, April 30–May 1, 1976.

SEAWAY, College of St. Rose, Albany, April 30–May 1, 1976.

SOUTHEASTERN, Central Piedmont Community College, Charlotte, N. Carolina, March 26–27, 1976.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Texas A&M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), May 7–8, 1976.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 21–26, 1977.

AMERICAN MATHEMATICAL SOCIETY, University of Toronto, August 24–27, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, University of Nevada, Reno, April 23–24, 1976.

ASSOCIATION FOR WOMEN IN MATHEMATICS

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MU ALPHA THETA, West Chester State College, West Chester, Pennsylvania, August 8–11, 1976.

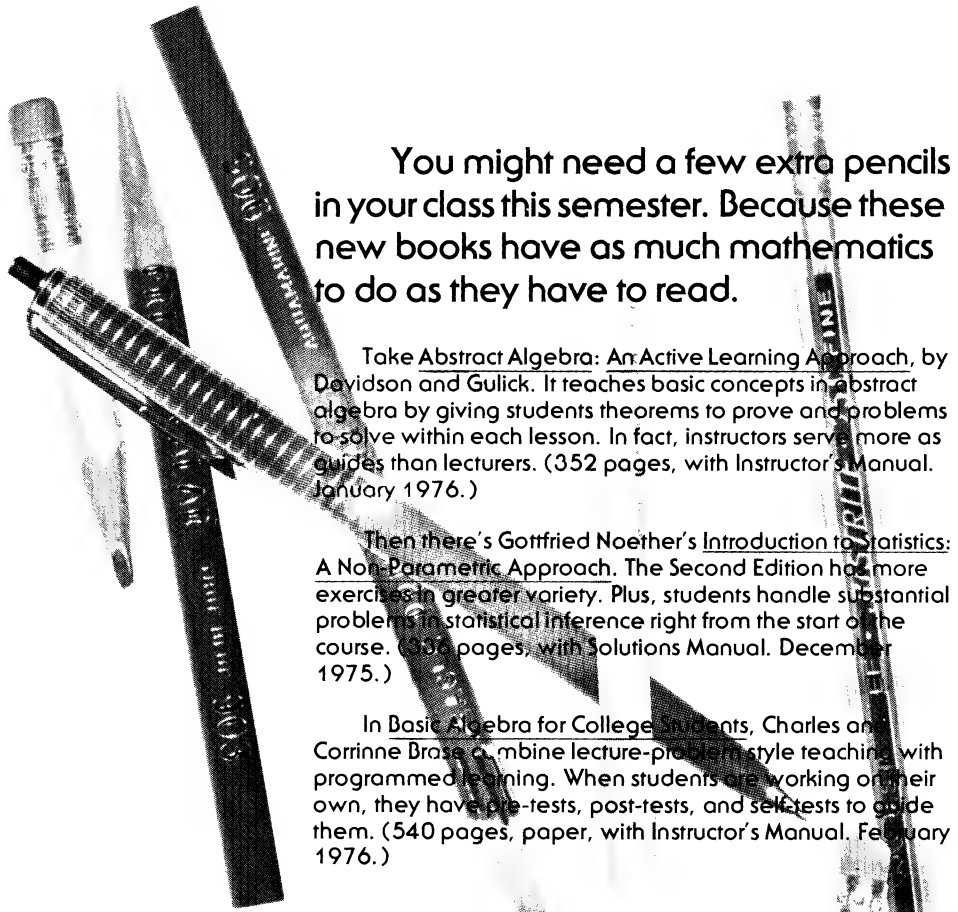
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, March 31–April 2, 1976.

PI MU EPSILON

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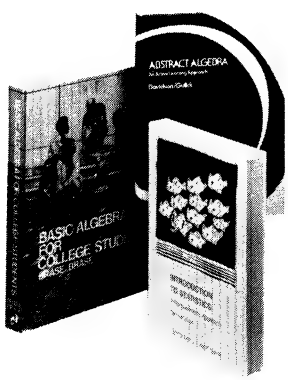
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| CALCULUS IN HIGHER DIMENSIONS | The amount of analytic geometry has been reduced; section on Lagrange multipliers has been simplified; least squares and differentials have been omitted. |
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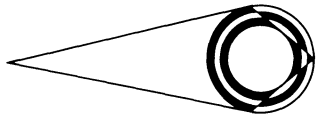
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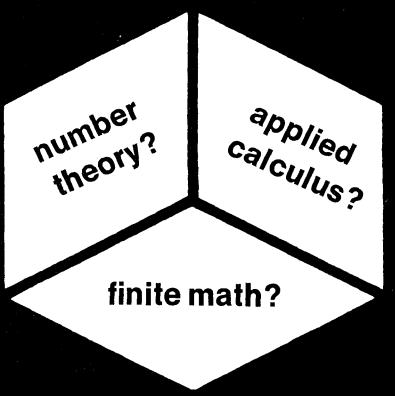
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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 4

CODEN: AMMYAE

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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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ALEX ROSENBERG, *Editor-in-Chief*

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# OPTIMAL CONTROL THEORY

LEONARD D. BERKOVITZ

**1. Introduction.** The development of the mathematical theory of optimal control began in the early to mid 1950's, partially in response to problems in various branches of engineering and economics. Despite its modern origins, optimal control theory, from a mathematical point of view, is a variant of one of the oldest and most important subfields in mathematics—the calculus of variations. We shall sketch some of the important aspects of optimal control theory, the story of their development, and their relationship to the calculus of variations and to other branches of mathematics.

Besides informing the reader about certain aspects of optimal control theory, we hope to impress him with yet another example of the fruitful interaction between developments in pure mathematics and applications and between concrete special problems and general theories. The great historical examples of such interactions and their contributions to the vigorous growth of mathematics are well known. The story of the development of optimal control theory in the last twenty years is a small and yet unfinished chapter in the history of such interactions. It illustrates how meaningful applied problems can lead to new developments in mathematics and to a revitalization of old areas of research. It also illustrates the utility of existing mathematics, which may have been developed in other contexts, for the solution of applied problems.

In order to keep the length of this paper within reason we must limit our discussion to a few selected topics. We trust, however, that these will adequately illustrate and document the assertions we made above.

**2. The servo problem.** In the early 1950's electrical engineers, influenced by a paper of McDonald [21], became interested in improving the performance of servomechanisms with respect to time of response. A typical problem is the following.

A control surface on an aircraft is to be kept at rest at a fixed position. A wind gust displaces the surface from the desired position. It is assumed that if nothing were done, then the control surface would behave as a damped harmonic oscillator. Thus if $\theta$ measures the deviation from the desired position, which we take to be zero, then the free motion of the surface satisfies the differential equation

$$\theta'' + a\theta' + \omega^2\theta = 0,$$

with initial conditions $\theta(0) = \theta_0$ and $\theta'(0) = \theta_0'$. Here $\theta_0$ is the displacement of the surface resulting

from the wind gust and $\theta_0'$ is the velocity imparted to the surface by the gust. On an aircraft the damped oscillation of a control surface cannot be permitted. Therefore, a servomechanism is to apply a restoring torque so as to bring the system to rest at the desired position in minimum time. The equation of motion of the control surface then becomes

$$\theta'' + a\theta' + \omega^2\theta = u(t) \quad \theta(0) = \theta_0 \quad \theta'(0) = \theta_0', \tag{2.1}$$

where $u(t)$ represents the restoring torque at time $t$. Since the voltages available are not unlimited, the magnitude of the torque is constrained by an inequality $|u(t)| \leqq c$, where $c$ is a constant. The problem is to apply the torque as a function of time in such a way that the system will be brought to $\theta = 0$, $\theta' = 0$ in minimum time.

It is clear that if $\theta_0 > 0$ and $\theta_0' > 0$ then the torque should initially be directed in the direction of negative $\theta$ and should have the largest possible magnitude. Thus $u(t) = -c$, initially. It is also clear that if the torque $-c$ is applied for too long a period then we shall overshoot the desired terminal condition of $\theta = 0$, $\theta' = 0$. Thus it appears that at some point there should be a torque reversal to $+c$ in order to brake the system. The question arises whether this is indeed so, and if it is, where should the switch occur. Or, is it better to remove the torque at some point, allow a small overshoot, and then apply $+c$. Still another possibility is to allow the system to overshoot under the influence of $-c$, then apply $+c$, and finally brake with $-c$. In this vein, one could ask whether a sequence $-c, +c, -c, \cdots, \pm c$ of $n$ steps is best, and if so, what is $n$, and where do the switches occur. Related to these questions is the question of what to do if the servo motor starts to act at $\theta_0 > 0$ and $\theta_0' < 0$?

In the engineering literature of the period it was always assumed—on the basis of intuition—that the optimal mode of operation was one in which the restoring torque only took on the values $+c$ and $-c$, and various incomplete analyses were made to determine the appropriate sequence of $\pm c$ to be used. A restoring torque, or control, $u$ that only takes on the extreme values $u(t) = \pm c$ is called a "bang-bang" control. A major contribution to the problem was made in 1952 by Bushaw in his Princeton doctoral thesis, later published in [8]. Bushaw also assumed that the bang-bang mode of operation was indeed optimal and then proceeded to determine for what values of $(\theta, \theta')$ one used $+c$ and for what values of $(\theta, \theta')$ one used $-c$. His arguments were elementary and non-variational. He also determined the optimal trajectories of the system in the $(\theta, \theta')$-plane.

**3. The linear time-optimal problem.** The servo problem of the preceding section can be rewritten in the following form. Let $x^1 = \theta$ and let $x^2 = \theta'$. The differential equation (2.1) is then equivalent to the system

$$\frac{dx^1}{dt} = x^2 \qquad x^1(0) = \theta_0$$

$$\frac{dx^2}{dt} = -ax^2 - \omega^2 x^1 + u, \qquad x^2(0) = \theta_0',$$

where $u$ is a function to be chosen from a specified class of functions and is required to satisfy the constraint $|u(t)| \leqq c$. The problem is to choose $u$ so as to minimize

$$\int_0^{t_1} 1\,dt,$$

where $t_1$ is the first time such that $x^1(t) = 0$ and $x^2(t) = 0$, provided such a first time exists.

The servo problem in the formulation just given is a special case of the following problem. The state of a system at time $t$ is described by an $n$-vector $x(t) = (x^1(t), \cdots, x^n(t))$. The system evolves according to a system of linear differential equations that we write in vector matrix form as follows:

$$dx/dt = A(t)x + B(t)u(t) \qquad x(t_0) = x_0.$$

Here $A$ is an $n \times n$ matrix whose entries are functions of $t$, $B$ is an $n \times m$ matrix whose entries are

functions of $t$, $u = (u^1, \cdots, u^m)$ is an $m$-vector whose components are functions of $t$ that satisfy $|u^i(t)| \leq c^i$, where $c^i$, $i = 1, \cdots, m$ are constants, $t_0$ is the initial time, and $x_0$ is the initial state. It is required to choose $u$ so as to bring the system to the origin in minimum time. In other words, we are to choose a $u$ that minimizes $t_1$, where $t_1$ is the first time such that $x(t) = 0$. The functions $u$ that we choose are called controls. A control $u^*$ that minimizes $t_1$ is called an optimal control.

With the problem formulated this way, Bellman, Glicksberg, and Gross [4] showed, using rather elementary functional analysis, that for special matrices $A$ and $B$ an optimal control does exist and is indeed bang-bang. Unfortunately, the assumptions made about the matrix $A$ in [4] were not fulfilled in many examples of interest, including the original servo problem. Nevertheless, this was the first general formulation of the linear problem and the methods introduced in this paper were expanded and exploited by others. Later, LaSalle [17] showed that for linear systems more general than those treated by Bellman, Glicksberg, and Gross it is true that an optimal control exists and that it is bang-bang. LaSalle also showed, under appropriate hypotheses, that if the system can be driven to the origin by a control $u$, then there is a bang-bang control that will drive the system to the origin in the same time. This is the so-called, "bang-bang principle." Note that this principle holds for any control, optimal or not.

The validity of the bang-bang principle is of importance in engineering for the following reason. One need only use "contact" or "on-off" servomechanisms, since one need only consider controls that take on the values $u^i(t) = \pm c^i$, $i = 1, \cdots, m$. Such servomechanisms are easier to build and operate than those in which all values in the intervals $[-c^i, c^i]$ are allowed.

The bang-bang principle can be established in varying degrees of generality by various arguments. As with other mathematical theorems, the proof has undergone a series of refinements and improvements. At present it is seen to follow with a minimum of technical detail from the Krein-Milman theorem and the characterization of the extreme points of convex sets in a certain function space. See [6].

In the Soviet Union, R. V. Gamkrelidze [11] and his collaborators also attacked the time optimal problem. Their methods were different, and we shall discuss their work later.

**4. The nonlinear optimal control problem.** The linear time optimal problem described in the preceding section is a special case of the following general nonlinear problem. The state of a system at time $t$ is described by a point or vector

$$x(t) = (x^1(t), \cdots, x^n(t))$$

in $n$-dimensional euclidean space, $n \geqq 1$. Initially, at time $t_0$, the state of the system is

$$x(t_0) = x_0 = (x_0^1, \cdots, x_0^n).$$

More generally, we can require that at the initial time $t_0$, the initial state $x_0$ is such that the point $(t_0, x_0)$ belongs to some pre-assigned set $\mathcal{T}_0$ in $(t, x)$-space. The state of the system varies with time according to the system of differential equations

$$\frac{dx^i}{dt} = f^i(t, x, z) \qquad x^i(t_0) = x_0^i \qquad i = 1, \cdots, n, \tag{4.1}$$

where $z = (z^1, \cdots, z^m)$ is a vector in real euclidean space $E^m$ and the functions $f^i$ are real valued continuous functions of the variables $(t, x, z)$.

By the 'system varying according to (4.1)' we mean the following. A function $u$ with values in $m$-dimensional euclidean space is chosen from some prescribed class $\mathcal{C}$ of functions, such as the piecewise continuous functions, the bounded measurable functions, the functions in $L_p$, etc. When the substitution $z = u(t)$ is made in the right hand side of (4.1) we obtain a system of ordinary differential equations:

$$\frac{dx^i}{dt} = f^i(t, x, u(t)) \qquad i = 1, \cdots, n. \tag{4.2}$$

For each $u$ in $\mathscr{C}$ it is assumed that there exists a point $(t_0, x_0)$ in $\mathscr{T}_0$ and a function $\phi = (\phi^1, \cdots, \phi^n)$ defined on an interval $[t_0, t_2]$ with values in $E^n$ such that (4.2) is satisfied. That is, we require that

$$(\phi^i)'(t) = \frac{d\phi^i}{dt} = f^i(t, \phi(t), u(t)), \quad \phi^i(t_0) = x_0^i \quad i = 1, \cdots, n.$$

The function $\phi$ describes the evolution of the system with time and is called a **trajectory**. The function $u$ is called a **control**.

The control $u$ is further required to be such that at some time $t_1$, where $t_0 < t_1 \leqq t_2$, the point $(t_1, \phi(t_1))$ belongs to a preassigned set $\mathscr{T}_1$ and for $t_0 \leqq t < t_1$ the points $(t, \phi(t))$ do not belong to $\mathscr{T}_1$. The set $\mathscr{T}_1$ is called the terminal set for the problem. In the servo problem the set $\mathscr{T}_0$ is the point $(t_0, x_0^1, x_0^2)$ and the set $\mathscr{T}_1$ is defined by:

$$\mathscr{T}_1 = \{(t_1, x_1^1, x_1^2): x_1^1 = 0, x_1^2 = 0\}.$$

The discussion in the preceding paragraphs is sometimes summarized in less precise but somewhat more graphic language by the statement that the controls $u$ are required to transfer the system from an initial state $x_0$ at time $t_0$ to a terminal state $x_1$ at time $t_1$, where $(t_0, x_0) \in \mathscr{T}_0$ and $(t_1, x_1) \in \mathscr{T}_1$. Note that to a given $u$ in $\mathscr{C}$ there will generally correspond more than one trajectory $\phi$. This results from different choices of initial points $(t_0, x_0)$ in $\mathscr{T}_0$ or from non-uniqueness of solutions of (4.2) if no assumptions are made to guarantee the uniqueness of solutions of (4.2).

It is often further required that a control $u$ in $\mathscr{C}$ and a corresponding solution $\phi$ must satisfy a system of inequality constraints

$$R^i(t, \phi(t), u(t)) \geqq 0 \qquad i = 1, 2, \cdots, r, \tag{4.3}$$

for all $t_0 \leqq t \leqq t_1$, where the functions $R^1, \cdots, R^r$ are given functions of $(t, x, z)$. In the linear time optimal problem the constraints are

$$u^i(t) + c^i \geqq 0$$

$$c^i - u^i(t) \geqq 0 \qquad i = 1, \cdots, m.$$

More generally, it is required that $u(t) \in \Omega(t, \phi(t))$, where $\Omega$ is a mapping that assigns to each point $(t, x)$ a subset $\Omega(t, x)$ of $E^m$.

Let $f^0$ be a real valued function of $(t, x, z)$ and let $g$ be a real valued function defined on $\mathscr{T}_0 \times \mathscr{T}_1$. For each control $u$ in $\mathscr{C}$ and corresponding trajectory $\phi$ we define a **payoff** as follows:

$$J(\phi, u) = g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(s, \phi(s), u(s))ds. \tag{4.4}$$

It is tacitly assumed here that the function $f^0$ and the class $\mathscr{C}$ are such that the integral in (4.4) exists.

In the servo problem we can write $J$ in two equivalent ways. We can either set $f^0 \equiv 1$ and $g \equiv 0$ or $g(t_0, x_0, t_1, x_1) = t_1$ and $f^0 \equiv 0$.

We define an **admissible control** $u$ to be a function $u$ in $\mathscr{C}$ such that a corresponding solution $\phi$ of (4.2) exists with the property that (4.3) holds, the integral in (4.4) is defined, and

$$(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathscr{T}_0 \times \mathscr{T}_1. \tag{4.5}$$

We call the solution $\phi$ an **admissible trajectory,** and we call the pair $(\phi, u)$ an **admissible pair.** The nonlinear optimal control problem that we consider is the following.

PROBLEM: *Find an admissible pair* $(\phi^*, u^*)$ *such that*

$$J(\phi^*, u^*) \leqq J(\phi, u)$$

*for all admissible pairs* $(\phi, u)$. *That is, minimize* $J$ *over the class of admissible pairs.*

**5. More about applied problems.** Concurrent with the interest of electrical engineers in the time optimal control problem, there was interest in other disciplines in optimization problems whose mathematical formulation was that of the general control problem. Notable among these disciplines were mechanical engineering, chemical engineering, aerospace engineering, and economics. Also, problems in electrical engineering other than the time optimal one were arising that could be formulated as a general control problem. A representative sample of these problems, let alone a complete catalogue, is not possible in this paper. A representative sample is given in [**6**]. We shall, however, present one simplified example from economics of a production planning problem.

Let $x(t)$ denote the rate of production at time $t$ of steel. The amount produced at time $t$ is to be allocated to one of two uses, the production of consumer products or investment. It is assumed that the steel allocated to investment is used to increase productive capacity by using steel to produce new steel mills, mining equipment, transport facilities, etc. Let $u(t)$, where $0 \leqq u(t) \leqq 1$, denote the fraction of the steel produced at time $t$ that is allocated to investment. Then $(1 - u(t))$ represents the fraction allocated to consumption. The assumption that the reinvested steel is used to increase the productive capacity means that

$$\frac{dx}{dt} = ku(t)x,$$

where $k$ is an appropriate constant. The problem is to choose $u(t)$ so as to maximize the total consumption over a fixed time period of length $T > 0$. That is, we are to maximize

$$\int_0^T (1 - u(t))x(t)dt.$$

The problem we are faced with is the following. Do we always consume everything produced? Or, do we invest some at present so as to increase capacity now so that we shall be able to produce more, and hence consume more later? Do we follow a "bang-bang" procedure of first investing everything and then consuming everything?

The production planning problem can be written in the control formulation of Section 4 as follows. Minimize $J(\phi, u)$, where

$$J(\phi, u) = -\int_0^T (1 - u(s))\phi(s)ds$$

over the set of pairs $(\phi, u)$ that satisfy the differential equation

$$\frac{dx}{dt} = ku(t)x \qquad x(0) = c$$

and constraints

$$0 \leqq u(t) \leqq 1 \qquad \phi(t) \geqq 0.$$

Here $c > 0$ is the initial capacity. The constraint $\phi(t) \geqq 0$ is present since we do not destroy capacity, and thus negative capacity is meaningless.

Although the problems considered in the various areas of application had the common mathematical formulation of the problem in Section 4, these problems were not so formulated at the time. Each was formulated in the language of its own field. For the most part, investigators were unaware of the work in other disciplines. Each discipline seemed to favor its own set of methods for attacking the problem.

In the next section we shall see that the control problem is variational in character. Almost all of the investigators in the areas of application were aware of the variational character of the problem. The constraint (4.3), however, prevented direct application of the known results in the calculus of variations. In many special problems the basic technique of the calculus of variations, namely that of

comparing the optimal solution with a perturbation thereof which still satisfies the side conditions, was carried out for the example in question, (see e.g. [2]). Others, tried various transformations of the problem into a variational problem without the constraint (4.3) with varying degrees of success (see e.g. [9], [19], [23]). As already noted, functional analysis was used effectively in the time optimal problem. This is because the differential equations (4.2) are linear. In some problems the dynamic programming formalism was used, (see e.g. [1]).

**6. The calculus of variations.** The "simple problem" in the calculus of variations is the following. A real valued function $f^0\colon (t, x, z) \to f^0(t, x, z)$ is given, where $t$ is a scalar, $x = (x^1, \cdots, x^n)$ is a vector in euclidean space $E^n$ and $z = (z^1, \cdots, z^n)$ is a vector in $E^n$. A point $(t_0, x_0)$ and a point $(t_1, x_1)$ in $E^{n+1}$ are given, as is a class $\Phi$ of functions $\phi$ such that $\phi(t_0) = x_0$, $\phi(t_1) = x_1$ and such that the integral

$$J(\phi) = \int_{t_0}^{t_1} f^0(t, \phi(t), \phi'(t))dt$$

is defined. The problem is to find a function $\phi^*$ in $\Phi$ that minimizes $J(\phi)$; i.e., find a $\phi^*$ in $\Phi$ such that $J(\phi^*) \leqq J(\phi)$ for all $\phi$ in $\Phi$. In connection with the simple problem, we call the reader's attention to two necessary conditions that a minimizing function must satisfy. Our statements will not be precise. The precise sense in which the conditions hold depend on the function class $\Phi$ considered, the properties of $f$, etc. For typographic purposes we now let $\phi$ designate the minimizing function.

The first necessary condition is the system of Euler equations

$$\frac{d}{dt} f^0_{z^i}(t, \phi(t), \phi'(t)) = f^0_{x^i}(t, \phi(t), \phi'(t)) \qquad i = 1, \cdots, n. \tag{6.1}$$

The second is the Weierstrass condition

$$\begin{aligned} &f^0(t, \phi(t), z) - \sum_{i=1}^{n} f^0_{z^i}(t, \phi(t), \phi'(t))z^i \\ &\geqq f^0(t, \phi(t), \phi'(t)) - \sum_{i=1}^{n} f^0_{z^i}(t, \phi(t), \phi'(t))(\phi^i)'(t) \end{aligned} \tag{6.2}$$

for all $z$ in the open domain of definition of $f^0$.

The **Bolza problem** in the calculus of variations is that of minimizing a functional of the form

$$g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(t, \phi(t), \phi'(t))\, dt$$

over those functions in an appropriate class $C$ that satisfy a system of differential equations

$$G^i(t, \phi(t), \phi'(t)) = 0 \qquad i = 1, \cdots, m$$

and end conditions

$$(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{T},$$

where $m < n$ and $\mathcal{T}$ is a preassigned set in $E^{2+2n}$. If $g \equiv 0$ then the problem is called **a Lagrange problem.** If $f^0 \equiv 0$ it is called **a Mayer problem.** It can be shown that all three problems are equivalent [7].

A fairly complete theory for the Bolza problem had been developed by 1940. Again, from the wealth of information about the Bolza problem we select two necessary conditions for the reader's attention. These can be summarized roughly as follows. Introduce the Lagrangian function $F$ by the formula

$$F(t, x, z, l^0, l) = l^0 f^0(t, x, z) + \sum_{i=1}^{m} l^i G^i(t, x, z).$$

If $\phi$ is the minimizing function then there exists a scalar $\psi^0 \geqq 0$ and a vector valued function $\psi = (\psi^1, \cdots, \psi^m)$ defined on $[t_0, t_1]$ such that equations (6.1) and (6.2) hold with $f^0$ replaced by $F$. Thus:

$$\frac{d}{dt} F_{z^i}(t, \phi(t), \phi'(t), \psi^0, \psi(t)) = F_{x^i}(t, \phi(t), \phi'(t), \psi^0, \psi(t)) \quad i = 1, \cdots, n \tag{6.3}$$

and

$$F(t, \phi(t), z, \psi^0, \psi(t)) - \sum_{i=1}^{n} F^*_{z^i}(t) z^i \geqq F(t, \phi(t), \phi'(t), \psi^0, \psi(t)) - \sum_{i=1}^{n} F^*_{z^i}(t)(\phi^i)'(t), \tag{6.4}$$

for all $z$ satisfying $G^i(t, \phi(t), z) = 0$, $i = 1, \cdots, m$, where

$$F^*_{z^i}(t) = F_{z^i}(t, \phi(t), \phi'(t), \psi^0, \psi(t)).$$

With (6.3) there is a set of conditions that the values $\psi^0$, $\psi(t_0)$ and $\psi(t_1)$ must satisfy in conjunction with the end values $(t_0, \phi(t_0), t_1, \phi(t_1))$. These conditions are known as the "transversality conditions." Equations (6.3) are called the Euler-Lagrange equations or the "Lagrange multiplier rule." Inequality (6.4) is called the **Weierstrass condition.**

The Euler equations and Weierstrass condition for the simple problem are relatively easy to establish. Although many engineering texts give the impression that the same is true for the multiplier rule, this is definitely not the case. The Lagrange multiplier rule was first stated by Lagrange for the problem that bears his name. His proof, however, contained two major errors. The first was corrected by A. Mayer in 1886 and the second by Kneser in 1900 and also by Hilbert in 1905. A satisfactory treatment of the Weierstrass condition for the Bolza problem was also long in coming forth. It was not until 1939 that such a treatment was first given by McShane [**22**]. Prior to this work the Weierstrass condition was established under conditions that were not entirely satisfactory. In his proof, McShane introduced a completely new idea involving convex sets of variations and the separation theorem for convex sets. We shall return to this point later.

In the preceding section we stated that the control problem of Section 4 is variational in nature. We now show this. If we introduce a new independent variable $y = (y^1, \cdots, y^m)$ and let $y' = u$, then the control problem of Section 4 can be written as follows.

Minimize

$$g(t_0, \phi(t_0), t_1, \phi(t_1)) + \int_{t_0}^{t_1} f^0(s, \phi(s), y'(s))\, ds$$

over the functions $(\phi, y)$ in an appropriate class that satisfy the system of differential equations

$$(\phi^i)'(t) - f^i(t, \phi(t), y'(t)) = 0 \qquad i = 1, \cdots, n,$$

the end conditions

$$(t_0, \phi(t_0), t_1, \phi(t_1)) \in \mathcal{T},$$

and inequality constraints

$$R^i(t, \phi(t), y'(t)) \geqq 0 \qquad i = 1, \cdots, r.$$

The problem without the inequality constraint is a Bolza problem in the calculus of variations. We have already noted that the early workers in applied problems recognized the variational character of the problems, but were prevented from applying the theory of the Bolza problem by the inequality constraint.

**7. The maximum principle.** Interest in problems of optimal control in the 1950's was not confined to the United States. These problems were also pursued in the Soviet Union, where they attracted the

attention of L. S. Pontryagin and his students V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko. They stated the general control problem essentially in the form given in Section 4, and in a series of papers in the late 1950's announced what has come to be known as the **Pontryagin maximum principle.** This principle is a set of necessary conditions that must hold along an optimal trajectory. The announcement of the maximum principle can properly be regarded as the birth of the mathematical theory of optimal control. Workers in different areas of application saw that their problems had a common mathematical formulation and were provided with a tool for attacking their problems.

We now give an imprecise statement of one form of the maximum principle. Other, more general forms are known, but we shall not consider them here. For a precise statement of the principle we refer the reader to [27], which is an account in English of the work of Pontryagin and his collaborators in the late 1950's, or to any other standard reference. As in [27], we shall suppose that the constraints on the control are of the form

$$u(t) \in \Omega,$$

where $\Omega$ is a fixed set in $E^m$. In the interests of simplicity we shall assume that the problem is in Lagrange form, i.e., $g \equiv 0$. As in the Bolza problem, there is no loss of generality in this assumption.

The statement of the maximum principle requires the introduction of a function $H$ defined as follows:

$$H(t, x, z, p^0, p) = p^0 f^0(t, x, z) + \sum_{i=1}^{n} p^i f^i(t, x, z).$$

The maximum principle states that if $(\phi, u)$ is an optimal pair, then there exist a constant $\psi^0 \leqq 0$ and a vector valued function $\psi = (\psi^1, \cdots, \psi^n)$ defined and absolutely continuous on $[t_0, t_1]$ such that $(\psi^0, \psi(t)) \neq 0$ for all $t$ in $[t_0, t_1]$, such that the following system of equations is satisfied

$$\frac{d\phi^i}{dt} = f^i(t, \phi(t), u(t)) \qquad i = 1, \cdots, n, \tag{7.1}$$

$$\frac{d\psi^i}{dt} = -\sum_{j=0}^{n} \psi^j(t) f^j_{x^i}(t, \phi(t), u(t)) \qquad i = 1, \cdots, n, \tag{7.2}$$

and such that

$$H(t, \phi(t), u(t), \psi^0, \psi(t)) \geqq H(t, \phi(t), z, \psi^0, \psi(t)) \tag{7.3}$$

for all $z$ in $\Omega$. If the set of initial values $\mathcal{T}_0$ and the set of terminal values $\mathcal{T}_1$ are $C^{(1)}$ manifolds and if certain other regularity conditions hold, then we also have that the vector

$$(H(t_0, \phi(t_0), u(t_0), \psi^0, \psi(t_0)), -\psi(t_0)) \tag{7.4}$$

is orthogonal to $\mathcal{T}_0$ at $(t_0, \phi(t_0))$ and the vector

$$(-H(t_1, \phi(t_1), u(t_1), \psi^0, \psi(t_1)), \psi(t_1)) \tag{7.5}$$

is orthogonal to $\mathcal{T}_1$ at $(t_1, \phi(t_1))$. The orthogonality conditions are the "transversality conditions."

Equations (7.1) are just the state equations. If we consider the constant $\psi^0$ and the pair $(\phi, u)$ as known in (7.2), then (7.2) is a linear system in $\psi$. Equations (7.1) and (7.2) when written in vector notation have the following Hamiltonian form

$$\frac{d\phi}{dt} = H_p(t, \phi(t), u(t), \psi^0, \psi(t))$$

$$\frac{d\psi}{dt} = -H_x(t, \phi(t), u(t), \psi^0, \psi(t)).$$

A pair $(\phi, u)$ that satisfies the maximum principle is called an **extremal pair.** Theoretically, we can use the maximum principle to determine all extremal pairs $(\phi, u)$, and from among these then determine the optimal ones by some other criteria. In problems where the dimension of the state vector is large it is difficult to determine extremal pairs. This is primarily because equations (7.1) and (7.2) and inequality (7.3) when taken in conjunction with the transversality conditions are such that the end values of $\phi$ and $\psi$ are not either all terminal or all initial. Rather some values are given at the initial time and others at the terminal time. In problems where the dimension of the state is small, one can either integrate (7.1)–(7.3) forward or backward and keep the unknown initial or terminal values as literal constants which can be adjusted to the data at the other end. For systems of high dimension, where one is trying to determine extremal pairs by numerical methods, this procedure is not feasible. Neither are procedures in which the missing values are guessed and later corrected. We shall not go into these problems. We mention them here merely to point out that the maximum principle has not done away with all difficulties.

Despite the limitations just mentioned, the maximum principle is important and useful. It does give information about the structure of the solution. It has yielded the solution in certain classes of problems. In many specific examples in which the dimension of the state is small, the maximum principle has given the complete solution. To appreciate this last statement the reader should compare the worked out examples in [**27**] with some of the early treatments of these problems in the engineering literature, say as collected in [**26**].

Pontryagin and his co-workers were aware of the variational character of the control problem and also believed that the presence of the constraints (4.3) made the existing theory of the Bolza problem inapplicable. They therefore undertook to develop necessary conditions for the control problem without reference to existing variational theory. They did, however, make essential use of the ideas introduced by McShane [**22**] in his proof of the Weierstrass condition. In the proof of the maximum principle convex sets of variations of the controls are introduced and it is shown that as a consequence of optimality, a certain pair of convex sets can be separated. The analytic consequences of this separation constitute the maximum principle. Further details are given in the appendix.

Shortly after the publication of the maximum principle the relationship between the calculus of variations and the maximum principle was clarified. At the end of Section 6 we showed how a control problem with inequality constraints could be transformed into a Bolza problem with inequality constraints involving the derivatives. If one introduces a new variable $\xi$ by means of the differential equations

$$R^i(t, x, y') - (\xi^i)^2 = 0 \qquad i = 1, \cdots, r, \tag{7.4}$$

then the inequality constraints are transformed into differential equation constraints and the problem is of standard Bolza type. Under reasonable assumptions on the functions $R^i$ and on the optimal trajectory, the necessary conditions for the variational problem apply, and when these are translated back into the control formulation language, one gets the maximum principle (see [**5**]). It is also quite easy to obtain the necessary conditions (6.3), (6.4) and the transversality conditions for the variational problem from the maximum principle. See [**27**] Chap. 5. Under these translations, the Euler-Lagrange equations (6.3) correspond to equations (7.2) of the maximum principle and the Weierstrass condition (6.4) corresponds to the maximum condition (7.3). Also, the transversality conditions of the two problems correspond.

The use of the variable $\xi$ as in (6.3) for Lagrange problems with inequality constraints on the derivatives is due to Valentine [**28**]. It was used by Hestenes [**15**] in a minimum time problem.

**8. A unified theory of necessary conditions.** Primarily in response to problems in economics and operations research, the theory of mathematical programming also began its development in the 1950's. One form of the mathematical programming problem is the following.

A set $\mathscr{C}$ is given in euclidean space $E^n$ and mappings,

$$F^0\colon \mathscr{C} \to E^1$$

$$F = (F^1, \cdots, F^\nu)\colon \mathscr{C} \to E^\nu$$

$$G = (G^1, \cdots, G^r)\colon \mathscr{C} \to E^r$$

are given. The problem is to minimize $F^0(x)$ over all $x$ in $\mathscr{C}$ such that

$$F^i(x) = 0 \qquad i = 1, \cdots, \nu$$

$$G^i(x) \geqq 0 \qquad i = 1, \cdots, r.$$

If the constraints $G^i(x) \geqq 0$ are absent, if the set $\mathscr{C}$ is open and the functions $F^0$ and $F$ are differentiable, then at a point $x^*$ in $\mathscr{C}$ at which the minimum is attained, the classical Lagrange multiplier rule holds. Various extensions of the classical multiplier rule were developed to handle the inequality constraints, and further extensions were developed to accommodate non-differentiable functions $F^0$, $F$, and $G$.

In the early to mid 1960's it was noted that the control problem of Section 4 can be cast as a programming problem in an appropriate function space. One way of so casting the control problem is the following. Let $\mathscr{C}$ denote the set of function pairs $(\phi, u)$ such that $\phi$ and $u$ belong to function spaces appropriate to the problem and such that

$$(\phi^i)'(t) = f^i(t, \phi(t), u(t)) \qquad i = 1, \cdots, n.$$

Let the mapping $F^0\colon \mathscr{C} \to E^1$ be defined by

$$F^0(\phi, u) = \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))\, dt.$$

We now suppose that the initial set $\mathscr{T}_0$ is a $C^{(1)}$ manifold, defined by a system of equations,

$$\chi^i(t_0, x_0) = 0 \qquad i = 1, \cdots, \gamma$$

and that the terminal set is a $C^{(1)}$ manifold, defined by a system of equations

$$\chi^i(t_1, x_1) = 0 \qquad i = \gamma + 1, \cdots, \nu.$$

We then define

$$F = (F^1, \cdots, F^\nu)\colon \mathscr{C} \to E^\nu$$

by

$$F^i(\phi, u) = \chi^i(t_0, \phi(t_0)) \qquad i = 1, \cdots, \gamma$$

$$F^i(\phi, u) = \chi^i(t_1, \phi(t_1)) \qquad i = \gamma + 1, \cdots, \nu.$$

Let

$$G = (G^1, \cdots, G^r)\colon \mathscr{C} \to E^r$$

be defined by

$$G^i(\phi, u) = \inf\{R^i(t, \phi(t), u(t))\colon t_0 \leqq t \leqq t_1\}.$$

If we write $x = (\phi, u)$, then formally the control problem of Section 4 becomes the following. Minimize $F^0(x)$ over $x$ in $\mathscr{C}$ subject to $F^i(x) = 0$, $i = 1, \cdots, \nu$ and $G^i(x) \geqq 0$, $i = 1, \cdots, r$. The control problem reads exactly as the programming problem, except that $x$ is now an element of a function space. The formulation just given of the control problem is essentially that of Neustadt **[24]**. Other writers have given slightly different formulations.

Not only were the similarities in the abstract formulation of control and programming problems noticed, but it was also noticed that the techniques and results concerned with necessary conditions in various optimization problems were similar. These similarities in the techniques, however, were not always on the surface. These observations prompted various writers, notably Neustadt [**24**], Halkin [**14**], Hestenes [**16**], Gamkrelidze and Haratisvili [**12**], and Dubovitskii and Milyutin to embark on the following program. (See [**13**] for an account in English of the work of Dubovitskii and Milyutin.)

First formulate a general optimization or programming problem that would include as special cases all optimization problems of interest, such as ordinary control problems, control problems with bounded states, control problems with lags, control problems with distributed parameters, mathematical programming problems, etc.

Second, develop a meaningful set of necessary conditions for the general problem under reasonable hypotheses. The necessary conditions must be such that one obtains useful necessary conditions for the special problems when appropriate specializations and identifications are made. One of the difficulties in the formulation of the general problem is that the hypotheses must be specific enough to yield necessary conditions with some structure, yet they should be general enough to include all the special problems of interest.

The program outlined above was carried the furthest by Neustadt. An account of his work will be found in his posthumous book [**25**].

**9. Other topics.** Although we have written about the time optimal problem and about the maximum principle, these were not the only developments in the mathematical theory of optimal control. An account of other developments is not possible in this paper. We shall, however, list a few developments.

In the course of studying existence questions for control problems, new methods and ideas were introduced which gave refinements, simplifications, and generalizations of classical results in the calculus of variations, as well as the sought for results in control theory. The field theory of the calculus of variations was significantly generalized by L. C. Young in his book [**31**] in order to cope with sufficiency problems in the calculus of variations. Generalized curves, introduced into the calculus of variations by L. C. Young in 1937 [**30**], were rediscovered in different form by investigators in control theory and were named "chattering controls" or "relaxed controls." Here again, new results were obtained as were improvements over the earlier work in this area.

Two other areas of investigation which arose in response to applied problems were the distributed parameter control problems and the stochastic control problems. In the distributed parameter problems the state of the system is governed by a system of partial differential equations and the functional to be minimized is taken over a region in the space of the independent variable. In stochastic control problems the state equations and payoff functional involve randomness.

A very important problem in terms of applications is that of developing computational methods. As we pointed out earlier, the pursuit of computational schemes to determine extremal trajectories and controls (those satisfying the maximum principle) has not been too fruitful. More promising techniques are those that involve direct methods. Although many specific applied problems of interest have been treated numerically with some success, there are no widely applicable algorithms with solid theoretical basis extant.

Although we could proceed with our list of topics, we shall stop at this point and suggest that the reader who wants to learn more consult some of the references [**6**], [**10**], [**16**], [**18**], [**20**], [**25**], [**27**], [**29**], [**31**].

**10. Appendix. A proof of the maximum principle.** We now sketch a proof of the maximum principle. Some of our statements will not be precise, some will be heuristic, and some will be formal. We hope, however, to convey the underlying ideas of the proof given by Pontryagin and his co-workers and the ideas behind the later generalizations and refinements of this proof.

In this section we shall use vector-matrix notation. Thus, the state equations will be written

$$\frac{d\phi}{dt} = f(t, \phi(t), u(t)).$$

The symbol $f_x$ will denote the matrix of partial derivatives $(\partial f^i/\partial x^j)$ $i = 1, \cdots, n$, $j = 1, \cdots, n$ and the symbol $f_u$ will denote the matrix $(\partial f^i/\partial u^j)$ $i = 1, \cdots, n$, $j = 1, \cdots, m$. The symbols $f_x^0$ and $f_u^0$ will denote vectors of partial derivatives. If $x$ and $y$ are vectors in $E^n$, then the symbol $\langle x, y\rangle$ will be used to denote their inner product. If $M$ is a matrix, then $M^T$ will designate the transpose of $M$.

In order to make the exposition as simple as possible we assume that the initial point $(t_0, x_0)$ and the terminal point $(t_1, x_1)$ are fixed. As in Section 7, we assume that the problem is to minimize

$$J(\phi, u) = \int_{t_0}^{t_1} f^0(t, \phi(t), u(t))dt$$

subject to the state equation (7.1) and to the control constraints $u(t) \in \Omega$. Let $\hat{x} = (x^0, x^1, \cdots, x^n)$ and let $\hat{f} = (f^0, f^1, \cdots, f^n)$. If we introduce a new state coordinate $x^0$ by means of the differential equation

$$\frac{dx^0}{dt} = f^0(t, x, u(t)) \qquad x^0(t_0) = 0,$$

then the problem becomes that of minimizing

$$\hat{J}(\hat{\phi}, u) = \phi^0(t_1) \tag{10.1}$$

subject to the state equations

$$\frac{d\hat{\phi}}{dt} = \hat{f}(t, \phi(t), u(t)), \tag{10.2}$$

the end conditions

$$\phi^0(t_0) = 0 \quad \phi(t_0) = x_0 \quad \phi(t_1) = x_1,$$

and control constraints $u(t) \in \Omega$. Note that in (10.2) the zero-th component of $\hat{\phi}$ does not appear in the right hand side.

Let $(\hat{\phi}, u)$ be an optimal trajectory-control pair. We perturb $u$ by $\delta u$ and obtain a trajectory $\hat{\phi} + \delta\hat{\phi} + e$, where $\delta\hat{\phi}$ represents the "first order" terms in the new trajectory and $e$ represents the "higher order" terms. If we substitute $u + \delta u$ and $\hat{\phi} + \delta\hat{\phi} + e$ into (10.2) and expand to first order, we get that $\delta\hat{\phi}$ satisfies

$$\begin{aligned} \delta\phi'(t) &= f_x(t, \phi(t), u(t))\delta\phi(t) + f_u(t, \phi(t), u(t))\delta u(t) \\ \delta\phi^{0\prime}(t) &= \langle f_x^0(t, \phi(t), u(t)), \delta\phi(t)\rangle + \langle f_u^0(t, \phi(t), u(t)), \delta u(t)\rangle, \end{aligned} \tag{10.3}$$

with $\delta\hat{\phi}(t_0) = 0$. Hence by the variation of parameters formula we get that

$$\delta\phi(t_1) = V(t_1, t_0)\int_{t_0}^{t_1} V^{-1}(s, t_0) f_u^*(s)\delta u(s)ds, \tag{10.4}$$

where $f_u^*(s) = f_u(s, \phi(s), u(s))$ and where $V(\cdot, t_0)$ is the fundamental matrix of solutions of the system

$$\frac{dz}{dt} = f_x(t, \phi(t), u(t))z$$

satisfying $V(t_0, t_0) = I$. Here $I$ is the $n \times n$ identity matrix. From (10.3) and (10.4) we get $\delta\phi^0(t_1)$ by a simple quadrature as follows:

$$\delta\phi^0(t_1) = \int_{t_0}^{t_1} [\langle f_x^0(s, \phi(s), u(s)), \delta\phi(s)\rangle + \langle f_u^0(s, \phi(s), u(s)), \delta u(s)\rangle]\, ds. \tag{10.5}$$

The perturbations $\delta u$ are functions in some function space. Therefore we can speak of a convex set of perturbations and of a convex cone of perturbations. Equations (10.4) and (10.5) define a linear map $L$ from the space of perturbations into $E^{n+1}$. Hence, if we restrict our attention to perturbations in an appropriate convex cone $\mathcal{U}$ with vertex at the origin in the space of perturbations, the set of corresponding points $\delta\hat{\phi}(t_1)$ in $E^{n+1}$ will be a convex cone in $E^{n+1}$ with vertex at the origin. Let $\mathcal{K}$ denote the translate of this cone by the vector $\hat{\phi}(t_1)$. Points in $\mathcal{K}$ are of the form

$$\hat{\phi}(t_1) + \delta\hat{\phi}(t_1), \tag{10.6}$$

where $\delta\hat{\phi}(t_1)$ is given by (10.4) and (10.5) for some $\delta u$ in $\mathcal{U}$.

Let $\mathcal{L}$ denote the half-line parallel to the $x^0$ axis with initial point at $\hat{\phi}(t_1)$ and in the direction of the negative $x^0$ axis. If the points of $\mathcal{K}$ were the true end points of the perturbed trajectories rather than the first order approximations thereto, then since $\phi^0(t_1)$ is the minimum value of the zero-th coordinate of all trajectories, it would follow that no point of $\mathcal{K}$ could also be a point of $\mathcal{L}$. Thus we would have that the two convex cones $\mathcal{K}$ and $\mathcal{L}$ have a common vertex and have no other points in common. It turns out that this is still true, even though the points (10.6) differ from the true end points by higher order terms. The proof of this assertion involves the use of either the implicit function theorem or of a fixed point theorem and will not be discussed here.

Since $\mathcal{K}$ and $\mathcal{L}$ are convex cones whose intersection is their common vertex $\hat{\phi}(t_1)$, they can be separated by a hyperplane $\Pi$ through $\hat{\phi}(t_1)$. We now consider the analytic consequences of this separation.

Let $t_0 \leqq \tau < t_1$ and let $\delta t > 0$ be such that $\tau + \delta t < t_1$. Define a control $\delta u$ as follows: $\delta u(t) = 0$ for $t$ not in $[\tau, \tau + \delta t]$ and $\delta u(t) = v - u(t)$ for $t \in [\tau, \tau + \delta t]$, where $v$ is a fixed vector in $\Omega$. Thus we perturb the trajectory by replacing $u(t)$ by $v$ over a small interval. We suppose that $\mathcal{U}$ is defined in such a way that $\delta u \in \mathcal{U}$. Let $\hat{\psi}$ denote the trajectory corresponding to $u + \delta u$. Then

$$\hat{\psi}(t) = \hat{\phi}(t) + \delta\hat{\phi}(t) + e(t),$$

where $e(t)$ represents "higher order terms." Since

$$\hat{\psi}(\tau + \delta t) = \hat{\phi}(\tau) + \int_{\tau}^{\tau+\delta t} \hat{f}(s, \psi(s), v)ds,$$

$$\hat{\phi}(\tau + \delta t) = \hat{\phi}(\tau) + \int_{\tau}^{\tau+\delta t} \hat{f}(s, \phi(s), u(s))ds,$$

it follows that for $\delta t$ small,

$$\delta\hat{\phi}(\tau + \delta t) = [\hat{f}(\tau, \phi(\tau), v) - \hat{f}(\tau, \phi(\tau), u(\tau))]\delta t, \tag{10.7}$$

assuming that $\hat{f}$ is continuous and that $u$ is continuous at $\tau$.

Let $\xi$ denote the right hand side of (10.7). Since $u(t) + \delta u(t) = u(t)$ for $t \geqq \tau + \delta t$, it follows that for $t \geqq \tau + \delta t$ we can consider $\delta\hat{\phi}$ as the first order approximation to the perturbation of the trajectory $\hat{\phi}$ that is obtained if we solve

$$\frac{d\hat{x}}{dt} = \hat{f}(t, x, u(t))$$

with initial condition $\hat{x}(\tau + \delta t) = \hat{\phi}(\tau + \delta t)$ perturbed to $\hat{\phi}(\tau + \delta t) + \xi$. It is then a consequence of theorems describing the dependence of solutions of differential equations on initial data that $\delta\hat{\phi}$ is a solution of

$$\frac{dz}{dt} = \hat{f}_{\hat{x}}(t, \phi(t), u(t))z \qquad z(\tau + \delta t) = \xi. \tag{10.8}$$

Note that $\xi$ and $z$ are vectors in $E^{n+1}$ and that $\hat{f}_{x^0} = 0$.

Let $\hat{n}$ be a normal to the hyperplane $\Pi$ that separates $\mathscr{K}$ and $\mathscr{L}$ and let $\hat{n}$ point into the half-space containing $\mathscr{L}$. Then $\langle \delta\hat{\phi}(t_1), \hat{n}\rangle \leqq 0$. We now construct the hyperplane through $(\tau, \hat{\phi}(\tau))$ with normal vector $\hat{\lambda}(\tau)$, where $\hat{\lambda}$ is the solution of

$$\frac{dw}{dt} = -\hat{f}_x(t, \phi(t), u(t))^T w \qquad w(t_1) = \hat{n}. \tag{10.9}$$

It is readily verified by differentiation and the use of (10.8) and (10.9) that $\langle \delta\hat{\phi}(t), \hat{\lambda}(t)\rangle$ is constant. Since $\hat{\lambda}(t_1) = \hat{n}$ and since $\langle \delta\hat{\phi}(t_1), \hat{n}\rangle \leqq 0$, it follows that

$$\langle \delta\hat{\phi}(\tau + \delta t), \hat{\lambda}(\tau + \delta t)\rangle \leqq 0.$$

From (10.7) and from the continuity of $\hat{\lambda}$ we then get that

$$\langle \delta t\,[\hat{f}(\tau, \phi(\tau), v) - \hat{f}(\tau, \phi(\tau), u(\tau))], \lambda(\tau) + o(1)\rangle \leqq 0,$$

where $o(1)$ is as $\delta t \to 0$. Since $\delta t > 0$, it follows on division by $\delta t$ that

$$\langle \hat{f}(\tau, \phi(\tau), v) - \hat{f}(\tau, \phi(\tau), u(\tau)), \lambda(\tau)\rangle \leqq 0.$$

This inequality and equation (10.9) with $w$ replaced by $\hat{\lambda}$ constitute the maximum principle for the problem in the form studied here.

The assertion that $\lambda^0$ is constant follows from (10.9) and from the fact that

$$\hat{f}_{x^0} = (\partial f^0/\partial x^0, \partial f^1/\partial x^0, \cdots, \partial f^n/\partial x^0) \equiv 0.$$

Since $\hat{n} = \hat{\lambda}(t_1)$, since $\mathscr{L}$ is the translate by $\hat{\phi}(t_1)$ of all nonnegative multiples of the $E^{n+1}$ vector $(-1, 0, \cdots, 0)$, and since $\langle \gamma, \hat{n}\rangle \geqq 0$ for all $\gamma$ in $\mathscr{L}$, it follows that

$$\langle (-1, 0, \cdots, 0), (\lambda^0, \lambda^1(t_1), \cdots, \lambda^n(t_1))\rangle \geqq 0,$$

and so $\lambda^0 \leqq 0$.

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## NOTE ON THE BERNOULLI-L'HOSPITAL RULE

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**1.** The Bernoulli-L'Hospital Rule on the convergence of $f(x)/g(x)$ in the case that $g(x)\to\infty$ can be formulated as follows (See [2], 45–46; [3], 128–129):

*I. Consider a fixed one of the following limiting processes*:

$$x \uparrow \infty, \quad x \downarrow -\infty, \quad x \uparrow x_0, \quad x \downarrow x_0, \tag{1}$$

*where $x_0$ is finite. Assume that $f(x)$ and $g(x)$, for the corresponding $x$, are continuous and have a derivative. Assume that $g'(x)$ is either always $< 0$ or always $> 0$ and that $|g(x)|\to\infty$. Then we have*

$$\underline{\lim} f'(x)/g'(x) \leqq \underline{\lim} f(x)/g(x) \leqq \overline{\lim} f'(x)/g'(x). \tag{2}$$

If this formulation has to be applied in the case where $f(x)$ or $g(x)$ are defined as integrals, the difficulty arises that the relation

$$\left(\int^x \varphi(x)dx\right)' = \varphi(x)$$

generally only holds **save at a set of measure 0**, unless $\varphi(x)$ is continuous. On the other hand the above formulation is no longer true if the existence of the derivatives is only assumed **save at a set of measure 0**, as will be shown by means of a counter-example in Section 2.

It is therefore of interest, that the above formulation remains true if the assumption that $f(x)$ and $g(x)$ are continuous is replaced with the assumption that $f(x)$ and $g(x)$ are absolutely continuous. We prove this in the Sections 3–7.

In the following sections, 8–12, we prove some results which can still be obtained if we drop the assumption of the absolute continuity of $g(x)$.

**2.** Consider a function $\gamma(x)$ which is strictly monotonically increasing and bounded in the interval $[0,1]$, and such that $\gamma(0)=0$ and, with the exception of a set of measure 0, $\gamma'(x)=0$. The existence of such a function $\gamma(x)$ follows for instance from the Theorem 12 in section 509 of [1]. (Instead of this reference, the reader may also consult [4].)

Continue the definition of $\gamma(x)$ for $x \geqq 1$ using recurrently in each interval $(\nu, \nu+1]$, $\nu = 1, 2, \cdots$, the definition

$$\gamma(x) := \gamma(\nu) + \gamma(x-\nu) \qquad (1 \leqq \nu < x \leqq \nu+1), \tag{3}$$

so that we have for $x>0$: $\gamma(x)\to\infty$ and almost everywhere $\gamma'(x)=0$.

Put

$$g(x) = -\frac{1}{x} + \gamma(x), \qquad f(x) = -\frac{1}{x}. \tag{4}$$

Then we have almost everywhere

$$f'(x) = g'(x) = 1/x^2, \qquad f'(x)/g'(x) = 1,$$

after a set of measure 0 has been removed, while obviously

$$f(x)/g(x)\to 0 \qquad (x\to\infty).$$

**3.** We prove first the

LEMMA. *Assume $f(x)$ absolutely continuous for $x \geqq a$ and $g(x)$ monotonically increasing to $\infty$ for $a \leqq x \to \infty$. Assume further that, save at a set, $\Omega$, of measure 0, $f'(x)$ and $g'(x)>0$ exist for $x \geqq a$. Finally assume that for a constant $B$:*

$$\frac{f'(x)}{g'(x)} < B \qquad (x \geqq a,\ x \notin \Omega). \tag{5}$$

*Then*

$$\overline{\lim_{x\to\infty}} \frac{f(x)}{g(x)} \leqq B, \tag{6}$$

*if either $g(x)$ is absolutely continuous for $x \geqq a$, or $B \geqq 0$.*

**4.** *Proof.* Observe that by well-known properties of monotonically increasing functions

$$g(x) = g(a) + \int_a^x g'(x)dx + P(x), \tag{7}$$

where $P(x)$ is non-negative (by a theorem due to de la Vallée Poussin, see [1], sec. 508, Theorem 10). If $g(x)$ is absolutely continuous, $P(x)$ vanishes. Thence, using (5), since $f(x)$ is absolutely continuous,

$$\frac{f(x)}{g(x)} - B = \frac{\int_a^x f'(x)dx + f(a)}{g(x)} - B < \frac{B\int_a^x g'(x)dx + f(a)}{g(x)} - B$$
$$= \frac{B\int_a^x g'(x)dx + f(a) - Bg(x)}{g(x)}.$$

This becomes, introducing the expression (7) of $g(x)$ into the numerator,

$$\frac{f(x)}{g(x)} - B < \frac{f(a) - Bg(a) - BP(x)}{g(x)}.$$

For a non-negative $B$ the numerator is here bounded from the right and therefore the whole expression has $\underline{\lim} \leqq 0$, since $g(x) \to \infty$. If $P(x) \equiv 0$, that is if $g(x)$ is absolutely continuous, the same argument holds for any $B$. Our lemma is proved.

**5.** II. *If both $f(x)$ and $g(x)$ are absolutely continuous, the assertion of Theorem I remains true, if in the computation of $\underline{\lim} f'(x)/g'(x)$ an arbitrary set of measure 0, $\Omega$, is left out, provided that in the remaining points of the range of $x$, $f'(x)$ and $g'(x)$ exist, and either always $g'(x) > 0$ or always $g'(x) < 0$.*

**6.** *Proof.* Proving II it is sufficient to consider only the first one, $x \uparrow \infty$, of the limiting processes enumerated in (1), since the three other cases are reduced to this one by one of the transformations

$$x = -y, \qquad x = x_0 \pm 1/y,$$

and the range of values of $f'(x)/g'(x)$ does not change by these transformations.

Further, we can assume, without loss of generality, that $g'(x) > 0$ almost everywhere for $x \geqq a$, since otherwise it would be sufficient to multiply $g(x)$ with $-1$.

Finally, it is sufficient to prove the *right side inequality* in (2) since, changing $f(x)$ into $-f(x)$, both inequalities in (2) are interchanged.

**7.** Put $\overline{\lim} f'(x)/g'(x) = \beta$. As there is nothing to prove for $\beta = \infty$, we can assume $\beta < \infty$. Then for any constant $B > \beta$ we have, save at a set of measure 0,

$$\frac{f'(x)}{g'(x)} < B \qquad (x \geqq a'), \tag{8}$$

choosing an $a' > a$, sufficiently large. But then the assumptions of the lemma of Section 3 are satisfied, replacing $a$ with $a'$. It follows therefore

$$\overline{\lim_{x \to \infty}} \frac{f(x)}{g(x)} \leqq B.$$

As this relation holds for any $B > \beta$, the assertion of the theorem follows with $B \downarrow \beta$.

**8.** If we drop in the hypotheses of Theorem II the assumption that $g(x)$ is absolutely continuous we can still obtain a partial result, not as strong as in II, but still of some interest.

III. *If in the assumptions of Theorem II the continuity of $g(x)$ is dropped, the relation*

$$\underline{\lim} \operatorname{Min}\left(0, \frac{f'(x)}{g'(x)}\right) \leqq \overline{\underline{\lim}} \frac{f(x)}{g(x)} \leqq \overline{\lim} \operatorname{Max}\left(0, \frac{f'(x)}{g'(x)}\right). \tag{9}$$

*holds, where in the computation of both extreme terms the points of $\Omega$ have to be omitted.*

**9.** *Proof.* Observe that in this case, again, we need only to consider the limiting process $x \uparrow \infty$ and to assume $g'(x) > 0$ $(x \notin \Omega)$. Further it is sufficient to prove the *right side inequality* in (9),

$$\overline{\lim} \frac{f(x)}{g(x)} \leqq \overline{\lim} \operatorname{Max}\left(0, \frac{f'(x)}{g'(x)}\right) = \beta. \tag{10}$$

Indeed, if we replace $f(x)$ with $-f(x)$ in (10), it is seen immediately that the result is equivalent with the left side inequality in (9).

**10.** Proving (10) we can obviously assume that $\beta < \infty$. Assume now a $B > \beta$. Then we have, for a convenient $a' > a$,

$$\frac{f'(x)}{g'(x)} < B \qquad (x \geqq a',\ x \notin \Omega).$$

Since here $B$ is $\geqq 0$, the lemma of Section 3 can be applied and it follows again that

$$\overline{\lim_{x \to \infty}} \frac{f(x)}{g(x)} \leqq B,$$

and, also, if we let $B$ go to $\beta$, the formula (10). Theorem III is proved.

**11.** As a corollary of III we give finally a theorem which, in an important special case, contains the essential parts of II and III in a particularly concise form.

IV. *Assume that for one of the limiting processes* (1), *save on a set* $\Omega$ *of measure* 0 *in the range of* $x$, $f'(x)$ *and* $g'(x)$ *exist,* $g'(x)$ *is either always* $> 0$ *or always* $< 0$ *and* $g(x)$ *tends monotonically to* $+\infty$ *or* $-\infty$. *Assume further that we have for a finite constant* $\alpha$,

$$\frac{f'(x)}{g'(x)} \longrightarrow \alpha, \quad |\alpha| < \infty \qquad (x \notin \Omega), \tag{11}$$

*and that* $f(x) - \alpha g(x)$ *is absolutely continuous. Then the relation*:

$$\frac{f(x)}{g(x)} \to \alpha \qquad \textit{holds.} \tag{12}$$

**12.** *Proof.* Putting $h(x) = f(x) - \alpha g(x)$, (11) becomes $h'(x)/g'(x) \to 0$ $(x \notin \Omega)$. If we replace in III $f(x)$ with $h(x)$, the assumptions of III are verified, while in the assertion (9) both the first and the last terms become $= 0$. From $h(x)/g(x) \to 0$, (12) follows immediately.

Sponsored in part by the Swiss National Science Foundation and in part by the Grant DA-ERO-75-G-035 of the European Research Office, United States Army, to the Institute of Mathematics, University of Basel.

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# GRADUATE STUDENT AT CHICAGO IN THE TWENTIES

W. L. DUREN, JR.*

As an undergraduate at Tulane in New Orleans, 1922–'26, I was programmed to go to the University of Chicago and study celestial mechanics with F. R. Moulton. My teacher, H. E. Buchanan, had been a student of Moulton. That was an example of the great strength of the University of Chicago. Its PhD graduates made up a large part of the faculties of universities throughout the Mississippi Valley, Midwest and Southwest. So they sent their good students back to Chicago for graduate work. I went there first in the summer of 1926 and came to stay in 1928. In the interim I studied Moulton's *Celestial Mechanics* and some of his papers in orbit theory. I met Moulton at a sectional meeting of the MAA where he was the invited speaker. He was a man of great charm and energy and was most encouraging to me. But by the time I got to Chicago in 1928 Moulton had resigned. I was told that he felt it was an ethical requirement, since he and his wife were getting a divorce. On the advice of T. F. Cope, another former student of Buchanan, who was working with Bliss, I turned to Bliss as an advisor in the calculus of variations.

It was a down cycle for mathematics at Chicago. All the great schools have their downs as well as ups, partly because great men retire, partly because their lines of investigation dry up. At Chicago at that time a young student could see the holdovers of the great period, 1892–1920, in Eliakim Hastings Moore, officially retired, Leonard E. Dickson, rounding out his work in algebra, Gilbert A. Bliss, busy with administration and planning for the projected Eckhart Hall. Also there was Herbert E. Slaught, teacher and doer, one of the original organizers of the Mathematical Association of America and its MONTHLY, even if he played only a supporting role in mathematics itself. He had an extrovert, friendly personality that reached out and got hold of you, whether he was organizing a department social or the Mathematical Association of America. He was the teacher of teachers and key figure in Chicago's hold on education in the midwest and south. Every graduate department needs a man like Slaught if it is fortunate enough to find one. He was being succeeded by Ralph G. Sanger, a student of Bliss, an outstanding undergraduate teacher, though not the organizer Slaught was.

The University of Chicago was founded in 1892 with substantial financial support from John D. Rockefeller. William Rainey Harper, the first president, had bold educational ideas, one of which was that the United States was ready for a primarily graduate university, not just a college with graduate school attached. Harper brought E. H. Moore from Yale to establish his department of mathematics. Moore's graduate teaching was done in a research laboratory setting. That is, students read and presented papers from journals, usually German, and tried to develop new theorems based on them. The general subject of these seminars was a pre-Banach form of geometric analysis that Moore called "general analysis." It was itself not altogether successful. But even if general analysis did not succeed, Moore's seminars on it generated a surprising number of new results in general topology, among them the Moore theorem on iterated limits and Moore-Smith convergence. Moore's seminars also produced some outstanding mathematicians. His earlier students had included G. D. Birkhoff, Oswald Veblen, T. H. Hildebrant and R. L. Moore, who took off in different mathematical directions. R. L. Moore developed the teaching method into an intensive research training regimen of his own, which was very successful in producing research mathematicians at the University of Texas.

I studied general analysis with other members of the faculty including R. W. Barnard, whom

* With the help of Antoinette Killen Huston, who earned her way as graduate student by serving as secretary to Mr. Bliss. She was Adrian Albert's first student, receiving her PhD degree in 1934.

Moore had designated as his successor and whose notes record the second form of the theory, [Am. Philosophical Soc., *Memoirs*, v. 1, Philadelphia, 1935]. Instead of taking the general analysis courses, my old friend E. J. McShane, from New Orleans, worked in Moore's small seminar in the foundations of mathematics. Although he was officially a student of Bliss, I think he was in a sense Moore's last student.

Moore himself was meticulous in manners and dress. He would stop you in the hall, gently remove a pen from an outside pocket and suggest that you keep it in the inside pocket of your jacket. Nobody thought of not wearing a jacket. But Moore was less gentle if you used your left hand as an eraser, and he displayed towering anger at intellectual dishonesty. To understand him and his times one must read his retiring address as President of the Society [*Science*, March 1903]. In those days the Society accepted responsibility for teaching mathematics and Moore's address was largely devoted to the organization of teaching, the curriculum, and the ideas of some of the great teachers of the time, Boltzman, Klein, Poincaré, and, in this country, J. W. A. Young and John Dewey, whose ideas Moore supported by proposing a mathematics laboratory. This address was adopted as a sort of charter by the National Council of Teachers of Mathematics and republished in its first Yearbook (1925). By the time I got to Chicago the Association had been formed to relieve the Society of concern for college education, and NCTM to relieve it of responsibility for the school curriculum and training teachers. In the top universities only research brought prestige, even if a few, like Slaught, upheld the importance of teaching.

L. E. Dickson's students tended to identify themselves strongly as number theorists or algebraists. I felt this particularly in Adrian A. Albert, Gordon Pall and Arnold Ross. All his life Albert strongly identified himself, first as an algebraist, later with mathematics as an institution and certainly with the University of Chicago. I remember him as an advanced graduate student walking into Dickson's class in number theory that he was visiting, smiling and self confident. He knew where he was going. Dickson was teaching from the galley sheets of his new *Introduction to the Theory of Numbers* [University of Chicago Press, 1929] with its novel emphasis on the representation of integers by quadratic forms. I think he requested Albert to sit in for his comments on this aspect. He was tremendously proud of Albert. I remember $A^3$ too with his beautiful young wife, Frieda, at the perennial department bridge parties. He had superb mental powers; he could read a page at a glance. One could see even then that as heir apparent to Dickson he would do his own mathematics rather than a continuation of Dickson's, however much he admired Dickson.

In the conventional sense Dickson was not much of a teacher. I think his students learned from him by emulating him as a research mathematician more than being taught by him. Moreover, he took them to the frontier of research, for the subject matter of his courses was usually new mathematics in the making. As Antoinette Huston said, "He made you want to be with him intellectually. When you are young, reaching for the stars, that is what it is all about." He was good to his students, kept his promises to them and backed them up. Yet he could be a terror. He would sometimes fly into a rage at the department bridge games, which he appeared to take seriously. And he was relentless when he smelled blood in the oral examination of some hapless, cringing victim. He was an indefatigable worker and in public a great showman, with the flair of a rough and ready Texan. An enduring bit in the legend is his blurt: "Thank God that number theory is unsullied by any application." He liked to repeat it himself as well as his account of his and his wife's honeymoon, which he said was a success, except that he got only two papers written.

The theme of beauty for its own sake was expressed more surprisingly by another Texan who worked in mechanics and potential theory, W. D. MacMillan. According to the story he had come to Chicago as a mature man, without a college education, to sell his cattle. Having sold them, he went to

Chicago's Yerkes Observatory to see the Texas stars through the telescope. He was so fascinated that he stayed on to get his degrees in rapid succession, all *summa cum laude.* Then he remained as a member of the faculty. One day in his course on potential theory he wrote some important partial differential equations on the board with obvious pleasure, drawing the partial derivative signs with a flourish. Standing back to admire these equations, he said: "That is just beautiful. People who ask, 'What's it good for?', they make me tired! Like when you show a man the Grand Canyon for the first time and you stand there as you do, saying nothing for a while." And we could see that old Mac was really looking at the Grand Canyon. "Then he turns to you and asks, 'What's it good for?' What would you do? Why, you would kick him off the cliff!" And old Mac kicked a chair halfway across the room. He was a prodigy, a good lecturer, an absolutely fascinating personality with a twinkling wit. Some of his work was outstanding, yet he had few doctoral students.

Celestial mechanics was being carried on by the young Walter Bartky, who was, I think, Moulton's last student. But celestial mechanics had gone into a barren period and Bartky with his superb talents turned to other applications of differential equations, to statistics and to administration.

Lawrence M. Graves was the principal hope of the department for carrying on the calculus of variations, which he did in the spirit of functional analysis. He was my favorite professor because he knew a lot of mathematics, knew it well, and in an unassuming way was glad to share it with you. Although he taught Moore's general analysis, he pointed out the difficulties in it to me. His own brand of functional analysis was more oriented towards the use of the Frêchet differential in Banach space.

Research in geometry at Chicago was a continuation of Wylczinski's projective differential geometry. There was no topology, though we heard that Veblen's students studied something called *analysis situs* at Princeton. I knew so little about the subject that years later when I wanted to prepare for Morse theory I spent months studying Kuratowski's point set topology before it dawned on me that what I wanted was algebraic topology. E. P. Lane and his students carried on the study of projective differential geometry using rather crude analytical methods, that is, expansions in which one neglected higher order terms. We who were not Lane's students tended to look on it with disdain as being non-rigorous. But the structure of the theory was beautiful, I thought. Lane was honest about the shortcomings of the methods, though he did not know how to overcome them.

Lane was a very fine man. I had come to Chicago in 1926 to run the high hurdles in the National Intercollegiate Track and Field Meet at Soldiers Field. I placed in the finals and some members of the U. S. Olympic Committee urged me to keep working for the 1928 Olympics. So I worked on the Stagg Field track until an accident set off a series of leg infections. I was very sick in Billings Hospital in the days before antibiotics and it was Lane who came to the hospital to see me and make sure that I got the best available care. The only way I was ever able to express my thanks to him was to do a similar service to some of my own students in later years. I guess that is the only way we ever thank our teachers.

Bliss was an outstanding master of the lecture-discussion. He could come into a class in calculus of variations obviously unprepared, because of the demands of his chairmanship, and still deliver an elegant lecture, drawing the students into each deduction or calculation, as he looked at us quizzically and waited for us to tell him what to write. His students learned their calculus of variations very thoroughly. Yet we did not work together, except in so far as we presented class assignments. Each research student reported to Bliss by appointment. The subject itself had come to be too narrowly defined as the study of local, interior minimum points for certain prescribed functionals given by integrals of a special form. Generalization came only at the cost of excessive notational and analytic complications. It was like defining the ordinary calculus to consist exclusively of the chapter on maxima and minima. A sure sign of the decadence of the subject was Bliss's project to produce a

history of it, like Dickson's *History of the Theory of Numbers.* The history reached publication only in the form of certain theses imbedded in *Contributions to the Calculus of Variations,* 4 vols, 1930–1944, University of Chicago Press.

It is perhaps surprising that this narrowly prescribed regimen turned out men who did important work in entirely different areas as, for example, A. S. Householder did in biomathematics and numerical analysis, and Herman Goldstine did in computer theory. Among all of us Magnus Hestenes has been most faithful to the spirit of Bliss's teaching in carrying on research in the calculus of variations. Yet when Pontryagin's Optimal control papers revived interest in the subject many years later, students of Bliss were easily able to get into it. Optimal control theory really contained relatively little that was correct and not in the calculus of variations. In fact, optimal control was anticipated by the thesis of Carl H. Denbow, *loc. cit.*

Quantum mechanics was breaking wide open in the twenties. Bliss himself got into it with his students by studying Max Born's elegant canonical variable treatment of the Bohr theory. While that was going on, Sommerfeld's *Wellenmechanische Ergänzungsband* to his *Atombau und Spektrallinien* [Vieweg, Braunschweig, 1929] came out. It was the first connected treatment of the new wave mechanics formulation of quantum mechanics due to de Broglie and Schrödinger. We dropped everything to study wave mechanics. Bliss was a remarkably knowledgeable mathematical physicist and quite expert in the boundary value problems of partial differential equations. That was not so remarkable in a mathematician of his generation. The narrowing of the definition of a mathematician and withdrawal into abstract specializations was just beginning. In fact Bliss had been chief of mathematical ballistics for the U.S. Government in World War I, and later was commissioned to do a mathematical study of proportionate representation for purposes of reassigning Congressional districts. Bliss did not follow up his move into quantum mechanics but returned to the classical calculus of variations.

There were always more students in summers with all the teachers who came. Visiting professors like Warren Weaver, E. T. Bell, C. C. Mac Duffee and Dunham Jackson came to teach. And there was the memorable visit of G. H. Hardy which was supposed to provide a uniting of Hardy's analytic approach to Waring's theorem with Dickson's algebraic approach. Even with this infusion of talent, the offerings of the department were rather narrow. Besides having no topology as such, more surprisingly, there was little in complex function theory. And I do not recall being in a seminar, either a research or journal seminar. Essentially all teaching was done in lectures. Yet the only one of the abler students who I remember taking the initiative to go elsewhere was Saunders MacLane, when he did not find at Chicago what he was looking for.

I once asked Edwin B. Wilson, a famed universalist among mathematicians, how he came to switch from analysis to statistics at Yale. With a humorous twinkle he said: "An immutable law of academia is that the course must go on, no matter if all of the substance and spirit has gone out of it with the passing of the original teacher. So when (Josiah Willard) Gibbs retired, his courses had to go on. And the department said: 'Wilson, you are it'." A graduate student at Chicago in the late twenties could see this immutable academic law in effect. In each line of study of the, then passing, old Chicago department, a younger Chicago PhD had been designated to carry on the work. If, in one's immaturity, this was not apparent, the point was made loud and clear in a blast from Dickson during a colloquium with graduate students present. Dickson charged the chairman with permitting the department to slide into second rate status. It was true that the spirit of original investigation had given way to diligent exposition in some of these fields. In some cases the fields themselves had gone sterile.

It was the lot of Bliss to preside over this ebb cycle of the department. He did an impressive best

possible with what he had, with high mathematical standards, firmly, kindly and quietly. Most of the difficulties he had inherited. Bliss was able to appoint some outstanding young men but, if he had asked for the massive financial outlay to bring in established leading mathematicians to make a new start like the original one under President Harper, the support would not have been forthcoming, even with a mathematician, Max Mason, as president and certainly not with the young Robert M. Hutchins, bent primarily on establishing his new college. It took the Manhattan Project, the first nuclear pile under the Stagg Field bleachers and Enrico Fermi to convince Hutchins of the importance of physical science and mathematics and to throw massive resources into the reorganization of the department near the end of World War II. Such reorganizations are necessary from time to time in every graduate department. They can be effective only when the time is right. It is the mark of a great university to recognize the necessity to break the immutable law of academia, and the opportunity, and to do it when the time is right. However, there were deep hurts, symbolized by Bliss's refusal ever again to set foot in Eckhart Hall to his death. But this is really getting ahead of my story.

It was no ebb cycle for the University of Chicago as a whole in the twenties. There was intellectual excitement in many places in the university. I attended the physics colloquia where the great innovators of the day came to talk. With Mr. Bliss's grudging consent, I took Arthur Compton's course in X-rays. He already had the Nobel Prize for his work on the phenomena of X-rays colliding with electrons. Yet he seemed so naïvely simple minded to me, far less expert and mentally profound than other physicists in the department. Somewhere in here Einstein came for a brief visit. He permitted himself to be escorted by the physics graduate students for a tour of their experiments. To one he offered a suggestion. The brash young man explained immediately why it could not work. Einstein shook his head sadly. "My ideas are never good," he said.

Michelson, another Noble Prizeman, was around, though retired. So was the great geologist, Chamberlin, with his planetesimal hypothesis in cosmology. In biology and biochemistry the great breakthroughs on the chemical nature of the steroid hormones and their effects on growth and development were excitingly unfolding. Young Sewall Wright was attracting students to his mathematical genetics. Economics promised a real breakthrough, though as it turned out, it was slow in coming. Linguistics was burgeoning. Anthropology and archeology were still actively following up the results of digs in Egypt, Turkey and Mesopotamia. The great debates over the truth of theories of relativity and quantum mechanics were raging. What was later to be planet Pluto had been observed as "Planet X" but heated arguments persisted on what it really was. On Sundays the University Chapel produced a succession of the leading Christian and Jewish spokesmen of the day. The textbook, *The Nature of the World and of Man*, H. H. Newman ed., University of Chicago Press, 1926, by illustrious Chicago faculty members was the best survey of physical and biological knowledge for college students that I have ever seen, though now dated, of course.

And outside the university the dangerous and ugly city of Chicago nevertheless had its charms, cultural and otherwise, that could take up all the time (and money) of a country boy. One could hear Mary Garden or Rosa Raisa at the Chicago Opera by getting a job as usher or super, or attend a fiesta in honor of the patron saint of some Halstead Street community that maintained its identity with the home village in the old country. One could drink wine at Alexander's clandestine speakeasy. For recall that it was Prohibition and the height of the bootlegging days of Al Capone and rival gangs. The famous Valentine Day massacre was just one of the lurid stories in the Chicago Tribune. We students formed an informal protective association to promulgate rules to optimize safety for oneself and date. One old boy from Georgia, a graduate student in history, was so impressed by our admonition never to approach a car asking him to get in, that, when a police car challenged him with order to stop, he just took off in a blaze of speed. Caught later, out of breath, his one phone call brought some of us to

police court to testify to his character. The officer who had made the arrest moved to dismiss the charges on the condition that "the defendant appear at Soldiers Field next Saturday and run for our company in the policemen's track meet." But it was grim business. Police, armed with machine guns, in such a car once arrested me on suspicion of rape on the Midway (not guilty!). Other students were mugged, raped, robbed and even killed.

Like today it was a time of inflation and most of us were poor. I had a full fellowship of $410, of which $210 had to be returned in tuition for three quarters. A dormitory room cost $135 out of what was left. We could get cheap meals at the Commons, and on Sundays one could go to the Merit Cafeteria and splurge on a plate-sized slab of roast beef. It cost 28¢ but it was worth it. We all looked forward to a teaching job, I think. Those jobs required 15 hours of teaching for about $2700. Soon the depression hit and, if we were lucky, we kept our jobs with salary cut to $2400. Some beginning salaries for Chicago PhD's were as low as $1800 in the early thirties.

Before closing these recollections I must write something about women as graduate students in those times, not long after the victory of women's suffrage. Only years later did I learn that it was considered unladylike to study mathematics. Many of the graduate students in mathematics were women. In fact there were 26 women PhD's in mathematics at Chicago between 1920 and 1935. I shall mention only a few by name. Mayme I. Logsdon (1921) was in the faculty of the department. Mina Rees (1931) was already showing the kind of ability that led her to a distinguished administrative career at Hunter College and CUNY. She did more than any other person to gain federal support for mathematics through her position as chief, Mathematics Branch ONR, when the National Science Foundation was established. Others included Abba Newton (1933), chairman at Vassar, and Frances Baker (1934) also of Vassar, Julia Wells Bower (1933), chairman at Connecticut College, Marie Litzinger (1934), chairman, Mt. Holyoke, Lois Griffiths (1927) Northwestern, Beatrice Hagen (1930) Penn State, and Gweneth Humphreys (1935) Randolph Macon. Graduate students married graduate students, though of necessity only after the man had his degree. In the department Virginia Haun married E. J. McShane. Emily Chandler, student of Dickson, married Henry Pixley and continued her publishing and teaching career at the University of Detroit. Antoinette Killen married Ralph Huston. They both later taught at Rensselaer Polytech. Aline Huke married a non-Chicago mathematician, Orrin Frink, and continued her teaching at Penn State. Jewel Hughes Bushey was in the department of Hunter College. These, and a number of others, were able to continue their professional work in spite of family obligations. Even intermarriage between departments was permitted! My wife to be, Mary Hardesty, was in zoology. We got our PhD degrees in the same commencement.

Looking back on those days, I wonder if the current women's liberation has even yet succeeded in pushing the professional status of women to the level already reached in the twenties. Maybe this time women can hold their gains in universities.

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# THE RELEVANCE OF MATHEMATICS

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Several years ago, I was asked by one of my colleagues in the Department of Mathematics at the University of Chicago to give a general nonmathematical lecture on mathematics to a nonspecialist audience consisting mainly of undergraduates. For this curiously unorthodox venture into semi-mathematical rhetoric, I chose the title: *Is mathematics relevant, and if so, to what?*

Even though the title is not facetious, it is mysterious. It is mysterious not because it contains exotic or technical terms far from our common experience of the use of language. Rather, it contains two ordinary terms, "mathematics" and "relevant" with which all but a small minority of us are familiar. The mystery lies in their ordinariness and their frequent use in ambiguous and unthoughtful ways. To get a meaningful answer to the question, we must clarify its meaning in a significant way.

Let me begin with the word "relevant." As it has been customarily used in the past few years, *relevant* refers to a relationship between some institution or mode of action, and a body of values or purposes. The modifying clause, *and if so, to what*, points to the basic vagueness of the customary usage of *relevant* by asking implicitly: What body of values or purposes? We know that different bodies of values or purposes have been emphasized or pursued within different social or historical contexts, on the basis of different perspectives of the important aspects of the human condition. If we presume an unambiguous meaning for the word *relevance*, the way in which the question is posed asks if we have reached a consensus with one another on the nature of the Good. It can also be taken as asking, if we wish, if we are willing to accept definitions of value and purpose arising out of particular aspects of human action as forming a spectrum of diverse human values, which hopefully can be tied together in a coherent view of the human condition.

What has all this to do with mathematics, aside from the practice which arose a few years ago of making public demands that all intellectual institutions should prove their relevance? To answer this question in a convincing way, I shall have to lay the appropriate foundation by clarifying what we mean by the word *mathematics*. Before I proceed to this task, let me note an interesting and important historical precedent for any discussion of the relation of mathematics and values. A little more than 2300 years ago, a very celebrated lecture was given in Athens on a closely related theme. This was the famous "Lecture on the Good" by Plato, the most influential of the world's philosophers, and it is the only lecture (or set of lectures) by Plato of which there is any objective evidence. Among his listeners was Aristotle, then a student in the Platonic Academy and later a critic of Plato's doctrines. Aristotle put down his testimony on the contents of Plato's lecture (or lectures) in a treatise in three books called "On the Good" of which no fragment remains, but which was paraphrased in the writings of Aristotles's disciples. "Many attended the lecture under the impression that they would obtain some of the human goods, such as riches, health, power, or above all a wonderful blissfulness. However, when the exposition began with mathematics, number, geometry, and astronomy and the Thesis 'The class of the Limit taken as One is the Good', the surprise became general. A part lost interest in the subject, the others criticized him." Plato is reported to have said: "The foundations of all things are the One and Indeterminate Magnitude, or the 'More or less'." In present day terms, what he seems to have said is that the *Archai*, the foundations of both the physical and moral orders (which he did not distinguish from one another) are the processes by which there is generated the sequence of integers or natural numbers and the continuum. Thus the report by Aristotle on Plato's most fundamental Unwritten Doctrine is that Plato solved the problem of the relationship between the foundations of the Good and of mathematics by identifying the two.

Plato's solution of the problem of value by identifying the Good with mathematics is one that very few of us nowadays would be willing to defend, especially in public. The report on Plato's Unwritten Doctrine by Aristotle and his followers has become a perennial scandal in the history of philosophy, and over the ensuing two thousand years continues to generate scholarly controversy to the present day as to whether such a respectable man as Plato could have held such raffish beliefs. Some of the psychological consequences are apparently so drastic that one extreme wing of the classical fraternity led by Professor Harold Cherniss of the Institute for Advanced Study in Princeton has tried to get rid of the problem by declaring on principle that Aristotle was a tendentious and misleading reporter on all his philosophical predecessors and contemporaries and therefore his testimony should be ignored. More weight (and the force of the evidence) can be assigned to views like those of W. D. Ross in his book *Plato's Theory of Ideas* that Plato did indeed put forward the views ascribed to him by Aristotle.

It seems clear that it was in the Academy, which Plato founded in Athens as the prototype of later institutions of higher education and research, that under the stimulus of Plato's emphasis upon the central role of mathematics and the new standards of logical rigor introduced by such logical critics as Zeno of Elea, the basic topics of Greek mathematics (geometry and the natural numbers) were first made the object of a reasoned penetrating logical development from first principles. Plato was himself a great patron of mathematics but not a mathematician. The Academy, however, sponsored and stimulated the work of some of the greatest mathematicians of its period, such as Theodorus, Thaetetus, and most important of all, Eudoxus who created the basic tools of Greek mathematical astronomy as well as resolving the logical problems of incommensurable magnitudes by creating the analogue of the modern theory of real numbers.

In one of his late Dialogues, the *Timaeus*, Plato expressed in writing the concept of a cosmos founded on mathematical principles. This concept was firmly opposed by Aristotle who tended to limit the role of mathematics to the heavenly bodies and, on the terrestrial level, to essentially the counting of individual instances of a phenomenon or type. The grand Platonic vision of the universe organized on mathematical principles had relatively little influence in the classical world or in the Middle Ages after the death of Plato and his immediate disciples. The Platonists or Neo-Platonists of succeeding ages (Plotinus, Porphyry, Iamblichus, Proclus, and others) had great intellectual influence during the Roman Empire and indeed seem to be the chief fountain-head of every variety of high mysticism which has flourished since their time, but they took over all the mystical elements in Plato's thought and relatively little of his mathematical interest.

It took the great Scientific Revolution of the 16th and 17th centuries (as the great historian of science, Alexandre Koyré, has pointed out) to vindicate Plato's dream and bring it to fruition. The central point of the greatest achievement of the modern scientific world-view was the creation of a completely successful mathematical physics and astronomy by Isaac Newton as the culmination of the beginnings made by Galileo and Kepler to try to justify the Copernican vision. The Newtonian mathematical astronomy was the model of a fulfilled Platonic vision of the cosmos.

Though there are few open believers in the explicit doctrine of Plato that mathematics is identical with the Good, our intellectual world in many of its most active and vital branches continues to be dominated by great Platonic visions of mathematical order. In mathematical physics, the Newtonian cosmos has been replaced by the even more perfected Platonic vision of general relativity theory. Mathematics itself has been consumed over the past hundred years as well as pricked, delighted, and tormented by Cantor's great Platonic vision of the theory of the accomplished infinite, the precise reasoned theory of infinite magnitudes. The main course of modern physics has moved through the channel of the revolutionary development of quantum mechanics in the late 1920's, yielding a new and precise mathematical formalism whose consequences were inexpressible in terms of classical physical intuitions. One consequence was a fundamental mathematicization of the basic principles of chemistry. In more recent decades, we have seen the development of molecular biology founding the basic machinery of biological inheritance upon the geometry of the DNA molecule and the combinatorics of sequences of amino acids. New cosmological visions of the Platonic order have been

formulated for the origin and possibly the vanishing of the whole physical cosmos. On a more speculative note, we have the suggestive Platonic program of the theory of catastrophes formulated by Rene Thom and his disciples for the mathematicization of developmental biology. As a final example, let us note the recent history of linguistics and the movement called structuralism in the humanistic and social disciplines in France which argues for formal mathematical structures underlying all the varied spheres of human action and meaning.

We have proceeded in the discussion of the Platonic tradition in Western thought and its relation to the steeping of crucial areas of modern science in their mathematical foundations and visions without further attention to the clarification of what we mean by the word *mathematics*. To repair this omission, I shall have to proceed with care and less in the style of Plato than of Aristotle. I believe and propose to show in detail that there are indeed four fundamentally different meanings in basic usage of the term *mathematics*. Thus, I propose to speak not about Mathematics simple but Mathematics I, Mathematics II, Mathematics III, and Mathematics IV.

Mathematics I refers to the mathematical practice imbedded in the common life of mankind in all civilized societies, and most intensively in the advanced industrial societies of which the U.S. is the leading example. This kind of mathematics includes all the counting, measuring, and calculation which is part of the life process for almost all human beings in our society as well as the systems of calculation and measurement which underlie the organization of every economic system beyond the most primitive stage when money is introduced. In its higher reaches, Mathematics I includes the use of mathematical techniques in such activities as accounting, engineering, and architecture, the collection of statistical data, the counting of votes, not to speak of the tremendous transformation of social practice that has been brought about in recent years by the use of electronic computers. The criterion of relevance or value with respect to Mathematics I is social utility and (even though the concept of social utility is not completely transparent especially in its relation to individual utility) we tend to assign such value in terms of the effectiveness or efficiency of the mathematical techniques or practices involved toward specific goals of social effort. Such effectiveness is often achieved through the use of relatively standardized techniques which are not especially interesting from an intellectual point of view.

Mathematics II refers to the use of known mathematical techniques and concepts to formulate and solve problems in other intellectual disciplines. In terms of day-to-day practice, this is the primary function of mathematics in the physical sciences, and more recently, in the biological and social sciences. The intellectual difficulties which must be resolved and the ingenuity which must be applied are often of a high order of magnitude in these applications, but the standard of relevance within the framework of these applications is the usefulness of the result for the discipline to which it applied rather than the intrinsic interest and fruitfulness of the processes by which that result is reached. Mathematics II represents an ever expanding area of intellectual activity taking over ever larger portions of many intellectual disciplines, sometimes to the despair of those disciplines' more old-fashioned practitioners. As an example less familiar than some, let me refer to the application of mathematical or statistical techniques in the historical field, known under names like cliometry or prosophography, which proposes to obtain more objective or accurate information about historical trends by a detailed analysis of data concerning life patterns of social or economic groups in a given historical period. Since the relevance of the mathematical activity under Mathematics II is its fruitfulness for the solution of the intellectual problems of the extramathematical discipline to which it is applied, its value is thereby linked to the relevance of that discipline in its own terms.

By Mathematics III, I refer to the body of what is usually called *mathematical research*, to the investigation of the concepts, methods, and problems of the diverse mathematical disciplines. Historically, these have developed over two thousand years from the classical problems of geometry, the theory of numbers, the solution of algebraic equations, the solution of the differential equations of mathematical physics, and the study of the formal methods of mathematical reasoning themselves. Though originating from this basic stock, new and extremely vital mathematical disciplines have

emerged, whose power and fruitfulness both in solving the problems of older fields and in generating important insights in newer directions, have made them new generating foci of basic mathematical developments. Four of the most important examples are the theory of groups, analytic functions of one or several complex variables, functional analysis, and algebraic and differential topology. Let me note explicitly that the division between Mathematics III and Mathematics II is not the same as the rough division which is sometimes made between pure and applied mathematics. Much that is done in the best varieties of applied mathematics falls under Mathematics III since the criterion of demarcation is whether the mathematical practitioner is seriously interested in the concepts and methods of the mathematical investigation in their own right or is mainly absorbed in the result of the particular application. Mathematics III denotes what mathematicians themselves refer to as "real mathematics."

The criterion of relevance for Mathematics III for all its components, whether pure or applied, is the intellectual criterion of the effectiveness of the mathematical activities in resolving the unsolved problems of their mathematical subdiscipline, in improving the power and fruitfulness of their concepts and methodological tools, and in clarifying the logical structure of calculations and proofs. As many great mathematicians such as Hadamard and Poincaré have described in detail, the practice of this intellectual discipline on the highest level takes the form of a creative art which works on the objective material of given problem and concepts by means of inventive jumps and intuitions. These free creations of the mind (to use a phrase of which Einstein was fond) find their judgment in their effectiveness in tangibly transforming the state of the discipline and solving its problems. They are subject to the most stringent analysis and criticism. The practitioners on the highest level judge their path forward, however, not by the prudential analysis of the safest path but by an unarticulated intuitive grasp, sometimes in aesthetic terms of the undeveloped potentialities of a given state of a mathematical field.

So far, so good. But what of Mathematics IV? Indeed, I suspect that many mathematicians of my acquaintance might resent the notion that any category might be put somewhere 'above' or 'beyond' the concrete practice of mathematical research. Yet, I propose to put forward such a category which I believe essential to a complete description of the nature of mathematics. Mathematics IV differs from the preceding three subdivisions in not being the full-time activity of any sector of the mathematically-employed population, and yet represents a major element of the impulse and vitality of the other three categories, and a large part of their unity as well. By Mathematics IV, I refer to the vision of mathematics as the ultimate and transparent form of all human knowledge and practice. We come here to the most radical question about the meaning of the word "mathematics." In classical Greece (and purportedly, among the Pythagoreans), the word *mathematics* came into existence with the original meaning, *that which can be taught.* From the classical age in Greece through the Renaissance, mathematics came to be identified with the great structure of deductive geometry as embodied in Euclid's *Elements* and the writings of Archimedes. In the 16th and 17th centuries, it came to be identified with the new analytical methods and the techniques of the differential and integral calculus. From the 17th century on, however, a broader vision of mathematics arose in the minds of such intellectual innovators as Leibniz and Descartes, a vision of mathematics as the total science of intellectual order, as the science of pattern and structure. It was in this form that the vital impulse of the Platonic vision of the world was reborn in its most permanent form.

This new vision of mathematics has had many significant fruits in the centuries that followed. Its most important characteristic has been the ability to detect and analyze significant form in one domain of human experience (often a relatively technical domain in classical mathematics) and then apply the insight so obtained to illuminate apparently unrelated contexts of human thought and action. As a first illustration, consider the development and application of the theory of groups. The basic algebraic concept of *symmetry,* originated in its explicit form as a consequence of the study of the roots of algebraic equations in the work of Lagrange and Galois in the late 18th and early 19th centuries. During the course of the 19th century group theory became one of the leading themes of mathematical

development which was interwoven with many of the other central themes of the developing mathematics of the time: analytic functions of a complex variable, the foundations of geometry, the theory of matrices, and the study of ordinary and partial differential equations. Group theory itself became a central building-block and starting point of new mathematical developments and has remained so till the present day. However, it has also become the fundamental conceptual and formal tool of the mathematical description of the physical world in the 20th century from its earliest uses as the basis of crystallography to its present role as the foundation of the description of the fundamental particles of high energy physics.

As another illustration of a more extreme sort, consider the consequences of the development of mathematical logic and the foundations of mathematics in the first three decades of the 20th century. In the early 1930's, Kurt Gödel, to the astonishment of the mathematical world, proved that the program put forward by the celebrated German mathematician Hilbert for justifying the framework of classical 19th century mathematics by showing its consistency as a formal system could not be carried through. To derive this conclusion, Gödel had to give a precise description of the formal structure of what is meant by a mathematical proof within the restrictions placed by Hilbert. In so doing, he created a theory which shortly afterwards was christened the theory of recursive functions. What is most remarkable as a historical consequence, but was already appreciated at the time by such farsighted mathematicians as Turing, was that the theory thus created was the basic theory of what could be accomplished by machines like the digital computer. Indeed, the theoretical development of the digital computer rested and still rests upon the foundation thus laid by Gödel's work.

This broader concept of mathematics' as the science of significant form has furnished the background and world view for the brilliant triumphs of mathematical research in all of its varied concrete forms, but it has also been the ultimate rationale for the belief in the power of mathematical methods in their application to all the varied intellectual disciplines beyond the boundaries of mathematics in its most restricted sense. Within the sweep of this concept, one sees the world as governed by objective laws of form which can be discovered in the last analysis only by the dialectic between the creative insight of the individual or the creative fantasy of the individual discoverer on the one hand and the objective testing of the consequences of the intuitive insight or fantasy on the other. When this dialectic works, and it works surprisingly often and with a surprising consistency, one obtains the experience of a concept of an order that enhances rather than stifles individual creativity and spontaneity. It is this experience that seems to have lain behind the original vision of Plato, and it might well be that in this kind of union of order with spontaneity and creativity, there resides an educational value of mathematics for those who will not be mathematicians that goes considerably beyond the technical utility of mathematics.

It was a thought of this sort that was expressed by the Anglo-American philosopher Alfred North Whitehead who in one of his last essays put forward his own rewriting of Plato's lecture on the Good in an intellectual credo entitled "Mathematics and the Good" written for the volume dedicated to him in the Library of Living Philosophers. Whitehead wrote: "The notion of the importance of pattern is as old as civilization. Every art is founded on the study of pattern. The cohesion of social systems depends on the maintenance of patterns of behavior, and advances in civilization depend on the fortunate modification of such behavior patterns. Thus the infusion of patterns into natural occurrences and the stability of such patterns, and the modification of such patterns is the necessary condition for the realization of the Good. Mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relation of patterns. Here, we reach the fundamental justification for the topic of Plato's lecture. Having regard to the immensity of its subject matter, mathematics, even modern mathematics is a science in its babyhood. If civilization continues to advance, in the next two thousand years, the overwhelming novelty in human thought will be the dominance of mathematical understanding."

It is this new and sophisticated version of the Platonic vision expressed in Whitehead's words that I have called Mathematics IV.

All the four modes of existence of mathematics as I have described them above are of great antiquity. Certainly Mathematics I and II both clearly existed in Babylonian civilization where numerical calculation and algebraic manipulation reached remarkable levels of proficiency while they were applied to both mercantile transactions and the religiously-oriented charting of the heavens. In ancient Greece, as already remarked, Mathematics III and IV appeared in a full-fledged mature form. By their nature, the four forms of mathematical activity and consciousness are mutually autonomous since in any realistic sense, no one of the four can absorb any of the others. All such attempts at absorption usually represent an effort to destroy the influence or activity of whatever aspect of mathematics is supposed to be absorbed. On the other hand, when all the aspects of the mathematical enterprise are flourishing, their mutual interaction is usually intensive. In modern times, they have only flourished together. In terms of social influence, mathematics as a tool in every-day life and in the technical aspects of social life tends to stimulate the application of mathematics as a tool in the very varied intellectual disciplines. In turn, the latter tends to stimulate the development of mathematical research and the success of research in generating new concepts and principles yields the transcendental ideal of mathematical knowledge. In terms of intellectual influence, the pattern of forces usually runs in the opposite direction. The transcendent ideal of mathematical knowledge gives meaning and force to the impulse of mathematical research, which in turn yields new tools and impulses for the application of mathematics, first in other intellectual domains and through their successes, in the practical life of mankind.

What can we say on the basis of this picture about the relation of mathematics and value in our contemporary historical context and in our own society? The complete answer is both exhilarating and alarming. On the one hand, we see the tremendous intellectual vitality of mathematics and the mathematicized sciences in the past several decades, their tremendous successes in solving their own problems and creating new tools for the solutions of both technical and practical problems. On the other hand, while the mathematicization of society goes on in leaps and bounds, as far as the every-day consciousness of the mass of human beings in our society is concerned, the forms of this development become more and more alien to their outlook. What might have been an increase in human powers and freedom becomes a vaguely defined and somewhat monstrous threat looming in the background of present-day life. The technical as well as the mathematical tools of society take an external and bureaucratic form which depresses human possibilities rather than raising them.

It is my suspicion that the discrepancy between the intellectual power of our mathematical and scientific disciplines and their negative impact upon the thinking of non-mathematicians and non-scientists even (or, perhaps, we should say, especially) among the well-educated in our society may be attributed in no small part to a profound defect in our fundamental concept of mathematical and scientific education for the non-specialists. To put the matter in mathematical terms, the defect lies in viewing the teaching of mathematics in practical, technical, or research terms exclusively, i.e., in terms of Mathematics I, II, or III. We have failed in large measure to find ways to convey the spirit of Mathematics IV, the transcendent ideal of mathematics as a fundamental and universal form of knowledge. This failure is in turn due to the failure or refusal of many of our contemporaries among the mathematicians to recognize the validity or even the meaningfulness of such an ideal.

It is my hope that this failure represents a challenge that will be overcome in the historical period in which we play a part. It can be overcome only, as Whitehead told us, in the spirit that even modern mathematics is still in its potential infancy and that the overwhelming novelty in human thought for many centuries to come will be the dominant role of mathematical understanding.

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## MATHEMATICAL NOTES

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### INDUCED TRANSFORMATIONS OF THE DERIVATIVE-VECTOR

RAY REDHEFFER

Let $y$ and $f$ be infinitely differentiable functions. By Leibniz' rule the transformation $y \to fy$ induces a linear transformation on the derivatives of $y$, namely, $y \to fy$, $y' \to f'y + fy'$, and so on. The latter transformation can be written in matrix form thus:

$$\begin{pmatrix} f & 0 & 0 & 0 & \cdots \\ f' & f & 0 & 0 & \cdots \\ f'' & 2f' & f & 0 & \cdots \\ f''' & 3f'' & 3f' & f & \cdots \\ \cdots\cdots & \cdots & \cdots & \cdots & \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \\ y''' \\ \cdots \end{pmatrix} = \begin{pmatrix} fy \\ (fy)' \\ (fy)'' \\ (fy)''' \\ \cdots \end{pmatrix}. \tag{1}$$

Let the matrix on the left be denoted by $D(f)$. Since $f(gy) = (fg)y$ it follows that

$$D(f)D(g) = D(fg). \tag{2}$$

The transformation $y \to fy$ is not the only one that induces a linear transformation on the derivatives. Another possibility is $y(t) \to y[f(t)] = z(t)$. Here the linear transfromation is

$$\begin{pmatrix} f' & 0 & 0 & \cdots \\ f'' & (f')^2 & 0 & \cdots \\ f''' & 3f'f'' & (f')^3 & \cdots \\ \cdots\cdots & \cdots & \cdots & \end{pmatrix} \begin{pmatrix} y' \\ y'' \\ y''' \\ \cdots \end{pmatrix} = \begin{pmatrix} z' \\ z'' \\ z''' \\ \cdots \end{pmatrix} \tag{3}$$

and the matrix product apparently corresponds to the composite function, $g \circ f = g(f)$. There is a complication, however, because the argument of $y$ on the left of (3) is $f(t)$, while the argument of $z$ is $t$. If the matrix in (3) is denoted by $C(f)$, then

$$C(f)|_t C(g)|_{f(t)} = C(g \circ f)|_t.$$

To get a simpler relation, let $t$ be a fixed point of $f$; that is, a value such that $f(t) = t$. In this case the result becomes

$$C(f)C(g) = C(g \circ f) \qquad \text{(at any fixed point for } f\text{)}. \tag{4}$$

The theory of these matrices is briefly developed here, with emphasis upon the matrix $D(f)$. It is left for the reader to decide which of the results have analogs for $C(f)$ and which do not.

**1. Inverses and powers.** Since $D(1)$ is the identity matrix the equation $f(1/f) = 1$ gives

$$[D(f)]^{-1} = D(1/f) \tag{5}$$

at all points $x$ where $f(x) \neq 0$. More generally, iteration of (2) gives

$$[D(f)]^n = D(f^n), \qquad n = 0, 1, 2, 3, \cdots \tag{6}$$

and an extension to negative $n$ follows from (5). Applying (6) to $f^{1/m}$ instead of $f$ gives (6) for rational exponents, and if a passage to the limit can be justified, we conclude that

$$[D(f)]^p = D(f^p) \tag{7}$$

for all real $p$. Besides the infinite differentiability of $f$ (which is always assumed in discussing $D(f)$ or $C(f)$) the only additional condition needed is $f(x) > 0$.

We shall find that (7) is in fact valid; it is the special case $g(t) = t^p$ of Theorem 1.

**2. The main theorem.** If $f_n$ and $f$ are infinitely differentiable at a given value $x$, we define $\lim * f_n(x) = f(x)$ to mean

$$\lim_{n\to\infty} f_n^{(j)}(x) = f^{(j)}(x), \qquad \text{for} \qquad j = 0, 1, 2, 3, \cdots.$$

A similar interpretation applies to limits involving $D(f)$ or $C(f)$; for example, the equation $\lim * D(f_n) = D(f)$ means that a corresponding limit relation holds (in the usual sense) for each element of the matrix.

If $g$ is infinitely differentiable at $x$ one can always find an *approximating sequence* of polynomials $\{g_n\}$; that is, a sequence such that $\lim * g_n(x) = g(x)$. For example, let $g_n(t)$ be the $n$th Taylor polynomial for expansion of $g(t)$ about the value $t = x$. Then

$$g_n^{(j)}(x) = g^{(j)}(x) \qquad \text{for} \qquad n \geqq j$$

and $\{g_n\}$ has the required property. It is not assumed that the Taylor series converges.

Given such a sequence of polynomials $\{g_n\}$, we define

$$g[D(f)] = \lim * g_n[D(f)]$$

whenever the limit exists. The fact that under suitable hypotheses the limit does exist, and is independent of $\{g_n\}$, is part of the content of the following theorem:

THEOREM. *Let $f$ be infinitely differentiable at $x$ and let $g$ be infinitely differentiable at $t = f(x)$. Then $g[D(f)]$ exists at $x$ independently of the approximating sequence $\{g_n\}$, and is given by the explicit formula*

$$g[D(f)] = D[g(f)].$$

**3. Proof.** The mapping $f \to D(f)$ is linear and hence, by (6),

$$g_n[D(f)] = D[g_n(f)] \tag{8}$$

for every polynomial $g_n$. If we choose $g_n$ so that $\lim * g_n(t) = g(t)$ at the argument $t = f(x)$, then it follows that $\lim * g_n(f) = g(f)$ at the argument $x$. This gives

$$\lim * D[g_n(f)] = D[g(f)]$$

at $x$, and Theorem 1 follows from (8).

**4. Discussion.** If $M$ is a matrix, the value $g(M)$ is often defined by assuming that $g(z)$ has a convergent power series for $|z| < \rho$, say, and that, in some suitable norm, $\|M\| < \rho$. By contrast, Theorem 1 does not assume that $g$ has a convergent expansion, still less does it assume any condition $\|D(f)\| < \rho$. The analytic content of Theorem 1 is that $g[D(f)]$ can be computed at $x$ under the sole hypothesis that $g \circ f$ is well behaved at $x$.

The algebraic content of Theorem 1 is that the operation $f \to D(f)$ commutes with $f \to g(f)$, where $g$ is an almost arbitrary function. It is a triviality that $D$ also commutes with the operations of differentiation and translation;

$$\frac{d}{dx} D(f) = D\left(\frac{df}{dx}\right), \qquad E_a D(f) = D(E_a f),$$

where $E_a f(x) = f(x + a)$. (Operations having the latter property are said to be *shift invariant* [5].) When combined with the fact that $D$ is linear, and satisfies $D(fg) = D(f)D(g)$, these commutativity properties show that an algebraic, functional, differential or difference equation for $f$ leads, in general, to a corresponding equation for $D(f)$.

For example if $T(x) = D(\tan x)$ the fact that $\tan x$ satisfies $y' = 1 + y^2$ gives $T' = I + T^2$, a matrix equation which is not trivially reducible to a scalar one. The addition theorem for $\tan(x + a)$ gives a corresponding formula for $T(x + a)$, which becomes

$$T(a) = [T(0) + I \tan a][I - T(0) \tan a]^{-1}$$

when we set $x = 0$. Other examples are discussed more fully below.

**5. Pascal's square.** For real $p$ let a matrix be defined by

$$P(p) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ p & 1 & 0 & 0 & \cdots \\ p^2 & 2p & 1 & 0 & \cdots \\ p^3 & 3p^2 & 3p & 1 & \cdots \\ \cdots\cdots & \cdots\cdots & & & \end{pmatrix}.$$

This is obtained from $D(e^{px})$ by setting $x = 0$. The equations

$$e^{px} e^{qx} = e^{(p+q)x}, \qquad (e^{px})^q = e^{pqx}, \qquad \log e^{px} = px$$

give respectively

$$P(p)P(q) = P(p + q), \qquad [P(p)]^q = P(pq), \qquad \log P(p) = pA, \tag{9}$$

where $A = D(x)$ at $x = 0$:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \cdots\cdots & & & & \end{pmatrix}.$$

The third relation (9) in the form $P(p) = \exp(pA)$ gives the other two by inspection.

These equations generalize and supplement a number of known results [2, 4] including, in particular, the familiar fact that $[P(p)]^{-1} = P(-p)$. The latter follows from $q = -p$ in the first relation or $q = -1$ in the second.

**6. Subfactorials.** The subfactorial is defined by

$$(a)_n = a(a-1)(a-2)\cdots(a-n+1), \qquad n = 1, 2, 3, \cdots.$$

If we set $x = 1$ in the formula for $D(x^a)$ the result is the matrix

$$Q(a) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ (a)_1 & 1 & 0 & 0 & \cdots \\ (a)_2 & 2(a)_1 & 1 & 0 & \cdots \\ (a)_3 & 3(a)_2 & 3(a)_1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{pmatrix}.$$

The relations $x^a x^b = x^{a+b}$, $(x^a)^b = x^{ab}$, $\log x^a = a \log x$ give respectively

$$Q(a)Q(b) = Q(a+b), \qquad [Q(a)]^b = Q(ab), \qquad \log Q(a) = aB, \tag{10}$$

where $B = D(\log x)$ at $x = 1$:

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & 0 & 0 & \cdots \\ 2 & -3 & 3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

The third equation (10) in the form $Q(a) = \exp(aB)$ gives the other two by inspection.

**7. Stirling numbers.** The coefficients $s(n, k)$ in the expansion

$$(a)_n = \sum_{k \geqq 0} s(n, k) a^k$$

are called *Stirling numbers* of the first kind. Upon expanding $Q(a)$ in powers of $a$ and equating coefficients one can transform the above results into equivalent results for $s(n, k)$. In particular, the exponential representation leads to

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ s(1,k) & 0 & 0 & 0 & \cdots \\ s(2,k) & 2s(1,k) & 0 & 0 & \cdots \\ s(3,k) & 3s(2,k) & 3s(1,k) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \frac{B^k}{k!},$$

for $k \geqq 1$; hence the powers of $B$ can be used to compute the Stirling numbers. If the matrix on the left is denoted by $s(k)$, the formula gives the equation

$$s(j)s(k) = \frac{(j+k)!}{j!\,k!}\, s(j+k)$$

whose direct verification is left to the reader. Discussion of Stirling numbers from a somewhat different point of view can be found in [**1**, **2**, **3**, **5**].

**8. Bell polynomials.** In (4) if we set $f(x) = x^a$ and then take $x = 1$, the resulting matrix

$$M(a) = \begin{pmatrix} a & 0 & 0 & \cdots \\ a(a-1) & a^2 & 0 & \cdots \\ a(a-1)(a-2) & 3a^2(a-1) & a^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

satisfies $M(a)M(b) = M(ab)$, because $(x^a)^b = x^{ab}$. (Note that 1 is a fixed point for $x^a$.) Identities of this sort arise in the theory of Bell polynomials [**3**], which are connected with the derivative of composite functions, $f(g)$. A study of such derivatives has also been made by Schwatt [**6**]. Since knowledge of these matters is not essential for understanding the matrix $M(a)$, we prefer to omit the details. Our sole objective is to suggest that $C(f)$, like $D(f)$, is connected with combinatorial identities of current interest.

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## THE NON-COMMUTATIVE TRIANGLE INEQUALITY FAILS

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We wish to show that even a weak form of the triangle inequality fails for the absolute value of operators on a Hilbert space $\mathcal{H}$. This points out the importance of commutativity for the inequality.

A weak form of the triangle inequality for operators would state: there is $\lambda > 0$ such that $\forall\ A, B$, $|A+B| \leqq \lambda(|A|+|B|)$.

A still weaker form would state:

* $\forall\ A, B$, there is $\lambda > 0$ such that $|A+B| \leqq \lambda(|A|+|B|)$.

In [2], R. G. Douglas proved a theorem about majorization of operators on Hilbert space, of which the following is a special case:

THEOREM [2]. *Let $P$, $Q$ be positive operators on $\mathcal{H}$. The following are equivalent*: (1) *Range* $(P) \subseteq$ *Range* $(Q)$; (2) $P^2 \leqq \lambda^2 Q^2$ *for some* $\lambda > 0$.

By taking square roots, (2) implies $P \leqq \lambda Q$; however, $0 \leqq P \leqq \lambda Q$ does not necessarily imply $P^2 \leqq \lambda^2 Q^2$ [1]. Thus, we can ask the following

QUESTION: Does $\overline{\text{Range}\,(P)} \subseteq \overline{\text{Range}\,(Q)}$ imply $P \leqq \lambda Q$, where $0 \leqq \lambda = \lambda(P, Q)$?

Now, let $P$, $Q$ be bounded positive operators on $\mathcal{H}$, $0 \leqq P,\ Q \leqq I$. Then:

(i) $$-(P+Q) \leqq P - Q \leqq P + Q.$$

Also, $\ker(P+Q) = \ker(P) \cap \ker(Q) \subseteq \ker(P-Q)$, so $\overline{\text{Range}\,(P-Q)} \subseteq \overline{\text{Range}\,(P+Q)}$. But, of course, $\ker(P-Q) = \ker|P-Q|$ (where $|X| = \sqrt{X^*X}$), so $\overline{\text{Range}\,|P-Q|} = \overline{\text{Range}\,(P-Q)}$.

Thus:

(ii) $$\overline{\text{Range}\,|P-Q|} \subseteq \overline{\text{Range}\,(P+Q)}.$$

We shall now show that there are positive operators $P$, $Q$, $0 \leqq P,\ Q \leqq I$ such that (*) fails, i.e., for no $\lambda > 0$ is $|P-Q| \leqq \lambda(P+Q) = \lambda(|P|+|-Q|)$. In this case, though, (i) and (ii) hold, so we also obtain a negative answer to the Question above.

Let $n \geqq 1$; let $P_n = \frac{1}{2}\begin{pmatrix} \left(\frac{n-1}{n}\right)^2 & \frac{n-1}{n} \\ \frac{n-1}{n} & 1 \end{pmatrix}$; let $Q_n = \frac{1}{2}\begin{pmatrix} 1 & \frac{n-1}{n} \\ \frac{n-1}{n} & \left(\frac{n-1}{n}\right)^2 \end{pmatrix}$;

then $0 \leqq P_n,\ Q_n \leqq I_2$. Also,

$$|P_n - Q_n| = \frac{1}{2}\begin{pmatrix} \frac{2n-1}{n^2} & 0 \\ 0 & \frac{2n-1}{n^2} \end{pmatrix}.$$

By direct calculation, $|P_n - Q_n| \not\leqq (2n-2)(P_n + Q_n)$, but $|P_n - Q_n| \leqq (2n-1)(P_n + Q_n)$. (It can be shown that $2n-1$ is the smallest $\lambda$ such that $|P_n - Q_n| \leqq \lambda(P_n + Q_n)$; finite-dimensionality, (ii), and Douglas' theorem guarantee the existence of such a $\lambda$.)

Now, let $P = \bigoplus_n P_n$, $Q = \bigoplus_n Q_n$; then $0 \leqq P,\ Q \leqq I$. Also, $|P - Q| = \bigoplus |P_n - Q_n|$. If there were $\lambda > 0$ such that $|P - Q| \leqq \lambda(P + Q)$, then by compression, $|P_n - Q_n| \leqq \lambda(P_n + Q_n)$ for all $n$, and this is impossible. Hence, $(*)$ fails.

REMARK: That $P > 0$, $S = S^*$, and $-P < S < P$ does not imply $|S| \leqq P$ can easily be seen from the example:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad P = \begin{pmatrix} \sqrt{2+\varepsilon} & 1 \\ 1 & \sqrt{2+\varepsilon} \end{pmatrix}, \qquad 0 < \varepsilon < 2.$$

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## A CHARACTERIZATION OF THE BALL IN $\mathbf{R}^3$

ROBERT SINE

The purpose of this note is to prove the following

THEOREM. *A smooth convex body in* $\mathbf{R}^3$ *is a ball iff every path of light internally reflected lies in a 2-plane.*

*Proof.* If $K$ is a ball in $\mathbf{R}^3$ it is easy to see that the entire past and future of any ray of light which is reflected internally by the surface of $K$ lies in a 2-plane. Indeed, suppose $l$ is a segment in $K$ and $p$ is a point of the intersection of $\partial K$ with the linear extension of $l$. Let $N(p)$ be the normal to $\partial K$ at $p$. Any 2-plane which contains both $N(p)$ and $l$ satisfies the condition.

Now consider a smooth convex body $K$ in $\mathbf{R}^3$ (so that each point of $\partial K$ has a (unique) supporting 2-plane). Select $p$ in $\partial K$ and denote the unique support plane at $p$ by $(p)$ and the (unique) outward normal at $p$ by $N(p)$. Let $L$ be any 2-plane through $N(p)$. Pick $q$ in $L \cap \partial K$ and let $l$ be the segment (of a light ray) from $p$ to $q$. It is clear that $l$ together with its entire reflected past and future must lie in $L$ itself if it lies in any 2-plane. Thus the normal $N(q)$ at $q$ lies in $L$. If we follow the normal segment $N(p)$ backwards through $K$, the point $p'$ which is intersection of $N(p)$ and $\partial K$ is an "antipodal" point of $p$. The normal $N(p')$ is on the same line at $N(p)$ since it lies in all 2-planes through $N(p)$.

Now let $w$ be a point of $\partial K$ so that $N(w)$ is perpendicular to $L$ and let $q$ be any point of $\partial K \cap L$.

Let $L^*$ be a 2-plane through $N(w)$ and $q$. Then $N(q)$ lies in both $L$ and $L^*$. Thus each 2-plane $L$ through a normal has the property that it contains the normals of all of the points of $L \cap \partial K$ and these normals pass through a center $c$ of the section; ($c$ is the intersection of the linear extension of $N(w)$ and $N(p)$). It is clear that the section $L \cap \partial K$ is a convex body in $\mathbf{R}^2$ which is of constant breadth about the center $c$. Thus it is a circle. It is also then clear that all planes through a normal have the same center and $K$ is a ball. This finishes the proof.

REMARK: The fact that each normal is a double normal is not sufficient for $K$ to be a ball. For let $K_1$ be any non-spherical figure of constant breadth in $\mathbf{R}^3$. Let $K_2$ be the parallel body defined by

$$K_2 = \{w: \operatorname{dist}(w, K_1) \leqq \varepsilon\}.$$

Then $K_2$ is smooth and still of constant breadth. Every convex body of constant breadth has the double normal property [**1**, p. 126] and [**2**, p. 158]. But it is clear that $K_2$ is a ball if and only if $K_1$ is a ball.

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## COMMUTING MAPPINGS AND FIXED POINTS

GERALD JUNGCK

**Introduction.** In [7] Pfeffer proves (Proposition 1) that any involution $r$ of a circle $S$ has a fixed point iff there exists a free involution ($\neq r$) of $S$ which commutes with $r$. This result depicts an interdependence between the commuting mapping and the fixed point concepts. We wish to highlight this interdependence, but in a more general context. So consider the following.

PROPOSITION. *Let $f$ be a mapping of a set $X$ into itself. Then $f$ has a fixed point iff there is a constant map $h: X \to X$ which commutes with $f$ (i.e., $h(f(x)) = f(h(x))$ for all $x$ in $X$).*

*Proof* (Sufficiency). By hypothesis there exists $a \in X$ and $h: X \to X$ such that $h(x) = a$ and $h(f(x)) = f(h(x))$ for all $x \in X$. We can therefore write, $f(a) = f(h(a)) = h(f(a)) = a$, so that $a$ is a fixed point of $f$. (The necessity portion of the proof is included in the proof of our main result.)

The intent of this note is to offer a fixed point theorem in the spirit of the above proposition which will have the Banach contraction principle as a consequence. A broadening spectrum of the mathematical community — including nonspecialists — has reason for interest in Banach's fixed point theorem and natural generalizations thereof. One such reason is that "advanced calculus" texts refer to this classic result in various contexts (see [**5**], pp. 338–39) and now commonly employ it to prove the inverse function theorem (see e.g., [**3**], [**5**], [7]).

Before stating our main result, we remind the reader that a contraction is a mapping of a metric space $(X, d)$ into itself such that $d(f(x), f(y)) \leqq \alpha d(x, y)$ for some fixed $\alpha \in (0, 1)$ and for all $x, y \in X$. The Banach contraction principle states that any contraction of a complete metric space has a unique fixed point.

**Main theorem & corollaries.** The proof of our theorem appeals to the following lemma, which is easily verified upon noting that condition (i) ensures that the sequence $\{y_n\}$ is Cauchy.

LEMMA. *Let $\{y_n\}$ be a sequence in a complete metric space $(X, d)$. If there exists $\alpha \in (0, 1)$ such that* (i) $d(y_{n+1}, y_n) \leqq \alpha d(y_n, y_{n-1})$ *for all $n$, then $\{y_n\}$ converges to a point in $X$.*

THEOREM. *Let $f$ be a continuous mapping of a complete metric space $(X, d)$ into itself. Then $f$ has a fixed point in $X$ iff there exists $\alpha \in (0,1)$ and a mapping $g : X \to X$ which commutes with $f$ and satisfies*

$$(*) \qquad g(X) \subset f(X) \text{ and } d(g(x), g(y)) \leqq \alpha d(f(x), f(y)) \text{ for all } x, y \in X.$$

*Indeed, $f$ and $g$ have a unique common fixed point if $(*)$ holds.*

*Proof.* To see that the stated condition is necessary, suppose that $f(a) = a$ for some $a \in X$. Define $g : X \to X$ by $g(x) = a$ for all $x \in X$. Then $g(f(x)) = a$ and $f(g(x)) = f(a) = a (x \in X)$, so $g(f(x)) = f(g(x))$ for all $x \in X$ and $g$ commutes with $f$. Moreover, $g(x) = a = f(a)$ for all $x \in X$ so that $g(X) \subset f(X)$. Finally, for any $\alpha \in (0,1)$ we have for all $x, y$ in $X$:

$$d(g(x), g(y)) = d(a, a) = 0 \leqq \alpha d(f(x), f(y)).$$

Thus $(*)$ holds.

On the other hand, suppose there is a mapping $g$ of $X$ into itself which commutes with $f$ and for which $(*)$ holds. We show that this condition is sufficient to ensure that $f$ and $g$ have a unique common fixed point.

To this end, let $x_0 \in X$ and let $x_1$ be such that $f(x_1) = g(x_0)$. In general, choose $x_n$ so that (1) $f(x_n) = g(x_{n-1})$. We can do this since $g(X) \subset f(X)$. The relation $(*)$ and (1) imply that $d(f(x_{n+1}), f(x_n)) \leqq \alpha d(f(x_n), f(x_{n-1}))$ for all $n$. The lemma yields $t \in X$ such that (2) $f(x_n) \to t$. But then (1) implies that (3) $g(x_n) \to t$.

Now since $f$ is continuous, $(*)$ implies that both $f$ and $g$ are continuous. Hence (2) and (3) demand that $g(f(x_n)) \to g(t)$ and $f(g(x_n)) \to f(t)$. But $f$ and $g$ commute so that $g(f(x_n)) = f(g(x_n))$ for all $n$. Thus $f(t) = g(t)$, and consequently $f(f(t)) = f(g(t)) = g(g(t))$ by commutativity. We can therefore infer

$$d(g(t), g(g(t))) \leqq \alpha d(f(t), f(g(t))) = \alpha d(g(t), g(g(t))).$$

Hence, $d(g(t), g(g(t)))\,(1 - \alpha) \leqq 0$. Since $\alpha \in (0,1)$, $g(t) = g(g(t))$. We now have: $g(t) = g(g(t)) = f(g(t))$; i.e., $g(t)$ is a common fixed point of $f$ and $g$.

To see that $f$ and $g$ can have only one *common* fixed point, suppose that $x = f(x) = g(x)$ and $y = g(y) = f(y)$. Then $(*)$ implies $d(x, y) = d(g(x), g(y)) \leqq \alpha d(f(x), f(y)) = \alpha d(x, y)$, or $d(x, y)(1 - \alpha) \leqq 0$. Since $\alpha < 1$, $x = y$.

COROLLARY 1. *Let $f$ and $g$ be commuting mappings of a complete metric space $(X, d)$ into itself. Suppose that $f$ is continuous and $g(X) \subset f(X)$. If there exists $\alpha \in (0,1)$ and a positive integer $k$ such that (i) $d(g^k(x), g^k(y)) \leqq \alpha d(f(x), f(y))$ for all $x$ and $y$ in $X$, then $f$ and $g$ have a unique common fixed point.*

*Proof.* Clearly, $g^k$ commutes with $f$ and $g^k(X) \subset g(X) \subset f(X)$. Thus the theorem pertains to $g^k$ and $f$, so there is a unique $a \in X$ such that $a = f(a) = g^k(a)$. But then, since $f$ and $g$ commute, we can write $g(a) = f(g(a)) = g^k(g(a))$, which says that $g(a)$ is a c.f.p. of $f$ and $g^k$. The uniqueness of $a$ implies $a = g(a) = f(a)$. (See [4].)

We obtain the Banach contraction principle as a consequence of Corollary 1 if we set $k = 1$ and let $f$ be the identity map $i(x) = x (x \in X)$. In fact, if we let $f$ be the identity map and keep the "general $k$", we obtain the generalization of Banach's theorem given in [1] and in [2] (Remark 2, pg 88). Note that Corollary 1 did not require that $g$ be continuous.

On the other hand, let $\alpha = 1/K$ and $f = g^{n+1}$ in the statement of our main theorem. We leave it to the interested reader to verify that this substitution yields:

COROLLARY 2. *Let $n$ be a positive integer and let $K$ be a real number $> 1$. If $g$ is a continuous mapping of a complete metric space $(X, d)$ onto itself such that $d(g^n(x), g^n(y)) \geqq Kd(x, y)$ for $x, y \in X$, then $g$ has a unique fixed point.*

With $n = 1$ in Corollary 2 we obtain the conclusion corollary of [6].

We conclude with an example of two functions neither of which is contractive (so that Banach's theorem doesn't pertain), or "expansive" in the sense of Corollary 2 (with $n = 1$), which nevertheless satisfy the hypothesis of our theorem and hence have a unique common fixed point.

EXAMPLE. Let $X = R^2$ — Euclidean two space with the usual metric which we denote by $d$. Define $f, g : X \to X$ for $p = (x, y)$ by $g(p) = (7x, (y/3) + 4)$ and $f(p) = (11x, (y/2) + 3)$. Then $f(g(p)) = (77x, (y/6) + 5) = g(f(p))$, so that $f$ and $g$ commute. Also it is easy to show that $d(f(p), f(q)) \geqq (3/2)d(g(p), g(q))$. We can therefore let $\alpha = 2/3$ in the statement of the theorem to conclude that $f$ and $g$ have a unique common fixed point. Now let $q = (x', y')$. If we set $y = y'$ (see def. of $p$), we have $d(f(p), f(q)) = 11d(p, q)$. Thus $f$ is not a contraction. Similarly, if we let $x = x'$, $d(f(p), f(q)) = (\frac{1}{2})d(p, q)$ so that $f$ is not "expansive." Of course, the same substitutions yield comparable conclusions for $g$.

In closing, we do observe that the requirement $(*)$ of our theorem can be appreciably weakened if we demand that $(X, d)$ be compact. Consider the following new result which we state here without proof. If we let $C_f$ denote the set of all mappings $g : X \to X$ which commute with $f$, we can say the following.

THEOREM. *Let $f$ be a continuous mapping of a compact metric space $(X, d)$ into itself. Then $f$ has a fixed point iff whenever $f(x) \neq f(y)$*

$$d(g(x), g(y)) < d(f(x), f(y)) \quad \textit{for at least one } g \in C_f.$$

(Note that the choice of $g$ need not be the same for different pairs $(x, y)$.)

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## GROUPS THAT ARE THE UNION OF FINITELY MANY PROPER SUBGROUPS

JACK SONN

What groups are equal to the set-theoretic union of finitely many proper subgroups? This question has been raised in earlier issues of this MONTHLY [1, 2] and is answered by the following theorem.

THEOREM 1. *A group is the set-theoretic union of finitely many proper subgroups if and only if it has a finite non-cyclic homomorphic image.*

It is clear that the condition in Theorem 1 is sufficient; for if $G$ has a finite non-cyclic homomorphic image $\bar{G}$, then since $\bar{G}$ is the union of its cyclic subgroups (finite in number), the union of their preimages in $G$ is equal to $\bar{G}$. The converse is a consequence of the following lemma.

LEMMA 1. *If a group $G$ is equal to the set-theoretic union of finitely many subgroups, then any of these subgroups which are of infinite index can be dropped, and the remaining ones still cover $G$.*

Indeed, it follows from Lemma 1 that a group $G$, which is the union of finitely many proper subgroups, is the union of finitely many proper subgroups of finite index. Each of these subgroups has only finitely many conjugates, so the intersection $N$ of all the conjugates of all these subgroups is normal and of finite index in $G$. $G/N$ is then finite and is the union of finitely many proper subgroups, hence is non-cyclic.

*Proof of Lemma* 1: Let $\{H, H_1, \cdots, H_n\}$ be a finite set of subgroups of $G$ whose union equals $G$, and suppose that the index $[G: H]$ of $H$ in $G$ is infinite. We shall show that $\{H_1, \cdots, H_n\}$ already covers $G$.

There is an infinite set of distinct (left) coset representatives of $H$ in $G$, and therefore infinitely many must lie outside of $H$ and in at least one $H_i$, say $H_1$ (renumber if necessary). Let $\{x_i \mid i = 1,2,3, \cdots\}$ be an infinite sequence of distinct such elements. Suppose that $H_1, \cdot\cdot, H_n$ does not cover $G$. We proceed to define recursively sequences $S_j = \{y_i^{(j)} \mid i = 1,2, \cdots\}$ in $G$, $j = 1,2, \cdots, n$, such that, with appropriate renumbering of the $H_j$, $S_j$ lies inside $\cap\{H_i \mid 1 \leqq i \leqq j\}$ and outside of $H$. $S_n$ will then lie in all the $H_i$ and outside of $H$, and if $h$ is an element of $H$ lying outside all the $H_i$, then any element $hs$, where $s \in S_n$, will lie outside $H$ and all the $H_i$, a contradiction.

Set $S_1 = \{x_i \mid i = 1,2, \cdots\}$. Now suppose recursively that $1 \leqq j < n$, and $S_j$ is an infinite sequence of distinct elements of $G$ lying outside $H$, inside $H_1, \cdots, H_j$, and that the elements of $S_j$ are pairwise incongruent mod $H$ ($a, b$ congruent mod $H$ means $a^{-1}b \in H$). Note that $S_1$ satisfies these conditions.

We construct a subsequence $\{z_k\} = \{y_{i_k}^{(j)}\}$ of $S_j$ such that mod $H$, each finite segment $z_k z_{k+1} \cdots z_{k+t}$ is congruent neither to 1 nor to any other such finite segment. Namely, set $z_1 = y_1^{(j)}$, and if $z_1, \cdots, z_m$ are defined, choose $z_{m+1}$ to be a $y_i^{(j)}$ with the property that for each $k = 1, \cdots, m$, the product $z_k z_{k+1} \cdots z_m y_i^{(j)}$ is congruent mod $H$ neither to 1 nor to any segment $z_r \cdots z_{r+s} (1 \leqq r \leqq r+s \leqq m)$. That such a $y_i^{(j)}$ exists follows from the fact that for each $k, m, r, s$, $1 \leqq k \leqq m$, $1 \leqq r \leqq r+s \leqq m$, the congruences

$$z_k \cdots z_m u \equiv 1 \pmod{H}, z_k \cdots z_m v \equiv z_r \cdots z_{r+s} \pmod{H}$$

have unique solutions $u, v$ respectively mod $H$, hence all solutions lie in a finite number of cosets of $H$. The hypothesis on $S_j$ of pairwise incongruence of its terms yields a $y_i^{(j)}$ lying outside these cosets. Moreover, since all $z_i$ lie in $H_1, \cdots, H_j$, so do all finite segments of the $z_i$.

Now consider the sequence

$$\{hz_1 z_2 \cdots z_i \mid i = 1,2, \cdots\}$$

of products of an element $h$ of $H$ lying outside all the $H_i$, with the initial segments $z_1 z_2 \cdots z_i$. This sequence lies outside of $H, H_1, \cdots, H_j$. Since $\{H, H_1, \cdots, H_n\}$ covers $G$, an infinite subsequence $\{hz_1 \cdots z_{i_k} \mid k = 1,2, \cdots\}$ lies in one of $H_{j+1}, \cdots, H_n$, say $H_{j+1}$ (renumbering if necessary). It follows that the segments

$$z_{i_k+1} \cdots z_{i_{k+1}} = (hz_1 \cdots z_{i_k})^{-1} hz_1 \cdots z_{i_{k+1}}$$

lie in $H_{j+1}$, hence in $H_1, \cdots, H_j, H_{j+1}$. Furthermore, they are congruent mod $H$ neither to 1 nor to each other, hence the desired sequence $S_j$ is obtained:

$$y_k^{(j+1)} = z_{i_k+1} \cdots z_{i_{k+1}}, \qquad k = 1, 2, \cdots.$$

The proof is complete.

REMARK 1. It was pointed out to me by J. J. Rotman that Lemma 1 is a special case of the following lemma of B. H. Neumann **[3]** (which predates **[1]** and **[2]**!).

LEMMA 2. *If a group $G$ is equal to the set-theoretic union of finitely many cosets of subgroups, then any of these cosets belonging to subgroups of infinite index can be dropped, and the remaining ones still cover $G$.*

It is interesting that the proof of Lemma 2 is by induction on the number of subgroups, and does not work for Lemma 1 as stated, because the induction hypothesis becomes too weak. At the same time, the proof of Lemma 1 does not seem to generalize to a proof of Lemma 2.

REMARK 2. It follows from Theorem 1 that infinite simple groups (such as the classical simple linear groups over infinite fields, for example the projective special linear groups $PSL(n, k)$, $n > 1$, $k$ an infinite field) cannot be finitely covered. Other examples are furnished by abelian groups such as the additive group of rational numbers. (For a detailed discussion of the abelian case, see [4].)

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## RESEARCH PROBLEMS

EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy; Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N IN4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge, England CB2 1SB.)*

### A BOUND FOR THE CHROMATIC NUMBER OF A GRAPH

CYRIEL VAN NUFFELEN

We deal with simple graphs $G$ [2]; i.e., finite, undirected, without loops or multiple edges. In the adjacency matrix $A = [a_{ij}]$ of $G$, $a_{ij} = 1$ if the vertices $x_i$ and $x_j$ are adjacent, otherwise $a_{ij} = 0$. $r(G)$ denotes the rank of $A$, calculated over the real numbers. $K_n$ is the complete graph with $n$ vertices, and the complementary graph $\bar{K}_n$ consists of $n$ isolated vertices. In a complete $k$-partite graph $G = (X, E)$, the set of vertices $X$, can be partitioned in $k$ disjoint classes $X_1, X_2, \cdots, X_k$; such that any two vertices in a same class are never adjacent, and any two vertices in different classes are always adjacent. The chromatic number $\gamma(G)$ of a graph $G$ is the smallest number of colours necessary to colour the set of vertices so that no two adjacent vertices have the same colour.

The work of A. J. Hoffman, [6], [7], implicitly contains the following upper bound for the chromatic number $\gamma(G)$. There exists a fixed function $f$ such that:

$$\gamma(G) \leqq f(r(G)).$$

We propose the following

CONJECTURE. *For a simple graph $G$, other than $\bar{K}_n$,*

$$\gamma(G) \leqq r(G);$$

*moreover, $\gamma(G) = r(G) = k$, if and only if the non-isolated vertices form a complete $k$-partite graph.*

We were able to confirm this conjecture in the cases: $\gamma(G) \leqq 3$; $G$ is $\gamma$-perfect, [1], [3]; $G$ is $\gamma$-imperfect-critical, [3], [9]; the conjecture either holds for $G$ or for the complementary graph $\bar{G}$, [4], [8].

Finally, it is easy to see that for a complete $k$-partite graph $G$, $r(G) = \gamma(G) = k$, [10], [11], [12].

Then by inspection it is clear that this bound is not included in the several known bounds on $\gamma(G)$, (for a survey of these bounds see e.g. [2] or [5]).

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## THE FIXED POINT PROPERTY FOR NON-EXPANSIVE MAPPINGS

SIMEON REICH

Let $C$ be a non-empty subset of a Banach space $(E, \| \|)$. A mapping $f: C \to C$ is said to be **Lipschitzian** if there is a constant $k \geqq 0$ such that $\| f(x) - f(y) \| \leqq k \| x - y \|$ for all $x$ and $y$ in $C$. Such a mapping is called **non-expansive** if $k = 1$, and a **strict contraction** if $k < 1$.

According to Banach's classical fixed point theorem [1, p. 160] a strict contraction has a (unique) fixed point (that is, a point $y$ such that $f(y) = y$) if $C$ is closed. This is no longer true in general when $f$ is assumed to be merely non-expansive. However, in 1965 Kirk [12, p. 1004] proved that a non-expansive self-mapping of a weakly compact convex $C$ has a fixed point provided $C$ possesses normal structure. (A convex $K \subset E$ is said to have **normal structure** [3] if each convex bounded subset $S$ of $K$ with positive diameter $d$ contains a point $x$ such that $\sup\{\| x - y \|: y \in S\} < d$.) In other words, such a subset $C$ has the fixed point property for non-expansive mappings. The same result

follows from Browder's proof of a less general theorem [**5,** p. 1041]. (Browder assumed that $E$ is uniformly convex.) Göhde [**10,** p. 252] established Browder's theorem (independently) by a different method which does not lead to Kirk's result. (Fixed points of non-expansive functions are known to be useful in the study of nonlinear evolution equations [**4,** p. 1101].)

PROBLEM. Does every weakly compact convex subset $C$ of a Banach space $E$ have the fixed point property for non-expansive mappings?

One cannot apply Tychonoff's fixed point theorem [**15,** p. 770] to obtain an immediate affirmative answer to this problem because a non-expansive mapping need not be weakly continuous. Simple examples [**12**] show that the answer is negative when "weakly compact" is replaced by "bounded and closed," or when "convex" is omitted, or when "non-expansive" is replaced by "Lipschitzian" (with a Lipschitz constant $k > 1$). On the other hand, the answer is affirmative for many Banach spaces $E$ (for example, those which satisfy Opial's condition [**11**] or are uniformly convex in every direction [**8**]) in which every weakly compact convex subset $C$ has normal structure. (In such spaces the non-empty fixed point set of a non-expansive $f: C \to C$ is a non-expansive retract of $C$ [**6,** Theorem 2].) Unfortunately, not all weakly compact convex subsets of Banach spaces possess normal structure [**2,** p. 439], and the above problem has remained open despite the recent extensive activity in fixed point theory [**7**]. Note that we have here a nonlinear problem in a linear setting.

Kirk's theorem, which is still the best positive result to date, can also be proved by replacing the normal structure assumption by another technical condition involving asymptotic centers [**14,** p. 254], a condition which is equivalent to normal structure by the results of [**3**] and [**13**].

Finally, let us mention another fact which may indicate that the answer to our problem is "yes". Let $F$ be a uniformly convex Banach space. There exists $t > 1$ with the following property: A weakly compact convex subset of a Banach space $E$, whose Banach-Mazur distance coefficient with respect to $F$ (that is, $\inf\{\|T\| \, \|T^{-1}\|: T: E \to F$ is an isomorphism$\}$) is smaller than $t$ has the fixed point property for non-expansive mappings. This is a consequence of [**9,** Theorem 1].

**Acknowledgment.** I am grateful to the referee and to the editor, Professor Richard Guy, for their helpful comments.

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## CLASSROOM NOTES

Edited by Richard A. Brualdi

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### THE MULTIPLICATION OF DETERMINANTS

D. C. Lewis

There are perhaps half a dozen different proofs given in the literature to the classical theorem asserting that the determinant of the product of two square matrices is equal to the product of the two determinants of the individual matrix factors. There is, of course, always the vague question as to how much variation in two proofs should entail a feeling that the two proofs are really essentially different. But nobody is likely to gainsay that the three proofs given in the standard texts of Bôcher, Kowalewski, and Birkhoff and MacLane are quite different from each other.

The purpose of this note is to present still another proof which I have used in the classroom and which has been particularly suitable from the point of view of conciseness and also from the point of view of what properties of determinants and matrices it is convenient to develop beforehand. Until recently I have never seen in the existing literature anything like it. But in the preparation of this note a more thorough search has revealed the idea in a book by G. Doster entitled *Éléments de la théorie des determinants*, pp. 65–71. The method is there incompletely developed; it is applied explicitly only to three rowed matrices and even so misses some points of rigor.

Our proof, as distinguished from most other proofs of the same theorem, uses analysis to obtain a purely algebraic result. Indeed the fundamental theorem of algebra (which despite its name is a theorem of analysis rather than of algebra) is fundamental to our method. We also use the further result from analysis, deduced from the implicit function theorem, to the effect that the roots of an algebraic equation of the $n$th degree are continuous functions of the coefficients, at least as long as the $n$ roots are distinct.

Theorem 1. *If $P$ and $Q$ are both $n \times n$ matrices, if $P$ is non-singular and if $Q$ has distinct eigen-values, none of which are zero, then the two equations*

$$\det(xI - Q) = 0 \tag{1}$$

*and*

$$\det(xP - PQ) = 0, \tag{2}$$

*which are both equations of the $n$-th degree in $x$, have the same roots all of which are simple and none of which is zero.*

There appears to be no way by which the methods of the present note can be modified so as to yield the Binet-Cauchy formula directly. However, even though Theorem 3 is generally deduced as a consequence of the Binet-Cauchy formula in most of the texts where the latter is discussed, it is also possible to deduce the Binet-Cauchy formula as a consequence of Theorem 3.

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## CERTAIN CONGRUENCES THAT HOLD IDENTICALLY

Edwin Hewitt

A simple exercise in elementary number theory shows that all squares of integers, written in decimal notation, have one of 0, 1, 4, 5, 6, or 9 as units digits, while cubes can have any units digit, and fourth powers can end only in 0, 1, 5, or 6. For fifth powers we have the curious identity $x^5 \equiv x \pmod{10}$ for all integers $x$. Mr. Michael D. Drew, in a class taught by the writer, has asked for what other moduli this phenomenon appears. The answer is simple, and may perhaps be of interest to other teachers and students. The writer is grateful to the referee for a great simplification and to Mr. Christopher Mueller for pointing out an error in the original proof.

Theorem. *Let $m$ be an integer greater than 1. There exists a positive integer $l$ such that*

(i) $x^{l+1} \equiv x \pmod{m}$ *for all integers $x$ if and only if*
(ii) *$m$ is square-free.*

*For square-free $m$, write $m = p_1 p_2 \dots p_r$, the $p_j$'s being distinct primes. The smallest value of $l$ for which* (i) *holds identically is the least common multiple of the integers $p_1 - 1, p_2 - 1, \dots, p_r - 1$.*

*Proof.* The necessity of (ii) is immediate. If $p$ is a prime such that $p^2 \mid m$, let $r = m/p$. We have $r^2 = m(m/p^2) \equiv 0 \pmod{m}$, while $r \not\equiv 0 \pmod{m}$. Thus we have $r^{l+1} \equiv 0 \not\equiv r \pmod{m}$ for all positive integers $l$.

To prove the sufficiency of (ii), suppose first that $p$ is any prime. For $(x, p) = 1$ and $h$ any positive integer, Fermat's theorem shows that $x^{h(p-1)} \equiv 1 \pmod{p}$ and so

$$x^{h(p-1)+1} \equiv x \pmod{p}. \tag{1}$$

If $(x, p) \neq 1$, then $p \mid x$ and (1) is obvious for the integer $x$.

If $p_1, p_2, \dots, p_r$ are distinct primes and $l$ is any common multiple of $p_1 - 1, p_2 - 1, \dots, p_r - 1$, then (1) shows that

$$x^{l+1} \equiv x \pmod{p_j} \tag{2}$$

for $j = 1, 2, \dots, r$, and so (i) holds since $m = p_1 p_2 \dots p_r$.

Finally, suppose that (i) holds for some $l$. Then we also have (2). Let $x_j$ be a primitive root modulo $p_j$, i.e., an integer prime to $p_j$ such that $x_j^s \equiv 1 \pmod{p_j}$ implies that $s$ is a multiple of $p_j - 1$. (See for

example [2], p. 47, Theorem 2.25.) For such $s$, (2) yields

$$x_j^{l+1} \equiv x_j \pmod{p_j}, \qquad x_j^l \equiv x_j^{l+1+s-1} \equiv x_j^s \equiv 1 \pmod{p_j},$$

and so $l$ must be a multiple of $p_j - 1$. □

We do not know the least positive value of $l$ for which there are $m$ distinct residues of $x^{l+1}$ modulo $m$ [e.g., $l = 2$ for $m = 10$]. A more detailed discussion of the integers modulo $m$ under multiplication may be found in [1].

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## TOPOLOGICAL OPERATORS, SUBSPACES, AND HEREDITARY NORMALITY

G. M. Rosenstein, Jr. and F. G. Slaughter, Jr.

The topology of a subspace $Y$ of a topological space $X$ is defined in such a way that the open and closed sets of $Y$ are obtained in a "nice" way from the open and closed sets of $X$; precisely, [1, Theorem 6.3]:

A subset $A$ of $Y$ is open [closed] in $Y$ if and only if there

$$\text{is an open [closed] subset } B \text{ of } X \text{ such that } A = B \cap Y.$$

It is natural to ask if other topological operators behave in a similar "nice" way. We show that the closure and interior operators do behave "nicely," and that a consideration of the boundary operator leads to a new characterization of hereditary normality.

We denote by Cl, Int, and Fr the closure, interior, and boundary operators respectively, and indicate the space on which the operator acts by an appropriate subscript. For the elementary properties of these operators and definitions of undefined terms, we refer the reader to any standard introduction to topology, for instance [1].

Theorem. *Let $X$ be a topological space, let $Y$ be a subspace of $X$, and let $A$ be a subset of $Y$.*

(i) *There is a set $B$ in $X$ such that $A = B \cap Y$ and $Cl_Y(A) = Cl_X(B) \cap Y$.*

(ii) *There is a set $B$ in $X$ such that $A = B \cap Y$ and $Int_Y(A) = Int_X(B) \cap Y$.*

*Proof.* (i) From [1, Theorem 6.3], $B$ may be chosen to be $A$.

(ii) There is a set $U$ open in $X$ such that $\mathrm{Int}_Y(A) = U \cap Y$. Let $B = A \cup U$. As $\mathrm{Int}_Y(A) \subset A$, $B \cap Y = A$; as $U \subset \mathrm{Int}_X(B)$, $\mathrm{Int}_Y(A) \subset \mathrm{Int}_X(B) \cap Y$. Finally, if $W$ is open in $X$ and a subset of $B$, $W \cap Y$ is open in $Y$ and a subset of $A$, so $\mathrm{Int}_X(B) \cap Y \subset \mathrm{Int}_Y(A)$.

A topological space $X$ is hereditarily normal or completely normal if and only if every pair of mutually separated sets may be separated by disjoint open sets. We do not assume that $X$ is a $T_1$-space.

Theorem. *For a topological space $X$, the following are equivalent*

(i) *$X$ is hereditarily normal.*

(ii) *If $Y$ is a subspace of $X$ and $A$ is a subset of $Y$, then there is a subset $B$ of $X$, which may be chosen to be open in $X$ if $A$ is open in $Y$, such that $A = B \cap Y$ and $Fr_Y(A) = Fr_X(B) \cap Y$.*

(iii) *If $Y$ is a subspace of $X$ and $A$ is a subset of $Y$ which is open in $Y$, then there is an open subset $B$ of $X$ such that $A = B \cap Y$ and $Cl_Y(A) = Cl_X(B) \cap Y$.*

(iv) *If $Y$ is a subspace of $X$ and $A$ is a subset of $Y$ which is closed in $Y$, then there is a closed subset $B$ of $X$ such that $A = B \cap Y$ and $Int_Y(A) = Int_X(B) \cap Y$.*

*Proof.* Statements (iii) and (iv) are equivalent by "taking complements." We prove if (i) then (ii); if (ii) then (iii); and if (iii) then (i).

*If* (iii) *then* (i). Suppose $A$ and $T$ are mutually separated sets in $X$. Let $Y$ be $A \cup \mathrm{Cl}_X(T)$. Then $A$ is $Y \backslash \mathrm{Cl}_X(T)$ as $A$ and $T$ are mutually separated, and $A$ is open in $Y$, so there is a set $B$, open in $X$, such that $A = B \cap Y$ and $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(B) \cap Y$. Since $A$ and $T$ are mutually separated, $\mathrm{Cl}_X(A) \cap T = \varnothing$, so $\mathrm{Cl}_X(B) \cap T = \mathrm{Cl}_X(B) \cap Y \cap T = \mathrm{Cl}_Y(A) \cap T \subset \mathrm{Cl}_X(A) \cap T = \varnothing$. Therefore $A$ and $T$ are separated by the disjoint open sets $B$ and $X \backslash \mathrm{Cl}_X(B)$.

*If* (ii) *then* (iii). Using (ii) to choose an open (in $X$) set $B$ such that $A = B \cap Y$ and $\mathrm{Fr}_Y(A) = \mathrm{Fr}_X(B) \cap Y$, we have

$$\mathrm{Cl}_X(B) \cap Y = [B \cup \mathrm{Fr}_X(B)] \cap Y = A \cup [\mathrm{Fr}_X(B) \cap Y] = A \cup \mathrm{Fr}_Y(A) = \mathrm{Cl}_Y(A).$$

*If* (i) *then* (ii) *(the only interesting part of the proof)*. Let $S = \mathrm{Int}_Y(A)$ and let $T = Y \backslash \mathrm{Cl}_Y(A)$. Then $S$ and $T$ are mutually separated in $X$, so there are disjoint sets $U$ and $V$ which are open in $X$ and which are such that $S \subset U$ and $T \subset V$. Let $A^* = \mathrm{Cl}_X(\mathrm{Fr}_Y(A))$; let $B = A \cup (U \backslash A^*)$.

We use two observations to simplify the proof that $B$ is properly chosen. The first is that, since $S \cap A^* = \varnothing$ and $U \backslash A^*$ is open in $X$,

$$S \subset U \backslash A^* \subset \mathrm{Int}_X(B). \tag{1}$$

Secondly, as $\mathrm{Fr}_Y(A) = \mathrm{Fr}_Y(Y \backslash A)$ and $\mathrm{Int}_Y(Y \backslash A) = Y \backslash \mathrm{Cl}_Y(A)$, we have $\mathrm{Cl}_Y(Y \backslash A) = (Y \backslash \mathrm{Cl}_Y(A)) \cup \mathrm{Fr}_Y(A)$. Both sets in this union miss $U \backslash A^*$, the former because it is contained in $V$, the latter because it is contained in $A^*$. Thus

$$\mathrm{Cl}_Y(Y \backslash A) \cap (U \backslash A^*) = \varnothing. \tag{2}$$

Notice now that if $A$ is open in $Y$, $A = S$, so from (1) we have $B = U \backslash A^*$ which is open in $X$.

We must show $A = B \cap Y$ and $\mathrm{Fr}_Y(A) = \mathrm{Fr}_X(B) \cap Y$. From (2) we have $(U \backslash A^*) \cap Y \subset Y \backslash \mathrm{Cl}_Y(Y \backslash A) \subset A$, so $B \cap Y = A \cup ((U \backslash A^*) \cap Y) = A$.

As $T \subset V$, $T \cap \mathrm{Cl}_X(U \backslash A^*) = \varnothing$, or $Y \cap \mathrm{Cl}_X(U \backslash A^*) \subset Y \backslash T = \mathrm{Cl}_Y(A)$. Thus, $Y \cap \mathrm{Fr}(B) \subset \mathrm{Cl}_Y(A) = \mathrm{Fr}_Y(A)$.

Finally, to show $\mathrm{Fr}_Y(A) \subset Y \cap \mathrm{Fr}_X(B)$, we need only show $\mathrm{Fr}_Y(A) \subset \mathrm{Fr}_X(B)$. Let $W$ be a neighborhood of an element of $\mathrm{Fr}_Y(A)$. Then $W$ meets $B$, for $W$ meets $A$. Using (2) we have

$$Y \backslash A \subset \mathrm{Cl}_Y(Y \backslash A) \subset Y \backslash (U \backslash A^*) \subset X \backslash (U \backslash A^*).$$

Since $Y \backslash A \subset X \backslash A$, $Y \backslash A \subset (X \backslash A) \cap (X \backslash (U \backslash A^*)) = X \backslash B$. Since $W$ meets $Y \backslash A$, $W$ meets $X \backslash B$, completing the proof.

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## A CHARACTERIZATION OF THE CONVERGENCE OF SUCCESSIVE APPROXIMATIONS

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Let $f$: $[0,1]\to[0,1]$ be a continuous function. It is easy to show that $f$ must have at least one fixed point, that is, a point $x$ in $[0,1]$ such that $f(x)=x$. One method of finding such a point is by successive approximation; i.e., for a point $x_1$ in $[0,1]$, define $x_{n+1}=f(x_n)$ and the resulting sequence $\{x_n\}$ is called the sequence of successive approximations of $f$. If the sequence $\{x_n\}$ converges, it is easy to show that it converges to a fixed point of $f$.

The usual method of showing that the sequence $\{x_n\}$ of successive approximations converges is to show that it satisfies the Cauchy Convergence Criterion; for every $\varepsilon>0$ there is an integer $N$, such that for all integers $j, k \geqq N, |x_j - x_k| < \varepsilon$. Chu and Moyer [2] have proven several other characterizations for the convergence of the successive approximations of $f$. Theorem 1 is a new characterization. It says that one may set $j=k+1$ in the Cauchy Criterion. Browder [1] has called this condition asymptotic regularity.

THEOREM. *Let $f$: $[0,1]\to[0,1]$ be a continuous function. Let $x_1$ be a point in $[0,1]$ and let $\{x_n\}$ denote the resulting sequence of successive approximations. Then the sequence $\{x_n\}$ converges to a fixed point of $f$ if and only if* $\lim(x_{n+1}-x_n)=0$.

*Proof.* Clearly $\lim x_{n+1}-x_n=0$ if $\{x_n\}$ converges to a fixed point. Suppose $\lim x_{n+1}-x_n=0$ and the sequence $\{x_n\}$ does not converge. Since $[0,1]$ is compact, there exist two subsequences of $x_n$ that converge to $\xi_1$ and $\xi_2$ respectively. We may assume $\xi_1<\xi_2$. It suffices to show that $f(x)=x$ for all $x$ in $(\xi_1,\xi_2)$. Suppose this is not the case and an $x^*$ exists in $(\xi_1,\xi_2)$ such that $f(x^*)\neq x^*$. Then a $\delta>0$ could be found such that $[x^*-\delta, x^*+\delta]\subset(\xi_1,\xi_2)$ and $f(\bar{x})\neq\bar{x}$ whenever $\bar{x}\in(x^*-\delta, x^*+\delta)$. Assume $\bar{x}-f(\bar{x})>0$ (the proof in the other case being analogous) and choose $N$ so that $|f^n(x)-f^{n+1}(x)|<\delta$ for $n>N$. Since $\xi_2$ is a cluster point, a positive integer $n>N$ must exist with the property that $f^n(x)>x^*$. Let $n_0$ be the smallest such integer. Then, clearly,

$$f^{n_0-1}(x)<x^*<f^{n_0}(x)$$

and since $f^{n_0}(x)-f^{n_0-1}(x)<\delta$ we must have

$$f^{n_0-1}(x)-f^{n_0}(x)>0 \text{ so that } f^{n_0}(x)<f^{n_0-1}(x)<x^*,$$

a contradiction.

The author is indebted to the referee for finding and correcting an error in the proof of this theorem as it was first presented.

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## ELEMENT OF AREA VIA EXTERIOR ALGEBRA

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Knowing that the element of arc-length $ds$ for a smooth curve in Euclidean three-space is given by the formula

$$ds^2 = dx^2 + dy^2 + dz^2, \tag{1}$$

I wondered why the calculus textbooks did not supply an analogous formula for the element of area $dS$ for a smooth surface. If a piece of surface is parametrized by parameters $u$ and $v$ from a plane domain, the textbooks did give (e.g., McAloon and Tromba, p. 752) the formula

$$dS = \left[\frac{\partial(y,z)^2}{\partial(u,v)} + \frac{\partial(z,x)^2}{\partial(u,v)} + \frac{\partial(x,y)^2}{\partial(u,v)}\right]^{1/2} du\,dv \tag{2}$$

for $dS$ in terms of the Jacobian determinants and the element of area $du\,dv$ in the Euclidean plane. Formula (1) is superior because it is independent of the parametrization.

The parameters can be eliminated from (2) by using the *exterior product* for differentials (see Flanders). If we replace $du\,dv$ in (2) with $du \wedge dv$ and recall the chain rule

$$dy \wedge dz = \frac{\partial(y,z)}{\partial(u,v)}\, du \wedge dv \tag{3}$$

etc., then the desired formula is

$$dS^2 = (dy \wedge dz)^2 + (dz \wedge dx)^2 + (dx \wedge dy)^2. \tag{4}$$

This formula generalizes nicely to arbitrary $p$-manifolds in Euclidean $n$-space (and not merely to hypersurfaces). The geometric meaning of (4) is that the square of the area of a parallelogram in 3-space is the sum of the squares of the areas of parallelograms obtained by orthogonal projections onto the coordinate planes. It is a nice exercise in linear algebra to generalize this "Pythagorean theorem for area" to $p$-dimensional volumes $dV_p$ of $p$-dimensional parallelipipeds in $n$-space; the result is

$$dV_p^2 = \sum_{i_1<i_2<\cdots<i_p} (dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p})^2 \tag{5}$$

(Schwartz, vol. 2, footnote p. 131).

Calculus textbooks also express the element of area $dS$ in terms of the normal vector to the surface. If $X = X(u,v)$ is a local parametrization of the surface, then $X_u = \partial X/\partial u$ and $X_v = \partial X/\partial v$ are linearly independent tangent vectors and their vector product $X_u \times X_v$ is the normal vector; $dS$ is then given by the formula

$$dS = \| X_u \times X_v \|\, du \wedge dv. \tag{6}$$

Here the length $\| X_u \times X_v \|$ of the normal vector is equal to the area of the parallelogram spanned by $X_u$ and $X_v$.

The analogue in higher dimensions of the vector product is the exterior product in the exterior algebra $\wedge E$ of the ambient Euclidean $n$-space $E$, for a vector $Y$ in $E$ belongs to the subspace $F$ spanned by linearly independent vectors $X_1, \cdots, X_p$ if and only if

$$Y \wedge (X_1 \wedge \cdots \wedge X_p) = 0 \tag{7}$$

(Greub, p. 183); in this sense $X_1 \wedge \cdots \wedge X_p$ is "normal" to $F$. Moreover, the inner product on $E$ induces an inner product on $\wedge^p E$ (Flanders, p. 14), and with respect to this inner product the length $\| X_1 \wedge \cdots \wedge X_p \|$ is just the $p$-dimensional volume of the $p$-dimensional parallelipiped spanned by $X_1, \cdots, X_p$. Formula (5) tells how to calculate $\| X_1 \wedge \cdots \wedge X_p \|^2$ in terms of the components of $X_1, \cdots, X_p$ with respect to the standard orthonormal basis of $E$. Hence, if a $p$-manifold in $n$-space is parametrized locally by $X = X(u_1, \cdots, u_p)$, and $\partial X/\partial u_i = X_i$ for $i = 1, \cdots, p$, we have the formula

$$dV_p = \|X_1 \wedge \cdots \wedge X_p\| \, du_1 \wedge \cdots \wedge du_p \tag{8}$$

as the generalization of formula (6).

Incidentally, in the case of a hypersurface ($p = n - 1$), we can recover the usual normal vector $X_1 \times \cdots \times X_{n-1}$ as follows: A vector $X$ is uniquely determined by its inner product $X \cdot Y$ with all other vectors $Y$, hence the equation

$$[(X_1 \times \cdots \times X_{n-1}) \cdot Y] E_1 \wedge \cdots \wedge E_n = X_1 \wedge \cdots \wedge X_{n-1} \wedge Y \tag{9}$$

defines the vector product $X_1 \times \cdots \times X_{n-1}$; here $E_1, \cdots, E_n$ is the standard orthonormal basis of Euclidean $n$-space $E$, and the basis vector $E_1 \wedge \cdots \wedge E_n$ for $\Lambda^n E$ determines the standard orientation of $E$.

As an application of this description, let us show that *a linear transformation of Euclidean n-space which preserves the volume of* $(n-1)$*-dimensional parallelipipeds must be an isometry.* For simplicity, we give the proof only when $n = 3$. Recall that for any linear transformation $T$,

$$TX \wedge TY \wedge TW = (\det T) X \wedge Y \wedge W. \tag{10}$$

Combined with (9) this gives

$$(TX \times TY) \cdot TW = (\det T) \, [(X \times Y) \cdot W]. \tag{11}$$

Let $T'$ denote the transpose of $T$; recalling that

$$T'A \cdot B = A \cdot TB \tag{12}$$

we get $T'(TX \times TY) \cdot W = (\det T)(X \times Y) \cdot W$ for all $W$, i.e.,

$$TX \times TY = (\det T)(T^{-1})'(X \times Y) \tag{13}$$

(transpose and inverse commute). If we now assume $T$ preserves area, we have

$$\|TX \times TY\| = \|X \times Y\|. \tag{14}$$

Combining (13) and (14) yields

$$|\det T| \, \|(T^{-1})'(X \times Y)\| = \|X \times Y\|. \tag{15}$$

Now every vector $Z$ can be written in the form $Z = X \times Y$; equation (15) says, therefore, that $(\det T)(T^{-1})'$ is an isometry. Since the transpose of an isometry is an isometry, we see that $T^{-1}$ is a constant times an isometry. That constant must be 1 if areas are to be preserved.

Note finally that in our formula (4) for the element of area $dS$, the rectangular orthonormal differentials $(dx, dy, dz)$ can be replaced with any other triple of orthonormal differentials (by the Pythagorean theorem for area). Thus for the cylindrical differentials $(dr, rd\theta, dz)$ we get

$$dS^2 = r^2(dr \wedge d\theta)^2 + r^2(d\theta \wedge dz)^2 + (dz \wedge dr)^2 \tag{16}$$

while for the spherical differentials $(d\rho, \rho d\phi, \rho \sin \phi d\theta)$ we get

$$dS^2 = \rho^2(d\rho \wedge d\phi)^2 + \rho^4 \sin^2 \phi \, (d\phi \wedge d\theta)^2 + \rho^2 \sin^2 \phi \, (d\theta \wedge d\rho)^2 \tag{17}$$

(compare Munroe, pp. 502–511, 669–673 and Flanders, p. 35).

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## A NOTE ON THE FERMAT PROBLEM

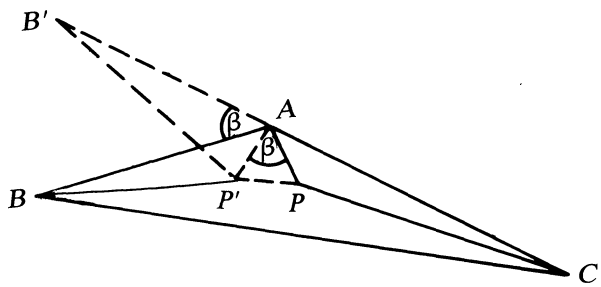
DANIEL SOKOLOWSKY

Fermat's Problem (to Torricelli) requires the determination of a point $P$ in the plane of a given $\triangle ABC$ which minimizes $f(P) = PA + PB + PC$. There are a variety of solutions of the problem for the cases (i) no angle of $\triangle ABC$ exceeds 120°; (ii) one angle of $\triangle ABC$ exceeds 120°. ([1], [2] deal with (i), leaving (ii) as an exercise; [3] deals extensively only with case (i); [4], [5] provide solutions for both cases.)

The main point of this note is to present a solution for (ii) suggested by the solution for (i) in [1]. (In connection with this latter solution we think it worth remarking that the method used therein works equally well when the domain of $P$, instead of being restricted to $\triangle ABC$, is enlarged to be the entire plane containing $\triangle ABC$. The same is true for our present solution of (ii).)

Thus, suppose $\angle A = \alpha > 120°$. We show the solution point $P$ required is the point $P = A$. All we need do is verify

$$PA + PB + PC > AB + AC \qquad \text{for } P \neq A. \tag{1}$$




To do so, let $\beta = 180° - \alpha$. Then $\beta < 60°$. Rotate $\triangle APB$ about the vertex $A$ through $-\beta$ to $AP'B'$. Then $B'$, $A$ and $C$ lie in a straight line. Clearly $PB = P'B'$, $AP = AP'$, and $PAP'$ is an isosceles triangle with vertex angle $\beta$. Since $\beta < 60°$, the equal base angles in this isosceles triangle exceed $\beta$, implying that the base $PP'$ is shorter than the arm $AP$. Consequently, the sum

$$\begin{aligned} PA + PB + PC &> PP' + P'B' + PC \\ &= \text{polygonal path } B'P'PC = B'C \\ &= B'A + AC = AB + AC, \end{aligned}$$

which is the sum $PA + PB + PC$ for $P = A$.

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## MATHEMATICAL EDUCATION

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### THE DESIGN AND USE OF AUDIO CASSETTES FOR A NUMERICAL METHODS COURSE

MYRON GINSBERG

**1. Introduction.** Within an undergraduate engineering curriculum it is imperative to provide a realistic view of the computer's influences on the efficiency, reliability, and accuracy of scientific calculations. Unfortunately, most traditional courses in numerical methods and introductory programming fail to deal adequately with these issues; little or no emphasis is placed upon practical error analysis, algorithmic strengths and weaknesses, factors affecting a method's computer implementation, use of mathematical software, program language limitations with respect to scientific computations, or language behavior variations across compilers and machines. A remedy for this situation is to develop inexpensive supplementary course materials which can be integrated into the curriculum with a minimum amount of teacher effort.

An attempt to make course revisions immediately provokes two questions: (1) What topics are to be removed from the classroom to make room for the new material? and (2) Are there sufficient reference sources available in the new areas to assist teachers? Our response to these problems is the development of an audio cassette package with accompanying text, exercises, supplementary readings, reference lists, programs and a teacher's guide. The tapes include a core of basic material along with individual modules discussing specific numerical methods and problems associated with various science and engineering disciplines. Our diverse and flexible modular cassette system can be easily incorporated into a variety of *existing* courses (either within or outside the classroom) or can be used on an individual, self-study basis. An audio cassette approach is convenient and relatively inexpensive; equipment is widely accessible with most students already owning a cassette player. Furthermore, the package of tapes and accompanying aids greatly relieves the burdens ordinarily placed on faculty by the introduction of new material.

In Section 2 we describe our efforts to produce an initial set of tapes. The modular design and extensions to a generalized package are discussed in Section 3 followed by possible uses of the system in Section 4 and summary and conclusions in Section 5.

**2. Initial efforts.** During the summer and fall of 1974 a feasibility study was conducted at SMU with funding by the Sloan Foundation to produce several audio cassettes and supplementary material to be used in connection with an undergraduate, introductory numerical methods course. To our knowledge, no attempt has been previously made with tape media applied to this subject area. We did, however, listen to sets of cassettes recorded by Holman [5] and Blum [1]; their objectives and the nature of their course materials differ substantially from our own. By analyzing their presentation styles and introducing our own attitudes toward the course material we developed our current format.

We have placed several restrictions on the content of the tapes. Each lesson is twenty minutes or less; this offers a relatively small learning unit. An effort is made to provide a smooth flowing and brisk presentation with an expressive voice; we assume this approach will help maintain student alertness and that if difficulties should arise, the listener can always playback the passage in question. The basic ideas of the recorded material are written in outline form and examined by the student as the tape is played; no prior reading of any kind is presently required. The taped lesson discusses a few ideas and analyzes some examples as the student follows along in an accompanying manual. With each lesson there is a set of exercises which requires the use of a computer. After the exercises are completed, the

student is issued a tape which discusses the problems; these tapes can be used to assist instructors as well as students.

Several aids have been developed to assist in understanding the presented material. Each lesson has sample computer output and a position paper summarizing work related to the area of the lesson (e.g., see Ginsberg [4]). Also references are given for supplementary and advanced reading assignments. Program decks or listings will be included in some lessons to assist students in acquiring practical experience; such routines may be models of various programming techniques or exemplify existing mathematical software. Accompanying the complete tape package is a comprehensive bibliography (see Ginsberg [3]) with approximately 875 references covering all the major specialities of numerical analysis, several related areas, computational comparison studies, and mathematical software; each section denotes "classics" in the field, "practical" or applied works, and sources with a large number of references. This guide should be particularly helpful for novices to the topics discussed in the tape units; it should also serve as a stimulus for individuals seeking increased depth in a particular area. These aids plus discussion with other students or a teacher should adequately round out the student's understanding of the taped material; he can always return to the original lesson, re-try the exercises, and/or read some of the suggested supplementary articles.

The student is also advised to maintain a diary of computer behavior observations as he performs the exercises; he is encouraged to create and perform his own experiments and, if possible, to make observations using two or more computers, compilers, or programming languages. Hopefully, astute observations will help the individual to build a profile of a computer's numerical behavior and environmental influences with respect to scientific calculations.

Production of the tapes and the student manual is an iterative process. Initially, the lecturer tapes each lesson as he looks at an outline; he then decides what the listener should be looking at as the tape is played. A written script is then composed from the tape. It is then modified along with manual material and the original tape is revised. After a few iterations, a final script is produced for the production tape. This process requires relatively minimal tape editing (usually more script revisions) and provides a smooth free-flowing natural presentation. The tapes are tested on students who evaluate them; necessary tape and text modifications then follow.

The subject matter of our initial tapes included three topics: floating-point arithmetic, subtractive cancellation, and error-bounding techniques. A brief outline of these lessons is given below:

1. Floating-Point Arithmetic (tapes 1 and 2)
    - A. Internal computer representation
    - B. Distribution over real line
    - C. Normalized and unnormalized arithmetic
    - D. Rounding vs chopping
    - E. Influence of number base and word length selection
    - F. In-and-out conversions
    - G. Effects on closure, associative, and distributive laws
    - H. Effects on inequality testing

II. Subtractive Cancellation (tape 3)
    - A. Examples of the phenomenon
    - B. Re-writing offending expressions
    - C. Use of double precision
    - D. Use of selected multiple precision

III. Error-Monitoring Methods (tape 4)
    - A. Motivation
    - B. Limitations of theoretical bounds
    - C. Difficulties in computing good, practical bounds
    - D. Interval analysis
    - E. Significant digit arithmetic
    - F. Multiple-precision arithmetic
    - G. Other specialized techniques

The above-mentioned tapes are still undergoing student testing and modification.

**3. Developing a general, comprehensive package.** The next step in building a basic core of tapes is to design modules in each of the areas usually covered in a numerical methods course: solution of algebraic equations (scalar and vector cases); interpolation and approximation; numerical differentiation and integration; numerical solution of ordinary differential equations. The material in these cassettes should supplement the traditional, in-class presentations and should focus attention on four aspects of each topic: (1) software influences; (2) hardware influences; (3) algorithmic behavior; and (4) error-monitoring techniques. For example, tapes in the solution of algebraic equations area could include the following subjects; encoding techniques for sparse matrices, data management for large systems, use of symbolic manipulation systems to find an exact solution (software influences); use of parallel machines for root-finding, assistance of a microprogramming facility, and effects of hardware paging on program efficiency (hardware influences); effects of pivoting and scaling, optimal methods, and bounds on total number of arithmetic operations (algorithmic behavior); use of interval analysis techniques and estimation of condition number (error-monitoring techniques). We could similarly define appropriate topics under our four category classification for each of the other areas included in the course. Thus our basic core of tapes would consist of introductory lessons in the four areas of computer influence, followed by one or more modules in each of the areas covered in a numerical methods course.

Once the core cassettes have been produced then the package can be extended to cover specific problems in a variety of science and engineering disciplines. As each new problem module is added to the system the user will have the option of either listening just to that specific tape or first listening to a sequence of suggested prerequisite core tapes; he would choose the former if he already had the equivalent of the prerequisite material and/or was not interested in a comprehensive, detailed view of the problem's background. These tapes would initially be arranged by fields but since each problem area cassette is dependent upon only a few prerequisite core tapes, users in one discipline could easily listen to specific problem tapes in another area without first having to listen to any of the other tapes in that area.

Prime candidates for problem application areas are physics and astronomy, electrical engineering, chemical engineering, mechanical engineering, computer science, mathematics, and numerical analysis. In physics and astronomy, potential topics include the use of large scale, parallel computers to study models of the upper atmosphere, nuclear fusion problems, and large scale simulations; applications of symbolic manipulation systems to problems in celestial mechanics, quantum electrodynamics, and general relativity; uses of error-bounding techniques to attack orbital mechanics problems. In electrical engineering, there is a need for a module dealing with the fast Fourier transform — its application and implementation. The chemical engineering area could probably use a unit on working with numerical solutions of stiff differential equations which arise in chemical kinetics problems; this topic is also of interest to electrical engineers working with network analysis problems. A module in the mechanical engineering field could discuss error-bounding approaches to problems involving flexural and torsional vibration of beams, continuous equilibrium, or self-sustained vibration of autonomous systems.

The computer science modules could concentrate on advanced discussions and expansion of the topics in the core tapes dealing with hardware, software, algorithmic, and error-monitoring factors: effects of compiler and operating system designs on numerical behavior and program efficiency; working with mini-computers for scientific computations; survey of the design and use of symbolic manipulation systems; extended precision arithmetic routines; analysis of specific mathematical software packages; survey of practical, computational complexity results; use of structured programming for numerical problems; limitations and influences of computer language design on scientific calculations.

The mathematics tapes could focus attention on efforts to formalize computer arithmetic behavior,

develop practical error-monitoring schemes, use theoretical error bounds, study stability problems, and prove program correctness for numerical algorithms. Also, cassettes could be introduced to present advanced topics in numerical analysis dealing with graduate and research level aspects of the basic subject areas covered in our numerical methods core tapes.

Many of the proposed topics in the above-mentioned problem application areas are on the frontiers of research. Our modular design would easily allow current, up-to-date additions to the package, thus keeping students closely informed about new advances related to the course and their own fields of interest.

**4. Applications of a cassette system.** Our initial tapes described in Section 2 are currently being used outside the classroom as part of the homework assignments for a sophomore-junior level numerical methods course. The tapes in the generalized package discussed in Section 3 could also serve as supplements for this course or for other courses in introductory programming, scientific programming, numerical analysis, mathematical physics, or engineering mathematics offered by computer science, mathematics, physics, or engineering departments. The modular system design makes such a package an attractive addition for these courses; the instructor can select from a diverse collection of tapes to find those applicable to his class needs and ignore most of the others (non-prerequisite material) if he chooses to do so.

We are considering use of the cassette approach in several other settings. If there are appropriate facilities, personnel, and sufficient interest, an experimental numerical mathematics laboratory (see Ginsberg [2]) can be organized in association with one or more existing numerical analysis or programming courses. Such a lab offers the student a first-hand opportunity to explore the numerical influences of various computer languages, compilers, machines (e.g., mini-computer, analog computer), and specialized mathematical software. Our tape system could be utilized during lab periods or as an outside supplement; the cassette set would increase the amount and/or depth of material which could be successfully covered.

Another application area is individualized instruction. For example, the package could form part of a self-study program to help engineers and scientists in industry to review a few areas or learn about new developments. The core tapes can assist in remedial work or be used to instruct transfer students who have had no previous exposure to many of the topics. Some of the tapes, particularly the more advanced problem application units, could serve as the basis for a reading project, or independent study course; part of this course could even include the production of new tapes for specialized topics.

The assistance a cassette system offers for many of the above-mentioned situations greatly reduces the burden on faculty resources without decreasing student guidance in new learning areas.

**5. Summary and conclusions.** We have described our current efforts and future plans to construct a collection of audio cassettes and related materials to fill the void between traditional, introductory courses in numerical methods and programming; these tapes have emphasized computational environmental influences: hardware, software, algorithmic behavior, and error-monitoring. The initial tapes are currently being tested and have been discussed in Section 2. We have proposed an extensive, generalized cassette package (in Section 3) consisting of a set of core tapes (environmental influences applied to the standard subject matter areas in a numerical methods course) and a set of specialized problem application tapes for various science and engineering fields. The latter sequence of tapes requires (optional) prior listening to certain pre-requisite core lessons. The modular system design permits use of the package in a wide variety of situations as described in Section 4 and allows easy modification and extension. The set can be implemented inexpensively and requires relatively little teacher resources.

The students in the class currently using the tapes are primarily sophomore and junior engineering students. They are enthusiastic about the adoption of an experimental approach similar to that used in most of their science and engineering courses. Many enter with the pre-conceived notion that it will be like most math courses, permeated by an atmosphere of theorems and proofs and a feeling that all the

major issues in the field are thoroughly resolved. The initial tapes and exercises shake their somewhat blind confidence in the computer's numerical reliability and in the application of several mathematical properties of the real line to the machine's discrete number structure. This situation tends to promote further individual experimentation and attempts to devise critical evaluating and testing procedures; it also motivates the use of theoretical tools which can assist in understanding the computer's behavior patterns. Although only a few tapes have been produced to date, the experimental approach is continued within the classroom throughout the entire course. When a more complete set of topics is available, it is hoped that they will evoke the same enthusiasm as in the current tape and lecture combination.

We plan to continue our developmental and testing activities with audio cassettes to determine their most appropriate applications and to extend the variety of available tapes. The desired net effect of the tape approach is to build a student's intution and critical evaluation powers in assessing computer influences and interactions on the numerical reliability and efficiency of his own programs and existing mathematical software. If this goal is achieved, then students should become more perceptive and intelligent computer users with respect to scientific computation.

**References**

1. H. A. Blum, A Compact Course in FORTRAN Programming, Audio Tutorial Associates, Inc., Dallas, Texas, 1970.

2. M. Ginsberg, The Design and Use of an Undergraduate Numerical Analysis Laboratory, Proc. Symposia Appl. Math., The Influence of Computing on Mathematics Research and Education, Amer. Math. Soc., 20 (1974) 135–140.

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4. ———, Introduction to the Study of Algorithms for Computing Upper and Lower Bounds to the Exact Solution of Problems in Numerical Analysis, Technical Report No. CP 73025, Department of Computer Science and Operations Research, Southern Methodist University, Dallas, Texas, September 1973.

5. J. P. Holman, Review and Problem Sessions to Accompany Heat Transfer (third edition), McGraw-Hill, New York, 1972.

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## USING ELECTRONIC PROGRAMMABLE CALCULATORS (MINI-COMPUTERS) IN CALCULUS INSTRUCTION

THOMAS O'LOUGHLIN

**Introduction.** A number of studies and projects have demonstrated that the use of a computer in conjunction with the teaching of calculus can be an effective means for improving student achievement. In 1973 a study was conducted at SUCC to investigate the effect of the incorporation of an electronic programmable calculator (mini-computer) in a first course in calculus. The rationale for the study was that the computational requirements for such a course are far below the capabilities of a computer (but well within the range of a mini-computer) and it is not reasonable to engage the computer to perform in a manner far beneath its potential (nor to have undergraduates compete for computer time with a faculty member or researcher whose studies require the capacity of the computer); the absence of a need for a formal programming language; the mini-computer is mobile and can be used in both the classroom and the mathematics laboratory; and over a period of a few years the cost of a mini-computer is less than the rental of a terminal.

**The mini-computer system.** The mini-computer system used in the study was a Hewlett-Packard 9810A programmable electronic calculator with a plug-in-mathematics function block, printer and

mark-sense card reader. In addition to the storage and programming capabilities of the 9810A, the additional equipment enabled the student to evaluate a number of commonly used mathematical functions ($\ln x, e^x, \sin x$, etc.), a print-out of all calculations and programs, and to enter previously verified programs into the mini-computer.

**The study.** A one-semester experimental course in beginning calculus was constructed that employed the mini-computer as an instructional aid in the classroom and as a student laboratory device. The experimental course is flexible, is not dependent on a specific text or brand of mini-computer and can be adapted to the traditional lecture method of teaching. The experimental course was specifically designed to be an improvement on the lecture method but to remain within the general format of the lecture method. The distinguishing characteristics of the experimental course and the incorporation of the mini-computer are:

(1) It provides the instructor with the opportunity to discuss numerous exemplars and non-exemplars of a concept.

(2) It includes topics normally omitted or reserved for a subsequent course.

(3) It includes more realistic applications of calculus.

Two sections of a first course in calculus were selected to serve as the control and experimental groups. The control group was taught in the traditional lecture method, and the course content consisted of the typical topics covered in beginning calculus. The treatment group was taught the experimental course with the content consisting of the topics covered in the control group plus additional topics in limits of functions, applications of the derivative and numerical integration. The method of instruction used with the treatment group was the lecture method augmented by the use of the mini-computer as a teaching aid. In addition to the regularly scheduled three lectures per week each student attended a recitation session. The recitation sessions for the control groups were devoted to discussing calculus problems while the treatment group's sessions also entailed exercises using the mini-computer. A series of tests were constructed to measure student achievement with respect to the following:

(1) Limits of functions.

(2) Continuity.

(3) Interrelationships between a function and its first two derivatives.

(4) Local extrema of a function.

(5) Solving verbal problems involving maxima-minima and related rates.

(6) The definite integral.

As it was impossible to exert any control over student registration in the two groups, it was decided to use a three by two analysis of variance with unequal cell frequencies to analyze the data. The entire sample was partitioned into three blocks on the basis of previously demonstrated achievement in mathematics in order to maintain some balance between the two groups with respect to mathematical ability.

**Results.** The data obtained from the analysis of variance implied that there was no significant difference in student achievement in the areas of limits of functions, continuity, and local extrema of functions. The data did imply that the method used in the treatment group did produce significantly higher student achievement in observing the interrelationships between a function and its first two derivatives ($F = 7.00$, $df = 1, 30$; $p < .05$), in solving verbal problems involving maxima-minima and related rates ($F = 3.87$, $df = 1, 29$; $p < .05$) and in interpreting the definition of the definite integral ($F = 3.40$, $df = 1, 29$; $p < .05$).

Examination of all the data obtained in testing student achievement confirms that the basic units underlying the construction of the experimental course are valid. Specifically, the described incorporation of the mini-computer into the beginning calculus course:

(1) permits the inclusion of topics normally omitted or reserved for subsequent calculus courses, without detracting from student performance on the usual topics covered;

(2) develops the student's understanding of the concept of a function and families of functions;

(3) promotes and improves the student's ability to apply sequential analysis in solving verbal problems;

(4) provides the student with a knowledge of the basic concepts of computer programming.

**Discussion.** The results of the study confirm the investigator's belief that the mini-computer can effectively be used to improve instruction in introductory mathematics courses, and that its capability is comparable to that of a computer when used as a teaching aid in beginning calculus. With respect to the capabilities of the mini-computer, it should be noted that all the exercises suggested by Hethcote and Schaeffer [1] can easily be executed on a mini-computer.

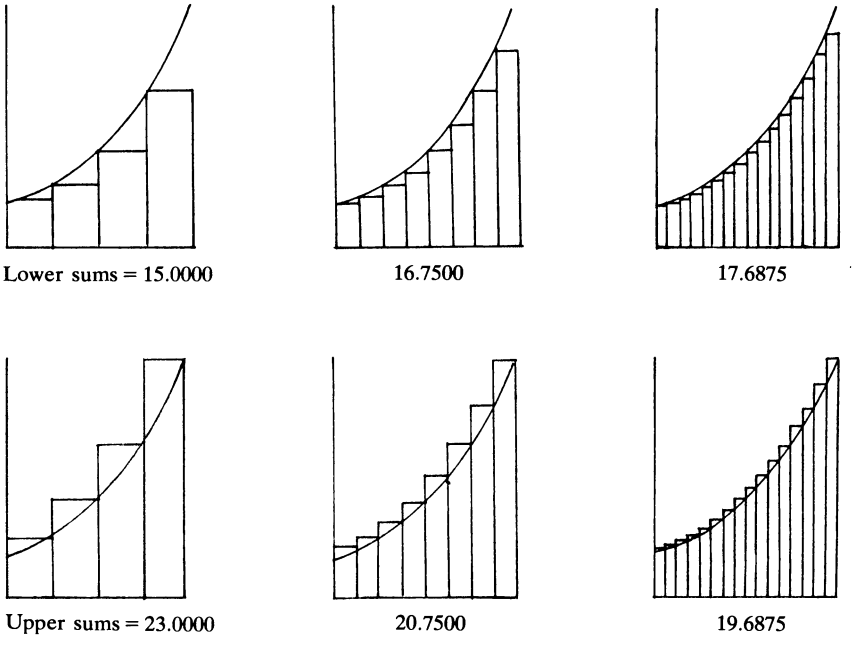



FIG. 1. Upper and Lower Sums for $f(x) = .5x^2 + 2$ on [0, 4]

Although it was not available at the time of the study, the mini-computer system used has been extended to include an $X - Y$ plotter. Initial investigations indicate that the addition of graphic representations of various problems and concepts increases the mini-computer's effect on student achievement. In particular, the simultaneous graphing and computing of lower and upper sums (see figure 1) aids the student in developing a strong intuitive understanding of the definition of the definite integral.

**Reference**

1. H. W. Hethcote and A. J. Schaeffer, A computer laboratory course for calculus and linear algebra, this MONTHLY, 79 (1972) 290–293.

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# PROBLEMS AND SOLUTIONS

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

## ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before July 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 2587. *Proposed by Bruno O. Shubert, Naval Postgraduate School*

Consider the system of $n$ equations

$$x_0 + x_k = \min_{j=1,\cdots,m} \max_{i=1,\cdots,n} (a_{ijk} + x_i), \quad k = 1, \cdots, n,$$

in $n+1$ unknowns $x_0, x_1, \cdots, x_n$, where the $a_{ijk}$ are given constants. Show that

(1) the system has always a solution, and

(2) the first component $x_0$ is unique.

E 2588. *Proposed by Stephen B. Maurer, Princeton University*

Let $A_n$ be the matrix of order $(2^n - 1) \times n$ where the $k$th row is the binary expression for $k$. Let $M_n = A_n A_n^t \pmod 2$. If $M_n$ is regarded as a matrix over the integers, what is its determinant?

E 2589. *Proposed by Joe Sunday, University of Guelph, Ontario*

Let $d_1, \cdots, d_n$ be distinct integers $> 1$. If $a_{ij} = \sin^2(j\pi/d_i)$ for $1 \leqq i, j \leqq n$, show that $\det(a_{ij}) \neq 0$.

E 2590. *Proposed by C. A. Nicol, University of South Carolina*

A natural number $n \geqq 2$ is said to be $\varphi$-subadditive if $\varphi(n) \leqq \varphi(k) + \varphi(n-k)$ for $1 \leqq k \leqq n-1$ and $\varphi$-superadditive if $\varphi(n) \geqq \varphi(k) + \varphi(n-k)$ for $1 \leqq k \leqq n-1$ ($\varphi$ denotes Euler's totient function). Show that there exist infinitely many $\phi$-subadditive numbers and infinitely many $\varphi$-superadditive numbers.

E 2591. *Proposed by Jan Mycielski, University of Colorado*

Prove that for every sequence $a_1, a_2, \cdots$ with $\lim a_n = 0$ there exists a sequence $b_1, b_2, \cdots$ with $b_1 \geqq b_2 \geqq \cdots \geqq 0$ such that $\Sigma b_n$ diverges and $\Sigma a_n b_n$ converges absolutely.

E 2592. *Proposed by Melvin Hausner, New York University*

Let $G$ be a finite group of even order $n = 2m$. Let $H$ be the set of all $x$ in $G$ with $x^m = 1$. Prove (a) $H$ is a subgroup of $G$, and (b) either $H = G$ or the index $[G: H]$ is 2.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### Six Congruent Conics

E 570 [1943, 260; 1974, 1026]. *Proposed by L. M. Kelly*

If the six conics, determined by each five of a set of six points, are congruent, must they coincide?

*Editor's comment.* The answer to this question is affirmative because the following theorem holds.

THEOREM. *There do not exist six points $P_i$ $(i = 1, \cdots, 6)$ in the plane and six congruent conic sections $C_i$ $(i = 1, \cdots, 6)$ such that $P_i \in C_j \Leftrightarrow i \neq j$ for all $i$ and $j$.*

The various cases of this theorem were proved as follows:

*Case* 1. $C_1$ is an equilateral hyperbola; proved by L. M. Kelly, *Covering Problems*, Math. Mag. 19 (1944), 123–130.

*Case* 2. $C_1$ is an ellipse; proved by J. J. Seidel and J. van Vollenhoven, *Zum Probleme Nr.* 43, Elem. der Math. 17 (1962), 85.

*Case* 3. $C_1$ is a pair of lines; proved in L. M. Blumentahl, *Theory and Applications of Distance Geometry*, 2nd ed., Chelsea, New York, 1970, p. 153, Theorem 61.1.

*Case* 4. $C_1$ is a parabola; proved in D. Pedoe, *Thinking Geometrically*, this MONTHLY 77 (1970), 711–721. The proof is attributed to B. Segre.

*Case* 5. $C_1$ is an arbitrary hyperbola; proved by R. C. Olson, *On the six conics problem*, Discrete Math. 10 (1974), 279–300.

In his paper Olson gives three interesting examples of points $P_i$ $(i = 1, \cdots, 6)$ and conic sections $C_i$ $(i = 1, \cdots, 6)$ such that $P_i \in C_j \Leftrightarrow i \neq j$ and

*Example* 1. $C_i$'s are two triples of congruent ellipses;

*Example* 2. $C_i$'s are two triples of congruent hyperbolas;

*Example* 3. $C_i$'s form a triple of congruent ellipses and a triple of congruent hyperbolas.

The above information was kindly supplied to us by R. C. Olson and O. P. Lossers.

### A Difficult Binomial Coefficient Summation

E 2384 [1972, 1034; 1974, 170]. *Proposed by H. W. Gould, West Virginia University*

Let $n$ be a nonnegative integer. For $p = 1, 2, \cdots$ define

$$S_p(n) = \sum_{k=0}^{[n/2]} \left[ \binom{n}{k} - \binom{n}{k-1} \right]^p,$$

where we make the usual conventions regarding binomial coefficients. Show that $S_3(n)$ is always divisible by $S_1(n)$.

*Solution and a generalization by Stuart Clary, City College of C.U.N.Y.* The key to this problem is replacing $S_p(n)$ with

$$S_p(m, x) = \sum_{k=0}^{m} \left[ \binom{x}{k} - \binom{x}{k-1} \right]^p.$$

Here and throughout, $x$ is a complex variable; $p$, $m$, $n$, and $k$ are integers with $p \geqq 1$ and $m \geqq 0$, and $\binom{x}{k} = x(x-1)\cdots(x-k+1)/k!$ with the conventions that $\binom{x}{k} = 0$ if $k < 0$ and that $\binom{x}{0} = 1$. We will prove that $S_3(m, n)$ is divisible by $S_1(m, n)$ whenever $S_1(m, n) \neq 0$, which is a generalization of the proposed problem.

If $p$ is odd, then $S_p(n+1, n) = 0$ for $n \geqq 0$, because the second half of the sum cancels the first half thanks to the identity

$$\binom{n}{k} = \binom{n}{n-k}.$$

Therefore $S_p(m, n) = 0$ when $0 \leqq n < m$, since the terms corresponding to $k = n+2, n+3, \cdots, m$ all vanish. In other words, the polynomial $S_p(m, x)$ has zeros at $x = 0, 1, \cdots, m-1$. In the case $p = 1$ these are the only zeros, and they are all simple; this is because the sum defining $S_1(m, x)$ telescopes, so $S_1(m, x) = \binom{x}{m}$. Thus for any odd $p$, $S_p(m, x)/S_1(m, x)$ is a polynomial in $x$; its degree is $(p-1)m$. Define $R_m(x) = S_3(m, x)/S_1(m, x)$, a polynomial of degree $2m$. We must prove that $R_m(n)$ is always an integer.

Here is a small table of $R_m(n)$. The numbers in parentheses are those which cannot be evaluated by direct substitution in the formula $R_m(x) = S_3(m, x)/S_1(m, x)$.

| $n$ | $R_0(n)$ | $R_1(n)$ | $R_2(n)$ | $R_3(n)$ | $R_4(n)$ |
|---|---|---|---|---|---|
| $-1$ | 1 | 7 | 1 | 7 | 1 |
| 0 | 1 | (3) | $(-6)$ | (9) | $(-12)$ |
| 1 | 1 | 1 | $(-3)$ | (9) | $(-18)$ |
| 2 | 1 | 1 | 1 | (3) | $(-12)$ |
| 3 | 1 | 3 | 3 | 1 | $(-3)$ |
| 4 | 1 | 7 | 6 | 7 | 1 |
| 5 | 1 | 13 | 19 | 19 | 13 |
| 6 | 1 | 21 | 57 | 49 | 57 |
| 7 | 1 | 31 | 141 | 163 | 163 |
| 8 | 1 | 43 | 298 | 541 | 472 |

The following three equations are suggested by the above table:

(1) $R_m(-1) = 1$ if $m$ is even, 7 if $m$ is odd.

(2) $R_m(n) = 3(-1)^{m+1}\binom{m}{n+1}$ if $0 \leqq n < m$.

(3) $R_m(n) = R_{n-m}(n)$ if $n \geqq m$.

Before proving these equations we show how they imply that $R_m(n)$ is always an integer. The proof is by strong induction on $m$. Fix $m$ and assume that $R_\mu(n)$ is an integer for all integers $\mu$ satisfying $0 \leqq \mu < m$ and every integer $n$. (This assumption is vacuous if $m = 0$.) We must prove that $R_m(n)$ is an integer for every integer $n$. By (1), $R_m(-1)$ is an integer. By (2), $R_m(n)$ is an integer for $0 \leqq n < m$. And if $m \leqq n < 2m$, then $0 \leqq n - m < m$, so $R_{n-m}(n)$ is an integer by the induction hypothesis, and therefore $R_m(n)$ is an integer by (3). Thus we know $2m+1$ *consecutive* values of $n$ at which $R_m(n)$ is an integer. Since $R_m(x)$ is a polynomial of degree $2m$, it follows that $R_m(n)$ is an integer for *every* integer $n$ (this is obvious from the difference table for $R_m(x)$, since the differences of order $2m$ all have to be equal), and that completes the induction.

All that remains is to prove (1), (2), and (3). Equation (1) can be verified by a direct calculation, using the equations $\binom{-1}{k} = (-1)^k$ if $k \geqq 0$, and $\binom{-1}{k} = 0$ if $k < 0$. Equation (3) is almost as easy. We have already noted that

$$\sum_{k=0}^{n+1}\left[\binom{n}{k} - \binom{n}{k-1}\right]^3 = 0$$

for $n \geqq 0$. If $0 \leqq m \leqq n$, rewrite this in the form

$$\sum_{k=0}^{m}\left[\binom{n}{k}-\binom{n}{k-1}\right]^3=\sum_{k=m+1}^{n+1}\left[\binom{n}{k-1}-\binom{n}{k}\right]^3.$$

Then the left hand side is $S_3(m, n)$, and the relation $\binom{n}{k}=\binom{n}{n-k}$ implies that the right hand side is $S_3(n-m, n)$. Also, $S_1(m, n)=S_1(n-m, n) \neq 0$ since $S_1(m, x)=\binom{x}{m}$, and (3) follows.

It seems to be somewhat more difficult to prove (2). If $0 \leqq n<m$ we have $R_m(n)=S_3'(m, n)/S_1'(m, n)$ by L'Hospital's rule, where the prime denotes differentiation with respect to the second variable. A direct calculation using $S_1(m, x)=\binom{x}{m}$ yields

$$S_1'(m, n)=(-1)^{m-n+1}\Big/\left[(n+1)\binom{m}{n+1}\right]$$

for $0 \leqq n<m$, and so we need to prove

$$S_3'(m, n)=\frac{3(-1)^n}{n+1} \quad \text{for } 0 \leqq n<m. \tag{4}$$

The infinite series $\Sigma_{k=0}^{\infty}\left[\binom{x}{k}-\binom{x}{k-1}\right]^3$ converges absolutely and uniformly on compact subsets of the half plane $\operatorname{Re} x>-2/3$ (as can be seen by using Raabe's test, for example), and it can be summed in closed form by means of a known formula for generalized hypergeometric series. This formula, which may be found on page 28 of the first edition of W. N. Bailey's book, *Generalized Hypergeometric Series* (Cambridge Tract No. 32, 1935), asserts that

$${}_4F_3\left[\begin{matrix}a, b, c, 1+a/2\\ 1+a-b, 1+a-c, a/2\end{matrix}; -1\right]=\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}.$$

We have

$$\begin{aligned}\sum_{k=0}^{\infty}\left[\binom{x}{k}-\binom{x}{k-1}\right]^3&=\sum_{k=0}^{\infty}\left[\binom{x}{k}^3-3\binom{x}{k}^2\binom{x}{k-1}+3\binom{x}{k}\binom{x}{k-1}^2-\binom{x}{k-1}^3\right]\\
&=3\sum_{k=0}^{\infty}\binom{x}{k}^2\left[\binom{x}{k+1}-\binom{x}{k-1}\right]=3\sum_{k=0}^{\infty}\binom{x}{k}^3\frac{(x+1)(x-2k)}{(k+1)(x-k+1)}\\
&=3x\sum_{k=0}^{\infty}\frac{(-x)_k(-x)_k(-x-1)_k(1-x/2)_k(-1)^k}{k!(1)_k(2)_k(-x/2)_k}\\
&=3x\cdot{}_4F_3\left[\begin{matrix}-x, -x, -x-1, 1-x/2\\ 1, 2, -x/2\end{matrix}; -1\right]=\frac{3x}{\Gamma(1-x)\Gamma(2+x)},\end{aligned}$$

where we have set $a=b=-x$, $c=-x-1$ in Bailey's formula. (The notation $(z)_k$ means $z(z+1)\cdots(z+k-1)$, with $(z)_0=1$.) Using the functional equations $\Gamma(x+1)=x\Gamma(x)$ and $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$, we obtain

$$\sum_{k=0}^{\infty}\left[\binom{x}{k}-\binom{x}{k-1}\right]^3=\frac{3\sin(\pi x)}{(x+1)\pi}.$$

Differentiating this relation term by term (which is justified by the uniform convergence) and then setting $x=n \geqq 0$, we find

$$3\sum_{k=0}^{\infty}\left[\binom{n}{k}-\binom{n}{k-1}\right]^2\left[\binom{n}{k}'-\binom{n}{k-1}'\right]=3(-1)^n/(n+1),$$

where the primes denote differentiation with respect to the upper variable. But on the left all the terms with $k \geq n+2$ vanish. Therefore, replacing the upper limit with any integer $m \geq n+1$, we see that $S_3'(m,n) = 3(-1)^n/(n+1)$ for $0 \leq n < m$, which is (4). This completes the proof.

The formula for ${}_4F_3$ used above may also be found in Lucy Joan Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, 1966), page 56, formula (2.3.4.7), and the Bateman Manuscript Project, *Higher Transcendental Functions*, vol. 1, page 190, formula 4.5(4).

Also solved by G. E. Andrews, Robert Breusch (New Zealand), L. Carlitz, I. Dodes, J. S. Frame, C. L. Mallows, and O. G. Ruehr. Carlitz has shown that $S_1(m,n)$ divides $S_p(m,n)$ for $p = 3,5,7,9,\cdots$.

### When $2^m - 2^n$ divides $3^m - 3^n$

E 2468* [1974, 405]. *Proposed by Harry Ruderman, Hunter College Campus School*

Suppose that $m > n \geq 0$ are integers such that $2^m - 2^n$ divides $3^m - 3^n$. Show that $2^m - 2^n$ divides $x^m - x^n$ for all natural numbers $x$.

*Remarks by W. Y. Vélez, Sandia Laboratories, Albuquerque, New Mexico.* Write $k = m - n$ and $2^m - 2^n = 2^n(2^k - 1)$. We first constructed examples where $2^n(2^k - 1)$ divides $3^n(3^k - 1)$. The following are the only values we found:

| $k$ | 1 | 2 | 4 | 6 | 12 |
|---|---|---|---|---|---|
| $n$ | 1 | 1,2,3 | 1,2,3,4 | 2,3 | 2,3,4 |

In these cases it is true that $2^n(2^k - 1)$ divides $x^n(x^k - 1)$ for all natural numbers $x$. Indeed we have the following theorem.

THEOREM. *Let $k \geq 2$, $2^k - 1 = \prod_{i=1}^{r} p_i^{e_i}$ where $p_i$ are distinct primes and $e_i \geq 1$. Write $k = 2^s t$ where $t$ is odd. Then*

$$2^n(2^k - 1) \mid x^n(x^k - 1), \qquad x = 1,2,3,\cdots \tag{1}$$

*if and only if*

$$\phi(p_i^{e_i}) \mid k, \qquad 1 \leq i \leq r \tag{2}$$

$$e_i \leq n \leq s+2, \qquad 1 \leq i \leq r \tag{3}$$

*hold.*

*Proof.* Assume (1). By taking $x$ to be a primitive root modulo $p_i^{e_i}$ we obtain from (1) that $x^k \equiv 1 \pmod{p_i^{e_i}}$ and consequently (2) holds. By taking $x = p_i$ in (1) we obtain $e_i \leq n$. By taking $x = 3$ we obtain $n \leq s+2$ because

$$2^{s+3} \nmid 3^k - 1. \tag{4}$$

Indeed, by (2) $k$ is even, $s \geq 1$. It is clear that (4) holds for $k = 2$. If $k \geq 4$ then (4) follows from

$$3^k - 1 = \sum_{j=1}^{k} \binom{k}{j} 2^j$$

$$2^{s+3} \left| \binom{k}{j} 2^j \quad \text{for } j = 3,5,6,\cdots,k; \right.$$

$$\binom{k}{1} 2 + \binom{k}{2} 2^2 = 2k^2 = 2^{2s+1} t^2;$$

$$\binom{k}{4} 2^4 = 2^{s+2} \cdot \frac{t}{3}(k-1)(k-3)(2^{s-1}t - 1).$$

Now assume that (2) and (3) hold. If $p_i \mid x$ then $p_i^{e_i} \mid x^n$ because by (3) $e_i \leqq n$. If $p_i \nmid x$ then it follows from (2) that $x^k \equiv 1 \pmod{p_i^{e_i}}$. Hence, in both cases $p_i^{e_i} \mid x^n(x^k - 1)$ and consequently $2^k - 1 \mid x^n(x^k - 1)$.

It remains to show that $2^n \mid x^n(x^k - 1)$. This is so if $x$ is even. If $x$ is odd, say $x = 2u + 1$, we claim that $2^n \mid x^k - 1$. This follows from $n \leqq s + 2$ and the fact that

$$2^{s+2} \mid x^k - 1. \tag{5}$$

Indeed, (5) follows from

$$x^k - 1 = \sum_{j=1}^{k} \binom{k}{j} (2u)^j; \qquad 2^{s+2} \left| \binom{k}{j} 2^j \quad \text{for } j = 3, 4, \cdots, k; \right.$$

$$\binom{k}{j} 2u + \binom{k}{2} 4u^2 = 2^{s+1} tu(1 + (k-1)u).$$

Further comments were received from Robert Breusch (New Zealand), Jeff Lagarias, Carolyn MacDonald, Carl Pomerance, Charles Wexler and David Zeitlin.

### A Difficult Triangle Inequality

E 2504 [1974, 1111]. *Proposed by Jack Garfunkel, Forest Hills High School, Flushing, New York*

Let $a$, $b$, $c$ be the sides of a triangle $ABC$, and let $h_a$, $m_b$, and $t_c$ be the altitude to side $a$, the median to side $b$, and the bisector of angle $C$ respectively. On the basis of a computer check of 500 random triangles, it is conjectured that

$$h_a + m_b + t_c \leqq \tfrac{1}{2}\sqrt{3}(a + b + c).$$

Prove or disprove this conjecture.

*Solution by C. S. Gardner, University of Texas at Austin.*

LEMMA 1. *Let $a + b \leqq 2c$. Then $m_a + m_b + t_c \leqq \frac{1}{2}\sqrt{3}(a + b + c)$ and the equality holds if and only if $a = b = c$.*

*Proof.* Write $a = u + x$, $b = u - x$, $c = 2v$, where $-v < x < v$ and $v < u \leqq 2v$. Then

$$t_c^2 = ab\left(1 - \left(\frac{c}{a+b}\right)^2\right) = (u^2 - x^2)\left(1 - \frac{v^2}{u^2}\right) \leqq u^2 - v^2 \tag{1}$$

and the equality holds if and only if $x = 0$, i.e., $a = b$. We have

$$2m_a = \sqrt{2b^2 + 2c^2 - a^2} = \sqrt{(3u - x)^2 + 8v^2 - 8u^2} = f(x)$$

and $2m_b = f(-x)$. Since $f''(x) < 0$ we have

$$m_a + m_b = \tfrac{1}{2}(f(x) + f(-x)) \leqq f(0) = \sqrt{u^2 + 8v^2}. \tag{2}$$

Hence it suffices to show that

$$\sqrt{u^2 + 8v^2} + \sqrt{u^2 - v^2} \leqq \sqrt{3}(u + v) \tag{3}$$

and the equality holds if and only if $u = 2v$. Writing $u/v = y$ and squaring, (3) is replaced by

$$2\sqrt{(y^2 + 8)(y^2 - 1)} \leqq y^2 + 6y - 4.$$

Since $1 < y \leqq 2$ this is equivalent to $4(y^2 + 8)(y^2 - 1) \leqq (y^2 + 6y - 4)^2$, i.e., to $(2 - y)^3(2 + y) \geqq 0$. The lemma is proved.

Since $h_a \leqq t_a \leqq m_a$ it is clear from Lemma 1 that

$$h_a + m_b + t_c \leqq \tfrac{1}{2}\sqrt{3}(a+b+c) \tag{4}$$

holds if $a+b \leqq 2c$ or $b+c \leqq 2a$.

From now on we shall assume that $a+b \geqq 2c$ and $b+c \geqq 2a$. By adding these inequalities we get $a+c \leqq 2b$. Since

$$4a+2c \leqq 2(b+c)+(a+b) = a+3b+2c,$$

$$2a+4c \leqq (b+c)+2(a+b) = 2a+3b+c,$$

we have $a \leqq b$ and $c \leqq b$. Also $b \leqq 2a+2c-b \leqq 2a+(a+b)-b = 3a$ and similarly $b \leqq 3c$.

It suffices to prove that

$$h_a - h_b \leqq m_a - m_b \tag{5}$$

since then Lemma 1 gives

$$h_a + m_b + t_c \leqq m_a + h_b + t_c \leqq m_a + t_b + m_c \leqq \tfrac{1}{2}\sqrt{3}(a+b+c).$$

Without loss of generality we assume that $b = 1$ and then we have

$$1/3 \leqq a \leqq 1, \quad 1/3 \leqq c \leqq 1, \quad 2a \leqq 1+c, \quad 2c \leqq 1+a, \quad a+c > 1.$$

If $S$ is the area of the triangle then

$$\begin{aligned} 16S^2 &= (1+a+c)(1+a-c)(c+1-a)(c-1+a) \\ &= 2(1+a^2)c^2 - c^4 - (1-a^2)^2 \end{aligned}$$

is a strictly increasing function of $c$ in the interval $1/3 \leqq c \leqq 1$. Hence, replacing $c$ by $(1+a)/2$ we get $16S^2 \leqq \frac{3}{16}(3-a)(3a-1)(1+a)^2$. Therefore

$$h_a - h_b = 2S(1-a)/a \leqq \frac{\sqrt{3}}{8}\,\frac{1-a^2}{a}\sqrt{(3-a)(3a-1)}. \tag{6}$$

Since $2m_a = (2+2c^2-a^2)^{1/2}$, $2m_b = (2a^2+2c^2-1)^{1/2}$ and $\sqrt{s}-\sqrt{t} \geqq (s-t)/\sqrt{2(s+t)}$, $(s,t>0)$, we find that

$$m_a - m_b \geqq (3/2)(1-a^2)[2(a^2+4c^2+1)]^{-1/2}.$$

Replacing $c$ by $(1+a)/2$ we get

$$m_a - m_b \geqq \tfrac{3}{4}(1-a^2)(1+a+a^2)^{-1/2}. \tag{7}$$

Inequality (5) follows from (6), (7) and the inequality

$$2\sqrt{3}a \geqq [(1+a+a^2)(3-a)(3a-1)]^{1/2}$$

which is equivalent to $(1-a)^2(3a^2-a+3) \geqq 0$.

This completes the proof of (5) when $a+b \geqq 2c$ and $b+c \geqq 2a$, and at the same time the proof of (4) with no restriction. It is clear from this proof that the equality in (4) holds if and only if $a = b = c$.

Also solved by Lu Ting & Richard Lo (jointly). Computer assisted solutions were submitted by G. S. Lessells & M. J. Pelling (Nigeria), and by W. A. Sentance (England).

*Editor's comment.* Ting and Lo obtain also the following:

$$\frac{1}{2} \leqq \frac{t_a+m_b+t_c}{a+b+c} \leqq \frac{\sqrt{3}}{2}, \qquad \frac{1}{4} \leqq \frac{h_a+m_b+t_c}{a+b+c} \leqq \frac{\sqrt{3}}{2}, \qquad \frac{3}{8} \leqq \frac{h_a+m_b+m_c}{a+b+c} \leqq 1,$$

where all the bounds are the best possible.

### The Diophantine Equations $x^2+1=2^r5^s$

E 2511 [1975, 73]. *Proposed by Morris Olitsky, Indianapolis, Indiana*

We observe that

$$\tfrac{1}{3} = 0.333333\cdots = 3[(0.1)+(0.1)^2+(0.1)^3+\cdots],$$

$$\tfrac{1}{7} = 0.142857\cdots = 7[(0.02)+(0.02)^2+(0.02)^3+\cdots].$$

Are there any other positive integers $x$ for which

$$\frac{1}{x} = x\left[\sum_{j=1}^{\infty}(m\cdot 10^{-n})^j\right]$$

for suitable integers $m$, $n$?

*Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands.* We have $1 \leqq m < 10^n$ and we may assume that $m$ is not divisible by 10. The problem reduces to solving the Diophantine equation

$$x^2+1 = 2^r5^s, \tag{1}$$

where $r, s \geqq 0$, $x \geqq 1$. If we have a solution of (1) then $m$ and $n$ are determined by

$$n = \max(r, s), \qquad 2^r5^s m = 10^n. \tag{2}$$

We cannot have $r \geqq 2$ since then (1) gives a contradiction $x^2 \equiv -1 \pmod 4$. Thus $r = 0$ or $r = 1$.

Assume first $r = 0$. Then $s$ must be odd because otherwise $s = 2t$, and (1) gives $1 = y^2 - x^2 = (y-x)(y+x)$ where $y = 5^t$. Since $y + x \geqq 2$ this is impossible. The case $s = 1$ gives $x = 2$. If $s \geqq 3$ is odd then $x^2 + 1 = 5^s$ has no solution (see L. J. Mordell, *Diophantine Equations*, Academic Press, 1969, p. 301 for the proof).

Now let $r = 1$. The equation $x^2 + 1 = 2 \cdot 5^s$ has only solutions $(x, s) = (1, 0), (3, 1), (7, 2)$ as shown by C. Engelman, *On close-packed double-error-correcting codes on p-symbols*, I.R.E. Trans. on Information Theory, Correspondence (Jan. 1961), 51–52.

Using (2) we find that $(x, n, m)$ must be one of the following: $(2, 1, 2), (1, 1, 5), (3, 1, 1)$ or $(7, 2, 2)$.

Also solved by L. Kuipers and Gerald Leibowitz. Partially solved by Marcia Asher, L. C. Bourburgh, Brother Alfred Brousseau, Paul Bruckman, Sarah Christiansen, Miltiades Demos, Edward Eby, James Foley, Samuel Glidewell, Michael Goldberg, M. G. Greening (Australia), Robert Hashway, Edwin Hoefer, J. P. Hoyt, John Jensik, Dennis Jesperson, Clayton Knoshaug, Peter Lindstrom, Carolyn MacDonald, William Markel, L. E. Mattics, Harry Nelson, Barry Powell, Richard Redfield, Eric Rosenthal, Michael Skalsky, Karl Stoop (Switzerland), F. B. Strauss, Charles Wexler, and the proposer.

*Editor's Comment.* Most of the contributors have found all four solutions but did not succeed in proving that there are no others.

The above reference to Mordell's book was supplied by Leibowitz. Lossers quotes L. E. Dickson, *Theory of Numbers* II, p. 239. Both Lossers and Leibowitz refer to the paper of V. A. Lebesgue, *Sur l'impossibilité en nombres entiers de l'équation* $x^m = y^2 + 1$, Nouv. Ann. Math. 9 (1850), 178–181.

### An Infimum for Polynomials

E 2518 [1975, 169]. *Proposed by Derek A. Zave, Sperry UNIVAC, Roseville, Minnesota*

Let $F$ be the set of real polynomials $f$ with nonnegative coefficients for which $f(1) = 1$. Let $0 < x_0 < 1$ and $0 < \alpha \leqq 1$ be fixed; compute

$$m(x_0; \alpha) = \inf\{f'(1): f \in F \text{ and } f(x_0) \leqq \alpha\}.$$

*Solution by Allen Stenger, Student, Pennsylvania State University.* Actually this set has a minimum. Denote by $n$ the integer such that $x_0^n \geqq \alpha > x_0^{n+1}$. Then the minimum occurs at the polynomial

$$\frac{x_0^n - \alpha}{x_0^n - x_0^{n+1}} x^{n+1} + \frac{\alpha - x_0^{n+1}}{x_0^n - x_0^{n+1}} x^n$$

and equals $n + (x_0^n - \alpha)/(x_0^n - x_0^{n+1})$.

To see this, suppose $f$ has the form

$$f(x) = \sum_{k=0}^{m} a_k x^k, \quad \text{with} \quad a_k \geq 0.$$

Write $g(x)$ for $f(x)/x^n$. Then

$$g''(x) = \sum_{k=0}^{m} (k-n)(k-n-1) a_k x^{k-n-2};$$

since $(k-n)(k-n-1) \geq 0$ for all $k$ and $n$, we have that $g(x)$ is convex for $x > 0$. Thus

$$g'(1) \geq \frac{g(1) - g(x_0)}{1 - x_0} \geq \frac{1 - \alpha/x_0^n}{1 - x_0} = \frac{x_0^n - \alpha}{x_0^n - x_0^{n+1}}$$

and so

$$f'(1) = ng(1) + g'(1) \geq n + \frac{x_0^n - \alpha}{x_0^n - x_0^{n+1}}.$$

This gives a lower bound on $f'(1)$, which is actually attained by the function mentioned above.

Also solved by Aage Bondesen (Denmark), Michael Ecker, G. A. Heuer, Carl Hurd, Dennis Jesperson, Kenneth Levasseur, Joel Levy, O. P. Lossers (Netherlands), Carl McCarty, Wayne Park, David Rubin, Michael Skalsky, and the proposer.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before July 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6084. *Proposed by Theodore J. Rivlin, IBM, Yorktown Heights*

Let $T_n(x) = t_0 + t_1 x + \cdots + t_n x^n$ denote the Chebyshev polynomial of degree $n$ (that is, $T_n(x) = \cos n\theta$ where $x = \cos\theta$). Suppose that $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is real-valued. Show that if $|p(\cos(j\pi/n))| \leq 1$ for $j = 0, 1, \cdots, n$ then

$$|a_{n-2m}| + |a_{n-2m-1}| \leq |t_{n-2m}|,$$

$m = 0, 1, \cdots, (n-1)/2$. (The fact that $|a_{n-2m}| \leq |t_{n-2m}|$ is well known.)

6085*. *Proposed by William J. Sánchez, Courant Institute of New York University*

Call a family $F$ of functions uniformly integrable (UI) if there exists $k(\varepsilon)$ such that $\int \{|f| d\mu : |f| > k\} < \varepsilon$ for all $f \in F$. If there exists integrable $h$ such that $|f| \leq h$ (a.e.) for all $f \in F$, then $F$ is UI (Burrill, *Measure, Integration, and Probability*, p. 183). Is the converse true?

6086. *Proposed by Raymond M. Redheffer, University of California, Los Angeles*

Let $d_{ij}$ be the number of divisors common to $i$ and $j$. Then the determinant $|d_{ij}|$ for $2 \leq i, j \leq n$ equals the number of square-free integers from 1 to $n$.

6087. *Proposed by Nathaniel Grossman, University of California, Los Angeles*

A loxodrome on a Riemannian surface is a curve meeting members of a specified one-parameter family of curves at a constant angle. For example, the bagel (round torus or anchor-ring) has two special families, the meridians and the parallels, each defining the same family of loxodromes. Prove that a loxodrome on a bagel is either periodic or dense.

6088*. *Proposed by Nathaniel Grossman, University of California, Los Angeles*

The functional-differential equation $f' = f^{-1}$ has a solution satisfying $f(0) = 0$ and $f'(x) > 0$ for $x > 0$, namely $f(x) = (1/\alpha)^{1/\alpha} x^{\alpha}$, where $\alpha = (1+\sqrt{5})/2$, the golden section. Is this the only solution satisfying the given conditions for $x \geqq 0$?

6089*. *Proposed by E. Ehrhart, Strasbourg, France*

Let $K$ be a convex body in $R_n$ of Jordan content $V(K) > (n+1)^n/n!$ and with centroid at the origin. Does $K \cup (-K)$ contain a convex body $C$, symmetric in the origin, for which $V(C) > 2^n$?

## SOLUTIONS OF ADVANCED PROBLEMS

### Rational Approximations to $\sqrt{2}$ and $\pi$

5983 [1974, 785]. *Proposed by H. D. Ruderman, Hunter College Campus School*

Let $\|x\|$ be the distance of $x$ to the nearest integer, i.e.,

$$\|x\| = \min\{x - [x], 1 + [x] - x\}$$

for real $x$. For positive integers $N$ find the greatest lower bound for $N \cdot \|N\pi\|$ and for $N \cdot \|N\sqrt{2}\|$.

I. *Partial solution by O. P. Lossers, Technological University, Eindhoven, The Netherlands.* We prove that the greatest lower bound of

$$N\|N\sqrt{2}\| = 2\|2\sqrt{2}\| = 6 - 4\sqrt{2} \approx 0.343146.$$

Since

$$N(1 + [N\sqrt{2}] - N\sqrt{2}) = \frac{N\{(1+[N\sqrt{2}])^2 - 2N^2\}}{1 + [N\sqrt{2}] + N\sqrt{2}} \geqq \frac{N}{1 + [N\sqrt{2}] + N\sqrt{2}}, \tag{1}$$

$$N(N\sqrt{2} - [N\sqrt{2}]) = \frac{N\{2N^2 - ([N\sqrt{2}])^2\}}{N\sqrt{2} + [N\sqrt{2}]} \geqq \frac{N}{N\sqrt{2} + [N\sqrt{2}]}, \tag{2}$$

$$\frac{N}{N\sqrt{2} + [N\sqrt{2}]} \geqq \frac{N}{1 + [N\sqrt{2}] + N\sqrt{2}} \geqq \frac{N}{1 + 2N\sqrt{2}} \geqq 6 - 4\sqrt{2}, \tag{3}$$

for $N > 6 + 4\sqrt{2}$, i.e., $N \geqq 12$. It follows that $N\|N\sqrt{2}\| \geqq 2\|2\sqrt{2}\|$ for $n \geqq 12$. It can now easily be checked by hand that

$$N\|N\sqrt{2}\| \geqq 2\|2\sqrt{2}\| \text{ for all } N.$$

II. *Comment by Simeon Reich, Tel Aviv University, Israel.* The value of $\inf N \cdot \|N\pi\|$ depends on the (as yet unsolved) question of whether the sequence $\{a_n\}$ of partial quotients in the regular continued fraction expansion of $\pi$ is bounded or not. In fact $\inf N \cdot \|N\pi\| = 0$ if and only if it is unbounded (see, e.g., A. Y. Khintchine, *Continued Fractions*, Noordhoff, Groningen, 1963, p. 44). If it is indeed unbounded, then also $\inf N|\sin N| = 0$ (see Problem 5849 [1973, 701]). Since $a_{431} = 20776$, we have $\inf N \cdot \|N\pi\| < 1/20776$.

III. *Note by Paul Erdös.* Mahler or Popken proved $|\pi - a/b| > 1/b^{42}$ and this may have been improved a bit but it has not been determined whether $|\pi - a/b| < \varepsilon/b^2$ is solvable for every $\varepsilon$, i.e., if $N \cdot \|N\pi\|$ has lim inf 0. The lim inf $N \cdot \|N\pi\|$ can be achieved by continued fractions as noted above.

Also solved (partially) by Michael Barr (Canada), H. D'Alarcao & I. P. Scalisi & G. Sethares, Thomas Foregger, Michael Goldberg, F. G. Schmitt, Jr., and J. L. Winter.

Schmitt also refers to Problem 5849* [1972, 399], and to an article, *An unsolved approximation question for* $\pi$, this MONTHLY, vol. 80 (1973), p. 701.

**Convolution of Null Sequences**

5984 [1974, 785]. *Proposed by Clark Kimberling, University of Evansville*

Suppose $\{x_i\}$ and $\{y_i\}$ are sequences of positive real numbers such that $\lim_{n\to\infty} \Sigma_{i=0}^{n} x_i x_{n-i} = 0$ and $\lim_{n\to\infty} \Sigma_{i=0}^{n} y_i y_{n-i} = 0$. Do we have $\lim_{n\to\infty} \Sigma_{i=0}^{n} x_i y_{n-i} = 0$? (The proposal is equivalent to a question about convolutions

$$f^*g(x) = \int_0^x f(t)g(x-t)dt$$

of positive-valued functions $f$ and $g$ defined on $[0,\infty)$; namely, if $f^*f(x)\to 0$ and $g^*g(x)\to 0$, then must $f^*g(x)\to 0$?)

*Solution by Allen Stenger, student, Pennsylvania State University.* No, not necessarily. My counterexample has non-negative terms, not positive terms; but the formulae

$$\sum (x_i + c_i)(x_{n-i} + c_{n-i}) = \sum x_i x_{n-i} + \sum c_i(2x_{n-i} + c_{n-i}),$$

$$\sum (x_i + c_i)(y_{n-i} + c_{n-i}) > \sum x_i y_{n-i}$$

show that such an example can be converted into one of the desired type by adding terms from a convergent positive-term series $\Sigma_{n=0}^{\infty} c_n$ (provided $x_n \to 0$ and $y_n \to 0$).

Set

$$x_n = \begin{cases} a_k & \text{if } n = 2^k \text{ for some } k > 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$y_n = \begin{cases} b_k & \text{if } n = 2^k - 2^l \text{ for some } k \text{ and } l, k > l > 0, \\ 0 & \text{otherwise;} \end{cases}$$

where each $a_k$ and each $b_k$ is a positive number to be determined later (our choices in the calculation are $a_k = k^{-1/4}$ and $b_k = k^{-2/3}$).

The sum $\Sigma x_i x_{n-i}$ will be zero unless $n$ has the form $2^k + 2^l$, $(k, l > 0)$; then it is $2a_k a_l$ (if $k \neq l$) or $a_k^2$ (if $k = l$) and so $\to 0$, if $a_r \to 0$ when $r \to \infty$.

Next, the sum $\Sigma y_i y_{n-i}$ is $\Sigma b_r b_t$ where $r, t$ run over all values so that there are $s$ $(< r)$ and $u$ $(< t)$ such that $n = 2^r - 2^s + 2^t - 2^u$, $(u, s > 0)$. By symmetry, the sum is $\leqq 2\Sigma b_r b_t$ if in addition we require $r \geqq t$.

From the requirement $r \geqq t$ follows that $2^{r-1} < n < 2^{r+1}$ so that $r$ has at most two values, say $r = m$ or $r = m + 1$; and $m \to \infty$ as $n \to \infty$. Since $n - 2^r - 2^t = 2^s + 2^u$, for each fixed $r$ and $t$ there are at most two values of $s, u$. Thus the sum is

$$\leqq 2(b_m + b_{m+1}) \sum_{t \leqq m+1} 2b_t.$$

This can be made to approach zero by a suitable choice of $b$, say $b_k = k^{-2/3}$

Now, if $n = 2^r$ for some $r$, then

$$\sum x_i y_{n-i} = \sum_{r=1}^{r-1} a_r b_r = r^{-2/3} \sum_{r=1}^{r-1} a_r.$$

Choosing (for instance) $a_t = t^{-1/4}$ causes this sum to approach $\infty$ while $a_t \to 0$ as required.

Also solved by Cal Poly Solution Group, Paul Erdös (Hungary), G. H. Fricke, Seymour Haber, Yoshinobu Kato (Japan), O. P. Lossers (Netherlands), L. E. Mattics, Stuart Nelson, Kim Penrose, Stuart Sidney, Oto Strauch (Czechoslovakia), and Peter Ungar.

**Subsets of the Plane**

5985 [1974, 785]. *Proposed by Siemion Fajtlowicz, University of Houston*

Let $f$ be a continuous function of the plane $E$ into itself. Does there exist a closed proper subset $X$ of $E$ such that $f(X) \subseteq X$?

*Solution by G. H. Fricke, Wright State University, Dayton, Ohio.* Let $f: E \to E$ be continuous. Define $f_n: E \to E$ by $f_0(X) = X$ and $f_n(X) = f(f_{n-1}(X))$ for $X \in E$ and $n = 1, 2, \cdots$. Let $S_X = \{f_n(X) \mid n = 0, 1, \cdots\}$. Since $f$ is continuous, $f(\bar{S}_X) \subset \bar{S}_X$ for any $X \in E$. Thus we only need to show there exists $X \in E$ such that $\bar{S}_X \neq E$.

Let $D$ be the closed unit disk and let

$$D_n = \{X \in D \mid \|f_i(X)\| \geqq 1 \quad \text{for all} \quad 1 \leqq i \leqq n\}.$$

If $D_n = \varnothing$ for some $n$ then $S_X \subset \bigcup_{i=0}^{n} f_i(D)$ for any $X \in D$. Thus,

$$\bar{S}_X \subset \bigcup_{i=0}^{n} \overline{f_i(D)} = \bigcup_{i=0}^{n} f_i(D) \neq E.$$

Now, if $D_n \neq \varnothing$ for all $n$ then, since $D_n \supset D_{n+1}$ and $D_n$ is closed for all $n$, we have $\bigcap_{n=1}^{\infty} D_n \neq \varnothing$. Thus, there exists $X \in \bigcap_{n=1}^{\infty} D_n$ such that $\|f_n(X)\| \geqq 1$ for $n = 1, 2, \cdots$, and therefore, $\bar{S}_X \neq E$.

Also solved by Bruce Aubertin & Peter Borwein, O. P. Lossers (Netherlands), and Thomas Sibley.

**Triangle Contractive Self Maps**

5986 [1974, 785]. *Proposed by D. E. Daykin, Reading University, England*

Let $E$ be the real Euclidean plane and $0 < \alpha < 1$. What can be said about maps $f: E \to E$ which send each triangle $T$ into a triangle $fT$ with $\text{area}(fT) \leqq \alpha \, \text{area}(T)$?

*Solution by the proposer.* First we remove the condition $\alpha < 1$ as it turns out to be redundant. Also we let $\mathcal{T}$ denote the set of the maps in the problem. If $f: E \to E$ is affine, using Cartesian coordinates $(\lambda, \mu)$ there are constants $a, b, \cdots, k$ such that

$$f(\lambda, \mu) = (\lambda, \mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (h, k).$$

It is easy to check that this affine map is in $\mathcal{T}$ if and only if $|ad - bc| < \alpha$.

Next we call $f: E \to E$ trivial if $fE$ is a subset of a line. Clearly $\mathcal{T}$ contains all trivial maps. Also by [1] Theorem 3 if $f \in \mathcal{T}$ is non-trivial then $f$ is continuous. Let $\mathcal{L}$ be the set of maps $f: E \to E$ which send collinear points into collinear points. Then $\mathcal{T} \subset \mathcal{L}$ and using Theorem 1 below, the solution is that $\mathcal{T}$ consists of all trivial maps and all affine maps satisfying $|ad - bc| < \alpha$. [It would be interesting to know what happens in an infinite dimensional Hilbert space. (See Problem 5913 [1974, 786].)]

THEOREM 1. *If $f \in \mathcal{L}$ is continuous but non-trivial then $f$ is affine.*

*Proof. Part* (i). Let $D$ be a closed disk of positive radius in $E$. We assume $fD$ is a point $p$ and derive a contradiction. Since $f$ is non-trivial there are $x, y \in E$ such that $p, fx, fy$ is a non-degenerate triangle. Then $x \neq y$ and the line $L(x, y)$ through $x$ and $y$ does not meet $D$ because $f \in \mathcal{L}$. Since $f$ is continuous, the set $(f^{-1}fx) \cap L(x, y)$ is closed, but it does not contain $y$. Let $w$ be a point in this set at minimum possible distance from $y$, and then $fw = fx$. Also let $(y_n)$ be a sequence of points of $L(x, y)$ lying between $w$ and $y$ and converging to $w$. Put

$$W = \bigcup_{d \in D} L(d, w) \quad \text{and} \quad Y_n = \bigcup_{d \in D} L(d, y_n),$$

then $fW \subset L = L(p, fx)$ and $fY_n \subset L_n = L(p, fy_n)$. If $v_n$ is a point of $W \cap Y_n$ at minimum possible distance from $w$, then $(v_n) \to w$ as $n \to \infty$. Also because $L \neq L_n$ we have $fv_n = p$. Thus our contradiction is that $f$ is discontinuous at $w$.

*Part* (ii). Next we show that $f$ is injective. Suppose $u \neq v$ but $fu = fv$ and choose $x, y$ such that $fv, fx, fy$ is a non-degenerate triangle. If $x, u, v$ happen to be collinear, since $f$ is continuous and $f(L(x, y)) \subset L(fx, fy)$ we can change our choice of $x$ so that they are not collinear. Thus we may assume no three of $u, v, x, y$ are collinear. Again by the continuity of $f$ there is an open interval $I$ of $L(u, x)$ containing $x$ such that $fi \neq fu$ for $i \in I$. Similarly there is an open interval $J$ of $L(v, y)$ containing $v$ such that $fj \neq fv$ for $j \in J$. Let

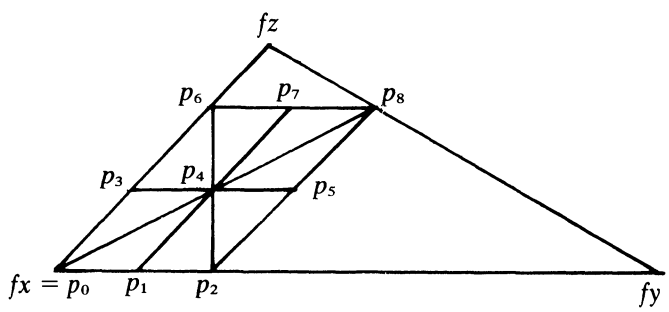
$$X = \bigcup_{i \in I} L(i, v) \quad \text{and} \quad Y = \bigcup_{j \in J} L(j, u).$$

Then $fX \subset L(fx, fv)$ and $fY \subset L(fy, fu)$ so $fZ = fu = fv$ where $Z = X \cap Y$. Since $Z$ contains a closed disk $D$ we have our contradiction by part (i).

*Part* (iii). Assume $x, y, z$ is a non-degenerate triangle but $fx, fy, fz$ are contained in a line $L$. By the last part we may assume $fy$ lies between $fx$ and $fz$. Then by continuity of $f$ there is a point $w$ on $L(x, z)$ with $fw = fy$ contradicting part (ii). Thus we have shown that the inverse images of collinear points are collinear.

*Part* (iv). It is easy to see that $fE$ is convex.

*Part* (v). Let $L$, $M$ be parallel but not collinear line segments in $fE$ and let $L', M'$ be lines containing $f^{-1}L$, $f^{-1}M$ respectively. Then $L', M'$ are parallel but not collinear, for otherwise they intersect in a point $x$ and $fx \in L \cap M$ which is impossible.




*Part* (vi). Let $x, y, z$ be such that $fx, fy, fz$ is a non-degenerate triangle, and choose $p_8 \in L(fy, fz)$ between $fy$ and $fz$. Then we get 9 lattice points $p_i$ of $fE$ as shown in the figure. Since $f$ is continuous and injective $f^{-1}p_8$ lies on $L(y, z)$ between $y$ and $z$, and using part (v) and the fact that $f \in \mathcal{L}$ we see that the inverse image of the 9 lattice points $p_i$ is also a lattice of 9 points. Arguing in this way shows that if the two lattices are extended over $E$ then $f$ is affine on the lattice points. Next we can refine the

lattice by inserting the mid-lattice points and $f$ will be affine over the refined lattice. Repetition of this process leads to a set of points dense in $E$ over which $f$ is affine, so $f$ is affine on the whole of $E$ by continuity.

**Reference**

1. D. E. Daykin and J. K. Dugdale, Triangle contractive self maps of a Hilbert space, Fund. Math., 83 (1974) 187–195.

*Editor's comment.* (1) Simeon Reich (Israel) points out that T. Zamfirescu has considered maps similar to those of the problem in a paper, *Area contractions in the plane*, Rend. Sem. Mat. Univ. Padova, 46 (1971) 49–52; MR # 9954, Vol. 46 (1973).

(2) Using the concept of triangle contractive self maps as defined by Daykin & Dugdale (*loc. cit.*), the proposer has proved that each such map in the plane has either a point $p$ with $fp = p$ or a line $L$ with $fL \subseteq L$ or both.

**A Trigonometric Product**

5987 [1974, 785]. *Proposed by H. Kestelman, University College, London, England*

If $\{n_r\}$ is an increasing sequence of positive integers, prove that $\prod_{r=1}^{\infty} \cos n_r x = 0$ for all real $x$ apart from a set of Lebesgue measure zero. Can the exceptional set be uncountable?

*Solution by Ron Evans, University of Wisconsin, Madison.* Since the partial products of $\prod_{r=1}^{\infty} \cos^2(n_r x)$ form a nonnegative decreasing sequence, $f(x) = \prod_{r=1}^{\infty} \cos^2(n_r x)$ exists and $f(x) \geqq 0$ for each $x \in \mathbb{R}$. Let $E = \{x \in \mathbb{R}: f(x) > 0\}$. Since $f$ is measurable, so is $E$. It suffices to show that $E$ has measure 0. Suppose that $E$ has positive measure. For each $x \in E$, $\cos^2(n_r x) \to 1$ as $r \to \infty$. Let $g(y)$ denote the distance from $y$ to the nearest integral multiple of $\pi$. Then $g(n_r x) \to 0$ as $r \to \infty$, for each $x \in E$. Let $D = \{x - y: x, y \in E\}$. Then $g(n_r x) \to 0$ as $r \to \infty$, for each $x \in D$. Steinhaus proved in 1920 that the set of distances between points of a set of positive Lebesgue measure contains an interval. Thus $D$ contains an interval. In particular $D$ is second category in $\mathbb{R}$. Let $D_R = \{x \in D: g(n_r x) < 1/10$ for each $r \geqq R\}$. Then $D = \bigcup_{R=1}^{\infty} D_R$. Hence $D_R$ is second category in $\mathbb{R}$ for some $R \geqq 1$. In particular, $D_R$ is dense in some interval $(\alpha, \beta)$. For a sufficiently large value of $r \geqq R$, $(n_r\alpha, n_r\beta)$ has length greater than $\pi$. Since $\{n_r x: x \in D_R\}$ is dense in $(n_r\alpha, n_r\beta)$, there exists $x \in D_R$ such that $g(n_r x) > 1/10$, a contradiction.

Although $E$ has measure 0, $E$ may be uncountable. For example, let $n_r = 10^{2^r} (r = 1, 2, \cdots)$. Then $E$ contains the uncountable set of reals of the form $x = \pi \sum_{r=1}^{\infty} \varepsilon_r 10^{-2^r}$, where $\varepsilon_r = 0$ or 1. To see this, note that for such $x$, $g(n_r x) < (2\pi)10^{-2^r}$. Hence

$$f(x) = \prod_{r=1}^{\infty} (1 - \sin^2(n_r x)) = \prod_{r=1}^{\infty} (1 - \sin^2(g(n_r x)))$$

$$= \prod_{r=1}^{\infty} (1 + O(10^{-2^{r+1}})) > 0.$$

This completes the proof.

Also solved by M. B. Gregory, O. P. Lossers (Netherlands), L. E. Mattics, and the proposer.

# REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.*

*Introduction to Computers and Computer Science.* By Richard C. Dorf. Boyd and Fraser, San Francisco, California, 1972. x + 628 pp. $12.95. (Telegraphic Review, August–September 1972.)

Occasionally one finds a text which seems to fit a course so well that no thought is given to finding another text. This book is one such. I have used it three times in a 3 semester hour course designed for students with no background in the computer field.

The material is appropriate for undergraduate mathematics and science students and can easily be adapted for social science and liberal arts majors.

All chapters, except for the sequence on programming (Chapters 4, 5, 6, 7, 8), can be covered in any order, which enhances the book's utility as a class text. The introductory chapters, including a history of computers, are well presented without being overdone. The chapter on computer components is the best I have seen. It includes many charts and graphs relating the hardware capabilities to each other and to various physical phenomena. All in all it is a very cohesive presentation. The sequence of chapters on problem solving and programming avoids the common pitfall of trying to be a programming manual within a general text yet manages to cover a wide variety of topics extremely well. Some of the exercises included here will be challenging to the instructor as well as to the student. The non-science students may have difficulty with a few of the mathematically oriented examples in Chapter 4 but this is a minor drawback.

Applications are often slighted in a beginning course but not in this book. There are separate, complete and well presented chapters on business applications, government uses, data banks, computers and the arts and more. The final chapter is one I feel no student of computer science should miss. It deals with the philosophical concepts of artificial intelligence and the social consequences of computers. The papers by Turing, Wiener and Samuel are particularly appropriate and should be discussed as well as read.

I supplemented the text with some transparencies, recorded computer music, and a variety of calculators with incrementally increasing capabilities. In addition, a number of problem solving and programming assignments were given.

It is difficult to find something to criticize in this book. The printing and format are excellent, with ample pictures and charts. The subjects chosen are comprehensive and well presented. Perhaps the biggest problem is that the more difficult exercises are not so identified and may be too difficult for the beginning student. But then these challenge the student to become more involved and hence may not be completely out of place.

I would not hesitate to recommend this book to anyone considering an introductory course in computer science.

NILO NICCOLAI, University of North Carolina at Charlotte

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P, L*. *Proceedings of the International Congress of Mathematicians*. Ed: R.D. James. Canadian Math Congress, 1975. *V. 1:* xlix + 552 pp; *V. 2:* viii + 600 pp. Proceedings of the 17th ICM meeting in Vancouver, 1974. Surveys of the work of the Fields medalists Enrico Bombieri and David Mumford; 16 "expository" addresses; several score invited papers. LAS

GENERAL, T(15-16), L. *Cours de Mathématiques, Tome 1-4*. Jacqueline Lelong-Ferrand, Jean-Marie Arnaudiès. Dunod. *Tome 1: Algèbre*, 1974, x + 534 pp, 79 F; *Tome 2: Analyse*, 1974, x + 638 pp, 89 F; *Tome 3: Géométrie et cinématique*, 1975, x + 733 pp, 120 F; *Tome 4: Equations Différentielles Intégrales Multiples Fonctions Holomorphes*, 1974, ix + 454 pp, 85 F. These four books constitute an excellent traditional junior-senior level program in mathematics. Volume I is an abstract and linear algebra text. Volume II covers topics in advanced calculus (primarily single variable), including topology and Fourier analysis. Volume III is a thorough geometry course--Euclidean, affine and projective. Volume IV is differential equations and multiple integrals, along with chapters on differential forms and complex analysis. Exercises are collected at the end of each volume, and each volume has its own bibliography and index. PJM

GENERAL, L. *Index of Mathematical Papers, V. 5*. AMS, 1975. *Part 1:* 530 pp; *Part 2:* 341 pp, $80 set (P). Part I is the subject index to *Math Reviews* for 1973; Part II is the author index. LAS

GENERAL, T(13; 1). *Mathematics, An Everyday Experience*. Charles D. Miller, Vern E. Heeren. Scott F, 1976, xxi + 519 pp, $10.95. A "cultural" approach to basic mathematics--counting, geometry, probability, statistics, computers, etc. In each chapter an attempt is made to present the material in a context familiar to the reader. Many fine illustrations. Good for a math for liberal-arts-students course. Exercises and answers. References. Index. RJA

BASIC, T(13; 1, 2). *Series in Mathematics Modules*. Leon J. Ablon, *et al*. Cummings. *Module I, Operations on Numbers*, 1973, vi + 115 pp, (P); *Module I-A, Practical Mathematics, Decimals, Ratios, Proportions, and Percents*, 1975, 104 pp, (P); *Module II, Operations on Polynomials*, 1973, vi + 111 pp, (P); *Module II-A, Practical Mathematics, Signed Numbers, Formulas, Graphs, and the Metric System*, 1975, 122 pp, (P); *Module III, Linear Equations and Lines*, 1973, vi + 113 pp, (P); *Module IV, Factoring and Operations on Algebraic Fractions*, 1973, vi + 120 pp, (P); *Module V, Quadratic Equations and Curves*, 1973, vi + 117 pp, (P); *Module VI, Basic Trigonometry, Second Edition*, 1976, 124 pp, (P); *Module VII, Trigonometry with Applications*, 1976, 121 pp, (P); *Module M, Medical Dosage Calculations*, 1974, viii + 120 pp, (P). A collection of modules which starts at a very low level and eventually gets into standard precalculus topics. The text is characterized by many examples and long lists of problems, with complete solutions given for many of the problems. No prize for style. CEC

BASIC, T(13; 1). *Beginning Algebra, Second Edition*. Margaret L. Lial, Charles D. Miller. Scott F, 1976, 334 pp, $9.95; *Study Guide for Beginning Algebra*, 208 pp, $4.95 (P); *Intermediate Algebra, Second Edition*, 432 pp, $10.50; *Study Guide for Intermediate Algebra*, 244 pp, $4.95 (P). The number of exercises and examples has been increased substantially in these editions. An appendix on the metric system is also new. Study guides can be used as accompanying workbooks. CEC

BASIC, T?(13; 1). *Business Mathematics for Calculating Machines*. Bonnie D. Phillips, Dale A. Storey. Macmillan, 1976, xi + 210 pp, $8.95 (P). Little more than an instruction manual for the Frieden, the Marchant, the Olivetti, and electronic calculators. Many of the exercises are also quite dated. LCL

PRECALCULUS, T(13; 1, 2). *Essentials of Mathematics*. Vernon E. Howes. Wiley, 1975. *Precalculus I, A Programmed Text, Algebra I*, xix + 677 pp, $10.95; *Precalculus II: A Programmed Text, Algebra II*, xiv + 291 pp, $7.95; *Precalculus III, A Programmed Text, Trigonometric Functions and Applications*, xiv + 529 pp, $9.95 (P). A three volume one-year course in intermediate algebra and precalculus. This is a programmed text which covers all the standard topics, and occasionally goes deeper. There is a good blend of the methods and the techniques of mathematics. CEC

PRECALCULUS T*(13; 1). *Modern College Algebra and Trigonometry With Applications*. Ronald D. Jamison. HarBrace J, 1976, xi + 628 pp, $12.95; *Modern College Algebra With Applications*, x + 462 pp, $11.95. Well written texts which cover the standard topics of college algebra. The problem sets are outstanding. There are lots of true-false questions to test reading comprehension, applications, and recreational problems along with basic skills problems. CEC

EDUCATION, S(16-17), P. *Elementary Mathematics Today: A Resource for Teachers Grades 1-8*. Elizabeth Williams, Hilary Shuard. A-W, 1970, x + 462 pp, $15. First published in London. Motivated by research of Piaget and others, the book is concerned with the mathematics of the British primary school. Integrates key mathematical concepts with insights into how children learn and uses of manipulative materials. Excellent supplement to any elementary text series. PSJ

EDUCATION, S(16-18), P. *The Teaching of Mathematics: From Intermediate Algebra through First Year Calculus.* Roy Dubisch. Krieger, 1975, xi + 124 pp, $7.50. Reprint of a 1963 Wiley text, itself based on (revised) notes prepared in 1951. Largely a guide to the relevant literature. LAS

EDUCATION, S(15-16), P. *Going Metric, Grades K-8: Guidelines for the Mathematics Teacher.* Walter W. Leffin. NCTM, 1975, i + 47 pp, $1.50 (P). Brief history of metric system, explanation of SI units, guidelines for teaching SI. Also classroom activities, list of materials, and student-made learning aids. PSJ

EDUCATION, P. *Logik, Mengen, Relationen.* H. Freund, P. Sorger. Teubner, Stuttgart, 1976, 191 pp, (P). One of a series of texts written for prospective teachers. Chapters on propositional calculus, predicate calculus, the algebra of sets, proofs, relations, mappings and functions. JD-B

EDUCATION, P, L. *Activity-Oriented Mathematics, Readings for Elementary Teachers.* Ed: William E. Schall. Prindle, 1976, xii + 481 pp, $8.95 (P). More than 100 reprinted articles (mostly from *The Arithmetic Teacher)* on activities for primary school mathematics. Includes a list of sources of free classroom material. LAS

HISTORY, P. *Ernst Eduard Kummer Collected Papers, V. II: Function Theory, Geometry and Miscellaneous.* Ed: André Weil. Springer-Verlag, 1975, xi + 877 pp, $42.20. Contains Kummer's early papers on function theory, his work in algebraic geometry, articles on aerodynamics and ballistics, as well as a handful of book reviews and several speeches delivered by Kummer as an officer in the Berlin Academy of Science. SG

ALGEBRA, T*(18: 1, 2), L. *Sheaf Theory.* B.R. Tennison. London Math. Soc. Lect. Notes, No. 20. Cambridge U Pr, 1975, vii + 164 pp, $8.95 (P). A very nice introduction to the subject. Two chapters on the concrete construction of presheaves and sheaves, and then a chapter on the categorical viewpoint. The book concludes with one chapter on ringed spaces (sheaves of algebras) and a chapter on cohomology. Exercises, good references, hints to some exercises. PJM

ALGEBRA, T(14-15: 1). *Abstract Algebra: A First Course.* Violet Hachmeister Larney. Prindle, 1975, xiii + 367 pp, $14.95. Slow development. An initial overview establishes language and formal notation for succeeding chapters on integers and polynomials. Latter third devoted to groups, rings, fields. Historical notes on 22 mathematicians. LCL

FINITE MATHEMATICS, T(13-14: 1), L. *Finite Mathematics.* David A. Sprecher. Har-Row, 1976, x + 366 pp, $12.95. Chapters on combinatorics, probability, and Markov chains, as well as the traditional coverage of set theory and logic. The text includes many worked examples. JG

FINITE MATHEMATICS, T(13: 2). *Modern Mathematics with Applications to Business and the Social Sciences, Second Edition.* Ruric E. Wheeler, W.D. Peeples, Jr. Brooks/Cole, 1975, xii + 607 pp, $13.95. Easy to read background for business mathematics. Matrices, finance, linear programming, probability, calculus. LH

FINITE MATHEMATICS, T*(13-14: 1). *Finite Mathematics: An Introduction.* Bodh R. Gulati. Har-Row, 1975, x + 389 pp, $12.95. Examples and exercises plentiful but all straightforward. Slanted to OR-type problems. Clear introduction to probability, matrices, Markov chains, game theory. Simplex method applied to matrix games. LH

CALCULUS, T(14-15: 2), S, L*. *Introduction to Calculus and Analysis, V. 2.* Richard Courant, Fritz John. Wiley, 1974, xxiii + 954 pp, $17.95. A thorough revision of Courant's *Differential and Integral Calculus, V. 3.* Retains the geometrical flavor, challenging examples, and applications of the original classic. Has more exercises (prepared by Albert Blank and Alan Solomon). More rigorous development of integration. More linear algebra so line, surface, and volume integrals can be developed in terms of differential forms. *The* book for obtaining geometrical and physical understanding of Gauss-Green-Stokes theorems. Contains a wealth of information. Includes, in addition to multivariable calculus, chapters on differential equations, calculus of variations, and functions of a complex variable. RBK

CALCULUS, T(13: 1), L. *Beginning Calculus with Applications.* Richard J. Maher. Hafner Pr, 1976, xvi + 436 pp, $11.95. A beginning text on calculus with applications to economics, business, the life sciences, the social and behavioral sciences. Written for non-science students. Each chapter contains a section devoted to specific applications. Good balance between theory and example. Appendices. Tables. Answers. Index. RJA

CALCULUS, T*(13: 1). *Calculus for Management, Economics and the Life Sciences, Revised Edition.* Kenneth Loewen. Prindle, 1975, xii + 436 pp, $12.50. Assumes little proficiency yet gets quickly to optimization in several variables. Lagrange multipliers, linear programming compared for constrained case. Differentials introduced clearly; integration almost an appendix. LH

CALCULUS, T*(13-14: 2, 3), L*. *Elementary Calculus.* H. Jerome Keisler. Prindle, 1976, xviii + 941 pp, $18.95. An unusually lucid and detailed treatment of the typical calculus topics from the refreshing perspective of Robinson's infinitesimal structure--here called hyperreal numbers. (The suspect term 'nonstandard' is used only once--as an historical reference on p. 875.) Classical $\varepsilon$-$\delta$ methods are introduced, both for completeness and for contrast with the intuitively appealing infinitesimal approach. A bold experiment that has been class tested for six years, this text is one of very few landmarks in the history of calculus textbook publishing. LAS

CALCULUS, T(13-14: 2, 3). *Basic Mathematics for Engineers and Technologists.* Alan Jeffrey. Nelson (U.S. Distr: Crane, Russak), 1974, ix + 555 pp, $6.75 (P). A practical treatment, containing only the essentials of an amazing range of topics--differential and integral calculus (some multivariable), vectors and matrices, complex variables, first order differential equations, probability and statistics. LCL

REAL ANALYSIS, P. *Lecture Notes in Mathematics-487: Funktionen beschränkter mittlerer Oszillation.* Hans Martin Reimann, Thomas Rychener. Springer-Verlag, 1975, vi + 141 pp, $7.40 (P). Seminar notes on functions of bounded mean oscillation. JD-B

REAL ANALYSIS, P. *Lecture Notes in Mathematics-481: Differentiation of Integrals in $R^n$.* Miguel de Guzmán. Springer-Verlag, 1975, xii + 225 pp, $9.50 (P). Generalizations of Lebesgue's theorem that the derivative of the integral of $f \in L^1(R^n)$ with respect to a sequence of balls centered at x with radii tending to zero equals f(x) almost everywhere. For example, if balls are replaced by intervals, the result holds for $f \in L^p(R^n)$, $1 < p < \infty$ but not for $f \in L^1(R^n)$. If balls are replaced by rectangles, the result fails even for characteristic functions of measurable sets. Interplay between covering theorems, differentiation properties, and Hardy-Littlewood maximal operators.RBK

COMPLEX ANALYSIS, T(18:1), P. *Lecture Notes in Mathematics-478: Univalent Functions--Selected Topics.* Glenn Schober. Springer-Verlag, 1975, v + 200 pp, $9.90 (P). Extremal problems for univalent functions studied from linear space-convex set, and variational points of view. Includes quasiconformal mappings and quasiconformal extensions of univalent functions. Appendices on convexity, coefficient and distortion theorems, and Schiffer's boundary variation and fundamental lemma. A useful reference. RBK

COMPLEX ANALYSIS, P. *Theorie der Funktionen mehrerer komplexer Veränderlicher.* C. Andreian Cazacu. Math. Reihe, B. 51. Birkhauser, 1975, 359 pp, $29. An introduction to the theory of functions of several complex variables. A translation and revision of the original Romanian edition of 1971. A considerable but not complete bibliography, listing publications up to 1972. No problems. JD-B

COMPLEX ANALYSIS, P*. *Entire Holomorphic Mappings in One and Several Complex Variables.* Phillip A. Griffiths. Annals of Math. Stud., No. 85. Princeton U Pr, 1976, x + 99 pp, $11.50; $4.50 (P). This monograph grew out of the Herman Weyl lectures, whose goal is to provide a survey of a broad area for nonspecialists. This goal is admirably realized in this book. Topics: orders of growth, curvature, defect relations. The style and topic make this an ideal source for a seminar. Contains a substantial bibliography. TAV

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Economics and Mathematical Systems-113: Stabilitätsaussagen über Klassen von Matrizen mit verschwindenden Zeilensummen.* Gerhard A. Aschinger. Springer-Verlag, 1975, v + 102 pp, $7.40 (P). Treats a special type of Liapunov stability of solutions of certain systems of linear homogeneous differential equations with constant coefficients which arose in the author's study of social choice. JD-B

DIFFERENTIAL EQUATIONS, P. *Stochastic Differential Equations and Applications, V. 2.* Avner Friedman. Prob. and Math. Stat., V. 28. Acad Pr, 1976, xiii + 299 pp, $32.50. Topics include: stochastic differential games, stopping time problems; Dirichlet problem for degenerate elliptic equations, singular perturbations, fundamental solutions for degenerate parabolic equations, behavior of sample paths of solutions of stochastic differential equations. Includes exercises and bibliographical remarks. SG

DIFFERENTIAL EQUATIONS, T*(14-15). *Elementary Differential Equations with Applications.* William R. Derrick, Stanley I. Grossman. A-W, 1976, x + 602 pp, $14.95. A fairly complete introductory text. Topics (in order): first order equations; linear second order differential and difference equations, power series solutions, Laplace transforms, systems of differential equations, numerical methods, Fourier series and boundary value problems, partial differential equations, existence and uniqueness, nonlinear equations and stability. Includes table of integral and Laplace transforms, an appendix on determinants, and a Fortran primer. Contains many exercises and applications, and is mathematically honest. SG

NUMERICAL ANALYSIS, P. *Proceedings of the 1975 Army Numerical and Computer Analysis Conference.* U.S. Army Research Office, Research Triangle Park, NC, 1975, xiii + 560 pp, $13.50 (P). Contains papers of those attending as well as some of those not attending the conference. RJA

NUMERICAL ANALYSIS, S(17-18), P. *A Rational Finite Element Basis.* Eugene L. Wachspress. Acad Pr, 1975, xiii + 331 pp, $15. A monograph on the use of rational functions in the finite element method resulting in greater accuracy and more general applicability. Employs algebraic geometry. Embodies some of the author's works. Interestingly written. RWN

NUMERICAL ANALYSIS, T(16-17: 1, 2), S, P, L. *Splines and Variational Methods.* P.M. Prenter. Wiley, 1975, xi + 323 pp, $19.95. Fine exposition. Proceeds smoothly from Lagrangian and Hermitian interpolation to splines and the least squares, Rayleigh-Ritz, Galerkin, finite element collocation methods. Includes some preliminaries, applications to ordinary and partial differential equations, and exercises. RWN

FUNCTIONAL ANALYSIS, T(17-18: 1), S, P. *Sobolev Spaces.* Robert A. Adams. Pure and Appl. Math., V. 65. Acad Pr, 1975, xviii + 268 pp, $24.50. Self-contained, beginning with review of functional analysis, $L^p$-spaces. Emphasises theory most applicable to PDE's. Trace-interpolation approach to definition of fractional-order spaces. Includes Orlicz-Sobolev theory. LH

FUNCTIONAL ANALYSIS, T(18:1). *Geometric Functional Analysis and its Applications.* Richard B. Holmes. Grad. Texts in Math., V. 24. Springer-Verlag, 1975, x + 246 pp, $16.80. Functional analysis via convexity. Applications are to optimization theory in general and to best approximation theory in particular. No discussion of spectral theory. Over 200 problems. A welcome text. DFA

FUNCTIONAL ANALYSIS, T(17-18: 2), S, P. *Funktionalanalysis.* Harro Heuser. Teubner, Stuttgart, 1975, 416 pp, (P). An introduction to functional analysis written for physicists and engineers as well as mathematicians. Many problems, bibliography. JD-B

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-486: Representations of AF-Algebras and of the Group U(∞).* Şerban Strătilă, Dan Voiculescu. Springer-Verlag, 1975, viii + 169 pp, $8.20 (P).

OPTIMIZATION, S(17-18), P. *Optimierung und Approximation.* Werner Krabs. Teubner, Stuttgart, 1975, 208 pp, (P). On the relations between approximation and optimization. Deals with linear, convex and nonlinear problems. Said to be an introduction, but requires considerable mathematical knowledge. JD-B

OPTIMIZATION, T(16-17: 1), S, P, L*. *Optimization Theory, The Finite Dimensional Case.* Magnus R. Hestenes. Wiley, 1975, xiii + 447 pp, $24.95. Careful presentation of the mathematical theory--only occasional references to computational aspects or applications outside of mathematics. Provides a natural transition to infinite dimensional situations (e.g., the author's *Calculus of Variations and Optimal Control Theory*). LCL

OPTIMIZATION, T(16-17: 2), S, P*. *Linear and Nonlinear Programming.* V.A. Sposito. Iowa St U Pr, 1975, x + 269 pp, $12.95. Provides the main ideas of linear programming and goes on into the Kuhn-Tucker theory and quadratic duality. Presents the computational techniques (sensitivity analysis and parametric programming with special attention given to IBM's linear programming package MPSX). Contains a chapter on statistical applications not usually found in comparable texts. I-CH

OPTIMIZATION, T(17-18), S, P, L. *Pursuit Games: An Introduction to the Theory and Applications of Differential Games of Pursuit and Evasion.* Otomar Hájek. Math. in Sci. and Eng., V. 120. Acad Pr, 1975, xii + 266 pp, $10.50. Lecture notes, introduced by several enticing examples, of general procedures for finding winning positions and strategies in games of pursuit. LAS

OPTIMIZATION, T*(16-17: 1), L. *Linear Programming: Methods and Applications, Fourth Edition.* Saul I. Gass. McGraw, 1975, x + 406 pp, $18.50. One of the classics in the field--it continues to be one of the best. Balanced presentation of theoretical, computational, and applied areas. Only substantial change is the inclusion of new material on the generalized upper-bound problem. Fine bibliography. LCL

OPTIMIZATION, T(18), P, L. *Theory of Optimal Search.* Lawrence D. Stone. Math. in Sci. and Eng., V. 118. Acad Pr, 1975, xiv + 260 pp, $29.50. A unified presentation of core results, selected in part for their versatility or computability. The basic search problem is to find an allocation of effort that maximizes the probability of detecting the target, subject to various constraints and difficulties (like false or moving targets). Requires sophistication in both probability theory and real analysis. LAS

OPTIMIZATION, P. *Lecture Notes in Mathematics-477: Optimization and Optimal Control.* Ed: R. Bulirsch, W. Oettli, J. Stoer. Springer-Verlag, 1975, vii + 294 pp, $11.50 (P). Proceedings of the conference at Oberwolfach in November 1974. JAS

OPTIMIZATION, T(16-17: 1), S, P. *Mathematical Programming Via Augmented Lagrangians: An Introduction with Computer Programs.* Donald A. Pierre, Michael J. Lowe. Appl. Math. and Comp., No. 9. A-W, 1975, xxi + 436 pp, $14.50 (P); $24.50. First discusses nonlinear programming problems, optimality conditions and sensitivity analysis. Then the Lagrangian is augmented by some penalty-like terms and conditions for the solution to the augmented problem are related to those of the original problem. Includes exercises, test problems, applications, implementation details and a Fortran subroutine. RWN

ANALYSIS, P. *Lecture Notes in Mathematics-471: Harmonic Maps of Manifolds with Boundary.* Richard S. Hamilton. Springer-Verlag, 1975, 168 pp, $8.60 (P). Generalization of 1964 results of Eells and Sampson; contains three major results corresponding to the Dirichlet, Neumann and mixed boundary value problems. LAS

ANALYSIS, T(18: 1), P. *Banach Algebras.* Richard D. Mosak. U of Chicago Pr, 1975, viii + 172 pp, $9; $3.95 (P). Comfortable presentation of material selected as necessary background for a course on harmonic analysis of locally compact groups. LCL

ANALYSIS, P. *Lecture Notes in Mathematics-470: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Rufus Bowen. Springer-Verlag, 1975, 108 pp, $7.80 (P). "The main purpose of these notes is to present the ergodic theory of Ansov and Axiom A diffeomorphisms. These diffeomorphisms have a complicated orbit structure that is perhaps best understood by relating them topologically and measure theoretically to shift spaces." Bibliography, index. TAV

ANALYSIS, P. *Sophus Lie's 1880 Transformation Group Paper.* Ed: Robert Hermann. Trans: Michael Ackerman. Math Sci Pr, 1975, xxiv + 563 pp, $28 (P). The first translation of Lie's seminal paper, preceded by four chapters of mathematical prerequisites, interspersed by commentary from a modern perspective, and followed by chapters applying Lie's ideas to physics, differential geometry and systems theory. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-474: Séminaire Pierre Lelong (Analyse) Année 1973-74.* Ed: Pierre Lelong. Springer-Verlag, 1975, vi + 182 pp, $8.20 (P).

ANALYSIS, P. *Fourier Series with Respect to General Orthogonal Systems.* A.M. Olevskiĭ. Transl: B. P. Marshall, H.J. Christoffers. Ergebnisse der Math., B. 86. Springer-Verlag, 1975, ix + 136 pp. $33.60. Four chapters, with these titles: Convergence of Fourier Series in the Classical Sense; Lebesgue Functions of Bounded Systems. Convergence Almost Everywhere; Conditions on the Coefficients. Properties of Complete Systems; The Role of the Haar System. Series from $L^2$ and Peculiarities of Fourier Series from the Spaces $L^p$. DFA

ANALYSIS, P. *Lecture Notes in Mathematics-466: Non-Commutative Harmonic Analysis.* Ed: J. Carmona, J. Dixmier, M. Vergne. Springer-Verlag, 1975, vi + 231 pp, $9.90 (P). Papers from the colloquium in July 1974 at Marseille-Luminy. JAS

GEOMETRY, P. *Regular Dirichlet-Voronoĭ Partitions for the Second Triclinic Group.* M.I. Štogrin. Proc. of Steklov Inst. of Math., No. 123. AMS, 1975, iii + 116 pp, $21.50 (P). A complete study of the decompositions of 3-space into convex polyhedra so that the second triclinic group acts transitively on the polyhedra of the decomposition. JAS

GEOMETRY, T(13-15: 1), S, L. *Geometry by Transformations.* E.A. Maxwell. Cambridge U Pr, 1975, xii + 276 pp, $19.50. Primarily for school level courses in Britain; plane Euclidean geometry is developed intuitively with transformations as central theme. Virtually all important topics treated. Algebra of transformations and geometric consequences follows the naive introduction and constitutes the second major portion of book, while third dwells on use of matrices. Many concrete exercises. Should be of interest to college audiences in U.S., particularly prospective school teachers. SS

GEOMETRY, P. *Gauge Fields and Cartan-Ehresmann Connections, Part A.* Robert Hermann. Interdisciplinary Mathematics, V. X. Math Sci Pr, 1975, ix + 515 pp, $25 (P). Continues the development of differential geometry and the associated topology with interpretations in physics. The notation depends heavily on earlier volumes, and typographical errors are all too easy to find. JAS

GEOMETRY, T(13-14: 1), S, L. *Rudiments of Plane Affine Geometry.* P. Scherk, R. Lingenberg. U of Toronto Pr, 1975, ix + 110 pp, $10. Begins with axioms for affine planes and related properties, and then studies successively restricted subfamilies of affine planes: translation, desarguesian, and Pappus planes. Concludes with desarguesian projective planes. Book is viewed by authors as an alternative to the traditional course on euclidean geometry. Exercises. Appendix. References. Index. RJA

GEOMETRY, S*(14-16), L**. *Symmetry in Science and Art.* A.V. Shubnikov, V.A. Koptsik. Trans: G.D. Archard. Plenum Pr, 1974, xxv + 420 pp, $35. Translation of the expanded 1972 second edition of a 1940 Russian classic. Nine chapters of profusely illustrated description of the classical space symmetries are followed by details of the symmetry groups, by generalized (colored) symmetry, and by diverse applications in art and science. LAS

GEOMETRY, S, L. *Polyhedron Models for the Classroom, Second Edition.* Magnus J. Wenninger. NCTM, 1975, viii + 43 pp, $1.40 (P). A brief how-to-construct-it pamphlet. This edition features templates to aid in construction. LAS

GEOMETRY, P. *Lecture Notes in Mathematics-484: Differential Topology and Geometry.* Ed: G.P. Joubert, R.P. Moussu, R.H. Roussarie. Springer-Verlag, 1975, ix + 287 pp, $11.50 (P). Proceedings of a colloquium held at Dijon, France in June 1974. JAS

GEOMETRY, T*(16: 1), S, L. *Deductive Transformation Geometry.* R.P. Burn. Cambridge U Pr, 1975, xi + 121 pp, $11.95. Defines the plane via geometric axioms and concisely uses transformations to develop geometric properties with emphasis on the inter-relationship between the plane and the real number system. Graded references. A nice text for mathematics or education students with moderate undergraduate algebra background. JNC

TOPOLOGY, P*. *Connections, Curvature, and Cohomology, V. III: Cohomology of Principal Bundles and Homogeneous Spaces.* Werner Greub, Stephen Halperin, Ray Vanstone. Pure and Appl. Math., V. 47-III. Acad Pr, 1976, xxi + 593 pp, $49.50. Volume I (TR, January 1973) defined DeRham cohomology. Volume II (TR, February 1974) discussed Lie groups and bundles. In Volume III, some more algebraic machinery is provided to help compute the cohomology of bundles and homogeneous spaces. Includes a self-contained chapter on spectral sequences. This book was worth waiting for. PJM

TOPOLOGY, P. *Bibliography on Quasi-uniform Spaces.* M.G. Murdeshwar (U. of Alberta, Edmonton, Alberta, Canada T6G 2G1), 1974, iii + 21 pp, free (P). A two part bibliography. Part one contains papers; part two books. Author index. RJA

TOPOLOGY, T(15-17: 1, 2), L. *Fundamentals of Topology.* Benjamin T. Sims. Macmillan, 1976, x + 179 pp, $12.95. Substantial, sophisticated, terse, and moderately detailed point set topology is followed with chapters on both homotopy and singular homology. The book contains a lot of topology but no philosophy or history. Lots of exercises and a good index and bibliography. JAS

TOPOLOGY, T*(15-17: 1), S**, L*. *Algebraic Topology, A First Course.* Max K. Agoston. Pure and Appl. Math., V. 32. Dekker, 1976, ix + 360 pp, $23.50. Emphasis on geometry and history and de-emphasis on algebra make this an unusual offering. The topological prerequisites are made minimal by working in $R^n$. It has an excellent bibliography and an adequate index for its intended purposes. However, there are too few problems and very few that are both simple and helpful to the intuition. As a book for advanced students to use as "culturally broadening" material, it may well be unique (in English), and even worth the fairly stiff price for typescript. JAS

TOPOLOGY, T(18: 1, 2), S, P*. *Theory of Shape.* Karol Borsuk. Mono. Matematyczne, T. 59. PWN, 1975, 379 pp. An extensive exposition of this new area of topology, by one of its leading developers. Only a minimum of topological background is assumed but the presentation is thorough and demanding. Contains a comprehensive bibliography up to early 1975, with references to the citations of the bibliography in the text. Good index. JAS

TOPOLOGY, T*(13-15: 1), S*, L*. *Surfaces.* H.B. Griffiths. Cambridge U Pr, 1976, xi + 120 pp, $12.50. A delightful, "naive" approach to the classification of (two dimensional) surfaces: ears, bridges, twists, handles are manipulated in a (sophisticated) school child's vernacular to prove several basic theorems. Intended to motivate the teaching of three-dimensional geometry. LAS

TOPOLOGY, P. *Lecture Notes in Mathematics-468: Dynamical Systems--Warwick 1974.* Ed: Anthony Manning. Springer-Verlag, 1975, x + 405 pp, $15.10 (P). Proceedings of the Symposium held at the University of Warwick in 1973-4. JAS

PROBABILITY, P*, L. *The Theory of Stochastic Processes II.* I.I. Gihman, A.V. Skorohod. Grund. math. Wissenschaften, B. 218. Springer-Verlag, 1975, vii + 441 pp, $48.20. Devoted principally to Markov processes including jump processes, semi-Markov processes, branching processes and processes with independent increments. Very carefully written, this volume contains a substantial number of new results. An important book at an inflated price. TAV

PROBABILITY, T*(17: 2), P*, L*. *Stochastic Convergence, Second Edition.* Eugene Lukacs. Prob. and Math. Stat., V. 30. Acad Pr, 1975, xi + 200 pp, $14. Differs from the first edition (1970) in the addition of a chapter on random power series, the inclusion of exercise sets, and minor editorial changes. This is an extremely well written treatment of convergence of numerical valued random variables. Contains an extensive bibliography. TAV

STATISTICS, S(15-18), P, L. *Sample Size Choice: Charts for Experiments with Linear Models.* Robert E. Odeh, Martin Fox. Statistics, V. 14. Dekker, 1975, viii + 190 pp, $18.50. Formulas and charts to determine sample size for many levels of significance and power. Examples of use. 1 and 2-way layouts, randomized blocks, regression models. F, chi-square, T-distribution. LH

STATISTICS, T(13-14), L. *Essentials of Statistics in Marketing.* C.S. Greensted, A.K.S. Jardine, J.D. Macfarlane. Wiley, 1974, xi + 257 pp, $12.50. Decision making for sales managers. Good examples but few problems. Very low mathematical level. LH

STATISTICS, S, P. *Bayesian Decision Problems and Markov Chains.* J.J. Martin. Krieger, 1975, xii + 202 pp, $13.50. Markov chains with uncertain transition probabilities. Sequential sampling and fixed sample-size. Emphasis on existence and convergence. Reprint of 1967 Wiley edition. LH

STATISTICS, S*(13), L. *Statistics in the Real World: A Book of Examples.* Richard J. Larsen, Donna Fox Stroup. Macmillan, 1976, xxi + 245 pp, $6.95 (P). Fifty ordinary, everyday, rather unsophisticated illustrations of statistical inference, taken from recently published research in a wide variety of subjects. Lively, attractive presentations: description, objectives, procedure, data, analysis, followed by several pertinent questions with space provided for answers. Most valuable as a workbook supplement for a precalculus course. LCL

STATISTICS, S*(14-17), P, L**. *A Primer of Multivariate Statistics.* Richard J. Harris. Acad Pr, 1975, xiv + 332 pp, $13.95. An attempt to "make multivariate statistical techniques available to a much broader audience by concentrating on heuristic descriptions of what all the math was designed to accomplish." Multiple regression, Hotelling's $T^2$, analysis of variance, canonical correlation, principal component analysis, factor analysis. Appendices on matrix algebra, using canned computer programs, and some program listings. FLW

STATISTICS, L. *Selected Tables in Mathematical Statistics, V. III.* Ed: H.L. Harter, D.B. Owen. AMS, 1975, viii + 419 pp, $21.60. Tables of the two factor and three factor generalized incomplete modified Bessel distributions; sample size requirements for classification experiments; passage time distributions for Gaussian-Markov processes; probability levels for the Kruskal-Wallis test; confidence limits for linear functions of the normal mean and variance. (Vol. I: TR, November 1974; Vol. II: TR, March 1975.) LCL

COMPUTER SCIENCE, T*(13-14: 1, 2), S*, L. *Introduction to Computer Science.* Alan J. Perlis. Har-Row, 1975, xii + 385 pp, $16.95. Introductory text which stresses underlying concepts and principles. Uses Algol 60 and APL to illustrate various ideas in detail. Also contains a good but brief survey of a number of widely used programming languages. Includes a chapter on data structures and introductory material on the theory of computation. Chapter references and problems. Index. RJA

COMPUTER SCIENCE, T(16-17: 1, 2), S, L. *Compiler Design Theory.* Philip M. Lewis II, Daniel J. Rosenkrantz, Richard E. Stearns. A-W, 1976, xx + 647 pp, $18.95. Uses automata and formal language concepts as a basis for discussing compiler design. Contains the complete theory for the lexical and syntax portions while including material on code generation. To demonstrate theoretical ideas a subset of BASIC is used. Appendices. Bibliography. Index. RJA

APPLICATIONS (BUSINESS), T(13: 1). *Mathematics of Investment, Fifth Edition.* William L. Hart. Heath, 1975, xiii + 152 pp, $11.95. For general business student. Bonds, depreciation, annuities, life insurance. Discrete math only. Good problems. LH

APPLICATIONS (ECONOMICS), S(16-17), P. *Mathematical Planning of Structural Decisions, Second Enlarged Edition.* János Kornai. North-Holland, 1975, xxxii + 644 pp, $49.95. Case studies of actual long-term planning in Hungary. 2nd edition enlarged to describe multi-level, economy-wide, mathematical programming efforts during 1964-1972. LH

APPLICATIONS (ENGINEERING), P. *Physical Structure in Systems Theory: Network Approaches to Engineering and Economics.* Ed: J.J. van Dixhoorn, F.J. Evans. Acad Pr, 1974, x + 305 pp, $20.75. Papers from the February 1973 colloquium at Twente University of Technology, Enschede, The Netherlands. JAS

APPLICATIONS (ENGINEERING), T(13: 1). *Mathematics for Scientific and Technical Students.* H.G. Davies, G.A. Hicks. Longman, 1975, v + 454 pp, $16.50; $7.95 (P). Computational. Contents from British technical colleges. Algebra, trigonometry, calculus, statistics. A quote: "(infinity) stands for the statement that a number is extremely large." LH

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Judith N. Cederberg, St. Olaf; Jennifer Galovich, St. Olaf; Steven Galovich, Carleton; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Paul S. Jorgensen, Carleton; Roger B. Kirchner, Carleton; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

*Arizona State University*: Mrs. Wilma Thompson has been promoted from Instructor to Assistant Professor; Professor Lucas N. H. Bunt retired with the title of Professor Emeritus on May 18, 1975.

*Bard College*: Dr. Martin Flashman, Brandeis University, has been appointed Assistant Professor; Assistant Professor Fred Kollett has been promoted to Associate Professor.

*Bowling Green State University*: Dr. Martin Fox, Michigan State University, has been appointed Visiting Professor; Dr. G. O. Kenny, University of Kansas, has been appointed Assistant Professor; Dr. T. P. Dence, New Mexico State University, has been appointed Assistant Professor; Assistant Professors Josef Blass, J. L. Hayden, C. S. Johnson, Jr., and J. G. Williams have been promoted to Associate Professors; Associate Professor J. L. Hayden is on leave for 1975–76 at Michigan State University; University Professor Eugene Lukacs retired in September 1975.

*Florida State University*: Associate Professors Christopher Lacher and J. L. Mott have been promoted to Professors; Professor Thomas L. Wade retired with the title of Professor Emeritus on June 12, 1975.

*Hanover College*: Dr. J. H. Driggs, University of Michigan, has been appointed Instructor; Dr. W. W. Dunham, Ohio State University, has been appointed Assistant Professor; Professor John E. Yarnelle retired on September 1, 1975, with the title of Professor Emeritus.

*Hofstra University*: Assistant Professor Harold Hastings has been promoted to Associate Professor; Assistant Professor Edward Ostling has been promoted to Associate Professor and appointed Chairman of the Mathematics Department; Associate Professor Alexander Weiner retired with the title of Associate Professor Emeritus on September 1, 1975.

*Illinois State University*: Assistant Professor John Dossey has been promoted to Associate Professor; Associate Professor Albert Otto has been promoted to Professor.

*Kean College of New Jersey*: Assistant Professor S. H. Lipson has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics and Computer Science; Dr. Susan G. Marchand has been promoted from Instructor to Assistant Professor.

*Mohawk Valley Community College*: Assistant Professors Sarah Brooks and Ivan Doszpoly have been promoted to Associate Professors; Professor L. A. Trivieri has been appointed Head of the newly created Department of Mathematics.

*New Mexico State University*: Instructor Darell Johnson, MIT, has been appointed Assistant Professor; Associate Professor Charles Swartz has been promoted to Professor.

*Northwest Missouri State University*: Assistant Professor Merry McDonald has been promoted to Associate Professor; Assistant Professor Vida Dunbar retired on July 1, 1975.

*Ohio Northern University*: Assistant Professor J. T. McLean, Ohio State University, Lima, has been appointed Associate Professor and Chairman of the Mathematics Department; Assistant Professor R. A. Hovis, Ohio University, Chillicothe, has been appointed Assistant Professor.

*Pennsylvania State University*: Assistant Professor William Waterhouse, Cornell University, has been appointed Associate Professor; Assistant Professors Paromita Chowla and Torrance Parsons have been promoted to Associate Professors; Professor Hugo Ribeiro retired with the title of Professor Emeritus on June 30, 1975.

*Shippensburg State College*: Dr. F. R. Bernhart, University of Waterloo, has been appointed Assistant Professor; Associate Professor W. G. McArthur has been promoted to Professor.

*Tarleton State University*: Associate Professor J. E. Cude, Head of the Department of Mathematics, has been promoted to Professor; Assistant Professor Linda M. Neal has been promoted to Associate Professor.

*University of Houston*: Dr. H. F. Walker has been appointed Associate Professor; Assistant Professor W. E. Fitzgibbon has been promoted to Associate Professor.

*University of Montana*: Assistant Professor Patrick Ewing, Ohio State University, has been appointed Visiting Assistant Professor; Associate Professor Don Loftsgaarden has been promoted to Professor.

*University of Rhode Island*: Assistant Professor James Lewis has been promoted to Associate Professor; Professor Gerald Haggerty retired with the title of Professor Emeritus on July 1, 1975.

*University of Santa Clara*: Assistant Professor D. H. Mugler, Syracuse University, has been appointed Assistant Professor; Mr. Dennis Smolarski, University of California, Santa Barbara, has been appointed Lecturer.

*University of South Florida*: Assistant Professors George Michaelides and Kenneth Pothoven have been promoted to Associate Professors; Associate Professor Arunava Mukherjea has been promoted to Professor.

*Waynesburg College*: Dr. D. S. Tucker, Florida State University, has been appointed Assistant Professor; Associate Professor L. O. Hagglund has been appointed Chairman of the Department of Mathematics.

*Wheeling College*: Dr. A. T. Basil, formerly Chairman of the Mathematics Department, has been appointed Assistant Dean of the College; Dr. Dorothy T. Coon has been appointed Chairman of the Mathematics Department.

Assistant Professor B. O. Baylis, Jr., King's College, has been appointed Acting Chairman of the Department of Mathematics.

Assistant Professor Bob Ducharme, Baldwin-Wallace College, has been appointed Assistant Professor at the University of Texas, San Antonio.

Assistant Professor Clifton Ealy, Northern Michigan University, has been promoted to Associate Professor; he is on leave at the University of Chicago with a fellowship granted by the National Fellowship Fund.

Dr. R. B. Feinberg, National Bureau of Standards, has been appointed Assistant Professor at Clarkson College of Technology.

Professor Frank Harary, University of Michigan, was awarded the honorary degree of Doctor of Science by the University of Aberdeen, Scotland, on July 10, 1975.

Dr. H. T. Karnes, Louisiana State University, recently received the Pi Mu Epsilon C. C. MacDuffee Award. This is the highest award from Pi Mu Epsilon; it has been conferred only five other times in Pi Mu Epsilon's 60 year history.

Professor W. I. Layton, Head of the Mathematics Department at Stephen F. Austin State University, was named Alumni Distinguished Professor for 1975. He also received a Certificate of Award and Honor from the mathematics faculty of Stephen F. Austin State University in May 1975.

Assistant Professor David Moxness, Dakota State College, has been promoted to Associate Professor.

Assistant Professor Jack Narayan, State University College at Oswego, has been promoted to Associate Professor.

Assistant Professor Kenneth Pacholke, Northland College, has been promoted to Associate Professor.

Associate Professor E. L. Roetman, University of Missouri, Columbia, has been promoted to Professor.

Associate Professor Bruce Rothschild, UCLA, has been promoted to Professor.

Dr. R. M. Tardiff, University of Massachusetts, has been appointed Assistant Professor at the University of Minnesota-Morris.

Assistant Professor Sally H. Thomas, Indiana University, has been appointed Associate Professor at Orange Coast College.

Professor Olga Taussky Todd, California Institute of Technology, has been elected a corresponding member of the Austrian Academy of Sciences in recognition of her accomplishments in several areas of mathematics.

Associate Professor D. W. VanderJagt, Chairman of the Mathematics Department at Grand Valley State College, has been promoted to Professor.

Assistant Professor W. G. Vick, Jr., Broome Community College, has been promoted to Associate Professor.

Professor Emeritus Ray E. Gilman, Brown University, died on January 26, 1975, at the age of 87. He was a member of the Association for fifty-five years.

Professor Emeritus George E. Raynor, Lehigh University, died on September 25, 1975, at the age of 80. He was a Charter Member of the Association.

Professor Emeritus J. R. K. Stauffer, University of Rhode Island, died on September 3, 1975, at the age of 72. He was a member of the Association for forty-eight years.

Dr. J. Dean Swift, UCLA, died on September 16, 1975, at the age of 56. He was a member of the Association for twenty-seven years.

Associate Professor Donald Warncke, Monmouth College, died on September 21, 1975. He was a member of the Association for seventeen years.

Dr. Frederick H. Young, Evergreen State College, died on December 28, 1974, at the age of 57. He was a member of the Association for twenty-seven years.

## LIST OF BOOKS SUITABLE FOR COLLEGE LIBRARIES

A list of all books that were classified as especially suitable for college libraries by the editors of Telegraphic Reviews for volumes 74–82, inclusive, (1967–75) is now available. The 522 titles are listed alphabetically by author, together with 1975 prices, ISBN and LC numbers, volume and page number of the review, and the classification given ($L*$, $L**$, $L***$).

Order copies by sending $1.00 to: Mathematics Department, Wabash College, Crawfordsville, IN 47933.

The list was prepared under a bequest from the late Professor George E. Carscallen of Wabash.

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*Official Reports and Communications*

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## ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in San Antonio, Texas, on January 23, 1976, elected to membership the twenty-seventh set of applicants for academic membership (for election of the other twenty-six sets, see the December 1975 issue of the MONTHLY, page 1041, the April 1975 issue, pages 448–9, and the references cited there). Approval of election was given to the following applicants for academic membership:

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Southwest Texas State University
SUNY Agricultural and Technical College at Alfred
Whitworth College
Ohio Northern University
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Heidelberg College
University of Vermont
California Polytechnic State University
Inter American University
Mills College

DAVID P. ROSELLE, *Secretary*

## NOVEMBER MEETING OF THE SEAWAY SECTION

The fall meeting of the Seaway Section of the MAA was held at the State University of New York, College at Cortland, on November 1, 1975, with a registered attendance of 117 people, including 89 members of the Association. Professor Mabel Montgomery of the State University College at Buffalo, Chairman of the Section, presided.

W. T. Corey, Vice-President for Academic Affairs at the College at Cortland, welcomed the members of the Section to the College.

Locations of the next three meetings of the Section were announced as College of St. Rose, Albany, N. Y., Broome Community College, Binghamton, N. Y., and State University College, Buffalo, N. Y.

During the morning and afternoon sessions, the following contributed papers were presented:

*Transition matrices*, by Donald Fama, Cayuga County Community College.
*Differential algebra and asymptotic expansions*, by Walter Strodt, St. Lawrence University.
*Retracts of Cantor Spaces*, by Carl Kohls, Syracuse University.
*Mortgages and recurrence relations*, by F. A. Hacker, Mohawk Valley Community College.
*Mathematics education or vice versa*, by Larry Copes, Ithaca College.
*Applications of graph theory and other mathematical models*, by Alwin Green, State University College, Buffalo, N. Y.
*A combinatorial property of* $2 \times 2$ *matrices*, by H. E. Bell, Brock University.
*Instructional television in College Calculus Courses*, by Sister John Frances Gilman, Niagara University.
*Meaning and precision in geometry*, by D. W. Henderson, Cornell University.
*The Mordell-Weil theorem*, by T. W. Lamm, Cornell University.

EMMET STOPHER, *Secretary-Treasurer*

## CALENDAR OF FUTURE MEETINGS

Fifty-sixth Summer Meeting, University of Toronto, August 26–28, 1976.

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia University, Morgantown, April 23–24, 1976.

FLORIDA, early March. Deadline for paper titles 2 wks. bef. mtg.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Franklin College, Franklin, May 1, 1976.

IOWA, Clarke College, Dubuque, April 9, 1976.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, University of Kentucky, Lexington, April 23–24, 1976.

LOUISIANA-MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 mths. bef. mtg.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Bowie State College, Bowie, Maryland, April 24, 1976.

METROPOLITAN NEW YORK, Hofstra University, Hempstead, April 25, 1976.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, Southwest Missouri State University, Springfield, April 9–10, 1976.

NEBRASKA, Kearney State College, Kearney, April 23–24, 1976.

NEW JERSEY, Montclair State College, Upper Montclair, May 8, 1976.

NORTH CENTRAL, St. Cloud State University, St. Cloud, Minnesota, April 30–May 1, 1976.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Youngstown State University, Youngstown, May 7–8, 1976

OKLAHOMA-ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 wks. bef. mtg.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, April 30–May 1, 1976.

SEAWAY, College of St. Rose, Albany, April 30–May 1, 1976.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, Eastern New Mexico University, Portales, New Mexico, April 1976.

TEXAS, Texas A&M University, College Station, 1st or 2nd weekend of April 1976.

WISCONSIN, Beloit College, Beloit (Friday), May 7, 1976, and University of Wisconsin, Rock County Center, Janesville (Saturday), May 8, 1976.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 21–26, 1977.

AMERICAN MATHEMATICAL SOCIETY, University of Toronto, August 24–27, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, University of Nevada, Reno, April 23–24, 1976.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, West Chester State College, West Chester, Pennsylvania, August 8–11, 1976.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlanta, Georgia, April 21–24, 1976.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Sheraton Hotel, Philadelphia, March 31–April 2, 1976.

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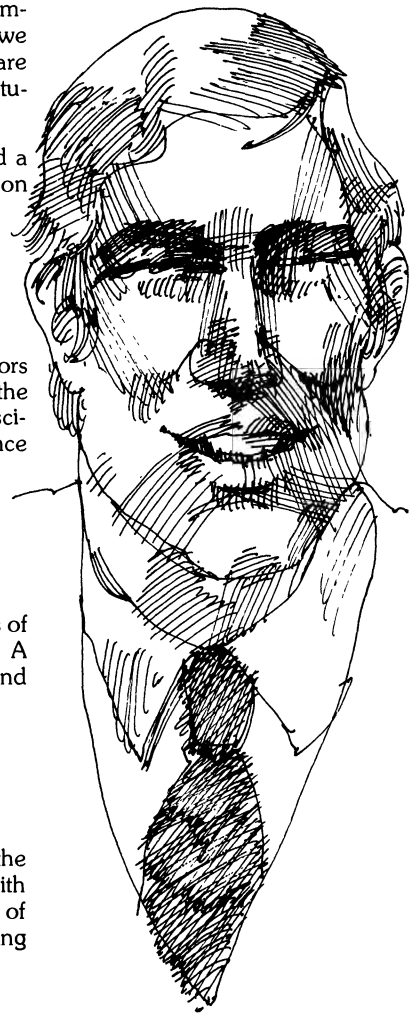
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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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# SOME APPLICATIONS OF THE WREATH PRODUCT CONSTRUCTION

CHARLES WELLS

**1. Introduction.** Any two semigroups can be combined, usually in more than one nonequivalent way, into a third semigroup known as a wreath product of the two semigroups. This construction has been used by group theorists for many years (see Section 16), and in the last fifteen years it has come to be used widely in the study of semigroups.

It is the aim of this article to state and prove a theorem of Krohn and Rhodes, which says that a semigroup action can be decomposed (in a certain technical sense) into the wreath product of simple groups and certain special semigroups. This theorem was motivated by the fact that putting two automata in series with no feedback corresponds roughly to taking the wreath product of certain semigroups associated with the automata. A number of other applications of the wreath product are also described in the course of the article.

Section 2 provides an introductory discussion giving a natural setting for the wreath product. A number of necessary facts about semigroups are given in Section 3. The wreath product involves semigroup actions in an intimate way; they are described in Section 4, and a brief description of the closely-related idea of an automaton is in Section 5.

The wreath product is defined in Section 6, and in a more general way in Section 7. After some necessary preliminaries in the next two sections, the structure of the wreath product is described explicitly as a semi-direct product in Section 10. The following three sections provide a number of ways of embedding (in a sense more general than usual) a semigroup action in a wreath product. In one of these sections (Section 12), the important Kaloujnine-Krasner Theorem for groups is proved. These results are then used to prove the Krohn-Rhodes Theorem in Section 14. The last two sections are occupied with descriptions of other applications of the wreath product and with historical remarks.

The Bibliography is fairly extensive. A reference to Rumpelstiltskin [64] means the article written in 1964 by Rumpelstiltskin. Two articles written in the same year are distinguished by letters; e.g., [64a], [64b].

**2. Triangular actions.** The kind of situation to be studied involves a set $X$ and a set $S$ of operators on $X$: for each $s \in S$, there corresponds a function $s\phi: X \to X$. In group character theory, $X$ is a complex vector space and $S$ is a group, so that $s\phi$ may be regarded as an invertible matrix. In many applications to abstract group theory, $X$ is the underlying set of a group $S$ and $s\phi$ is right multiplication by $s$. In combinatorics, $X$ is often a combinatorial structure (e.g., a graph) and $S$ its group of automorphisms. For automaton theorists, $X$ is a set of states and $S$ an input alphabet. A scientific theory can often be cast in this mold, too, as observed by Krohn, Langer and Rhodes [67] (from which interesting article I have drawn the discussion in this section): $X$ is the set of possible states allowed by the theory and $S$ is the set of possible inputs (perturbations). An ambitious working out of this idea is in Rhodes [71].

To study such a system, it is desirable to parametrize the set $X$ in such a way that the action by $S$ is analyzed into simpler actions on the parameters. Another way of saying this is: Regard $X$ as a set of vectors, in the broad sense of $n$-tuples from some Cartesian product of sets, and describe the action of $S$ via the changes it makes in the coordinates.

It is too much to hope that $S$ would act coordinatewise: that is, that for $s \in S$, the effect of $s\phi$ on a coordinate would depend only on that coordinate. In general, it will depend on some or all of the other coordinates. What can frequently be done is that coordinates can be found which are ordered so that the effect on a given coordinate depends *only on that coordinate and the coordinates following it.* The action is then said to be **triangular**.

In most applications, it is convenient to take $S$ to be a semigroup. If it is not already a semigroup, $S$ is usually replaced by the free semigroup on the elements of $S$ (the set of finite strings of elements of $S$

with concatenation as operation) or sometimes by the semigroup generated by the set of functions $s\phi$ with functional composition as operation. Moreover, one usually wants the property that the successive action of two elements of $S$ is the same as the action of the product of those elements.

These considerations mean that the proper objects to study are "semigroup actions"; these are described in succeeding sections, and the wreath product is defined in terms of them.

**3. Semigroups.** A **semigroup** is a set $S$ together with an associative multiplication on $S$. A **unity** in $S$ is a (necessarily unique) element $1 \in S$ satisfying $1s = s1 = s$ for all $s \in S$. A semigroup with unity is a **monoid**. A **subsemigroup** of a semigroup $S$ is a subset of $S$ closed under multiplication. $T$ is a **subgroup** of $S$ if $T$ is a subsemigroup of $S$ which is a group. Warning: The unity of $T$ may not be the unity of $S$; $S$ may not even have a unity. $T$ is a **unitary subgroup** of a monoid $S$ if $T$ is a subgroup of $S$ and its unity is the same as that of $S$.

If $S$ has no unity, a unity may be adjoined: let 1 be an element not in $S$, and define $S^1 = S \cup \{1\}$ to be the semigroup with $S$ as a subsemigroup and $1s = s1 = s$ for $s \in S$. If $S$ has a unity, $S^1$ denotes $S$.

If $X$ is a set, Trans$X$ is the set of all functions from $X$ to $X$, together with the binary operation, composition of functions. A function in Trans$X$ is called a **transformation** of $X$. Sym$X$ is the unitary subgroup of Trans$X$ of all **permutations** of $X$ (bijections from $X$ to $X$). Trans$X$ is contained in the larger monoid PF($X$) of all **partial functions** from $X$ to $X$. An element $f \in \mathrm{PF}(X)$ is by definition a function from a subset of $X$ to $X$. If $f, g \in \mathrm{PF}(X)$, $f \circ g$ is the function with domain $\{x \in \mathrm{dom}\, f \mid xf \in \mathrm{dom}\, g\}$ (which may be empty), with $x(f \circ g) = (xf)g$. (Observe that I write functions on the right and compositions from left to right. This is undoubtedly the Wave of the Future. It makes functional diagrams easier to read and corresponds to the natural order of doing things on a pocket calculator.)

Let $S$ be a semigroup. If $z \in S$ satisfies $zs = sz = z$ for all $s \in S$, then $z$ is a **zero** of $S$. PF($X$) has a zero (the empty function), but not Trans$X$. If $e \in S$ satisfies $e^2 = e$, $e$ is **idempotent**. A unity of a semigroup is idempotent; so is a zero. A function $f \in \mathrm{Trans}X$ is idempotent if and only if its image is contained in its set of fixed points.

If $V$ is a subset of $S$ and $s \in S$, then $sV = \{sv \mid v \in V\}$. $Vs$ and more complicated symbols like $sVt$ are defined analogously.

A subset $I$ of a semigroup $S$ is a *right ideal* if $is \in I$ for every $i \in I$ and $s \in S$. *Left ideal* is defined similarly. A right or left ideal is necessarily a subsemigroup. If $h \in S$, then $hS$ is a right ideal of $S$. $S$ is a left and right ideal in $S^1$. A group has only the empty set and the whole group as ideals. Some authors do not count the empty set as an ideal or subsemigroup.

If $S$ and $T$ are semigroups, a function $\phi\colon S \to T$ is a **homomorphism** if $(ss')\phi = (s\phi)(s'\phi)$ for all $s, s' \in S$. If $e$ is idempotent, then $e\phi$ is idempotent. If $\phi$ is surjective and 1 is a unity of $S$, then $1\phi$ is a unity of $T$; the same statement is true of zeroes. $T$ is a **homomorphic image** of $S$ if $S\phi = T$; i.e., if $\phi$ is surjective.

If $S$ and $T$ are semigroups, $S \mid T$ ($S$ "divides" $T$) if $S$ is a homomorphic image of a subsemigroup of $T$. "Divides" is a reflexive, transitive relation. Two *finite* semigroups divide each other if and only if they are isomorphic.

The following technical lemmas will be used in the proof of the Krohn-Rhodes Theorem and may be skipped on first reading. The reader unfamiliar with semigroups will find their proofs a good illustration of elementary semigroup-theoretic proof techniques.

LEMMA 3.1. *Let $S$ be a finite semigroup, and let $x \in S$. Then some power of $x$ is an idempotent.*

*Proof.* Let $k$ and $n$ be the minimal integers such that $0 < k < n$ and $x^k = x^n$. Then $x^{k+n-k} = x^k$, so by induction $x^{k+m(n-k)} = x^k$ for all integers $m \geq 0$. Let $s$ be a positive integer for which $k + s = m(n-k)$ for some $m$. Then it is easy to see that $(x^{k+s})^2 = x^{k+s}$.

LEMMA 3.2. *Let $S$ be a finite semigroup, $\phi$ a semigroup homomorphism with domain $S$, and $S\phi$ a group. Then there is a subgroup $G$ of $S$ such that $G\phi = S\phi$.*

*Proof.* Let $G$ be a minimal subsemigroup of $S$ with the property that $G\phi = S\phi$. Let $e \in G$ be an arbitrary idempotent. Then $e\phi$ is idempotent in $S\phi$, hence is the unity of $S\phi$ since the only idempotent in a group is its unity. It follows that $(eGe)\phi = G\phi = S\phi$, so by the minimality of $G$, $eGe = G$: in other words, every element of $G$ is of the form $ege$ for some $g \in G$. But $e(ege) = (ege)e = ege$, so that $G$ is in fact a monoid with unity $e$. We started with an arbitrary idempotent $e$; hence $e$ is the unique idempotent of $G$.

Now if $x \in G$, $x^k = e$ for some $k$ by Lemma 3.1. Then $x \cdot x^{k-1} = x^{k-1} \cdot x = e$, so $x$ has an inverse. This proves Lemma 3.2.

**4. Semigroup actions.** An **action on the right** by a semigroup $S$ on a set $X$ is a function $(x, s) \mapsto xs$ from $X \times S$ to $X$ satisfying

$$s(st) = (xs)t \qquad (x \in X, s, t \in S). \tag{4.1}$$

Such an action determines a homomorphism $\phi\colon S \to \mathrm{Trans}X$ defined by

$$x(s\phi) = xs \qquad (x \in X, s \in S). \tag{4.2}$$

Conversely, any homomorphism from $S$ to Trans$X$ determines an action on $X$ by $S$. Actions will be denoted by boldface capitals (usually by the boldface form of the letter which denotes the semigroup) or, when necessary, by the ordered triple $(S, \phi, X)$ with $\phi\colon S \to \mathrm{Trans}X$. **Trans$X$** and **Sym$X$** denote the actions by Trans$X$ and Sym$X$ respectively on $X$. (In their case $\phi$ is an inclusion function and (4.1) is just the definition of composition of functions.)

An **action on the left** of $S$ on $X$ is a function $(s, x) \mapsto sx$ satisfying

$$(st)x = s(tx) \qquad (s, t \in S, x \in X). \tag{4.3}$$

An action on the left determines and is determined by an *antihomomorphism* $\phi\colon S \to \mathrm{Trans}X$. If functions were written on the left and composition from right to left then a left action would determine a homomorphism, but then a right action would determine an antihomomorphism. This difficulty is not avoidable!

Actions in this article are on the right except where specified.

Actions are called by many other names in the literature. If $\boldsymbol{S} = (S, \phi, X)$ is an action, $\boldsymbol{S}$ is also called a **semi-automaton** or **machine** and $X$ may be called an **$S$-set** or **$S$-operand.**

I have already given **Trans$X$** and **Sym$X$** as examples of actions. Here are some other examples.

*Example* 4.1. Any semigroup of transformations of a set $X$ acts on $X$; in particular, any permutation group on $X$ is an example of an action. Thus the automorphism group of a structure (algebraic structure, graph, topological space, etc.) acts on the underlying set of the structure.

*Example* 4.2. Any semigroup $S$ acts on its own underlying set by right multiplication. This action will be denoted $\boldsymbol{S}_S$. More generally, if $I$ is a right ideal of $S$, then $S$ acts on $I$ by right multiplication; this action is denoted $\boldsymbol{S}_I$.

*Example* 4.3. A group $G$ with normal subgroup $N$ acts on $N$ on the right by setting

$$n^g = g^{-1}ng \qquad (g \in G, n \in N), \tag{4.4}$$

where the action is written as exponentiation to avoid confusion with multiplication in $G$. There is also an action on the left taking $n \mapsto gng^{-1}$ which is best denoted by writing $G$ additively and setting

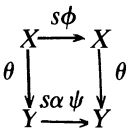
$$gn = g + n - g \qquad (g \in G, n \in N). \tag{4.5}$$

This notation will be used in Section 9.

Automata, discussed in the next section, provide another example of actions.

A pair $(X, \theta)$ is a **morphism of actions** from $S = (S, \phi, X)$ to $T = (T, \psi, Y)$ if $\alpha\colon S \to T$ is a homomorphism, $\theta\colon X \to Y$ is a function, and, for every $s \in S$,

(4.6)




commutes: in other words, $(x\theta)(s\alpha) = (xs)\theta$ for $x \in X$ and $s \in S$. Notice that then $(S, \alpha\psi, X\theta)$ is an action by $S$ on $X\theta$. The category of actions and morphisms of actions is called Act. The analogous category for left actions is $\mathrm{Act}_L$.

The morphism $(\alpha, \theta)$ is **injective** (resp. **surjective**) if $\alpha$ and $\theta$ are both injective (resp. surjective). $\boldsymbol{S}$ is a **subaction** of $\boldsymbol{T}$ if $\alpha$ and $\theta$ are inclusions. The morphism $(\alpha, \theta)$ is an isomorphism (= invertible in Act) if and only if $\alpha$ and $\theta$ are both bijective. (This is a theorem, not a definition!)

If $\boldsymbol{S} = (S, \phi, X)$ and $\boldsymbol{T} = (T, \psi, Y)$ are actions, and there is a surjective morphism of actions from a subaction of $\boldsymbol{S}$ onto $\boldsymbol{T}$, then $\boldsymbol{T}$ **divides** $\boldsymbol{S}$. In that case, $T$ divides $S$ in the sense of Section 3.

Of special importance is the case when $\alpha = id_S$. An $\boldsymbol{S}$**-morphism** or **equivariant map** from $(S, \phi, X)$ to $(S, \psi, Y)$ (*same* $S$) is a function $\theta: X \to Y$ for which

$$(xs)\theta = (x\theta)s \qquad (x \in X, s \in S) \tag{4.7}$$

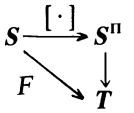
(it looks like commutativity).

In the case of left actions, this becomes $(sx)\theta = s(x\theta)$ (it looks like associativity). The category of actions by $S$ and $S$-morphisms is denoted $S$-Act. $S$-Act is a much better-understood category than Act; it is a topos, which means that in many ways it resembles the category of sets. See Wraith [**75**].

If the action $\boldsymbol{G}$ is a permutation group $G$ on a set $X$, the group of $G$-automorphisms of $\boldsymbol{G}$ is traditionally called the **centralizer of** $G$ in $\mathrm{Sym}X$. The group of automorphisms in Act of $\boldsymbol{G}$ is the **normalizer of** $G$ in $\mathrm{Sym}\, X$.

If $\boldsymbol{S} = (S, \phi, X)$ is an action, an $\boldsymbol{S}$**-partition** is a partition $\Pi$ of $X$ with the property that, if $x \equiv x' \pmod{\Pi}$ and $s \in S$, then $xs \equiv x's \pmod{\Pi}$. Any $\boldsymbol{S}$**-partition** $\Pi$ gives rise to a natural $S$-action on the set $\Pi$ by setting $[x]s = [xs]$, where $[x]$ denotes the block of $\Pi$ containing $x$. This action is denoted $\boldsymbol{S}^{\Pi}$.

The fact just mentioned has a converse. The **quotient** of a function is the partition of its domain whose blocks are the inverse images of elements of the image of the function. The quotient $\Pi$ of an $S$-homomorphism $F: \boldsymbol{S} \to \boldsymbol{T}$ is then an $S$-partition. Furthermore, the class map $x \to [x]$ is a surjective $S$-morphism from $\boldsymbol{S}$ to $\boldsymbol{S}^{\Pi}$, and there is a unique injective $S$-morphism from $\boldsymbol{S}^{\Pi}$ to $\boldsymbol{T}$ so that




commutes.

Dual to the notion of action on an $S$-partition is the idea of a **stable subset**: a subset $Y \subset X$ for which $ys \in Y$ for all $y \in Y$ and $s \in S$. The induced action is denoted $\boldsymbol{S}_Y$ (cf. $\boldsymbol{S}_I$ in example 4.2).

If a semigroup $S$ is regarded as a category with one object, an action $\phi: S \to \mathrm{Trans}X$ is just a functor from $S$ to the category of sets. Some of the embedding theorems below hold in this more general case. See Elkins and Zilber [**73**] and Wells [tbp].

**5. Automata.** An automaton is a semigroup action with certain additional structure. A definition suited to our use is:

DEFINITION 5.1. An **automaton** is a septuple $(X, A, B, \phi, \mu, i, C)$, where $X$, $A$ and $B$ are sets, $(FA, \phi, X)$ is an action ($FA$ denoting the free semigroup on the set $A$), $\mu: A \times X \to B$ is a function, $i \in A$ and $C \subset B$. $X$ is the set of **states**, $A$ the **input alphabet**, $B$ the **output alphabet**, $i$ the **initial state**, and $C$ the set of **acceptor states**.

The part of the automaton of interest to us is the action $(FA, \phi, X)$; in this context, the action is often called a **semiautomaton** or **machine** (sometimes a machine has an output alphabet). There are variations on the definition in the literature, as for example in Arbib [68b], Ginzburg [68], Hopcroft and Ullman [69], and Eilenberg [74]. Davis [70] gives a neat categorical description of machines. (Note: I object to the nearly universal use of the phrase "automata theory"; after all, one does not say "groups theory.")

Both computer programs and computers themselves can be thought of as automata, but this requires care to spell out precisely. See Scott [71] and Eilenberg [74].

Although this article focuses on semigroup actions, I want to emphasize that the major concerns of automaton theorists do not lie in that direction. The major emphasis in automaton theory may be described this way: Say a subset $L$ (a "language") of $FA$ is **accepted** by the automaton $(X, A, B, \phi, \mu, i, C)$ if $w \in L$ if and only if $iw \in C$. The problem is to describe connections between properties of an automaton and properties of the language which it accepts.

**6. Wreath products.** It is now possible to define the basic object of study of this article. Let $\boldsymbol{S} = (S, \phi, X)$ and $\boldsymbol{T} = (T, \psi, Y)$ be actions. The **wreath product** $\boldsymbol{S}$ wr $\boldsymbol{T}$ of $S$ by $T$ (in that order) is a semigroup action of a certain semigroup $S\mathrm{wr}T$ on the set $X \times Y$. To define it, let $S^Y$ denote the set of all functions from $Y$ to $S$. The underlying set of $S\mathrm{wr}T$ is $S^Y \times T$ and the action is given by

$$(x, y)(F, t) = (x(yF), yt) \qquad (x \in X, y \in Y, F \in S^Y, t \in T). \tag{6.1}$$

Thus the action on the first coordinate depends on both coordinates, whereas the action on the second coordinate depends only on the second coordinate.

To calculate the composition of two such functions, some notation is needed. For $F, G \in S^Y$, let $F + G$ be defined by

$$y(F + G) = yF \cdot yG \qquad (y \in Y), \tag{6.2}$$

the product on the right side taking place in $S$. This is fairly common notation, and one of only a few places in mathematics where $+$ denotes a not necessarily commutative operation. It will fit with the notation of (4.5).

Now suppose $(G, u) \in S^Y \times T$ also. Then

$$\begin{aligned}[(x, y)(F, t)](G, u) &= (x(yF), yT)(G, u) \\ &= ([x(yF)](ytG), (yt)u) \\ &= (x, y)(F + tG), tu),\end{aligned}$$

where $tG \in S^Y$ is defined by

$$y(tG) = (yt)G. \tag{6.3}$$

Thus multiplication in $S\mathrm{wr}T$ is given by the formula

$$(F, t) \circ (G, u) = ((F + tG), tu) \qquad (F, G \in S^Y;\ t, u \in T), \tag{6.4}$$

which incidentally shows that $S\mathrm{wr}T$ is closed under composition. It is straightforward to show that multiplication is associative. It is apparent from (6.4) that the semigroup $S\mathrm{wr}T$ depends on $S$, $T$, and the action of $T$ on $Y$, but not on the action of $S$ on $X$. $S\mathrm{wr}T$ will sometimes be written "$S\mathrm{wr}(\psi)T$" to emphasize this. It is the **abstract wreath product** corresponding to $\boldsymbol{S}\mathrm{wr}\boldsymbol{T}$.

Because of the independence of $S\mathrm{wr}(\psi)T$ from $\phi$, many authors define $S\mathrm{wr}(\psi)T$ given only $S$, $T$, and $\boldsymbol{T}$, without reference to an action by $S$. Such authors then define an action of $S\mathrm{wr}(\psi)T$ on $X \times Y$ by (6.1).

NOTE 1. If $Y$ is finite, $S\mathrm{wr}T$ can be represented as a semigroup of matrices. Let $Y = \{1, \cdots, n\}$. If

$(F, t) \in S \text{wr} T$, define the $n \times n$ matrix $M(F, t)$ by

$$M(F,t)_{ij} = \begin{cases} iF & \text{if } j = it \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

These matrices have exactly one entry in a given row. The mapping $(F, t) \to M(F, t)$ is a semigroup homomorphism which is an isomorphism when the action by $T$ is faithful. (Two matrices $M(F, t)$ are multiplied in the obvious way.) Furthermore, if $\boldsymbol{x} = (x_1, \cdots, x_n)$ is any list of $n$ elements of $X$, then the list $\boldsymbol{x}' = \boldsymbol{x}[M(F, t)^T]$ obtaining by "multiplying" (in the obvious way) $\boldsymbol{x}$ by the transpose of $M(F, t)$ has the property

$$(\boldsymbol{x}')_{it} = (\boldsymbol{x}_i)(iF). \tag{6.6}$$

Because of the transpose, this defines an action *on the left* of $S \text{wr} T$ on $X^n$. A special case of this action is described by Ore [42]; the general case (not in matrix-theoretic terms) by Harary [59b]. Another special case appears in Section 10 below, preparatory to describing the structure of $S \text{wr} T$ as a semigroup. A nice application of wreath products as matrix groups appears in Wiegold [63].

NOTE 2. Let $\mathcal{A}_1 = (Y, A, B, \psi, \lambda, i, D)$ and $\mathcal{A}_2 = (X, B, C, \phi, \mu, j, E)$ be automata, with the output alphabet of the first equal to the input alphabet of the second. If they are hooked up in series:

$$\text{input} \to \mathcal{A}_1 \to \mathcal{A}_2 \to \text{output}, \tag{6.7}$$

then a word $w$ in the input acts on $Y$ by $\psi$, and it also causes $\mathcal{A}_2$ to change state; but this latter action *depends on the state* $\mathcal{A}_1$ *was in*, because the output from $\mathcal{A}_1$ depends on that state. In other words, $w$ acts on $X$ by a function $yF$: $X \to X$ depending on $y \in Y$. Regarding (6.7) as one big automaton with state set $X \times Y$, we have that $w$ induces

$$(x, y) \to (x\,(yF), yw)$$

(compare (6.1)); whence the connection between automata and wreath products.

**7. The wreath product of an ordered set of actions.** In this section, I shall describe how to construct the wreath product of a set of semigroup actions indexed by an ordered set $I$. The construction is a special case of a construction of Holland [69]. When $I$ has two elements, the construction reduces to that given in section 6. If the reader follows through the discussion below with $I = \{1, 2, 3\}$ (usual ordering), he will discover that the action given on $X$ is triangular in the sense of section 2.

Let $I$ be a partially ordered set, and $(\boldsymbol{S}_i)_{i \in I}$ a set of actions indexed by $i$; let $\boldsymbol{S}_i = (S_i, \phi_i, X_i)$. Define:

(a) $X = \Pi_{i \in I} X_i$.

(b) For each $i \in I$, $M_i = \Pi_{j > i} X_i$. (If there are no $j > i$, take $M_i$ to be a one-element set.)

(c) $W = \Pi_{i \in I} S_i^{M_i}$.

(d) The $i$th coordinate of $x \in X$, $m \in M$, $w \in W$ to be $x_i$, $m_i$, $w_i$ respectively.

(e) If $x \in X$, $x^i$ is the element of $M_i$ which agrees with $x$ (the projection of $x$ on $M_i$). Thus, if $x \in X$, $(x^i)_j = x_j$ for all $j > i$. (If there are no $j > i$, $x^i$ is the unique element of $M_i$.)

$W$ acts on $X$ by an action $\omega$ defined this way:

$$(xw)_i = x_i(x^i w_i) \qquad (i \in I, x \in X, w \in W). \tag{7.1}$$

If $v, w \in W$, then $[x(vw)]_i = x_i(x^i v_i)[(xv)^i w_i]$ so that

$$(vw)_i = v_i + v^i w_i, \tag{7.2}$$

where $v^i w_i$: $M_i \to S_i$ is defined by

$$u(v^i w_i) = (uv)^i w_i \qquad (i \in I, U \in M_i, v, w \in W). \tag{7.3}$$

$(W, \omega, X)$ is then the **wreath product of the system** $\{\boldsymbol{S}_i\}_{i\in I}$.

If $I = \{1, \cdots, n\}$ with the usual ordering, $(W, \omega, X)$ is denoted $\mathrm{Wr}_{i=1}^{n}\boldsymbol{S}_i$.

PROPOSITION 7.1. *Let* $\boldsymbol{S}_i$, $i = 1, 2, 3$, *be actions. Then* $(\boldsymbol{S}_1\mathrm{wr}\boldsymbol{S}_2)\mathrm{wr}\boldsymbol{S}_3 \cong \boldsymbol{S}_1\mathrm{wr}(\boldsymbol{S}_2\mathrm{wr}\boldsymbol{S}_3) \cong \mathrm{wr}_{i=1}^{3}\boldsymbol{S}_i$.

The proof is straightforward.

**8. Some constructions.** In this section and the next, some properties which semigroup actions may have, are defined, and some constructions are made. These will then be used in Section 10 to describe the abstract wreath product as a semigroup.

Let $\boldsymbol{S} = (S, \phi, X)$ be a semigroup action. Then $S\phi$ is a semigroup of transformations on $X$ called the **constituent** of $\boldsymbol{S}$.

If $Y$ is a stable subset of $X$, there is a natural $S$-morphism res (for "restriction") from the constituent of $\boldsymbol{S}$ to that of $\boldsymbol{S}_Y$ such that $\phi_Y = \phi \circ \mathrm{res}$, where $\phi_Y$ is the action of $S$ on $Y$.

If $S$ is a monoid with unity $1_S$, then $(S, \phi, X)$ is **unitary** if

$$x1_S = x \qquad (x \in X). \tag{8.1}$$

Then $1_S\phi = \mathrm{id}_X$, where $\mathrm{id}_X$ denotes the identity function from $X$ to $X$.

*It will be assumed throughout this article that, if S is a group, then the action by S is unitary.*

A semigroup action $\boldsymbol{S} = (S, \phi, X)$ is **faithful** if $\phi$ is injective; thus

$$(xs = xt \text{ for all } x \in X) \Rightarrow s = t. \tag{8.2}$$

In this case, $S$ is isomorphic to the constituent of $\boldsymbol{S}$. Observe that if $Y$ is a stable subset of $X$, then $\boldsymbol{S}_Y$ may not be faithful even if $\boldsymbol{S}$ is, since two elements of $S$ may agree on $Y$ without agreeing on all of $X$.

Any transformation semigroup acts faithfully. If $S$ is any monoid, $S$ acts faithfully on itself by right multiplication.

$\boldsymbol{S}$ is **transitive** if for every ordered pair $(x, x') \in X \times X$, there is $s \in S$ for which $xs = x'$. If $x \in X$, write

$$xS = \{xs \mid s \in S\}. \tag{8.3}$$

Then $\boldsymbol{S}$ is transitive if and only if $xS = X$ for every $x \in X$.

*Warning*: It is not necessary that $x \in xS$ if $S$ is not a monoid.

If $X$ is a set and $z \in X$, the **constant function** $c_z\colon X \to X$ satisfies

$$xc_z = z \qquad (x \in X). \tag{8.3}$$

If $\boldsymbol{S} = (S, \phi, X)$ is a faithful action, $S_X^c$ denotes the semigroup $S\phi \cup CX$, where $CX$ is the semigroup of constant functions on $X$. $S$ will be regarded as a subsemigroup of $S_X^c$ by the obvious identification between elements of $S$ and $S\phi$. The natural action of $S_X^c$ on $X$ is denoted $\boldsymbol{S}^c$.

Similarly, $S_X^{-c}$ denotes the subsemigroup of $S\phi$ generated by the nonconstant functions in $S\phi$, and $\boldsymbol{S}^{-c}$ the corresponding action. If every function in $S\phi$ is constant, $S_X^{-c}$ will be $\{\mathrm{id}_X\}$ by convention.

The proof given in Section 14 of the Krohn-Rhodes theorem requires some delicate maneuvering among $\boldsymbol{S}$, $\boldsymbol{S}^c$, and $\boldsymbol{S}^{-c}$.

The following facts are useful and easy to prove.

PROPOSITION 8.1. *If S is a group and* $\boldsymbol{S} = (S, \phi, X)$ *an action, then*

(a) $\boldsymbol{S}$ *is faithful if and only if the only element of S fixing everything in X is the unity of S.*

(b) $\boldsymbol{S}$ *is transitive if and only if* $xS = X$ *for at least one* $x \in X$.

PROPOSITION 8.2. *If S is a semigroup, then*

(a) *S acts faithfully on* $S^1$ *by right multiplication.*

(b) *Right multiplication is transitive if and only if S is a group.*

PROPOSITION 8.3. *For P any of the four words below, if* $\boldsymbol{S}$ *and* $\boldsymbol{T}$ *are both P, then so is* $\boldsymbol{S}\mathrm{wr}\boldsymbol{T}$:

(a) *unitary* (b) *faithful*

(c) *transitive* (d) *group.*

**9. Semidirect products.** It is natural to consider semigroup actions $\boldsymbol{S} = (S, \phi, E)$ where $E$ has additional structure and each map in the constituent of $\boldsymbol{S}$ is an endomorphism of the structure. For example, $S$ is a semigroup of matrices and $E$ is a vector space, or $S$ is a group acting continuously on a topological space $E$. Here, one special case is needed, that where $E$ is also a semigroup and $S$ acts on the left.

This causes notational difficulties; the device of Example 4.3 will be followed here. A semigroup $S$ **acts** on a semigroup $E$ **on the left** if there is a function $(s, e) \mapsto se$ from $S \times E$ to $E$ for which

$$(st)e = s(te) \qquad (s, t \in S, e \in E), \text{ and} \tag{9.1}$$

$$s(e + f) = se + sf \qquad (s \in S, e, f \in E), \tag{9.2}$$

where $E$ is written additively.

Such an action gives rise to an antihomomorphism $\phi\colon S \to \operatorname{End} E$ ($\operatorname{End} E$ is the endomorphism monoid of the semigroup $E$), and conversely any such antihomomorphism yields an action of $S$ on the left on $E$.

Morphisms are defined as in Act, yielding a category Sem-Act.

If the semigroup $S$ acts on the semigroup $E$ on the left, a new semigroup $E\mathrm{sd}S$, the **semidirect product of** $E$ **by** $S$, may be constructed. (Some authors would say: "The semidirect product of $S$ by $E$".) The underlying set of $E\mathrm{sd}S$ is $E \times X$, and multiplication satisfies

$$(e, s)(f, t) = (e + sf, st) \qquad (e, f \in E, s, t \in S). \tag{9.3}$$

(One must check associativity.) Mapping the action of $S$ on $E$ to $E\mathrm{sd}S$ is the object map of a functor $SD$ from Sem-Act to the category of semigroups; the definition of the morphism map is left to the reader.

When $S$ and $E$ are monoids with unities $1_S$ and $0_E$ respectively, and the action of $S$ on $E$ is unitary, then $(0_E, 1_S)$ is a unity for $E\mathrm{sd}S$ which is therefore a monoid. Then $E\mathrm{sd}S$ has submonoids $E \times \{1_S\}$ and $\{0_E\} \times S$ which are isomorphic to $E$ and $S$ respectively, and it is easy to check that if $s$ is invertible in $S$, then it has $(0_E, s^{-1})$ as inverse in $E\mathrm{sd}S$, and that for $e \in E$, conjugating $(e, 1_S)$ by $(0_E, s)$ yields $(se, 1_S)$. (That is conjugating as in (4.5), not as in (4.4).) Thus any left action by a monoid on a monoid can be embedded in a bigger monoid in which the invertible elements act by conjugation. In particular, the semidirect product of groups $N\mathrm{sd}G$ with $G$ acting on $N$ contains a copy of $N$ as a normal subgroup, a copy of $G$ as a subgroup, and the action of the copy of $G$ on the copy of $N$ by conjugation is the same as the given action of $G$ on $N$. Thus group actions on groups can always be internalized in a semidirect product.

**10. The structure of the abstract wreath product.** Let $Y$ be a set, $S$ a semigroup. Then $S^Y$ is a semigroup under the operation $+$ defined by (6.2). If $(T, \psi, Y)$ is a right semigroup action, there is a canonical left semigroup action of $T$ on $S^Y$ defined this way: If $t \in T$, $G \in S^Y$, then $tG$ is defined as in Section 6 by

$$s(tG) = (st)G \qquad (s \in S). \tag{10.1}$$

To show that this gives a semigroup action, it is necessary to check that $(tt')G = t(t'G)$ for $t, t' \in T$ and $G \in S^Y$, and that $t(G + G') = tG + tG'$ for $t \in T$ and $G, G' \in S^Y$. Both are easy.

For a fixed $T$, the map which associates the action $(T, \psi, Y)$ and the semigroup $S$ to the action defined above is the object map of a bifunctor $E(Y, S)$: $\boldsymbol{T}$-**Act** $\times$ **Sem** $\to$ **Sem-Act** which is contravariant in $(T, \psi, Y)$ and covariant in $S$. The definitions of the morphism maps are again left to the reader (copy the definitions of the morphism maps of a hom-functor).

If $Y = \{1, 2, \cdots, n\}$, $S^Y$ may be identified with the Cartesian power $S^n$ by regarding $F$: $Y \to S$ as the $n$-tuple $(1F, 2F, \cdots, nF)$. The action of $T$ on $S^Y$ can be written this way:

$$t(s_1, \cdots, s_n) = (s_{1t}, \cdots, s_{nt}) \qquad (s_1, \cdots, s_n \in S, t \in T). \tag{10.2}$$

Then

$$(tt')(s_1, \cdots, s_n) = (s_{1tt'}, \cdots, s_{ntt'}) = t(s_{1t'}, \cdots, s_{nt'}).$$

It is tempting, but mistaken, to think that

$$t(s_{1t'}, \cdots, s_{nt'}) = (s_{1t't}, \cdots, s_{nt't}).$$

The action is on the positions the elements occupy (alibi) rather than the elements themselves (alias). I recall two days of bafflement when reading the paper by Ore [42] before I caught onto this.

The structure of the abstract wreath product can now be described.

THEOREM 10.1. *Let* $(S, \phi, X)$ *and* $(T, \psi, Y)$ *be actions. Then* $S\mathrm{wr}(\psi)T \cong S^Y \mathrm{sd} T$.

*Proof.* This follows immediately from (6.4), the definition of semidirect product in Section 10, and the definition of $S^Y$.

$S$ is called the **bottom semigroup** of $S\mathrm{wr}T$, and $T$ the **top semigroup**. $S^Y$ is the **basis semigroup**. When $S$ and $T$ are monoids, $S^Y$ and $T$ are canonically subsemigroups of $S\mathrm{wr}T$, and $S$ is embedded in $S\mathrm{wr}T$ in many ways. One embedding in particular is the **diagonal embedding**, taking $s \in S$ onto $(c_s, id_T)$. The diagonal embedding takes $S$ onto the scalar matrices when $S\mathrm{wr}T$ is written as a matrix semigroup. $S^Y$ corresponds to the diagonal matrices.

When $S$ and $T$ are groups, then $S^Y$ is a normal subgroup of $S\mathrm{wr}T$, and the action $\psi$ is conjugation of $S^Y$ by the canonical copy of $T$ in $S\mathrm{wr}T$, as described in Section 9.

The functor which takes $(S, \phi, X)$ and $(T, \psi, Y)$ to $S\mathrm{wr}T$ is $A \circ SD$, where $A$ and $SD$ are described at the beginning of this section and in Section 9 respectively. Since $S\mathrm{wr}T$ does not depend on $\phi$ and $X$, this functor factors through a functor from **Sem** $\times$ $T$-**Act** to **Sem** which takes $S$ and $(T, \psi, Y)$ to $S$ wr $T$.

When $Y$ is the underlying set of $T$ *and* $T$ acts by left multiplication, $S\mathrm{wr}T$ is called the **standard wreath product**. A good deal is known about the structure of standard wreath products of groups (P. Neumann [64]). Kerber [68] studied representations of standard wreath products, and Houghton [63] their automorphisms. See also Kappe and Parker [70] and Huppert [67], pp. 94 ff and 101. *Warning*: If $S$, $T$ and $U$ are semigroups, then $(S\mathrm{wr}T)\mathrm{wr}U$ and $S\mathrm{wr}(T\mathrm{wr}U)$ (standard wreath products) are both defined, but are *not* isomorphic, except in trivial cases. This does not contradict Proposition 7.1!

When $S$ is a group and $T$ is **Sym** $\boldsymbol{X}$, where $X = \{1, 2, \cdots, n\}$, then $S\mathrm{wr}T$ is the **complete monomial group of degree** $\boldsymbol{n}$ **over** $\boldsymbol{S}$. The properties of this group have been studied by Ore [42]. See also Crouch [55] and [60]. There are important differences between complete monomial groups and standard wreath products of groups. For example, all the complements of the basis subgroup are conjugate in the standard wreath product, but not necessarily in a complete monomial group.

When $Y$ is infinite and $S$ is a monoid, the semigroup $S^Y$ has a subsemigroup $S^{(Y)}$ consisting of all those functions $F$: $Y \to S$ for which $yF = 1$ for all but a finite number of $y \in Y$. It is easy to see that $S^{(Y)}$ is taken into itself as a set by any $t \in T$ under the action of $T$ on $S^Y$ defined by (10.1); the corresponding semidirect product is the **restricted wreath product** of $S$ by $T$. Taking the restricted wreath product preserves various properties. For example, if $G$ and $H$ are solvable groups, so is $G\mathrm{wr}H$ for any action of $H$. This is not true of nilpotence, however (Baumslag [59]).

**11. A decomposition schema.** Let $\boldsymbol{S} = (S, \phi, X)$ be a semigroup action. Parametrizing $\boldsymbol{S}$ so that the action becomes triangular means precisely to find actions $\boldsymbol{M} = (M, \psi, Y)$ and $\boldsymbol{N} = (N, \rho, P)$ so that $\boldsymbol{S}$ divides $\boldsymbol{M}\mathrm{wr}\boldsymbol{N}$; that is, so that there is a surjective morphism $(\alpha, \theta)$: $T \to \boldsymbol{S}$ where $\boldsymbol{T}$ is some subaction of $\boldsymbol{M}\mathrm{wr}\boldsymbol{N}$. This is called a **decomposition** of $\boldsymbol{S}$, or an **embedding** of $\boldsymbol{S}$ in $\boldsymbol{M}\mathrm{wr}\boldsymbol{N}$.

The latter word, although common in the literature, can be misleading; it need not happen that $(\alpha, \theta)$ has a right inverse in Act. If it does, then $S$ is isomorphic to a subaction of $M\text{wr}N$; we say $S$ is **homomorphically embedded** in $M\text{wr}N$. Even if $(\alpha, \theta)$ does not have a right inverse, however, it is still true (because $(\alpha, \theta)$ is surjective) that each $x \in X$ and $s \in S$ corresponds to $(y, p) \in Y \times P$ and $(F, n) \in M\text{wr}N$ respectively in such a way that $(y, p)\theta = x$, $(F, n)\alpha = s$, and (*a fortiori*) $[(y, p)(F, n)]\theta = xs$. The point is that it might happen that $x's' = xs$, yet no choice of $(y', p')$ and $(F', n')$ yields $(y', p')(F', n') = (y, p)(F, n)$. In other words, the "embedding" may be multivalent as well as nonhomomorphic. This should not bother the automaton theorist, who can filter out the multivalence with the output map. He will be more concerned with keeping $Y$ and $P$ a reasonable size, and with ensuring that $M$ and $N$ bear some reasonable relation to $S$.

In the next three sections, I shall describe a number of ways of decomposing semigroup actions into wreath products. These methods will then be used to give a proof of the Krohn-Rhodes Theorem (Section 14) and more briefly in some other applications. Some of the methods will give a homomorphic embedding of the action into the wreath product; others only an embedding or decomposition in the weaker sense just discussed.

In this section a general Schema for such decompositions is described. It includes many but not all of the methods described below. It is a Schema rather than a Theorem because it gives a general way for decomposing an action into a wreath product, but provides no control over the sizes or properties of the semigroups or sets involved in the wreath product.

DEFINITION. Let $S = (S, \phi, X)$ be a semigroup action and $\Pi$ an $S$-partition of $X$. A **coordinate system** for $\Pi$ is an ordered pair $(M, \{\beta_B\}_{B \in \Pi})$ where $M$ is an action by a semigroup $M$ on a set $Y$, and for each block $B \in \Pi$, $\beta_B : B \to Y$ is an injection satisfying the following requirement: For each $s \in S$ and $B \in \Pi$ there is $m_B^s \in M$ such that for all $x \in B$,

$$(x\beta_B)m_B^s = (xs)\beta_C, \tag{11.1}$$

where $C$ is the block of $\Pi$ to which $s$ takes elements of $B$; i.e., $C = Bs$. In other words, *the partial function induced on $Y$ by $\beta_B$ and $s$ extends in at least one way to an action by an element of $M$.*

THEOREM 11.1 (Decomposition Schema). *Let $S = (S, \phi, X)$ be a faithful semigroup action with an $S$-partition $\Pi$ and coordinate system $(M, \{\beta_B\}_{B \in \Pi})$. Then $S$ divides $M\text{wr}S^{\Pi}$.*

*Proof.* Let $\lambda : X \to Y \times \Pi$ be defined by

$$x\lambda = (x\beta_B, B) \qquad (x \in B \in \Pi) \tag{11.2}$$

and let $\theta = \lambda^{-1} : 1m\lambda \to X$ (observe that $\lambda$ is injective). For each $s \in S$, let $F_s : \Pi \to M$ be defined by

$$BF_s = m_B^s \qquad (B \in \Pi), \tag{11.3}$$

where $m_B^s$ is an element of $M$ satisfying (11.1). Let $\bar{\phi}$ be the action of $S$ on $\Pi$. Then for all $x \in B$, $B \in \Pi$,

$$[(x\beta_B, B)(F_s, s\bar{\phi})]\theta = (xs, (x\beta_B m_B^s, Bs\bar{\phi})\theta) \tag{11.4}$$

by (11.1) and definition of the action $M$ wr $\Pi$. Since $S$ is a faithful, $s$ is determined uniquely by (11.4), so defines $(F, s\bar{\phi})\alpha = s$, for all elements $(F, s\bar{\phi})$ with $F$ satisfying (11.3). This set of elements $(F, s\phi)$ is clearly a subsemigroup of the semigroup of $M$ wr $S^{\Pi}$, and $(\alpha, \theta)$ is a surjective morphism of actions by (11.4). This proves the Theorem.

NOTE. Given a set $Y$ and injections $\beta_B : B \to Y$ for each $B \in \Pi$, one can always find a semigroup action $M$ of a semigroup $M$ on $Y$ making $(M, \{\beta_B\}_{B \in \Pi})$ a coordinate system, even in such a way that the covering $(\alpha, \theta)$ has a right inverse. For example, take an element $e \notin Y$ and let $\tilde{Y} = Y \cup \{e\}$. For

$s \in S$ and $B \in \Pi$, define a transformation $m_B^s \in \operatorname{Trans} \tilde{Y}$ by

$$ym_B^s = \begin{cases} xs\beta_{Bs} & \text{if } y = x\beta_B,\ x \in B \\ e & \text{otherwise.} \end{cases}$$

Let $BF = m_B^s$ for $B \in \Pi$. Then $(s \mapsto (F_s, s\bar{\phi}), x \mapsto (x\beta_B, B))$ for $s \in S$, $x \in B \in \Pi$, is an injective morphism from $\boldsymbol{S}$ to a wreath product of $\mathbf{Trans}\ \tilde{\boldsymbol{Y}}$ by $N$ acting on $\tilde{Y} \times X$. The trouble is that $\mathbf{Trans}\ \tilde{\boldsymbol{Y}}$ is so large.

COROLLARY 11.2. *Let $\boldsymbol{S}$ and $\boldsymbol{M}$ be as in Theorem* 11.1, *and suppose that $\boldsymbol{M}$ is faithful and all the $\beta_B$ are bijections. Then there is a homomorphic embedding $\boldsymbol{S} \to \boldsymbol{M}\mathrm{wr}\Pi$.*

*Proof.* The conditions of the corollary force $m_B^s$ to be uniquely determined by (11.1), and hence $(F_S, s\bar{\phi})$ to be uniquely determined by $s$. Hence $(\alpha, \theta)$ is not only surjective, but injective as required.

If $\boldsymbol{S}$ is transitive, $\mathrm{Aut}_s X$ satisfies the requirements on $\boldsymbol{M}$ in this corollary, with $\Pi$ the set of orbits of $\mathrm{Aut}_S X$ and $Y$ one of the orbits. (Krohn, Langer and Rhodes [67].) This corollary will be used to prove the Kaloujnine-Krasner Theorem in Section 12.

COROLLARY 11.3. *Let $S$ be a faithful semigroup action with a stable subset $Y$. Then the set $\Pi = \{Y\} \cup \{\{x\}, x \in X - Y\}$ is an $S$-partition, and $\boldsymbol{S}$ divides $(\boldsymbol{S}_Y)^c \mathrm{wr} S^\Pi$.*

*Proof.* Let $y_0 \in Y$ and let $\beta_Y = id_Y$, $x\beta_{\{x\}} = y_0$ for $x \in X - Y$. If $s \in S$, let

$$m_B^s = \begin{cases} s\phi_Y & \text{if } B = Y \\ C_{xs} & \text{if } B = \{x\},\ x \in X - Y,\ xs \in Y \\ C_{y_0} & \text{if } B = \{x\},\ x \in X - Y,\ xs \in X - Y, \end{cases}$$

where $\phi_Y$ is the action of $S$ on $Y$.

Then $m_B^s \in S_{\phi_Y}^c$ and it satisfies (11.1). The corollary thus follows from the decomposition schema.

This lemma will be needed later:

LEMMA 11.4. *If $\boldsymbol{S} = (S, \phi, X)$ has an $S$-partition $\Pi$ and coordinate system $(\boldsymbol{M}, \{\beta_B\}_{B \in \Pi})$, then $(\boldsymbol{M}^c, \{\beta_B\}_{B \in \Pi})$ is a coordinate system for $\boldsymbol{S}^c$. Hence $\boldsymbol{S}^c$ divides $\boldsymbol{M}^c \mathrm{wr}(S^\Pi)^c$.*

*Proof.* Easy.

**12. Decomposition of group actions.** In this section a sequence of propositions will be given which culminate in the important Kaloujnine-Krasner Theorem, which says roughly that a finite group can be homomorphically embedded in an iterated wreath product of the simple groups which occur as factors in a composition series of the group. Enroute, a brief look will be taken at an application to character theory.

PROPOSITION 12.1. *Let $\boldsymbol{G} = (G, \phi, X)$ be a faithful group action and $\Pi$ a $G$-partition. Suppose that the induced action of $G$ on $\Pi$ is transitive, let $Y \in \Pi$ and let $G_Y$ denote the constituent on $Y$ of the subgroup of $G$ which fixes $Y$ setwise. Then $\boldsymbol{G}$ is homomorphically embedded in $\boldsymbol{G}_Y \mathrm{wr} G^\Pi$.*

*Proof.* Let $B \in \Pi$, and let $g_B \in G$ carry $B$ to $Y$. Define $\beta_B : B \to Y$ by

$$x\beta_B = xg_B \qquad (x \in B).$$

Then $\beta_B$ is a bijection. Now let $h \in G$, $x \in B \in \Pi$, $xh \in C \in \Pi$. Then

$$(xh)\beta_C = xhg_C = xg_B g_B^{-1} hg_C = (x\beta_B) g_B^{-1} hg_C.$$

But if $y \in Y$, then $yg_B^{-1} hg_C \in Y$. Hence by (11.1), $\boldsymbol{G}_Y$, together with the set $\{\beta_B\}_{B \in \Pi}$, forms a coordinate system for $\boldsymbol{G}$ which satisfies the requirements of Corollary 11.2.

If $\boldsymbol{G}$ is transitive, then certainly the action of $G$ on $\Pi$ is transitive. Thus the proposition applies in particular to any $G$-partition of any transitive group action.

If $G$ is a group with subgroup $H$, then $G$ acts transitively on the set of right cosets $Hg$ of $H$ by

$$(Hg)g' = Hgg' \qquad (g, g' \in G). \tag{12.1}$$

(This is clearly well-defined.) Indeed, every transitive action by $G$ is $G$-isomorphic to an action of this form (take $H$ to be the subgroup fixing an element). Let $G^H$ denote the constituent of $G$ on the set of cosets of $H$.

COROLLARY 12.2. *Let $G$ be a group with subgroup $H$ and $G/H$ the set of right cosets of $H$ in $G$. Then $\boldsymbol{G}_G$ is homomorphically embedded in $\boldsymbol{H}_H \mathrm{wr} \boldsymbol{G}^H$.*

*Proof.* $\boldsymbol{G}_G$ ($G$ acting on its own underlying set by right multiplication) is faithful and transitive, and $G/H$ is a $G$-partition. Let $Y = H$ (which is a block of $G/H$). The subgroup of $G$ which fixes $H$ setwise is $H$ itself. The corollary then follows from Prop. 12.1 (see Wells [69]).

Observe that Cor. 12.2 embeds $G$ in a wreath product of $H$ by the constituent of $G$ on $G/H$. If any action of $H$ on a set $X$ is given, then $G$ acts on $X \times (G/H)$ via this embedding (see Section 6).

Now suppose a $k$-dimensional complex representation $\tau$ (a homomorphism into a multiplicative group of $k \times k$ matrices with complex entries) of $H$ is given, and suppose $H$ has finite index $n$ in $G$. By using the construction in Corollary 12.2 and the matrix representation in Section 6, $G$ may be represented by $n \times n$ monomial matrices whose entries are $k \times k$ complex matrices. By what may best be described as erasing parentheses, an $nk$-dimensional complex representations of $G$ is obtained; it is the representation of $G$ *induced* by the given representation of $H$. Because of the relative ease of calculating the character of such a representations, this construction is a fundamental tool used by finite group theorists in constructing character tables. See Gorenstein [68], chapter 4, section 4, or Lang [65], chapter VIII, section 6. Of special interest in this connection is the study of "$M$-groups"; see Huppert and Wielandt [62], Kerber [70], and the references therein. This construction is a special case of the left adjoint construction in Wraith [75] 53, example (iii).

COROLLARY 12.3. *Let $G$ be a group with normal subgroup $H$. Then $G$ is isomorphic to a subgroup of the standard wreath product $H\mathrm{wr}(G/H)$.*

*Proof.* Observe that the assumption that $H$ is normal means that $G/H$ is a group. The corollary will follow from Corollary 12.2 if it is shown that $\boldsymbol{G}^H$ is Act-isomorphic to $(\boldsymbol{G}/\boldsymbol{H})_{G/H}$. But that is immediate from (12.1).

A direct proof of Cor. 12.3 is in H. Neumann [67], p. 46.

A group $G$ is an **extension of $\boldsymbol{H}$ by $\boldsymbol{B}$** if there is an exact sequence

$$1 \to H \to G \to B \to 1.$$

(This means that there is a monomorphism from $H$ into $G$ and an epimorphism from $G$ onto $B$ such that the image of the monomorphism is exactly the kernel of the epimorphism.) The preceding corollary says that *every extension of $H$ by $B$ is embedded in the standard wreath product of $H$ by $B$.*

Now let $G$ be a finite group. It is well known that there is a sequence (not necessarily unique) of subgroups $1 = N_0, N_1, N_2, \cdots, N_k = G$ of $G$ with the property that for $i = 0, 1, \cdots, k-1$, $N_i$ is a normal subgroup of $N_{i+1}$ and $N_{i+1}/N_i$ is simple (has no nontrivial normal subgroups). Furthermore, the number of simple groups of a given isomorphism type is determined uniquely. This sequence is called a *composition series* for $G$.

Thus $G$ is an extension of $N_{k-1}$ by $N_k/N_{k-1}$. Furthermore, $N_{k-1}$ is an extension of $N_{k-2}$ by $N_{k-1}/N_{k-2}$, and in general $N_i$ is an extension of $N_{i-1}$ by $N_i/N_{i-1}$ for $i = 1, \cdots, k$. The following theorem then follows from Corollary 12.3 by induction.

THEOREM 12.4. (Kaloujnine-Krasner [51a], p. 47). *A finite group can be embedded in an iterated standard wreath product of the simple groups occurring in a composition series of the group.*

REMARKS. (1) In the notation preceding the statement of Theorem 12.4, let $N_1$ act on $X$, and let

$E_i = N_{i+1}/N_i$ for $i \geqq 1$. Then the embedding of the theorem gives a triangular action of $G$ on the set $X \times E_1 \times \cdots \times E_n$. This fact will be used in the proof of the Krohn-Rhodes Theorem.

(2) Observe that the simple groups in the theorem divide the given group.

(3) "Kaloujnine" is frequently transliterated "Kalužnin".

(4) Actually, Kaloujnine and Krasner proved a more general theorem, allowing any tower of subgroups instead of a composition series — but then the wreath product may not be standard. See also Marchionna-Tibiletti [57].

**13. Other decomposition theorems.** In this section, two related decomposition techniques are given which are needed for the proof of the Krohn-Rhodes Theorem. These two techniques are in a way dual to the Schema of Section 11, since they decompose $(S, \phi, X)$ by breaking up $S$ instead of breaking up $X$.

THEOREM 13.1. *Let $\boldsymbol{S}$ be a monoid and $S = (S, \phi, X)$ be a faithful unitary action, and suppose $S = I \cup T$, where $I$ is a left ideal and $T$ a unitary submonoid. Let $\boldsymbol{I}^1$ denote the induced unitary action by $I^1$ on $X$, where $I^1 = I \cup \{1_S\}$ even if $I$ has a unity. Let $\boldsymbol{T}$ be the action by right multiplication of $T$ on itself. Then $\boldsymbol{S}$ divides $\boldsymbol{I}^1 \mathrm{wr} \boldsymbol{T}^c$.*

*Proof.* Let $W = I^1 \mathrm{wr} T^c$, and define mappings $E: T \to I^1$, $F_s: T \to I^1$ (for $s \in I^1$) by

$$tE = id_X \qquad (t \in T) \tag{13.1}$$

and

$$tF_s = ts \qquad (t \in T, s \in I^1). \tag{13.2}$$

Let $\bar{S}$ denote the subsemigroup of $W$ consisting of elements of the form $(E, t)$ $(t \in T)$ and $(F_s, c_t)$ $(s \in I^1,\ t \in T)$. One checks that $\bar{S}$ is indeed a subsemigroup:

$$(E, t)(E, t') = (E, tt') \qquad (t, t' \in T)$$

$$(E, t)(F_s, c_{t'}) = (F_{ts}, c_{t'}) \qquad (s \in I^1, t, t' \in T)$$

$$(F_s, c_t)(E, t') = (F_s, c_{tt'}) \qquad (s \in I^1, t, t' \in T)$$

and

$$(F_s, c_t)(F_{s'}, c_{t'}) = (F_{sts'}, c_{t'}) \qquad (s, s' \in I^1, t, t' \in T).$$

Define $\alpha: \bar{S} \to S$ by

$$(E, t)\alpha = t, \quad (F_s, c_t)\alpha = st \qquad (t \in T, s \in I^1), \tag{13.3}$$

and $\theta: X \times T \to X$ by

$$(x, t)\theta = xt \qquad (t \in T, x \in X). \tag{13.4}$$

Then $\alpha$ and $\theta$ are surjective, and

$$[(x, t)\theta](E, t')\alpha = xtt' = [(x, t)(E, t')]\theta$$

by (6.1), (13.1), (13.3), and (13.4), and

$$[(x, t)\theta](F_s, c_{t'})\alpha = xtst' = [(x, t)(F_s, c_{t'})]\theta$$

by (6.1), (13.2), (13.3) and (13.4). By (4.6), this proves Theorem 13.1.

THEOREM 13.2. *Let $M$ be a finite monoid and $\boldsymbol{M} = (M, \phi, X)$ a unitary faithful action. Let $G$ be the group of units of $M$. Then $I = M - G$ is a two-sided ideal of $M$, and $\boldsymbol{M}$ divides $\boldsymbol{I} \mathrm{wr} \boldsymbol{G}_G$, where $\boldsymbol{I}$ is the induced action of $I$ on $X$.*

*Proof.* First a proof that $I$ is a two-sided ideal: If $h \in I$ then by right multiplication $h$ induces a mapping from $M$ to $M$ which is neither injective nor surjective, because if it were one it would have to

be the other ($M$ is finite!) and an element inducing a bijection is invertible. But then if $m \in M$, neither $mh$ nor $hm$ can induce bijections, so neither is in $G$.

Now the remainder of the proof is sketched; it is similar to that of Theorem 13.1.

Define $E: G \to I$ and $F_s: G \to I$ for $s \in I$ by

$$gE = id_X \qquad (g \in G)$$

and

$$gF_s = gsg^{-1} \qquad (g \in G, s \in I).$$

The subset $\bar{S}$ of elements $(E, g)$ and $(F_s, g)$ for $g \in G$ and $s \in I$ is then a submonoid of $I\text{wr}G$. Define $\alpha: \bar{S} \to S$ by

$$(E, g)\alpha = g, \qquad (F_s, g)\alpha = sg$$

and $\theta: X \times G \to X$ by

$$(x, g)\theta = xg.$$

Then $(\alpha, \theta)$ is a surjective morphism of actions onto $(S, \phi, X)$.

**14. The Krohn-Rhodes Theorem.** The Krohn-Rhodes Theorem says that simple groups and certain semigroups $U_1$, $U_2$ and $U_3$ are the fundamental semigroups into which finite semigroup actions may be decomposed. $U_3$ is the transformation semigroup on a two-element set consisting of the two constant functions, and the identity function. $U_1$ is the subsemigroup of constant functions, and $U_2$ is a subsemigroup (either one of them) consisting of one constant function and the identity function.

THEOREM (Krohn-Rhodes [65]). *Let $\boldsymbol{S} = (S, \phi, X)$ be a faithful semigroup action with $S$ and $X$ finite. Then $\boldsymbol{S}$ divides an iterated wreath product of a finite number of semigroup actions $\boldsymbol{T} = (T_i, \Psi_i, Y_i)$, where $T_i$ is either $U_1$, $U_2$ or $U_3$ or a finite simple group dividing $S$.*

NOTE: It can happen that the decomposition require $U_3$ even though $U_3$ does not divide $S$ (Meyer-Thompson [69], section 4). On the other hand, it is obvious that any occurrence of $U_1$ or $U_2$ in the covering can be replaced by $U_3$. Much more is known about the role the semigroups $U_1$, $U_2$ and $U_3$ play in the Krohn-Rhodes Theorem. For example, finite simple groups and $U_1$, $U_2$, $U_3$ are exactly the **irreducible** semigroups: $T$ is irreducible if whenever $T|A\text{wr}B$ then $T|A$ or $T|B$. This was included by Krohn and Rhodes [65] in their statement of the theorem. More recent information about this is in Stiffler [73].

Every cyclic group of prime order is simple. An automaton whose constituent is a cyclic group is called a **counter.** The famous result of Feit-Thompson [63] thus implies that a group action by a group of odd order can be decomposed into counters. There are many other finite simple groups, of course; the discovery of finite simple groups is an active area of research (Carter [72], Gorenstein [68]).

The least number of groups needed in a covering of $\boldsymbol{S}$ by a wreath product is called the **group complexity** of the actions. This has been the subject of active research lately. See Rhodes and Tilson [72], Rhodes [74], Rhodes [to appear] and Tilson [to appear], the articles in *Advances in Mathematics* Vol. 11 (1973), and the references therein.

The proof below is in essence due to Meyer and Thompson [69]. A very different proof is that of Ginzburg [68] based on a proof by Zeiger which contained errors.

*Proof of Krohn-Rhodes Theorem.* The proof is by transfinite induction on a measure $\|\boldsymbol{S}\|$ of $\boldsymbol{S}$ defined this way:

$$\|\boldsymbol{S}\| = (|S_X^c|, |X|, |S|). \tag{14.1}$$

Thus $\|\boldsymbol{S}\|$ is an ordered triple of positive integers; note that $|S_X^c| > 0$ by convention (Section 8).

The set of measures is lexicographically ordered.

$$(m_1, m_2, m_3) < (n_1, n_2, n_3) \Leftrightarrow \begin{cases} m_1 < n_1 \text{ or} \\ m_1 = n_1 \text{ and } m_2 < n_2 \text{ or} \\ m_1 = n_1,\ m_2 = n_2 \text{ and } m_3 < n_3. \end{cases}$$

This is a well-ordering.

The smallest measure is $(1,1,1)$, which makes $S$ the one-element group, for which the Theorem is true (if we take the one-element group to be simple, which not all authors do). Because of the faithfulness assumption, the next occurring measure is $(1,2,1)$, for which the same comment applies. The next two measures are $(1,2,2)$ and $(1,2,3)$, which correspond to the actions defined above for $U_1$, $U_2$ or $U_3$.

If $(S, \phi, X)$ is a group, the Theorem is true by the Kaloujnine-Krasner Theorem. (Note that the latter theorem is a bit stronger, since it says that the simple groups occurring in the wreath covering are factors in a composition series for the group, rather than merely dividing the group. That is a genuine distinction; for example, every alternating group on more than 4 letters is simple and contains every smaller alternating group.)

I shall now show that if $\boldsymbol{S}$ is not a group and has measure larger than $(1,2,3)$, then it divides a wreath product $\boldsymbol{T}_1 \mathrm{wr} \boldsymbol{T}_2$ such that for $i = 1,2, |\boldsymbol{T}_i| < |\boldsymbol{S}|$ and $\boldsymbol{T}_i^{-c}$ divides $\boldsymbol{S}$. The theorem will then follow by transfinite induction: Suppose the theorem is true of all actions with measure strictly less than that of $\boldsymbol{S}$. Then $\boldsymbol{T}_i$ is covered by a wreath product of semigroup actions $\boldsymbol{T}_{ij}$ where $\boldsymbol{T}_{ij}$ is either $\boldsymbol{U}_1$, $\boldsymbol{U}_2$ or $\boldsymbol{U}_3$ or a finite simple group dividing $T_i$. Then $\boldsymbol{S}$ is covered by $\boldsymbol{T}_{11}\mathrm{wr}\cdots\mathrm{wr}\boldsymbol{T}_{21}\mathrm{wr}\cdots$. Furthermore, if $\boldsymbol{T}_{ij}$ is a finite simple group, it divides $T_i$ and hence $T_i^{-c}$ (Lemma 3.1), and hence $S$.

CASE 1. $\boldsymbol{S}^{-c}$ has a proper nontrivial $S_X^{-c}$-subaction. That is a subset $Y \subset X$ closed under action by any nonconstant element of $\boldsymbol{S}$. "Nontrivial" means that $Y$ has more than one element.

Then by Corollary 11.3 and Lemma 11.4. $\boldsymbol{S}^c$, and hence $\boldsymbol{S}$ divides $(\boldsymbol{S}_Y)^c \mathrm{wr}(\boldsymbol{S}^{\Pi})^c$, where $\Pi$ is the congruence on $X$ which has $Y$ as the only class with more than one element. Observe that $(\boldsymbol{S}_Y)^c \mathrm{wr}(\boldsymbol{S}^{\Pi})^c$ is an action on the set $Y \times \Pi$.

It is easy to verify that $S_Y^{-c}$ divides $S_X^{-c}$, so $|S_Y^{-c}| \leqq |S_X^{-c}|$. Since $|Y| < |X|$, it follows that $\|\boldsymbol{S}_Y^c\| < \|\boldsymbol{S}\|$. (Implicitly used here is the fact that $(S_X^c)^{-c} = S_X^{-c}$.)

Similarly, $(\boldsymbol{S}^{\Pi})^{-c}$ divides $\boldsymbol{S}^{-c}$, and $|\Pi| < |X|$, so that $\|(\boldsymbol{S}^{\Pi})^c\| < \|\boldsymbol{S}\|$.

CASE 2. $S$ is a monoid, $\boldsymbol{S}$ is unitary, and the subgroup $G$ of invertible elements of $S$ satisfies $1 < |G| < |S|$. By Theorem 13.2, $\boldsymbol{S}$ divides $\boldsymbol{I}\mathrm{wr}\boldsymbol{G}_G$ acting on $X \times G$, where $I = S - G$.

Clearly $|I^{-c}| \leqq |S^{-c}|$ and $|I| < |S|$, so $\|I\| < \|S\|$. Moreover, $I^{-c}$ divides $S$. On the other hand, $\boldsymbol{G}_G$ is a group action and $G$ divides $S$.

CASE 3. $S$ is not a group and does not satisfy Case 1 or Case 2. I also assume that $S$ is a monoid and $\boldsymbol{S}$ is unitary, because if the Krohn-Rhodes Theorem is true for finite unitary monoid actions it is necessarily true for all finite semigroup actions. (If $G$ is a nontrivial group, $S$ a finite semigroup, and $G \mid S^1$, then certainly $G \mid S$.)

$S - \{1_S\}$ has no permutations, so is an ideal of $S$ (no non-permutation times any element of $S$ can be a permutation, but $1_S$ is a permutation). $S - \{1_S\}$ cannot consist entirely of constant functions, for then $S^{-c} = \{id_X\}$, $\boldsymbol{S}^{-c}$ has a nontrivial subaction, and Case 1 holds (unless $|\boldsymbol{X}| = 2$, but then $\|\boldsymbol{S}\| \leqq (1,2,3)$).

Thus there is $t \in S$ with $t \neq 1_S$, $t$ nonconstant. Then $1 < |Xt| < |X|$. Then $St$ is a proper subset of $S - \{1_S\}$ or else $Xt$ would be a subaction making Case 1 hold. Hence $St$ is a proper left ideal of $S - \{1_S\}$.

Let $I$ be a maximal proper left ideal of $S - \{1_S\}$. Let $x \in S - I^1$ and let $T = Sx \cup \{1_S\}$. Then $I \cup Sx$ is a left ideal properly containing $I$, so $I \cup T = S$. $T$ is a unitary submonoid, so Theorem 13.1 implies that $S$ divides $\boldsymbol{I}^1 \mathrm{wr} \boldsymbol{T}^c$, where $\boldsymbol{I}^1$ acts on $X$ and $\boldsymbol{T}^c$ on $T$.

Clearly $(I^1)^{-c}$ and $(T^c)^{-c} = T^{-c}$ both divide $S$. It follows from the facts that $\boldsymbol{I}^1$ acts on $X$ and that $|I^1| < |S|$ that $\|\boldsymbol{I}^1\| < \|\boldsymbol{S}\|$. To show that $\|\boldsymbol{T}\| < \|\boldsymbol{S}\|$ it is necessary to show that $|T^{-c}| < |S^{-c}|$ since

we have no control over the relative sizes of the sets $T$ and $X$ being acted on. Since $|S^{-c}| \geqq 2$, if $T-\{1_S\}$ consists entirely of constant functions, we are done. Otherwise, $T-\{1_S\}$ contains a nonconstant element $x$, so by the same argument as for $t$, $Sx$ is a proper subset of $S$ and $XSx = Xx$ is a proper subset of $X$. If $S-T$ consisted only of constants, this would imply that $Xx$ is a proper nontrivial $Sx^{-c}$ subaction, so Case 1 would hold. Hence $S-T$ contains a nonconstant, so $|T^{-c}| < |S^{-c}|$, which completes the proof of the Krohn-Rhodes Theorem.

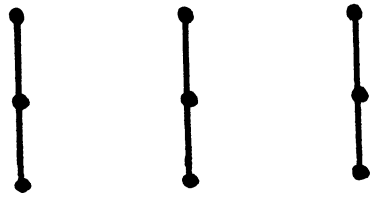


FIG. 15.1

**15. Applications to automorphism groups and combinatorics.** An automorphism of the disconnected graph $G$ in Fig. 15.1 must take one of the three connected components into the same one or another one. Thus Aut $G$ has a $G$-partition $\Pi$ consisting of the three components. It follows easily from Prop. 12.1 that Aut$G$ is the wreath product of the cyclic group of order 2 (the automorphism group of one component) by the symmetric group of degree 3 (the action of Aut$G$ on the components). (Observe that Aut$G$ is not transitive.)

This is an instance of a general theorem known to graph-theorists which dates back at least to Frucht [49]:

THEOREM 15.1. *Let a finite graph $G$ be the disjoint union $\cup_{i=1}^{m} \cup_{j=1}^{n_i} A_{ij}$, with $A_{ij}$ connected and $A_{ij} \cong A_{rs}$ if and only if $j = s$. Then*

$$\mathrm{Aut}G \cong \prod_{i=1}^{m} (\mathrm{Aut}A_{i1} \times \mathrm{Sym}(n_i)).$$

It is necessary that the components to be connected, and the theorem won't work for infinite graphs unless each component has no proper subgraph isomorphic to it.

An analogous theorem is true of partially ordered sets and of other kinds of structures for which the disjoint union of two instances of that kind of structure is another structure of that kind.

Going back to a graph $G$, Aut$G$ sometimes has a fixed point which, when removed, puts one in the situation of Theorem 15.1. Thus the graph $G$ in Fig. 15.2. has Aut $G \cong C_2 \mathrm{wr} \mathrm{Sym}4$. Erasing the edges $e$ and $f$ would yield a graph with automorphism group $C_2 \mathrm{wr}(C_2 \times C_2)$.

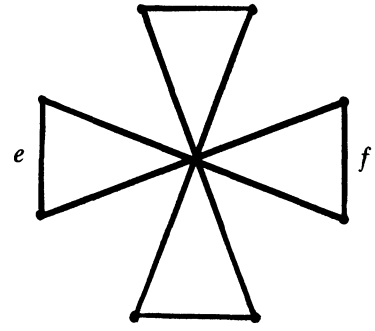



FIG. 15.2

See Harary [59] for another situation in which the automorphism group of a graph decomposes as a wreath product.

This is a useful device for studying rooted trees, the automorphism groups of which are direct products of wreath products of symmetric groups, if one insists that the automorphisms preserve roots and levels (distances from roots). Thus the automorphism group of the tree in Fig. 15.3 is (Sym2wrSym3) × (Sym2wrSym3wrSym2).

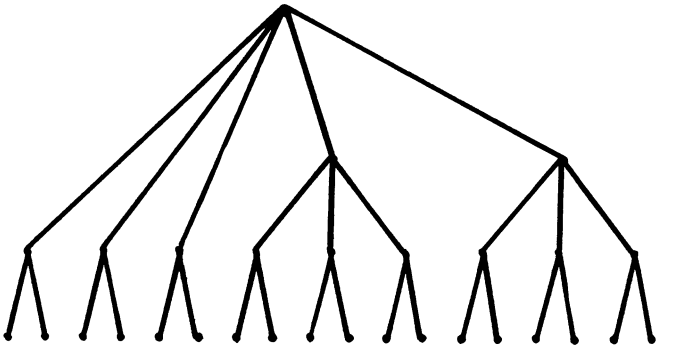


FIG. 15.3

For more on this, see McCarthy [tbp]. The opposite, synthetic problem of constructing a graph with a given wreath product as automorphism has been studied by Frucht and Harary [70] and Uebelacker [tbp].

This sort of analysis can be made for certain structures that appear in counting-problems.

Thus suppose one wants to segregate 8 distant objects into two groups of 3 and a group of 2. If the two groups of 3 were to be kept distinct, so that for example (naming the objects by the first eight digits) $\{\{1,2,3\},\{4,5,6\},\{7,8\}\}$ were different from $\{\{4,5,6\},\{1,2,3\},\{7,8\}\}$, the number of ways to do this is the well-known multinomial coefficient

$$m(2,3,3) = \frac{8!}{2!3!3!} = 560. \tag{15.5}$$

But suppose the two groups of 3 are indistinguishable. How many ways can the objects be arranged now? There are 8! ways to permute the objects, but some permutations do not yield a new arrangement. For example, the permutation (1 4) (2 5 3 6) (7 8) applied to the arrangement $\{\{1,2,3\}\ \{4,5,6\},\{7,8\}\}$ leaves the arrangement unchanged.

Formally, define an **arrangement** as a partition of the set of integers from 1 to 8 into a set of two and two sets of three integers. An **automorphism** of the arrangement is a permutation of the set which leaves the partition unchanged. By the same sort of argument as for the graphs and trees before, the automorphism group of any arrangement is $(\mathrm{Sym}(3)\mathrm{wr}C_2) \times C_2$, which has $6^2 \cdot 2 \cdot 2 = 144$ elements. Thus there are $8!/144 = 280$ possible arrangements. In this simple case, one could have deduced this directly from (15.5), but it is clear that the method is quite general. This problem is treated by Pólya [37], in the paper in which he gave the wreath product its name.

In that paper, he introduced his general theory of counting which is a fundamental tool in combinatorial theory. Many of its applications involve the wreath product. Good recent expositions of his theory and generalizations are in Beckenbach [64] and Read [68].

A particular application which dates back to the turn of the century (see Burnside [55]) is to centralizers of permutations.

THEOREM 15.2. *Let $\phi$ be a permutation of a finite set, and suppose $\phi$ has $n_i$ cycles of length $r_i$, for*

*$i = 1, 2, \cdots, m$. Then the permutations which commute with $\phi$ form a group isomorphic to*

$$\prod_{i=1}^{m} (C_{n_i} \text{wr} \text{Sym}(n_i)).$$

The formal similarity between Theorems 15.1 and 15.2 is no accident. One can represent a permutation as a directed graph; for example as in Fig. 15.4, which represents the permutation (1 2) (3 4 5) (6 7 8).

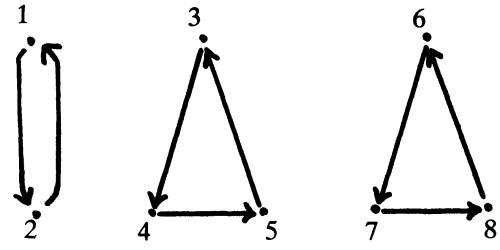



FIG. 15.4

A permutation commutes with this permutation if and only if it is an automorphism of the directed graph, and the automorphisms of an $n$-cycle form a group isomorphic to $C_n$. Theorem 15.2 then becomes a special case of Theorem 15.1.

The more difficult task of constructing the centralizer in $\text{Sym}(n)$ of a permutation group $G$ on $n$ letters can be simplified by using Theorem 15.2 and a theorem of Kuhn [04]. The normalizer of $G$ can be described as a "twisted wreath product" (see Section 16). Certain sets of polynomials over finite fields appear as wreath products; see Carlitz and Hayes [72], Lausch and Nöbauer [73, Section 4.9] (and the references there in), and Wells [74].

**16. History, other applications and generalizations.** Group actions with nontrivial $G$-partitions (called **imprimitive** by group theorists) have been understood as long as groups have been known. The earliest reference I have is Ruffini, *Teoria generale delle equazioni*, published in Bologna in 1799; but I only know of it secondhand (Kuhn [04]). Other works in which the concept appeared include Cayley [78], and Miller [01].

The wreath product in something like its present form appears in Loewy [29]. This apparently came to the attention of I. Schur, who suggested questions related to it to his students B. H. Neumann and W. Specht (see Specht [33]). The name *Kranzgruppe* is due to Pólya [37] in a long and influential paper already mentioned in Section 15. P. Hall developed the standard wreath product as a tool for studying abstract groups, a technique continued by the Neumanns and many other group theorists and in wide use today. This line of development has been ignored almost completely here, a lack I have tried to make up for by including a sizeable selection of relevant papers in the Bibliography. Two excellent expositions in specific areas are H. Neumann [67] and Kerber [71].

The abstract wreath product construction was extended to semigroups by B. H. Neumann [60]. The automaton theorists began studying wreath product decomposition of semigroups in the early sixties, their initial major result being the Krohn-Rhodes Theorem (Section 14).

The wreath product is also used in other studies of semigroups not connected with the Krohn-Rhodes Theorem, particularly in the work of Petrich [73a], [73b], [74] (and the references therein).

The wreath product has been generalized in various ways. The wreath product of ordered groups is discussed by B. H. Neumann [49], [60]. The **verbal wreath product** (where the basis group is a verbal product rather than a direct product) is introduced by Smelkin [64]; see also Burns [66] and H. Neumann [67]. Petrich [73a, V. 3], and [73b, Section 2] (and in other papers referred to there) allows the functions in $S^Y$ to be defined only partly, yielding the **partial-mapping wreath product.** B. H. Neumann [56, Section 8] constructs the **crown product** by amalgamating subgroups in the basis group.

Various related constructions by B. H. Neumann [63], H. Neumann [67, p. 65], and Wong [67] have been called the **twisted wreath product**. Krohn, Mateosian and Rhodes [67] used this construction (they called it the $\#$ product) to characterize the concatenate of two "generalized machines", and Altinger [70] used a version almost identical with that of H. Neumann [67] just mentioned to describe the normalizer of a subgroup of a symmetric group.

The wreath product construction has been extended to groupoids by Houghton [74b] and to categories by Wells [tbp].

The author would like to thank C. Houghton, D. McCarthy, B. H. Neumann, J. Wiegold and the referee for suggestions and corrections made to an earlier version of this paper, for historical backgrounds, and for suggesting items to include in the Bibliography.

## Bibliography

This Bibliography is selective. I have included fundamental papers, papers referred to in the article, and papers published in obscure places of which the reader might never otherwise become aware. I have included a good many recent papers in which the wreath product is applied to problems in group theory in an effort to remedy the lack of attention given these areas in the body of the article. I have excluded many papers referred to in the following articles and books, which have extensive bibliographies of their own: H. Neumann [67], Kerber [71], Petrich [73a] and Rhodes and Allen [73].

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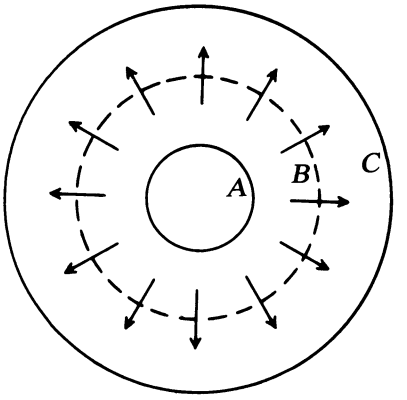
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## A SMALL STEP BACKWARDS

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A common description of mathematical research is that it expands the frontiers of our knowledge. This suggests the following sort of image:



The inner circle $A$ represents an existing body of mathematical knowledge which has already been extensively developed, the outer circle $C$ represents an area of mathematics about which relatively little is known, and the fluid boundary $B$ represents the frontier of research. The effort of the research mathematician, in this model, is directed at expanding the frontier $B$ towards the objective $C$.

One area of research which the model appears to describe fairly well is mathematical logic. Logicians study mathematical relationships from the perspective of the formal sentences in which those relationships can be expressed. If a certain body of mathematics can be described by the relatively simple sentences which are called "first-order," there is an impressive wealth of information which can be provided by logical theories that have been around for quite a while (circle $A$ in the drawing). But much of the important work in mathematics is being done with more sophisticated material. A class of formal sentences called "second-order" is fully capable of describing most of this material, but very little is known at present about second-order logic, making this the region $C$ of desired knowledge which we do not yet possess. A major research effort is presently being made in a loosely defined area known as "$1\frac{1}{2}$-order logic," developing the theory of formal sentences somewhere between the simple first-order and the complex second-order, in hopes of making more of mathematics accessible to analysis by formal logic.

The following is an example of a very small piece of this research. Aimed squarely along the arrows in the drawing, it was intended simply to develop the theory of a new class of sentences on the edge of circle $B$. But (since reality has a certain mischievous contempt for human models) the investigation took a turn that had not been planned at all and ended up in one of the oldest parts of region $A$, completing some unfinished business in the heavily-developed theory of first-order logic. The result is a modest one, but the course of work which led to it illustrates some of the limitations of the "frontier" model of research.

We shall begin with a few preliminary observations. Formal sentences are constructed from three basic kinds of ingredients. *Atomic formulas*, or *atoms*, are simple assertions of mathematical relationships such as "$x < y$", "$x + y = z$", or "$|f(x) - f(y)| < \varepsilon$". *Boolean connectives*, such as "&" (and), "$\vee$" (or), "$\sim$" (not), "$\rightarrow$" (if ... then), and "$\leftrightarrow$" (if and only if), can be used to combine atomic formulas into more complicated statements. Finally, *quantifiers*, such as "$\forall x$" (for all $x$), "$\exists y$" (there exists $y$ such that ...) "$\forall z$" (for all $z$), etc. make use of the variables ($x, y, z \cdots$) in these formulas to make statements of generality and existence.

Beginning with the atomic formula $y > x$, for example, we can add an existential ($\exists$) quantifier and obtain the statement $\exists y\ (y > x)$, i.e., something is larger than $x$. Using another atomic formula $x = +\infty$, and some Boolean connectives, we can say: $(\exists y\ (y > x)) \rightarrow \sim (x = +\infty)$; i.e., if something is larger than $x$, then $x$ is not $+\infty$. Finally, adding a universal ($\forall$) quantifier, we obtain the formal sentence

$$\forall x((\exists y\ \ (y > x)) \rightarrow \sim (x = +\infty)),$$

which is a true statement about the extended real numbers.

The above formal sentence could be expressed in another form which is logically equivalent: $\forall x \forall y\ (y > x \rightarrow \sim (x = +\infty))$. (Given any $x$ and $y$, if $y$ is bigger than $x$, then $x$ isn't $+\infty$.) This second form turns out to be much easier for logicians to work with because it is neatly separated into a string of quantifiers followed by a Boolean combination of atoms. A formal sentence in which the quantifiers are separated out in this way is called a *prenex* sentence, and it is an important theorem of logic that every formal sentence has an equivalent prenex form.

When a sentence is expressed in prenex form, all of its existential quantifiers can be interpreted as functions of the universal quantifiers appearing in front of them. For example, if we take $\mathcal{Q}(x, y, \varepsilon, \delta)$ to be the formal expression

$$\varepsilon > 0 \rightarrow \delta > 0 \,\&\, (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon),$$

then the statement $\forall x \forall \varepsilon \exists \delta \forall y \mathcal{Q}(x, y, \varepsilon, \delta)$ that the function $f$ is pointwise continuous could also be

written as $\exists g \forall x \forall \varepsilon \forall y \mathcal{Q}(x, y, \varepsilon, g(x, \varepsilon))$. The stronger statement that $f$ is *uniformly* continuous could be written:

$$\forall \varepsilon \exists \delta \forall x \forall y \mathcal{Q}(x, y, \varepsilon, \delta) \qquad \text{or:} \quad \exists g \forall x \forall \varepsilon \forall y \mathcal{Q}(x, y, \varepsilon, g(\varepsilon)).$$

The difference between these two concepts is that in the first one the choice of $\delta$ depends on $\varepsilon$ *and* on $x$ (hence $\delta = g(x, \varepsilon)$; but in the second, $\delta$ depends on $\varepsilon$ alone (hence $\delta = g(\varepsilon)$). Formally this difference is visible in the fact that in the quantifier-string $\forall x \forall \varepsilon \exists \delta \forall y$ the existential quantifier $\exists \delta$ is preceded by both $\forall x$ and $\forall \varepsilon$, whereas in $\forall \varepsilon \exists \delta \forall x \forall y$, $\exists \delta$ is preceded only by $\forall \varepsilon$.

The example of pointwise vs. uniform continuity suggests that prenex sentences with the quantifier pattern $\forall\forall\exists\forall$ are very different from those with the pattern $\forall\exists\forall\forall$. Experience with the effects of other quantifier patterns suggests that this is no isolated situation, and that every quantifier pattern has its own unique effect which no other pattern can duplicate. Stated more precisely, we have the following:

CONJECTURE $*$. *If $Q$ and $q$ are two different quantifier-patterns of the same length (e.g. $Q = \forall\forall\exists\forall$ and $q = \forall\exists\forall\forall$), then there is a prenex sentence with quantifier-pattern $Q$ which is not equivalent to any prenex sentence with pattern $q$.*

Although it is not of pressing importance for any particular application of logic, $*$ is a very simple conjecture which it would seem natural to explore in pursuit of the general objective of learning the limitations which the form of a mathematical statement can place on the kind of information the statement can express. And so it is somewhat surprising that the conjecture remained open until 1970. We shall return to it later.

In the late 1950's, the logician Leon Henkin used the idea of representing existential quantifiers by functions to define some very unusual "two-dimensional" arrangements of quantifiers. For a formula $\mathcal{Q}(x, y, a, b, c)$ involving the variables $x$, $y$, $a$, $b$, and $c$, for example, he defined the formal sentence $\exists c \begin{matrix} \forall x \exists a \\ \forall y \exists b \end{matrix} \mathcal{Q}(x, y, a, b, c)$ to mean: $\exists f \exists g \exists c \forall x \forall y \mathcal{Q}(x, y, f(x), g(y), c)$.

$$\forall y \begin{matrix} \forall x \exists a \\ \exists b \exists c \end{matrix} \mathcal{Q}(x, y, a, b, c)$$

was defined as $\exists f \exists g \exists h \forall x \forall y \mathcal{Q}(x, y, f(x), g(y), h(y))$, and so on, following the rule that an existential quantifier is influenced by all universal quantifiers which precede it in the (partial) ordering of the quantifier pattern.

The main problem that Henkin had in developing the theory of his new class of formal sentences is that their definition involved the use of second-order logic. The difference between first-order and second-order logic is a matter of the kinds of mathematical objects to which the quantifiers in formal sentences refer: In first-order logic, quantifiers may only refer to individual objects (real numbers, group elements, points in a topological space, etc.); in second-order logic they may also refer to sets and functions. This distinction may seem rather subtle, but in terms of finding useful ways to apply formal sentences to the study of mathematics, it makes all the difference between regions $A$ and $C$ in our model of research. Since the definitions of Henkin's new formal sentences involved function quantifiers ($\exists f$, $\exists g$, etc.), those definitions are second-order and very difficult to work with. In order to relate the new formal sentences to the large existing body of first-order theory, Henkin tried to extend first-order logic to include them, expressing the special properties of partially-ordered quantification with added logical axioms.

The following example, given by Ehrenfeucht, proved that Henkin's goal could not be achieved:

Let $S$ be the sentence

$$\exists c \begin{matrix} \forall x \exists a \\ \forall y \exists b \end{matrix} [(x = y \to a = b) \ \& \ (a = b \to x = y) \ \& \ a \neq c].$$

Henkin's second-order definition for $S$ is then:

$$\exists f \exists g \exists c \forall x \forall y [(x = y \rightarrow f(x) = g(y)) \ \& \ (f(x) = g(y) \rightarrow x = y) \ \& \ f(x) \neq c].$$

A standard logical transformation gives us the following equivalent sentence, which is easier to interpret:

$$\exists f \exists g [\forall x \forall y (x = y \rightarrow f(x) = g(y))$$
$$\& \ \forall x \forall y (f(x) = g(y) \rightarrow x = y) \ \& \ \exists c \forall x (f(x) \neq c)].$$

The first line tells us that $f$ and $g$ are the same function; by substituting $f$ for $g$ in the second line, we see that $f$ is one-to-one; the last line says that there is something which is not included in the range of $f$. The sentence $S$ is therefore true on exactly those sets which can be put into one-to-one correspondence with some proper subset; i.e., on infinite sets.

Now, the idea of an infinite set can be axiomatized in first-order logic, using infinitely many sentences:

$$\exists x_1 \exists x_2 \ x_1 \neq x_2$$
$$\exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_2 \neq x_3)$$
$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 (x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_1 \neq x_4 \ \& \ x_2 \neq x_3 \ \& \ x_2 \neq x_4 \ \& \ x_3 \neq x_4) \cdots .$$

But once we have $S$ in our formal language, we also have the characterization $\sim S$ of *finite* sets, which bends some of the fundamental properties of first-order logic to the breaking point.

What's so complicated about the property of finiteness? Consider the following set of sentences:

$$\mathscr{C} = \{ \sim S, \ \exists x_1 \exists x_2 \ x_1 \neq x_2, \ \exists x_1 \exists x_2 \exists x_3 (x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_2 \neq x_3), \cdots \}.$$

Taken in its entirety, $\mathscr{C}$ describes a set which is both finite and infinite — a clearly contradictory notion. Indeed, there is no mathematical system which *satisfies* $\mathscr{C}$ (makes its sentences all true). But for an unsatisfiable set of sentences, $\mathscr{C}$ has some very unusual properties!

For one thing, $\mathscr{C}$ is what a logician would call *finitely satisfiable*, meaning that every finite subset of $\mathscr{C}$ *is* a satisfiable set of sentences. The subset of $\mathscr{C}$ consisting of the three explicitly listed sentences, for example, is satisfied by any finite set with at least three elements. But one of the most important properties of first-order logic is the *compactness theorem*, which says that every finitely satisfiable set of sentences must be satisfiable in its own right.

By violating the compactness theorem, $\mathscr{C}$ automatically violates another, better-known property of first-order logic: *completeness*. The sentences of $\mathscr{C}$ are intuitively self-contradictory, but $\mathscr{C}$ is *not* inconsistent! An inconsistent set of sentences is one from which it is possible to derive a formal contradiction, such as $P \ \& \ \sim P$ — and no formal proof will ever lead from the sentences of $\mathscr{C}$ to a contradictory conclusion. Proofs can be (and unfortunately often are) very long, but at least they always have a *finite* number of steps. So no formal proof could ever use more than a finite number of sentences from $\mathscr{C}$. And every finite set of sentences from $\mathscr{C}$ is consistent! The logic is *incomplete* because there is something which we clearly *should* be able to prove (the self-contradictory character of $\mathscr{C}$) but cannot. Another "missing" result is the fact that $S$ follows from the first-order axioms for an infinite set — a "theorem" which is unprovable for reasons which the reader is invited to discover by rethinking the example of $\mathscr{C}$.

Henkin's new formal language does not merely extend first-order logic by a few logical axioms. It changes the rules of the game altogether, and belongs in a higher order of logic.[1]

But it is far less than the full second-order language. Every Henkin sentence has a definition of the

1. Ehrenfeucht actually showed that Henkin's logic is nonaxiomatizable in a very fundamental sense: Its set of valid sentences cannot be generated by definable rules of inference from any definable set of axioms!

form:

| $\exists f_1 \cdots \exists f_m$ | $\forall x_1 \cdots \forall x_n$ | $\mathcal{Q}$ |
|---|---|---|
| (second-order existential quantifiers) | (first-order quantifiers) | (some Boolean combination of atoms) |

— a very limited kind of second-order sentence known as second-order existential, or $\Sigma_1^1$. Ehrenfeucht's example placed "Henkin logic" squarely on the early $1\frac{1}{2}$-order frontier.

After the publication of the initial results of Henkin and Ehrenfeucht in 1959 [1], very little was done with Henkin logic until 1967, when it was discovered that every $\Sigma_1^1$ sentence can be expressed by applying Henkin's partially-ordered quantifier patterns to suitably chosen Boolean combinations of atomic formulas, and hence that Henkin logic is fully equivalent to the part of second-order logic which had been used to define it. The same investigations (published in [2]) showed how to express all Henkin sentences in a few simple forms. The sentence

$$\begin{array}{ccc} \forall v \exists a & & \\ \forall u & \quad \forall x \exists b & \\ & \forall w & \quad \mathcal{Q}(u, v, w, x, y, z, a, b, c, d, e) \\ & \quad \forall y \exists c \exists d & \\ \forall z \exists e & & \end{array}$$

for example, has the second-order form

$$\exists f_1 \exists f_2 \exists f_3 \exists f_4 \exists f_5 \forall u \forall v \forall w \forall x \forall y \forall z \mathcal{Q}(u, v, w, x, y, z, f_1(u, v), f_2(u, w, x), f_3(u, w, y), f_4(u, w, y), f_5(z)),$$

and can be re-expressed as

$$\forall u_1 \forall v \exists a$$

$$\forall u_2 \forall w_1 \forall x \exists b$$

$$\forall u_3 \forall w_2 \forall y_1 \exists c \ (u_1 = u_2 = u_3 = u_4 \ \& \ w_1 = w_2 = w_3 \ \& \ y_1 = y_2 \rightarrow \mathcal{Q}(u, v, w, x, y, z, a, b, c, d, e)).$$

$$\forall u_4 \forall w_3 \forall y_2 \exists d$$

$$\forall z \exists e$$

Similarly, any partially-ordered quantifier prefix can be represented by a prefix made up of parallel rows with at most one existential quantifier in each row.

By allowing more than one existential quantifier per row we can represent any given quantifier prefix using only two rows of quantifiers, expressing the above example as:

$$\begin{array}{c} \forall u \forall v \forall w \forall x \forall y \forall z \exists a \exists b \exists c \exists d \exists e \\ \forall u_1 \forall v_1 \forall w_1 \forall x_1 \forall y_1 \forall z_1 \exists a_1 \exists b_1 \exists c_1 \exists d_1 \exists e_1 \end{array} [(u = u_1 \ \& \ v = v_1 \rightarrow a = a_1)$$

$$\& \ (u = u_1 \ \& \ w = w_1 \ \& \ x = x_1 \rightarrow b = b_1) \ \& \ (u = u_1 \ \& \ w = w_1 \ \& \ y = y_1 \rightarrow c = c_1 \ \& \ d = d_1)$$

$$\& \ (z = z_1 \rightarrow e = e_1) \ \& \ \mathcal{Q}(u, v, w, x, y, z, a, b, c, d, e)].$$

Separating a Henkin sentence into its partially-ordered quantifier pattern $Q$ and its Boolean combination $\mathcal{Q}$ of atoms, the new Henkin logic results and Ehrenfeucht's earlier example had the common feature of using elaborate choices of $\mathcal{Q}$ to express a surprising variety of relationships with rather simple choices of $Q$. The question then arose: Does the design of the pattern $Q$ impose any important restrictions at all on what the sentence $Q\mathcal{Q}$ can express?

Since representing a sentence $Q\mathcal{Q}$ which has a complicated quantifier pattern $Q$ with a simpler sentence $q\psi$ always seemed to require more quantifiers in $q$ than in $Q$, the conjecture * was posed for Henkin logic. A thorough search of the literature at that time would have uncovered the discouraging fact that this problem was still unsolved in the much simpler first-order setting, but Henkin logic seemed too far removed from first-order theory for the literature to be very helpful, and no extensive search was made. Instead, mathematical tools more appropriate to the higher-order frontier were employed, with mixed results.

The conjecture * turns out to be false in Henkin logic, but it is false in a very restricted way. It is shown in [2], for example, that every sentence of the form $\exists a \exists b \exists c \forall w \forall x \forall y \forall z\, \mathcal{Q}$ can be expressed in the form

$$\exists c \begin{matrix} \forall w \exists a \\ \forall x \forall y \forall z \exists b \end{matrix} \psi,$$

which says that the quantifier-pattern $\exists\exists\exists\forall\forall\forall\forall$ can be represented by the pattern $\exists \begin{matrix} \forall\exists \\ \forall\forall\forall\exists \end{matrix}$. Since the only part of the partial ordering of quantifiers which really matters is the question of which $\forall$s come before which $\exists$s, this is the same thing as saying that $\exists\exists\exists \begin{matrix} \forall \\ \forall\forall\forall \end{matrix}$ can be represented by $\exists \begin{matrix} \forall\exists \\ \forall\forall\forall\exists \end{matrix}$. Hence *a quantifier pattern $Q$ can sometimes be represented by a different pattern $q$ of the same size* (i.e., same number of quantifiers) *if $q$ can be obtained from $Q$ by moving some $\exists$s in front of additional $\forall$s.*

It is also shown in [2] that the pattern $\exists\exists\exists\exists\exists$ can be represented with the pattern $\exists \begin{matrix} \forall\exists \\ \forall\exists \end{matrix}$. Rewriting $\exists\exists\exists\exists\exists$ as $\exists \begin{matrix} \exists\exists \\ \exists\exists \end{matrix}$, we see that *a pattern $Q$ can sometimes be represented by a different pattern $q$ of the same size if $q$ can be obtained from $Q$ by simply changing some of the $\exists$s to $\forall$s.* But the main theorem of [2] shows that the underlined statements are the *only* ways in which one quantifier pattern can be represented by another of the same size.

What happens if we apply these results to ordinary first-order quantifier patterns? Is it possible, for example, to represent the pattern $\forall\exists\forall\forall$ by $\forall\forall\exists\forall$? This possibility seems quite reasonable because of the loophole in * about moving $\exists$s to the right, but a special feature of first-order logic gets in the way.

A prenex first-order formula always begins with a linearly ordered string of quantifiers. If we form the denial of such a formula by prefacing the symbol $\sim$, the prenex form of the new sentence is found by moving the denial symbol inside of the quantifiers, changing all $\forall$s to $\exists$s and all $\exists$s to $\forall$s in the process. The new quantifier pattern which results is called the *dual* of the original pattern. The sentence $\sim \forall x \exists y (x \cdot y = 1)$, for example, is equivalent to the prenex sentence $\exists x \forall y (x \cdot y \neq 1)$, and the pattern $\exists\forall$ is the dual of $\forall\exists$. This dualizing of quantifier patterns by denials is an important property of linearly-ordered quantifier patterns which partially-ordered patterns do not share.

Now suppose that $\forall\exists\forall\forall$ can be represented by $\forall\forall\exists\forall$. Then, given any Boolean combination $\mathcal{Q}$ of atoms, there must be a corresponding Boolean combination $\psi$ of atoms, such that $\forall w \exists x \forall y \forall z \mathcal{Q}$ is equivalent to $\forall w \forall y \exists x \forall z \psi$. It follows that $\sim \forall w \exists x \forall y \forall z \mathcal{Q}$ is equivalent to $\sim \forall w \forall y \exists x \forall z \psi$. Using the first-order rule for converting these denials to prenex form, $\sim \forall w \exists x \forall y \forall z \mathcal{Q}$ becomes $\exists w \forall x \exists y \exists z \sim \mathcal{Q}$, $\sim \forall w \forall y \exists x \forall z \psi$ becomes $\exists w \exists y \forall x \exists z \sim \psi$, and we see that $\exists w \forall x \exists y \exists z \sim \mathcal{Q}$ is equivalent to $\exists w \exists y \forall x \exists z \sim \psi$. Since $\mathcal{Q}$ could have been any Boolean combination of atoms, the denial of $\mathcal{Q}$ inside the quantifiers does not restrict our generality, and we have shown that the quantifier pattern $\exists\forall\exists\exists$ can be represented by the pattern $\exists\exists\forall\exists$. But this contradicts the main result of [2], because the relationship of these patterns involves moving an $\exists$ the wrong way. Hence, the assumption that $\forall\exists\forall\forall$ can be represented by $\forall\forall\exists\forall$ cannot be true.

This argument can be generalized. Whenever one linearly-ordered quantifier pattern $Q$ can be represented by another linear pattern $q$ of the same size, the application of denials as above shows that the dual of $Q$ must be representable by the dual of $q$. Hence, the set of transformations of $Q$ into

$q$ which allow this to occur must be very symmetrical. If we may move $\exists$s to the right, we must also be able to move them to the left; if $\exists$s may be changed to $\forall$s, then $\forall$s may also be changed to $\exists$s. But the set of transformations which could provide exceptions to the conjecture * possess no such symmetry. *Hence * is a valid theorem of first-order logic.*

This result first appeared in [2], as a corollary to the main theorem on representations of Henkin quantifier patterns. A simpler proof was later developed in [3]. But why hadn't the problem been solved much earlier, during the extensive development of first-order logic?

The main arguments used in [2] and [3] do not employ much first-order theory, since they grew out of the study of a higher-order logic. Apparently, the best way to prove * was to ignore the older theory and use the kind of *ad hoc* approach more typical of work in an unexplored area. And the way that finally happened was for some "frontier" research to develop in the "wrong" direction.

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## MATHEMATICAL NOTES

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### THE INVERSE AS A FIXED POINT IN FUNCTION SPACE

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The Contraction Mapping Principle [2, p. 338] or [3, p. 229] is widely used in proofs of the Inverse and Implicit Function Theorems [1], [2], [3], [6] to give a point-by-point construction of the local inverse or implicitly defined function. Thus in [2, p. 349] and [6, p. 14] the Inverse Function Theorem is proved for a function $f$ near a non-singular point $a$ by repeated use of the Principle to find a solution $x$ near $a$ to each of the equations $y = f(x)$, where $y$ varies in a neighborhood of $f(a)$. Here this standard proof of the Inverse Function Theorem is recast into a form which exhibits the local inverse "all at once" as a fixed point of a naturally occurring contraction map on a space of continuously differentiable functions.

This use of the Principle seems somewhat better motivated, and more consonant with modern viewpoints. Consider the normalized case where $a = 0 = f(0)$ and $d_0 f = I$, the identity map. It is certainly plausible to claim that the theorem is true because $f$ can be regarded as a small perturbation of the identity, and of course all direct proofs rely heavily on this fact. Thus if $f = I + p$ the local inverse $g$ will satisfy $I = f \circ g = g + p \circ g$ near 0. Simply rewriting this as

$$g = I - p \circ g \tag{1}$$

exhibits the local inverse as a fixed point of the operation

$$T: g \to I - p \circ g. \tag{2}$$

The task of designing a complete space of functions on which $T$ acts as a contraction turns out to be straightforward, and in places quite mechanical. Only the familiar ingredients are used, notably the Chain Rule [1, Theorem 8.2.1, p. 151] or [2, p. 194], the Mean Value Theorem in its general form [3, Theorem 7.4, p. 149], and the operator norm [3, Section 3.3].

The use of (2) is actually another way of regarding the standard proof in [2] or [6], in which a point $y$ is chosen near enough to 0 so that the map

$$x \to y - f(x) + x \tag{3}$$

is a contraction on a small ball centered at 0. The fixed point of this map is a solution to $y = f(x)$. The map (3) can be obtained from $(Tg)(y) = y - p(g(y)) = y - f(g(y)) + g(y)$, which is the pointwise statement of (2), by substituting $x = g(y)$. Thus the use of (2) really just transports the standard proof to a function space setting, where the entire argument will proceed in a more direct manner. In particular, it seems to me that (1) and (2) are less magical than (3), which appears to be a very clever trick.

Two versions of the Inverse Function Theorem will be given here. The first is Theorem 1, in which the derivative of $f$ is required to satisfy a uniform Lipschitz condition. Although this is somewhat stronger than the usual requirement of continuous differentiability, it permits the nicest use of (1), because then $T$ will be a contraction in a certain $C^1$ norm. All but one of the properties of $g$ will then follow from (1). The exception is the important fact that $g(W)$ is open for a suitable open neighborhood $W$ of 0. This requires additional argument, but even here a use of (1) speeds things up. The conclusion of Theorem 1 is also stronger, making use of the Lipschitz condition to give an easy computation of the size of the neighborhood on which an inverse for $f$ exists.

The second version is Theorem 2, the standard result which admits all continuously differentiable functions. Since in this much generality $T$ is only a contraction in the uniform norm, the conclusion merely gives a continuous inverse $g$. To prove that $g$ is in fact continuously differentiable, one must then make the usual [2, p. 352] additional argument. These extra steps are independent of (1), making this proof not as neat as the first version and less in keeping with a "function space" philosophy.

I would like to acknowledge that in [4] there is a proof of Theorem 2 by successive approximations in which (2) is implicit.

The setting for both results is quite general: $X$ is a Banach space with norm $\|\cdot\|$ and $U$ an open subset of $X$. $B_a(r)$ denotes the ball $\{x \in X: \|x - a\| < r\}$ and $B_a(r)^-$ its closure. The Fréchet differential of $f$ at $a$ is $d_af$, and $\|G\|$ denotes the operator norm of a bounded linear map $G: X \to X$.

THEOREM 1. *If $f: U \to X$ is differentiable on $U$ and*

(a) *$d_af$ has a bounded inverse $G$ at a point $a \in U$ and*

(b) *there exists $C > 0$ such that $\|d_xf - d_yf\| \leqq C\|x - y\|$, all $x$, $y$ in $U$,*

*then for each $r > 0$ such that $r \leqq 1/(6C\|G\|^2)$ and $B_a(2r\|G\|)^- \subset U$, there exists a unique continuously differentiable map $g: B_{f(a)}(r) \to B_a(2r\|G\|)$ such that*

(c) *$V = g(B_{f(a)}(r))$ is an open neighborhood of $a$, and*

(d) *$f \circ g$ = identity on $B_{f(a)}(r)$.*

The hypotheses will of course be satisfied by any twice continuously differentiable function—the constant $C$ in (b) will be obtained from the Mean Value Theorem and a local bound on the second derivatives.

The proof begins with the usual preliminary reductions to the case $a = 0 = f(0)$ and $d_0f = G = I$. Thus replacing $f$ by its composition with suitable translations will achieve the first requirement and leave (a) and (b) unchanged. Therefore $a = 0 = f(0)$ may be assumed. Moreover, under the

hypotheses $f \circ G$ is defined in a neighborhood of $B_0(2r)^-$, and by the chain rule $d_x(f \circ G) = d_{G(x)}f \circ d_xG = d_{G(x)}f \circ G$ for all $x$. Therefore $f \circ G$ satisfies (a) with $d_0(f \circ G) = I$, and (b) with $C$ replaced by $C\|G\|^2$, for

$$\|d_x(f \circ G) - d_y(f \circ G)\| = \|(d_{G(x)}f - d_{G(y)}f) \circ G\| \leqq \|G\| \, \|d_{G(x)}f - d_{G(y)}f\|$$
$$\leqq C\|G\| \, \|G(x) - G(y)\| \leqq C\|G\|^2 \|x - y\|.$$

Hence it may also be assumed that $d_0f = G = I$.

Now (b) is also satisfied by $p$, and since $d_0p = 0$ it follows that $\|d_xp\| \leqq 2rC \leqq 1/3$, $x \in B_0(2r)^-$. This and the Mean Value Theorem yield the familiar Lipschitz estimate

$$\|p(x) - p(y)\| \leqq (1/3)\|x - y\|, x, y \in B_0(2r)^-. \tag{4}$$

Let $BC^1$ be the space of continuously differentiable functions $g: B_0(r) \to X$ such that both $\|g\|_\infty = \sup\{\|g(x)\|: x \in B_0(r)\}$ and $\|dg\|_\infty = \sup\{\|d_xg\|: x \in B_0(r)\}$ are finite. It is standard that this is a Banach space under any of the (equivalent) $C^1$ norms $\|g\| = K_1\|g\|_\infty + K_2\|dg\|_\infty$, where $K_1$ and $K_2$ are both $> 0$. These constants will be chosen so that $T$ is a contraction on the space

$$Y = \{g \in BC^1: \|g\|_\infty \leqq 2r, \|dg\|_\infty \leqq 2, \text{ and } g(0) = 0\},$$

which is a closed and hence complete subspace of $BC^1$ (the functions $g \to \|g\|_\infty$ and $g \to \|dg\|_\infty$ are continuous on $BC^1$).

Assume that $g \in Y$. It is clear that $g(B_0(r)) \subset B_0(2r)^-$ and also that $p(B_0(2r)^-) \subset B_0((2/3)r)^-$ because of (4) and $p(0) = 0$. Therefore

$$\|Tg\|_\infty \leqq (1 + \tfrac{2}{3})r < 2r, \text{ if } g \in Y. \tag{5}$$

Moreover,

$$\|d_xTg\| = \|I - d_{g(x)}p \circ d_xg\| \leqq 1 + \|d_{g(x)}p\| \, \|d_xg\| \leqq 1 + (\tfrac{1}{3}) \cdot 2 = \tfrac{5}{3} < 2.$$

Since $Tg(0) = 0$ it follows that (2) defines a map $T: Y \to Y$ for any choice of $K_1$ and $K_2$.

The next step is to estimate $\|Tg - Th\| = K_1\|Tg - Th\|_\infty + K_2\|dTg - dTh\|_\infty$ for $g$ and $h$ in $Y$. First,

$$\|Tg(x) - Th(x)\| = \|p(g(x)) - p(h(x))\| \leqq (\tfrac{1}{3})\|g - h\|_\infty,$$

by (4). Second,

$$\begin{aligned}
&\|d_xTg - d_xTh\| = \|d_{g(x)}p \circ d_xg - d_{h(x)}p \circ d_xh\| \\
&\leqq \|d_{g(x)}p \circ d_xg - d_{h(x)}p \circ d_xg\| + \|d_{h(x)}p \circ d_xg - d_{h(x)}p \circ d_xh\| \\
&\leqq \|d_{g(x)}p - d_{h(x)}p\| \, \|d_xg\| + \|d_{h(x)}p\| \, \|d_xg - d_xh\| \\
&\leqq 2C\|g - h\|_\infty + (\tfrac{1}{3})\|dg - dh\|_\infty.
\end{aligned} \tag{6}$$

Thus

$$\|Tg - Th\| \leqq (K_1(\tfrac{1}{3}) + K_2 2C)\|g - h\|_\infty + K_2(\tfrac{1}{3})\|dg - dh\|_\infty \leqq (\tfrac{5}{6})\|g - h\|$$

if $K_1 = 1$ and $K_2 = 1/(4C)$, which shows that the choice of norm $\|g\| = \|g\|_\infty + 1/(4C)\|dg\|_\infty$ makes $T$ a contraction. It therefore has a fixed point $g \in Y$ which by (1) verifies (d).

It remains to verify (c). Note that (1) and (5) show that $V = g(B_0(r)) = Tg(B_0(r)) \subset B_0(2r)$ (mere membership of $g$ in $Y$ only yields $V \subset B_0(2r)^-$). Moreover, $f$ is one-to-one on $B_0(2r)^-$, by the usual estimate based on (4):

$$\|x - y\| - \|f(x) - f(y)\| \leqq \|p(x) - p(y)\| \leqq (\tfrac{1}{3})\|x - y\|,$$

or $\| f(x) - f(y) \| \geqq (\frac{2}{3}) \| x - y \|$ for $x, y$ in $B_0(2r)^-$. These facts and $f(V) = B_0(r)$ imply that $V = B_0(2r) \cap f^{-1}(B_0(r))$, which is clearly open. This completes the proof.

It seems to be true that $g$ and $f$ will always share the same smoothness properties. This principle holds in Theorem 1, which can be made more symmetric by adding to its conclusion

(e) $$\| d_x g - d_y g \| \leqq 6C \| G \| \, \| x - y \|, \text{ all } x \text{ and } y \text{ in } B_{f(a)}(r).$$

This can be proved nicely by differentiating (1) and using estimates similar to those in (6). It is attractive to consider the possibility of a more symmetric proof of Theorem 1 using a Lipschitz norm which would fully utilize the hypothesis, such as $\| g \| = K_1 \| g \|_\infty + K_2 \| dg \|_L$, where $\| dg \|_L = \sup \{ \| d_x g - d_y g \| / \| x - y \| : x \neq y \text{ in } B_0(r) \}$. One would then get (e) immediately. Unfortunately, $T$ does not appear to be a contraction in this type of norm, at least not by a straightforward argument.

However, the method used to prove Theorem 1 can be modified for a function with higher-order smoothness to yield greater smoothness for $g$. For example, by choice of appropriate weights for a norm on the space of bounded twice continuously differentiable functions with bounded derivatives, $T$ will be a contraction on a suitable analog of $Y$, if $f$ has Lipschitz continuous second derivatives. But verification of this is rather tedious due to the complexity of expressions for the second derivatives of $p \circ g$, a difficulty which rapidly becomes worse as the degree of smoothness increases and still higher chain rules must be employed.

The same use of (2) can be incorporated into a proof of the Theorem in its usual form, where $f$ has only a continuous derivative. It only yields a continuous inverse, and no ready estimate of the domain of $g$.

THEOREM 2. *If $f: U \to X$ is continuously differentiable and at $a \in U$ $d_a f$ has a bounded inverse, then there exist open neighborhoods $V$ of $a$ and $W$ of $f(a)$ such that $f \mid V$ is a homeomorphism of $V$ onto $W$.*

The proof will construct open neighborhoods $V$ and $W$ and a continuous map $g: W \to V$ such that $g(W) = V$ and $f \circ g = \text{identity on } W$. The same normalizations can be assumed, and the continuity of $dp$ and $d_0 p = 0$ are used to choose an $r > 0$ such that $B_0(2r)^- \subset U$ and $\| d_x p \| \leqq \frac{1}{3}$ on $B_0(2r)^-$. This yields (4), and under these conditions it has already been shown in the proof of Theorem 1 that (2) defines a contraction on the complete subspace

$$Y_0 = \{ g \in BC : \| g \|_\infty \leqq 2r \text{ and } g(0) = 0 \}$$

of the Banach space $BC$ of bounded continuous functions $g: B_0(r) \to X$, normed by $\| g \|_\infty$. It follows that there exists a function $g \in Y_0$ satisfying (1), which must be a right inverse for $f$ on $W \equiv B_0(r)$. Arguing just as before, $V = g(W)$ is open. This completes the proof.

The successive approximations $g_{n+1} = Tg_n$ to $g$ often provide an efficient way to compute its Taylor series. Since $p$ vanishes at 0 to order $k \geqq 2$ ($p(x) = O(\| x \|^k)$), it can easily be shown that if $g_0 = I$ the Taylor series of $g_{n+1}$ and $g_n$ agree in at least their terms of degree $\leqq (n+1)(k-1)+1$. These terms thus define a formal power series which is an obvious candidate for the Taylor series of $g$. It will be, at least when $f$ admits a convergent power series expansion near 0, for then $g$ and the $g_n$ will also be analytic near 0, and the $g_n$ converge uniformly to $g$ on a small compact neighborhood of 0. In [3, p. 234] this computation is carried out for a simple example.

There is a connection between (1) and the Neumann series [3, Theorem 8.1, p. 224] or [2, Theorem 1, p. 342] for the inverse of a bounded linear map. The successive approximations to $g$ beginning with $I$ are $I$, $TI = I - p$, $T^2 I = I - p \circ (I - p), \cdots, T^n I = I - p \circ (I - p \circ (I - p \circ (\cdots)))$. In a (relatively unusual) case where composition is distributive over sums from the left, this complicated expression becomes $T^n I = \Sigma_{k=0}^{n} (-p)^k$, where $p^k = p \circ \cdots \circ p$ ($k$ times). A general "Neumann" series for $g$ results: $g = \Sigma_{n=0}^{\infty} (-p)^k$.

Keeping this in mind, differentiate (1) to obtain $d_x g = I - d_{g(x)} p \circ d_x g$, which shows that $d_x g$ is a fixed point of the induced operation $\bar{T}: G \to I - d_{g(x)} p \circ G$ on $B(X)$, the Banach space of bounded linear maps of $X$. Now if $x \in B_0(r)$, $\| d_{g(x)} p \| \leqq 1/3$, so $\bar{T}$ is a contraction in the operator norm. The

distributivity of composition for linear maps shows that the successive approximations $\bar{T}^n I$ are the partial sums for the Neumann series of its fixed point $d_x g$.

One can therefore think of this proof as a generalization to the non-linear case of a method which recovers the Neumann series as an application of the Contraction Mapping Principle. In [5, p. 145] there is a discussion of integral equations in which a Neumann series solution to the Volterra equation is obtained with successive approximations using an operator similar to (2).

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## THE MEASURE OF RECOGNIZABLE SETS OF REAL NUMBERS

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**Abstract.** The sets of real numbers which can be recognized by certain finite state automata are (Lebesgue) measurable. A necessary and sufficient condition for such sets to have measure 0 is that from each state of the recognizer a failure state can be reached.

**Introduction.** A common variety of problem in elementary measure theory is typified by the following: "Show that the set of real numbers in [0, 1] whose decimal expansions contain no (or a finite number of) 9's has measure 0." In this paper, the definition of a finite state recognizer is slightly modified for semi-infinite strings, and used to provide a very broad generalization of the preceding problem.

DEFINITIONS. By *measure*, Lebesgue measure is intended throughout.

A *complete finite state machine* $M$ consists of an ordered 5-tuple $M = (S, s_1, F, A, t)$, where $S$ is a finite set of states $S = \{s_i \mid 1 \leqq i \leqq m\}$, $s_1$ is the initial state, $F \subseteq S$ is a set of failure states, $A = \{a_j \mid 1 \leqq j \leqq n\}$ is a set of symbols called the alphabet, and $t$ is a transition function on $S \times A$ into $S$. A state $s'$ is *reached* from a state $s$ by a string $X = x_1 x_2 \cdots x_k$ of symbols in $A$ if there is a sequence of states $r_1, r_2, \cdots, r_k$ such that $r_1 = s$, $r_k = s'$, and $r_{p+1} = t(r_p, x_p)$ for $1 \leqq p < k$. It is explicitly assumed that every state is reachable from the initial state.

A semi-infinite string $X = x_1 x_2 \cdots$ of symbols in $A$ is *accepted* by $M$, if failure states are reached from $s_1$ by (at most) a finite number of initial substrings of $X$, and otherwise *rejected*. A real number $x \in I = [0, 1]$ is *recognized* by $M$ if its digit expansion in some base $b$ is accepted by $M$, where $A$ is the set of digits in the base $b$. The set $R(b, M)$ of such real numbers is called a *recognizable set*.

LEMMA 1. *The set, $Q$, of real numbers in $I$ whose digit expansion in the base $b$ contains no occurrence of the digit $d$ after the $N$-th digit position, has measure* 0.

*Proof.* There are just $b^N(b-1)^p$ choices for the first $N+p$ digits. Hence $Q$ is contained in a finite union of intervals with measure $((b-1)/b)^p$ for all $p$. Hence $Q$ has measure 0.

Precisely the same argument applies when the choice of proscribed digit depends upon the $N+p-1$ preceding digits.

COROLLARY 1. *The set, Q, of real numbers in I, whose digit expansion in base b contains the digit d a finite number of times, has measure* 0.

*Proof.* For each such number, there is an $N$ such that the digit $d$ does not occur beyond the $N$th digit position. So $Q$ is contained in the countable union of sets whose measure is 0 by Lemma 1.

LEMMA 2. *If a state s' can be reached from a state s, then it can be reached by a string of at most m digits.*

*Proof.* Suppose not, and let $X = x_1x_2 \cdots x_k$, $k > m$, be any shortest string from $s$ to $s'$. Then $(r_1, \cdots, r_k)$ must have some repeated state, say $r_p = r_q$. But then $x_1 \cdots x_p x_{q+1} \cdots x_k$ is a shorter string from $s$ to $s'$. Contradiction.

THEOREM 1. *All recognizable sets $R(b, M)$ are measurable, and have measure* 0 *if and only if some failure state is reachable from every state of M.*

*Proof.* Assume that some failure state can be reached from every state. Then it can be reached by a string of at most $m$ digits by Lemma 2. Partition the digit expansion of each number $x$ into $m$-tuples, thus creating a representation of $x$ in base $b^m$. For each such digit position, there is some following digit which creates a failure state. So by Corollary 1 and the comment following Lemma 1, $Q$ has measure 0.

Next, assume that $M$ has some states from which no failure state can be reached. Each such state can be reached from the initial state by a countable set of digit strings. $M$ accepts every real number with such an initial sequence. All real numbers with other initial sequences are contained in sets of measure 0. So $Q$ consists of a countable union of non-empty intervals plus a countable union of sets of measure 0. Hence $Q$ is measurable with non-zero measure.

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## ANOTHER CHARACTERIZATION OF SEMIPRIME IDEALS IN $C(X)$

W. T. WHITLEY

In this note, $C(X)$ will denote the ring of continuous real-valued functions on a topological space $X$. A semiprime ideal is an ideal which is an intersection of prime ideals. Unless otherwise noted, all other terms will be as defined in [1].

An interesting property of semiprime ideals in $C(X)$ is that given any member $f$ of the ideal, there exist elements $g, h$ in the ideal such that $f = gh$. (Let $g = f^{1/3}$, $h = f^{2/3}$.) The purpose of this note is to prove that semiprime ideals are the only ideals in $C(X)$ with this factorization property.

DEFINITION. An ideal $I$ in a commutative ring is an *FP-ideal* if for each $x \in I$, there exist $y$ and $z$ in $I$ such that $x = yz$.

LEMMA. *If f and g are in $C(X)$ with $|f| \leqq |g|^r$ for some real $r > 1$, then f is a multiple of g.* (This is Exercise 1D.3 of [1].)

THEOREM. *An ideal I in $C(X)$ is an FP-ideal if and only if I is semiprime.*

*Proof.* Let $I$ be an $FP$-ideal in $C(X)$. We first show that $I$ is convex. Let $0 \leqq f \leqq g$, with $g$ in $I$. Since $I$ is an $FP$-ideal there exist $h$, $k$ in $I$ with $0 \leqq f \leqq hk \leqq (h + k)^2$. (Note that the last inequality follows since $hk$ is nonnegative.) Then by the lemma, $f$ is a multiple of $h + k$, and hence is in $I$. Therefore, $I$ is convex.

Next we show that $I$ is absolutely convex. Suppose $|f| \leqq |g|$ with $g$ in $I$. Again, $g = hk$ for some

$h, k$ in $I$. Thus, $0 \leqq |f| \leqq |hk| \leqq h^2 + k^2$. Then $|f| \in I$ by convexity, so $f \in I$. Therefore, $I$ is absolutely convex.

Finally, to show that $I$ is semiprime, we show that $f^2 \in I$ implies $f \in I$. Now $f^2 \in I$ implies $f^2 = gh$ for some $g, h \in I$. By absolute convexity, $|g|$ and $|h|$ are in $I$. Also, $|f| = |g|^{1/2}|h|^{1/2} \leqq |g| + |h|$. Then by absolute convexity, $f \in I$. Hence, $I$ is semiprime.

We note that neither implication in the theorem is true for commutative rings in general.

*Example* 1. Let $R$ be the ring of integers modulo 12. Let $I$ be the ideal in $R$ generated by 4. Thus, $I = \{0, 4, 8\}$. Then $I$ is an $FP$-ideal, but $I$ is not semiprime since $2^2 \in I$ but $2 \notin I$.

*Example* 2. Let $R$ be the ring of integers, $I$ the ideal generated by 2. Then $I$ is prime, and hence semiprime, but 2 cannot be expressed as a product of members of $I$. Hence, $I$ is semiprime, but $I$ is not an $FP$-ideal

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## A RELLICH COMPACTNESS THEOREM FOR SETS OF FINITE VOLUME

A. V. LAIR

Let $H_s(R^n)$ be the completion of $C_0^\infty(R^n)$ with respect to the norm $\|U\|_s = \{\int (1+|\xi|^2)^s |\hat{U}(\xi)|^2 d\xi\}^{1/2}$ where $\hat{U}$ is the Fourier transform of $U$ defined by $\hat{U}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi \cdot x} U(x) dx$. It is well known that any bounded subset of $H_s(R^n)$ consisting of functions with support in some compact set is relatively compact in $H_t(R^n)$ for $t < s$. In this note we give an elementary proof that for $s \geqq 0$ the result remains true if the support simply has finite volume. We also give an example to show that the finite volume condition is not sufficient when $s < 0$. Slightly stronger results for $s \geqq 0$ are given in [1] and [2] using more complicated methods.

THEOREM. *Given a number $s \geqq 0$, let $S$ be a bounded subset of $H_s(R^n)$ consisting of functions with support in some subset $D$ of $R^n$ with finite volume. Then $S$ is relatively compact in $H_t(R^n)$ for each $t < s$.*

*Proof.* Let $\{U_k\}$ be a sequence of functions in $S$. We wish to show first that the sequence $\{\hat{U}_k\}$ is equicontinuous. Using the Schwarz Inequality and the identity $|e^{ix \cdot h} - 1|^2 = 2(1 - \cos x \cdot h)$, we obtain the inequality

$$|\hat{U}_k(\xi) - \hat{U}_k(\xi + h)| \leqq \sqrt{2}(2\pi)^{-n/2} \|U_k\|_0 \left\{\int_D (1 - \cos x \cdot h) dx\right\}^{1/2}.$$

Since $s \geqq 0$ and there is a constant $M'$ such that $\|U_k\|_s \leqq M'$, we let $M = \sqrt{2}(2\pi)^{-n/2} M'$ and thus obtain

$$|\hat{U}_k(\xi) - \hat{U}_k(\xi + h)| \leqq M\left\{\int_D (1 - \cos x \cdot h) dx\right\}^{1/2}. \tag{1}$$

Now if $\varepsilon > 0$ is given, we may choose disjoint sets $D_1$ and $D_2$ such that (i) $D_1$ is bounded, (ii) $\operatorname{vol} D_2 < \varepsilon^2/4M^2$, and (iii) $D_1 \cup D_2 = D$. Then $\delta > 0$ may be chosen so that

$$\int_{D_1} (1 - \cos x \cdot h) dx \leqq \sup_{x \in D_1} (1 - \cos x \cdot h) \operatorname{vol} D_1 < \varepsilon^2/2M^2 \tag{2}$$

for $|h|<\delta$. We also have

$$\int_{D_2}(1-\cos x\cdot h)dx \leqq 2(\operatorname{vol} D_2)<\varepsilon^2/2M^2. \tag{3}$$

Combining (1), (2) and (3) we have that the sequence $\{\hat{U}_k\}$ is equicontinuous.

Since $|\hat{U}_k(\xi)|\leqq(\operatorname{vol} D)^{1/2}(M/\sqrt{2})<\infty$, the Arzela-Ascoli Theorem implies the existence of a subsequence $\{\hat{U}_{k_j}\}\subset\{\hat{U}_k\}$ such that $\{\hat{U}_{k_j}(\xi)\}$ converges pointwise uniformly on compact subsets of $R^n$.

Now for any $R>0$ and $t<s$,

$$\|U_{k_j}-U_{k_m}\|_t^2\leqq\left\{\sup_{|\xi|\leqq R}|\hat{U}_{k_j}(\xi)-\hat{U}_{k_m}(\xi)|^2\right\}\int_{|\xi|\leqq R}(1+|\xi|^2)^t d\xi$$
$$+(1+R^2)^{t-s}\int_{|\xi|>R}(1+|\xi|^2)^s|\hat{U}_{k_j}(\xi)-\hat{U}_{k_m}(\xi)|^2 d\xi.$$

Then if $\varepsilon>0$ is given, we can make the second term on the right less than $\varepsilon^2/2$ by taking $R$ large. For a fixed such $R$, the first term on the right is less than $\varepsilon^2/2$ for large $j$ and $m$. Consequently $\|U_{k_j}-U_{k_m}\|_t<\varepsilon$ for large $j$ and $m$, and the proof is complete.

We now give an example where the result fails for $s<0$. Let $n=1$, $t=-1$, and $-1<s<-\frac{1}{2}$. Consider $S=\{U_k\}_{k=1}^{\infty}$ where $U_k(x)=\delta(x-k)$, $\delta$ being the well-known Dirac delta-function. We have

$$\hat{U}_k(\xi)=\frac{1}{\sqrt{2\pi}}e^{-ik\xi}\qquad\text{and}\qquad\|U_k\|_s^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+\xi^2)^s d\xi\equiv M(s)<\infty.$$

The support of each function in $S$ is contained in $Z^+$, the set of positive integers, which has finite measure. However, $S$ is not relatively compact in $H_{-1}(R)$. To see this we observe that

$$\|U_n-U_m\|_{-1}^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+\xi^2)^{-1}|e^{-in\xi}-e^{-im\xi}|^2 d\xi$$
$$=\frac{1}{\pi}\int_{-\infty}^{\infty}(1+\xi^2)^{-1}[1-\cos(m-n)\xi]d\xi$$
$$=1-e^{-|m-n|}>\frac{1}{2}\quad\text{for}\quad m\neq n.$$

Thus, $\{U_k\}_{k=1}^{\infty}$ has no convergent subsequence in $H_{-1}(R)$.

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## A THEOREM ON DIFFERENTIAL OPERATORS

L. Carlitz

**1.** Let $\Omega$ denote a polynomial differential operator, that is, a sum of a finite number of monomials of the form

$$A_0(x)DA_1(x)D\cdots A_{n-1}(x)DA_n(x), \tag{1.1}$$

where the $A_j(x)$ are arbitrary functions of $x$. Frequently one encounters operational equations of the

type $\Omega_1 = \Omega_2$. For example, there is the well-known formula of Boole:

$$x^n D^n = xD(xD-1)\cdots(xD-n+1). \tag{1.2}$$

Such formulas are usually proved by induction. In some instances they can be proved very rapidly by means of the following theorem.

THEOREM 1. *Let $\Omega_1, \Omega_2$ denote two polynomial differential operators. Then $\Omega_1 = \Omega_2$ if and only if*

$$\Omega_1 x^k = \Omega_2 x^k \qquad (k = 0, 1, 2, \ldots). \tag{1.3}$$

*Proof.* Since $DA(x) = A(x)D + A'(x)$, it is clear that any polynomial differential operator can be written in the form

$$\Omega = \sum_{j=0}^{m} A_j(x) D^j.$$

By means of (1.2), this may be replaced by

$$\Omega = \sum_{j=0}^{m} B_j(x)(xD)^j. \tag{1.4}$$

Thus to prove the theorem, it suffices to show that $\Omega = 0$ if and only if

$$\Omega x^k = 0 \qquad (k = 0, 1, 2, \ldots). \tag{1.5}$$

The necessity is obvious. As for the sufficiency, it follows from (1.4) and (1.5) that

$$\sum_{j=0}^{m} B_j(x) k^j = 0 \qquad (k = 0, 1, 2, \ldots). \tag{1.6}$$

Hence

$$f(y) = \sum_{j=0}^{m} B_j(x) y^j$$

is a polynomial in $y$ that vanishes for $y = 0, 1, 2, \ldots$ . It follows that $B_j(x) = 0$ $(j = 0, 1, 2, \ldots)$.

This evidently proves the theorem. Note that it suffices to assume that (1.3) holds for finitely many values of $k$.

**2.** As an example, consider the operational formula [**1,** (1.4)]

$$(DxD)^n = \sum_{s=0}^{n} \binom{n}{s} \frac{n!}{s!} x^s D^{n+s}. \tag{2.1}$$

Since $(DxD)x^k = k^2 x^{k-1}$, it follows from the theorem that (2.1) is equivalent to

$$\left(\frac{k!}{(k-n)!}\right)^2 = \sum_{s=0}^{n} \binom{n}{s} \frac{n!}{s!} \frac{k!}{(k-n-s)!}. \tag{2.2}$$

The right hand side of (2.2) is equal to

$$\frac{n!\,k!}{(k-n)!} \sum_{s=0}^{n} \binom{n}{s} \binom{k-n}{s} = \frac{n!\,k!}{(k-n)!} \binom{k}{n} = \frac{k!}{(k-n)!} \frac{k!}{(k-n)!},$$

by Vandermonde's theorem.

Another example is furnished by [**1,** (6.8)]

$$(D(xD)^m)^n = (xD+1)^m (xD+2)^m \cdots (xD+n)^m D^n. \tag{2.3}$$

By the theorem this is equivalent to

$$\left(\frac{k!}{(k-n)!}\right)^{m+1} = \frac{k!}{(k-n)!}\prod_{j=1}^{n}(k-n+j)^m,$$

the truth of which is obvious.

This example illustrates the possibility of constructing operational formulas from simple algebraic identities. As another example take

$$k(k-1)^2(k-2)^3\cdots(k-n+1)^n = k^{(n)}(k-1)^{(n-1)}\cdots(k-n+1)^{(1)},$$

where $k^{(n)} = k(k-1)\cdots(k-n+1)$. We get

$$D\cdots x^{n-2}D^{n-1}x^{n-1}D^n = (Dx\cdots xD)\cdots(DxDxD)(DxD)D; \tag{2.4}$$

it is understood that the products are read from right to left.

**3.** M. S. Klamkin has proposed as a problem [2] the identity

$$x^{n-1}\int\frac{dx}{x^2}\int\frac{dx}{x^2}\cdots\int\frac{dx}{x^2}\int x^{n-1}F(x)dx = \int\cdots\int F(x)(dx)^n + P(x), \tag{3.1}$$

where there are $n$ integrals on each side and $P(x)$ is a polynomial of degree $n-1$. (Klamkin states the result for $n$ even.) By taking definite integrals, we may ignore the $P(x)$.

Denote the integral operators on each side of (3.1) by $\Omega_1$, $\Omega_2$, respectively. It will evidently suffice to prove

$$\Omega_1 D^n = \Omega_2 D^n. \tag{3.2}$$

However, it is evident that each of the operators in (3.1) is of the type covered by the theorem so that we need only verify $\Omega_1 D^n x^k = \Omega_2 D^n x^k$ $(k = 0, 1, 2, \ldots)$, or equivalently

$$\Omega_1 x^k = \Omega_2 x^k \qquad (k = 0, 1, 2, \ldots). \tag{3.3}$$

It is easily seen that each side of (3.3) is equal to

$$\frac{1}{(k+1)(k+2)\cdots(k+n)}.$$

Thus it is evident how to construct other identities like (3.1). For example we cite

$$\int xdx\cdots\int xdx\int F(x)dx = x^{2n-2}\int\frac{dx}{x^3}\cdots\int\frac{dx}{x^3}\int x^{2n-2}F(x)dx + P(x), \tag{3.4}$$

where there are $n$ integrals on each side and $P(x)$ is of degree $n-1$.

**4.** One can also state a theorem like Theorem 1 for difference operators. A polynomial difference operator is defined as the sum of a finite number of monomials

$$A_0(x)\Delta A_1(x)\Delta\cdots A_{n-1}(x)\Delta A_n(x),$$

where the $A_j(x)$ are arbitrary functions of $x$ and $\Delta f(x) = f(x+1) - f(x)$.

Also put $x^{(n)} = x(x-1)\cdots(x-n+1)$, $x^{(0)} = 1$. We now state

THEOREM 2. *Let $\Omega_1, \Omega_2$ denote two polynomial difference operators. Then $\Omega_1 = \Omega_2$ if and only if*

$$\Omega_1 x^{(k)} = \Omega_2 x^{(k)} \qquad (k = 0, 1, 2, \ldots). \tag{4.1}$$

As before, it suffices to assume that (4.1) holds for finitely many values of $k$.

As pointed out by a referee, Theorems 1 and 2 are not new (see, for example, [3, Ch. 5]). More generally, if

(*) $$L(D)F_i(x)=0 \text{ for } i=1,2,\cdots,n,$$

where $L(D)$ is of order $n$ and the $F_i(x)$ are linearly independent, then all solutions of (*) are of the form $\Sigma a_iF_i(x)$. Hence, if there is another linearly independent solution, $L(D)\equiv 0$.

Supported in part by NSF GP-37924.

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## THE TWO-STATE MARKOV PROCESS AND ADDITIONAL EVENTS

L. RÅDE

The additional event method (or method of collective marks) is an effective tool for deriving transforms in applications of probability theory. The method was originally devised by van Dantzig and has been applied to queuing problems by Runnenburg [2]. The method is also described in [3]. In this note we shall apply the method to a problem dealing with the two-state Markov process with states and transition intensities according to figure 1.

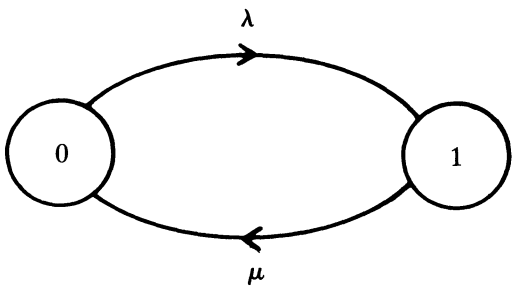



FIG. 1

We assume that the process starts in state 0 at time epoch $t=0$. Let $\xi$ be the total amount of time the process spends in state 0 during the time interval $[0,t]$ and let $\eta$ be the number of transitions from 0 to 1 during the same period of time. We shall derive the probability distribution of $(\xi,\eta)$ through the transform

$$\hat{F}(s,u)=E[e^{-s\xi}u^{\eta}].$$

The essential idea of the additional event method is to interpret a transform like $\hat{F}(s,u)$ as a probability which can be found directly by a simple probabilistic argument. In order to do so in this case we introduce possibilities for additional events (a.e.) as follows.

When the process is in state 0 additional events are generated by an independent Poisson process with intensity $s$, and in connection with each transition from 0 to 1 another kind of additional event is generated independently with probability $1-u$. If $F(v,n)=P(\xi\leqq v,\eta=n)$, then

$$\hat{F}(s,u)=\int_0^t \sum_{n=0}^{\infty} u^n e^{-sv}dF_v(v,n).$$

But this implies that $\hat{F}(s, u)$ is the probability that no a.e. of any kind occurs in the time interval $[0, t]$. Thus

$$\hat{F}(s, u) = P(\text{no a.e. in } [0, t]).$$

The introduction of possibilities for additional events means that we now have a three-state Markov process with transition intensities according to figure 2.

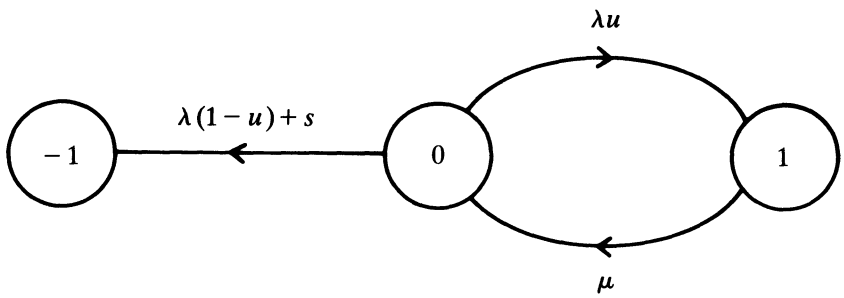



FIG. 2.

Here state 0 is the same as before but 1 indicates the previous state 1 with the condition that no a.e. occurred when the process entered the state. Furthermore, state $-1$ indicates the state that an a.e. occurs. This state is assumed to be absorbing. Let $P_0(t)$, $P_1(t)$ and $P_{-1}(t)$ be the probabilities of these states at time epoch $t$. Then

$$\hat{F}(s, u) = P_0(t) + P_1(t),$$

because $P_0(t) + P_1(t)$ is the probability that the process has not entered the state $-1$ at time epoch $t$, that is, that no a.e. has occurred at time epoch $t$.

The Kolmogorov forward differential equations of this process are

$$P'_{-1}(t) = (\lambda(1-u) + s)P_0(t)$$

$$P'_0(t) = -(\lambda + s)P_0(t) + \mu P_1(t)$$

$$P'_1(t) = -\mu P_1(t) + \lambda u P_0(t).$$

The solution for $P_0(t)$ and $P_1(t)$ can be written

$$P_0(t) = \frac{(a-\mu)e^{-at} + (\mu - b)e^{-bt}}{a-b},$$

$$P_1(t) = \frac{\lambda u(e^{-bt} - e^{-at})}{a-b},$$

where $-a$ and $-b$ are the roots of the equation $x^2 + x(\lambda + \mu + s) + u\lambda(1-u) + \mu s = 0$. By series expansion we obtain with $\rho = \lambda + \mu$, $v = 1 - u$

$$a = \rho + (\lambda s)/\rho + (u\lambda v)/\rho + (u\lambda(\mu - \lambda)sv)/\rho^3 + \cdots$$

$$b = (\mu s)/\rho - (\mu\lambda v)/\rho + (\mu\lambda(\lambda - \mu)sv)/\rho^3 + \cdots.$$

This can be used to expand $\hat{F}(s, u)$ in Taylor series, $\hat{F}(s, u) = 1 + As + Bs + Csv + \cdots$. From $E[\xi] = -A$, $E[\eta] = B$ and $E[\xi\eta] = -C$ we find

$$E[\xi] = \frac{\mu t}{\mu+\lambda} + \frac{\lambda}{(\mu+\lambda)^2}(1 - e^{-(\mu+\lambda)t})$$

$$E[\eta] = \frac{\mu\lambda t}{\mu+\lambda} + \frac{\lambda^2}{(\mu+\lambda)^2}(1 - e^{-(\mu+\lambda)t})$$

$$\mathrm{Cov}[\xi, \eta] = \frac{\mu\lambda(\mu - 4\lambda)}{(\mu+\lambda)^4} + \frac{\mu\lambda(\lambda-\mu)t}{(\mu+\lambda)^3} + \frac{\lambda(4\mu\lambda - \mu^2 + \lambda^2)}{(\mu+\lambda)^4} e^{-(\mu+\lambda)t}$$

$$+ \frac{\lambda^2(3\mu - \lambda)}{(\mu+\lambda)^3} te^{-(\mu+\lambda)t} - \frac{\lambda^3}{(\mu+\lambda)^4} e^{-2(\mu+\lambda)t}.$$

Of these formulas, the one for $E[\xi]$ can also be obtained by integration of the transition probability $P_{01}(t)$ which is derived in most textbooks on stochastic processes. In Bhat [1] the formula for $E[\xi]$ is derived by a limit process from the corresponding Markov chain. The distribution of $\xi$ is given by Takács in [4].

It follows from the expression for the covariance $\mathrm{Cov}[\xi, \eta]$ that

$$\mathrm{Cov}[\xi, \eta] \approx \frac{\mu\lambda(\lambda-\mu)}{(\mu+\lambda)^3} t \text{ for large } t$$

and $\mathrm{Cov}[\xi, \eta] \approx -\frac{1}{2}\lambda t^2$ for small $t$. Thus $\xi$ and $\eta$ are positively correlated for large $t$ if $\lambda > \mu$ and negatively correlated if $\lambda < \mu$.

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## RESEARCH PROBLEMS

EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge, CB2 1SB England.)*

### HOW EASY IS IT TO GENERATE A GROUP?

J. L. BRENNER AND JAMES WIEGOLD

We start with the observation that any nontrivial element of a group of prime order will generate that group. Call a set $S$ of nontrivial elements $\{x_1, x_2, \cdots\}$ of a group **independent** if no one of them is contained in the subgroup generated by all the others. The reader may wish to pause long enough to

characterize the category $C_r$ of groups $G$ in which every set $S$ of $r$ independent elements generates the group.

S. Picard [5, 6] studied generation of the symmetric group $\mathcal{S}_n$ by pairs of its elements; every nontrivial element in $\mathcal{S}_n$ is a member of a generating pair. Binder [2, 3] showed that given any pair $x_1, x_2$ of nontrivial elements in $\mathcal{S}_n$, there is a third element $y$ such that $\langle x_1, y\rangle = \langle x_2, y\rangle = \mathcal{S}_n$.

DEFINITION 1. *Let $\Gamma_r$ be the category of groups $G$ of* ***spread*** *$r$, i.e., with the property: for every set $S$ of $r$ nontrivial elements of $G$, $|S| = r < |G|$, there is an element $y$ such that $\{\forall x \mid x \in S\}(\langle x, y\rangle = G)$, i.e., $y$ mates with each of the $r$ elements of $S$ to form a generating pair.*

Binder proved that $\mathcal{S}_n \in \Gamma_2\backslash\Gamma_3$ if $n$ is even, and $\mathcal{S}_n \in \Gamma_3\backslash\Gamma_4$ if $n$ is odd, $n > 4$. Concerning the alternating groups, we have proved [4] that $\mathcal{A}_n \in \Gamma_4\backslash\Gamma_5$ if $n$ is even, $n \geqq 8$. By contrast, the spread of $\mathcal{A}_{19}$ exceeds $6 \cdot 10^9$. Many important groups, including essentially all known finite nonabelian simple groups, are two-generator groups [7]. Thus they may all lie in $\Gamma_1$. Albert and Thompson [1] showed that in fact $PSL(n, q) \in \Gamma_1$ with a few early exceptions. We have shown that every nontrivial element in $PSL(2, q)$, $q \neq 9$, $q > 3$, is a member of a two-element generating pair *in which one member of the pair has period* 2. The group $PSL(3, 2)$ has the same property. For $n = 3$, $q$ odd; or for $n = 3$, $q > 4$; or for $n > 4$ ($q$ arbitrary), $PSL(n, q)$ fails to have the property. The question concerning the group $PSL(3, 4)$, of order 20,160, has been settled affirmatively (unpublished).

THEOREM 2. *If the nonabelian group $G$ lies in $\Gamma_1$, then $G$ is monolithic.*

*Proof.* A group is **monolithic** if the intersection $K$ (the monolith) of all nontrivial normal subgroups is $\neq 1$. Let $H$, $1 \neq H \lhd G$ be any normal subgroup of $G$, and let $1 \neq h \in H$. By hypothesis, $y$ exists with $\langle h, y\rangle = G$. Thus $G/H$ is cyclic, and a typical element of $G$ is $y^\lambda h$, $h \in H$. The commutator of any two such elements lies in $H$, so that $H$ includes the derived group. If $G$ is nonabelian, $K$ clearly is this derived group $G'$ (generated by all the commutators in $G$).

COROLLARY 3. *If $G \in \Gamma_1$, $G/G'$ is cyclic.*

DEFINITION 4. *The category $\Delta_r$ of groups of* ***restricted spread*** *$r$ is like $\Gamma_r$, but with "noncentral" in place of "nontrivial."*

THEOREM 5. *If $G \in \Delta_r$, then $L = G/Z(G) \in \Gamma_s$, $s = \min(r, |L| - 1)$.* Here $Z(G)$ is the center of $G$.

Now we make some more intricate definitions.

DEFINITION 6. *$\Gamma_{t,u,v}$ is the category of groups $G$ with the following property. To every $t$ sets $\mathfrak{S}_i$ of $u$ independent elements each, $\mathfrak{S}_i = \{x_{i1}, x_{i2}, \cdots, x_{iu}\}$, $i = 1, 2, \cdots, t$, there corresponds a fixed set $\{y_1, \cdots, y_v\}$ of $v$ elements each such that, for $i = 1, 2, \cdots, t$, $\langle x_{i1}, x_{i2}, \cdots, x_{iu}, y_1, \cdots, y_v\rangle = G$.*

Given *any* finite group $G$, and any reasonable integers $t, u$, there is an integer $v$ such that $G$ lies in $\Gamma_{t,u,v}$. This gives a doubly infinite class of classification problems. $C_r$ is $\Gamma_{1,r,0}$.

Further classification problems suggest themselves, if some or all of the $x_{ij}$ and $y_i$ are restricted in some way, for instance, in period. One can also impose relative restrictions, for example by requiring that the periods be relatively prime, or all divisible by prescribed integers.

It is easy to see that, if $G_1, G_2$ lie in $\Gamma_2$, then $G_1 \times G_2$ lies in $\Gamma_1$. Under what hypotheses can stronger statements be made? If $G$ lies in $\Gamma_r$, then so does its quotient group $W$ (if $|W| < r$).

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## EXISTENCE OF NODAL LINES FOR SOLUTIONS OF HYPERBOLIC EQUATIONS

GORDON PAGAN

A classical question in the theory of ordinary differential equations concerns the oscillatory behavior of solutions of $u''+g(x)u=0$. Specifically one is interested in conditions on $g(x)$ for $a \leqq x < \infty$ which assures that every solution of $u''+g(x)u=0$ has an infinite number of zeros $x_1 < x_2 < \cdots$, with $x_n \to \infty$.

Analogous questions can be considered for partial differential equations when the problem is appropriately generalized. The present research problem deals with oscillation properties of solutions $u(x, y)$ of the characteristic initial value problem

$$\begin{aligned} &u_{xy}(x, y)+g(x, y)u(x, y)=0 \qquad (x \geqq 0,\ y \geqq 0) \\ &u(x, 0)=p(x);\ u(0, y)=q(y). \end{aligned} \tag{1}$$

Here, $g(x, y)$, $p(x)$ and $q(y)$ are given real-valued functions. Also, $g(x, y)$ is continuous, $p(x)$ and $q(y)$ are continuously differentiable, and $p(0)=q(0)$. The usual Picard existence theory [4] applies and establishes the existence of a unique solution $u(x, y)$ which, together with its derivatives $u_x$, $u_y$, and $u_{xy}$, is continuous in $x \geqq 0$, $y \geqq 0$. A **nodal line** for $u(x, y)$ is defined to be a curve in the $x$, $y$-plane implicitly defined by $u(x, y)=0$. We pose the following problem:

*Problem*: Under the conditions

$$\begin{aligned} &\text{(i)} \quad g(x, y)>0 && (x \geqq 0,\ y \geqq 0); \\ &\text{(ii)} \quad p(x)>0,\ q(y)>0 && (x \geqq 0,\ y \geqq 0); \\ &\text{(iii)} \quad p'(x) \leqq 0,\ q'(y) \leqq 0 && (x \geqq 0,\ y \geqq 0); \end{aligned}$$

does every solution $u(x, y)$ have an infinity of non-intersecting nodal lines $\Gamma_n$ each of which is asymptotic to the $x$- and $y$-axes?

The matter can be illustrated by the case in which $g(x, y)=xy$, $p(x)=1$, $q(y)=1$ [1, 2]. Here the solutions of (1) is given in terms of the Bessel function $J_0(z)$ by

$$u(x, y)=J_0(xy).$$

This solution has an infinity of nodal lines which are the rectangular hyperbolas $xy=j_n$, where $j_n$ is the $n$th zero of $J_0(z)$.

In attempting to generalize this situation for the general problem (1), I proved the following theorem in [3], using a Sturmian argument.

THEOREM. *Let* (ii) *and* (iii) *above hold and in place of* (i) *above let* (i') $g(x, y) \geqq K^2$, *where* $K$ *is a constant. Then, for every positive constant* $L$, $u(x, y)$ *changes sign in each region*

$$D_n(L)=\{(x, y) \mid x \geqq 0,\ y \geqq 0,\ Ln\pi \leqq y+L^2K^2x \leqq L(n+1)\pi\}.$$

As shown in [3], if we consider the cases $L \to 0$ and $L \to \infty$ in this theorem, we obtain the existence of a first nodal line $\Gamma_1$ which is asymptotic to the $x$- and $y$-axes. However, the arguments in [3] apply only to $\Gamma_1$ and do not allow us to predict the behavior of other nodal lines which may exist.

The relevance of condition (iii) is shown by the example $g(x, y) = k^2$, $p(x) = e^{kx}$, $q(y) = e^{-ky}$, $u(x, y) = e^{k(x-y)}$. We emphasize that (i), (ii) and (iii) are the *minimum* conditions under which it is reasonable to pose our problem. Any strengthening of these conditions which would give a positive result would also be of interest.

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## CLASSROOM NOTES

EDITED BY RICHARD A. BRUALDI

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### A GENERALIZATION OF BAUDET'S CONJECTURE (VAN DER WAERDEN'S THEOREM)

PETER G. ANDERSON

**Van der Waerden's Theorem.** Suppose that the integers, $\{0, 1, 2, \cdots\}$, are colored with $k$ colors, and that $m$ is any positive integer. Then there exists a monochromatic arithmetic progression of length $m$. Furthermore, such an arithmetic progression exists within a bounded subset of the integers whose bound depends only upon the numbers $k$ and $m$ and not upon the particular coloring used.

Several proofs of this theorem have been given; the original proof was by B. L. van der Waerden [4] who recently described the experience of its discovery in [5]. Proofs by A. Y. Khinchin [2] and R. L. Graham and B. L. Rothschild [1] are also available.

The purpose of this paper is to offer a new proof of a generalization of van der Waerden's theorem to $d$-dimensional space. This generalization is found in [3], where it is attributed to Dr. G. Grunwald.

Denote by *$d$-space* the set of $d$-tuples of non-negative integers; a *$k$-coloring* is any partition of $d$-space into $k$ classes. If $U$ and $V$ are subsets of $d$-space, let $U + V = \{u + v \mid u \in U \text{ and } v \in V\}$; if $v$ is a point in $d$-space, let $U + v = U + \{v\}$; and if $m$ is a positive integer, let $mU = \{mu \mid u \in U\}$. Denote by the *$n$-cube* $Q_n$ the subset of $d$-space consisting of all the points of $d$-space all of whose coordinates are less than $n$. We prove the following:

THEOREM. *Let $S$ be a finite subset of $d$-space. For any $k$-coloring of $d$-space, there is a positive integer $a$ and a point $v$ in $d$-space such that the set $aS + v$ — i.e., some translation of some dilation of $S$ — is monochromatic. Furthermore, the dilation factor $a$ and the coordinates of the translation point $v$ are bounded by a function that depends only upon the set $S$ and the number $k$ (and not upon the particular coloring used).*

Van der Waerden's theorem is obtained by setting $d = 1$ and $S = \{1, 2, \cdots, n\}$.

Throughout the discussion, $S$ will denote a finite subset of $d$-space, and $w$ will denote a point of $d$-space not contained in $S$; $S \cup w$ will, of course, stand for $S \cup \{w\}$. The theorem will be proved by induction on the cardinality of the set $S$. Let $A(S)$ be the following assertion about the set $S$.

$A(S)$: There is a positive integer $n = n(k)$ such that, for any $k$-coloring of the points of the cube $Q_n$, $Q_n$ contains a monochromatic subset of the form $aS + v$.

First note that $A(S)$ is true if $S$ is a one-point set, since one-point sets have to be monochromatic. This anchors the induction.

In order to prove that $A(S)$ implies $A(S \cup w)$ — which will complete the theorem by induction — we first introduce another statement, $C$.

$C(S, w, p)$: There is a positive integer $n_p = n_p(k)$ such that, for any $k$-coloring of $Q_{n_p}$, there are positive integers $a_0, \cdots, a_p$ and a point $v$ in $d$-space such that the sets

$$T_q = \sum_{i<q} a_i S + \sum_{i \geq q} a_i w + v, \qquad q = 0, \cdots, p,$$

are monochromatic subsets of $Q_{n_p}$.

LEMMA 1. *$A(S)$ implies $C(S, w, p)$ for all positive integers $p$.*

*Proof.* (By induction on $p$.)

*$A(S)$ implies $C(S, w, 1)$.* Choose $n_1$ so that whenever $aS + v$ is a subset of $Q_n$ then $a(S \cup w) + v$ is a subset of $Q_{n_1}$. If $Q_n$ contains a monochromatic subset $aS + v$, then $Q_{n_1}$ will contain the two monochromatic subsets $T_0 = aw + v$ (which is monochromatic since it is a one-point set) and $T_1 = aS + v$.

*$A(S)$ implies that $C(S, w, p)$ implies $C(S, w, p+1)$.* To prove this, we need the notion of $k^{n^d}$-coloring of $d$-space derived from a $k$-coloring of $d$-space; this works as follows. With $k$ colors, there are $k^{n^d}$ ways to color the cube $Q_n$ which contains $n^d$ points. Call each of the $k^{n^d}$ patterns a "color" for the derived coloring, and let the derived "color" of the point $v$ be the pattern on $v + Q_n$ in the $k$-coloring.

Let $k' = k^{n_p(k)^d}$ (where $n_p(k)$ is the integer specified in $C(S, w, p)$), and apply the hypothesis $C(S, w, 1)$ to the derived $k'$-coloring of $d$-space. Let $n' = n_1(k')$; then any $k'$-colored $Q_{n'}$ contains a subset of the form $a'(S \cup w) + v'$ with the portion $a'S + v'$ monochromatic in the $k'$-coloring. Since this $k'$-coloring was derived from a $k$-coloring, for any element $s$ in $S$, each of the cubes $Q_{n_p(k)} + a's + v'$ is colored in the same pattern of the $k$ colors.

Each of these $|S|$ cubes contains the $p+1$ monochromatic sets $T_q + a's + v'$, and the sets corresponding to the same suffix $q$ are of the same color. This provides us with monochromatic sets $a'S + T_q + v'$. The set $a'w + T_0 + v'$ contains just one point, so it is monochromatic. This concludes the verification of $C(S, w, p+1)$ and the proof of Lemma 1.

LEMMA 2. *If $C(S, w, p)$, for all positive integers $p$, then $A(S \cup w)$.*

*Proof.* In particular, $C(S, w, k)$ is true, where $k$ is the number of colors.

This guarantees $k+1$ monochromatic sets, $T_0, T_1, \cdots, T_k$, two of which must be the same color (Dirichlet's "box principle"). Suppose that $T_r$ and $T_q$ are the same color, with $r < q$.

$$T_r = \sum_{i<r} a_i S + \sum_{i=r+1}^{q} a_i w + \sum_{i=q}^{k} a_i w + v, \qquad T_q = \sum_{i<r} a_i S + \sum_{i=r+1}^{q} a_i S + \sum_{i=q}^{k} a_i w + v.$$

Let $s_0$ be any element in $S$; then

$$\left(\sum_{i<r} a_i\right) s_0 + \sum_{i=r}^{q-1} a_i (S \cup w) + \sum_{i=q}^{k} a_i w + v$$

is a monochromatic set of the desired form, concluding the proof of Lemma 2.

It follows from the two Lemmas, that $A(S)$ implies $A(S \cup w)$, concluding the proof of the main theorem.

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## CUTTING CERTAIN MINIMUM CORNERS

L. H. LANGE

**To my teacher, George Pólya, for his 89th birthday**

**A. A long (geometric minimum) story short.** An ancient geometer might simply present the following figure using only the word "Behold!"

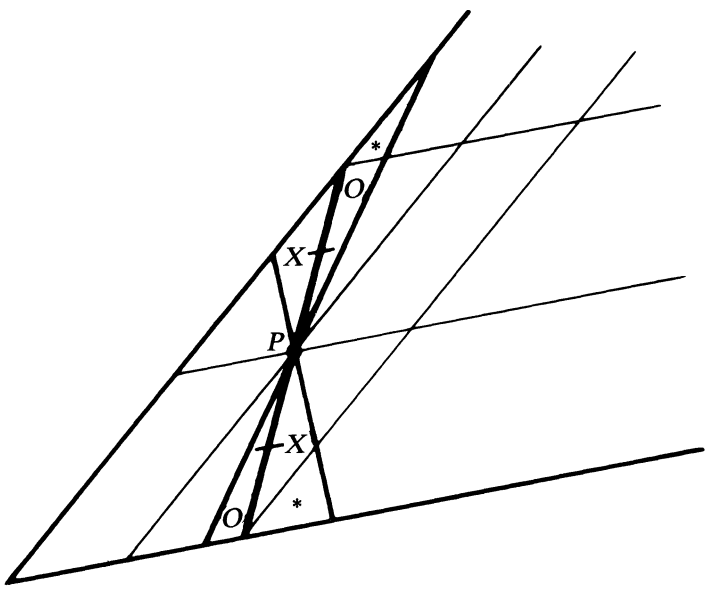



FIG. 1

Here is an explanatory note. If we wish to prove some theorems about polygons of minimum area circumscribed about a given convex set, as in [2], for example, we need a helping theorem which tells us *how to cut off a triangle of minimum area with a line passing through a given point P inside an angle.* What the figure tells us (*sans* calculus and *sans* any linear transformation theory) is that of all

segments through $P$ determining corner triangles, it is that segment for which $P$ is the *midpoint* which will yield the corner triangle of minimum area, for all other eligible triangles are seen to be bigger than that particular one. (The angle does not have to be acute.)

**B. Note from the prelude.** Years ago, after using "$f'(x) = 0$" to solve the related old calculus book problem where $P$ is a point $(a, b)$ in the first quadrant, a right angle, and then lingering over the geometric interpretation of the fact that the area of the minimum triangle in that first quadrant corner turns out to be $2ab$, I had noticed that I could solve that special old problem with a rather quick, direct thrust, *without* the tools of calculus, as indicated in Figure 2.

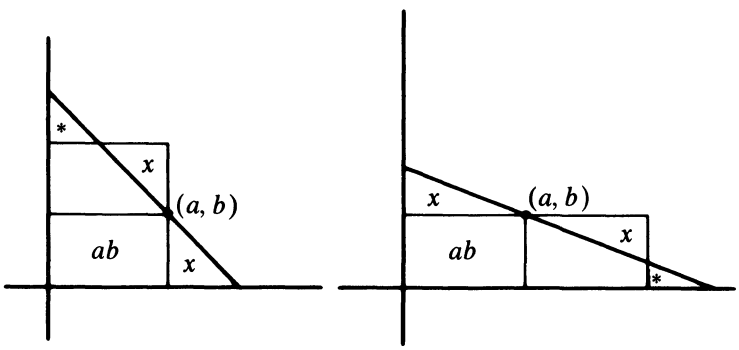



FIG. 2

Later, when I needed to deal with the non-right angle problem and brought to mind the familiar properties of affine transformations of the plane—e.g., "midpoints go into midpoints" and "ratios of areas are preserved" so that the property of being a minimum triangle in a collection is preserved—I knew I had the solution. It was still later that it occurred to me to make the delightfully *direct construction* of the proof as shown in Figure 1. (For a convenient listing of some of the planar affine properties involved here, see [2, p. 61]. The easy three-dimensional analogues are needed when we necessarily exploit affinity in our next section.)

**C. Some notes from the postlude.** It is natural to look for related problems and make guesses of possible theorems. (One of Professor Pólya's "synthetic proverbs" from [8] leaps to mind: "Look around when you have got your first mushroom or made your first discovery: they grow in clusters." See also [10].) The main one I settle on here is this particular three-dimensional generalization: *Through a fixed point $P$ in a given trihedral angle, identify the plane which will cut off a corner tetrahedron of minimum volume.* Solution? We yield to an irresistible analogy and pick that plane through $P$ which is such that $P$ is the *centroid* of the triangle which is determined when this plane cuts across that given trihedral angle.

For the proof, we would ignore affinity only with pain, for it is an unhappy circumstance that an easy analogue to the direct construction in Figure 1 is difficult. Relying on affinity, then we restrict our attention to the case where the given point $P$ finds itself in the right trihedral angle represented by the first octant of an $XYZ$ coordinate system. We shall use the fact that an equation for the tangent plane to a surface $F(X, Y, Z) = 0$ at the point $P = (x, y, z)$ is given (usually) by

$$\frac{\partial F}{\partial x}(X - x) + \frac{\partial F}{\partial y}(Y - y) + \frac{\partial F}{\partial z}(Z - z) = 0.$$

(Here, for example, $\partial F/\partial x$ is the value of $\partial F/\partial X$ at $P$, of course.)

Here first is yet another preparatory proof of the (right angle) case considered in Figure 2, using the two-dimensional analogues for the equations just mentioned. For sound heuristic reasons, we now

look at the family of hyperbolas $XY = a^2$. Each choice of $a$ ($a > 0$, for convenience) determines a specific hyperbola whose first quadrant branch we'll call $H(a)$. Then, in an $XY$ system, the tangent line at a point $(x, y)$ on $H(a)$ is given by $y(X - x) + x(Y - y) = 0$, or $yX + xY = 2xy = 2a^2$. This then yields $(X/2x) + (Y/2y) = 1$ as a convenient equation for this tangent line at $(x, y)$. The line intersects the axes at $(2x, 0)$ and $(0, 2y)$, and this tells us, in turn, that $(x, y)$ is the midpoint of the corner segment and that the resulting corner triangle has area

$$A(a) = \frac{(2x)(2y)}{2} = 2a^2.$$

We've thus encountered an heuristic prelude note from analytic geometry days: the hyperbola $xy = a^2$ is precisely the locus of the midpoints of all those corner segments which determine corner triangles of constant area $2a^2$.

Now, to finish the proof concerning the minimum triangle in the corner, let $\sigma$ be the tangential segment to the hyperbola $H(a)$ at the given point $P = (x, y)$. $P$ is the centroid of $\sigma$ and determines $H(a)$, we know. Furthermore, $H(a)$ cuts the plane into three natural separate pieces $D(a)$, $H(a)$, and $D'(a)$—where $D(a)$ is the maximal domain containing the origin and lying below $H(a)$, while $D'(a)$ is the maximal domain "out beyond" $H(a)$. It is useful to note that a first quadrant point $(X, Y)$ belongs to the planar portion $D(a)$, $H(a)$, or $D'(a)$, respectively, if and only if $XY < a^2$, $XY = a^2$, or $XY > a^2$.

Now consider any *other* line through $P$, and the resulting first quadrant line segment $\sigma'$ which this line determines. (That is, $\sigma'$ is simply the intersection of this other line through $P$ with the first quadrant.) There then exists a point $P'$, other than $P$, which is the centroid of $\sigma'$ and determines a hyperbola $H(a')$. Finally, after working a bit (we do this in two ways below) to observe that $P'$ belongs to $D'(a)$, we have $a' > a$, yielding the desired result $A(a') > A(a)$, and settling the two-dimensional case. It is $\sigma$ that gives us the minimum triangle in the corner.

Here is a way of assuring ourselves that the point $P'$, the centroid of $\sigma'$ in the preceding paragraph, belongs to $D'(a)$. There exists a segment $\sigma''$ which is parallel to $\sigma'$ and tangent to $H(a)$ at a point $P''$, which, by our earlier work, is the centroid of $\sigma''$. The ray from the origin through $P''$ will go out into $D'(a)$ and there intersect $\sigma'$ in *its* centroid, namely $P'$. (Happily, this reasoning readily applies to the higher dimensional cases.)

We record a few more observations concerning the present two-dimensional situation.

For example, we can prove with a pleasant simplicity and without our earlier appeal to certain calculus formulae, that at $P = (x, y)$, the segment $\sigma$, whose points $(X, Y)$ satisfy $(X/2x) + (Y/2y) = 1$, is *tangent* to our hyperbolic curve $H(a)$ given by $XY = a^2$. For, suppose that some point $P' = (x', y')$ is, like $P$, in the intersection of $\sigma$ and $H(a)$. Then we have $(x'/2x) + (y'/2y) = 1$ and also $x'y' = a^2 = xy$. This tells us that

$$1 = \left(\frac{x'}{x}\right)\left(\frac{y'}{y}\right) = \left[\frac{(x'/x) + (y'/y)}{2}\right]^2 = 1,$$

and so, from the geometric-arithmetic mean inequality theorem (in its simplest, two-dimensional form) we conclude that it must be that $x'/x = y'/y = 1$. Thus $P'$ and $P$ are one and the same point; our tangency proof is in hand. (See [7]; also [4].)

Now suppose $P' = (x', y')$, $P' \neq P$, is the centroid of the segment $\sigma'$ discussed in our minimum area proof above. We needed to know that $P' \in D'(a)$, a fact which we shall confirm once more if we manage to show, for example, that $x'y' > a^2$. Well, since $P$ is on $\sigma'$, we have $(x/2x') + (y/2y') = 1$; and since $P' \neq P$, we have

$$\frac{a^2}{x'y'} = \frac{xy}{x'y'} < \left[\frac{x}{2x'} + \frac{y}{2y'}\right]^2 = 1.$$

This yields $a^2 < x'y'$, and we are finished with our preparations for higher dimensional analogues.

For the proof of the three-dimensional case, finally, we may now merely mimic what we have just done. Here we consider the first-octant branches of the family of surfaces $XYZ = a^3$, $a > 0$. In the $XYZ$ system, the tangent plane to the surface $H(a)$ at a point $P = (x, y, z)$ in $H(a)$ is given by

$$yz(X - x) + zx(Y - y) + xy(Z - z) = 0;$$

$$yzX + zxY + xyZ = 3xyz;$$

$$\frac{X}{3x} + \frac{Y}{3y} + \frac{Z}{3z} = 1.$$

(Of course, the tangency of this plane can, here too, be established via the geometric-arithmetic mean theorem.) The corner tetrahedron thus is seen to have vertices at $(0, 0, 0)$, $(3x, 0, 0)$, $(0, 3y, 0)$, and $(0, 0, 3z)$; it has volume

$$V(a) = \frac{(3x)(3y)(3z)}{6} = \frac{9a^3}{2};$$

and $P = (x, y, z)$ is indeed the *centroid* of its outer face. Where we formerly had the tangential line segment $\sigma$, determined by the intersection with the first quadrant of a tangent line through $P$, we now have a triangle $\tau$, determined by the intersection with the first octant of a tangent plane through $P$. The hyperbolas $H(a)$ are here replaced by surfaces $H(a)$, the maximal domains $D(a)$ and $D'(a)$ take on three-dimensional meanings, areas $A(a)$ are replaced by volumes $V(a)$ of the convenient sort where $a' > a$ implies $V(a') > V(a)$, and "it all goes through, word for word." It is $\tau$ that gives us the minimum tetrahedron in the corner; we have solved completely the (general) trihedral angle problem with which we began this section. *The minimum does exist and we know how to identify it.*

On the other hand, there exists an inductive proof (due to the reviewer) which does not require the three axes to be orthogonal, but requires one assume existence of a minimizing section. To show how the three dimensional case follows from the two dimensional case, consider the three rays ***OX***, ***OY***, ***OZ*** with origin $O$ forming the edges of the solid angle, a point $P$ interior to the angle, and a plane section through $P$ intersecting ***OX***, ***OY***, ***OZ*** in points $A$, $B$, $C$, respectively. Assume this section cuts minimum volume from the solid angle. The ray ***CP*** intersects the side $AB$ of triangle $ABC$ in a point $Q$. If $Q$ is not the midpoint of $AB$, then the two dimensional case implies that there are points $A'$ on ***OX*** and $B'$ on ***OY*** with $Q$ the midpoint of $A'B'$ and triangle $OA'B'$ has smaller area than triangle $OAB$. Then the volume of tetrahedron $OA'B'C$ is smaller than the volume of $OABC$, a contradiction since the plane $A'B'C$ also contains $P$. Thus the ray ***CP*** must be along a median of triangle $ABC$. Similarly ***AP*** and ***BP*** are along medians. Hence $P$ is the centroid of triangle $ABC$. (Parts of this section are reminiscent of Pólya's treatment of "tangent level lines." See [9]; also [4], [11], [12].)

**D. A selection of guesses, problems, and parting comments.** How does one cut off a triangle of minimum *perimeter* with a line passing through a point $P$ inside an angle?

We can see quickly that this problem is probably more difficult than the minimum area problem, because affine transformations do not preserve lengths, only ratios of parallel lengths. It develops that this problem has been treated by Pólya in [11], and with his usual elegance. Chamberlain and Moore [3] have suggested and solved the following related problem: find the slope of the line through the first quadrant point $(a, b)$ which forms the corner triangle of minimum perimeter. And here is a problem which I leave to the reader: given a point $P$ in an $n$-hedral angle in three-space, find a plane through $P$ which will determine a corner solid with minimum *surface area.*

In [13, p. 47] Jakob Steiner observes that the planes which cut a constant volume from a given solid angle envelope a certain curvilinear surface $S$, and each of the resulting planar sections is tangent to $S$ at its centroid. A footnote then calls attention to the (rectangular) special case we have treated in the discussion of our minimum volume problem.

Peano later derived corresponding results for planes cutting a constant volume from an arbitrary

given solid, and Ascoli in [1, footnote on p. 128] refers to this to prove that *if* a plane cuts off an extremum volume of all planes through a given point $P$ in a given solid, then $P$ must be the centroid of that planar section. This result is also established in [**14**], where Professor Stewart motivates questions like these by suggesting that a gem-cutter might well be interested in removing an impurity at a point $P$ in a gem by making a planar cut which leaves a maximum amount of good quality gem.

Because of this work we have these theorems as special cases: given a point $P$ in an $n$-hedral angle (or in an elliptical cone) in three-space, with vertex at the origin, then the corner solid with minimum volume determined by a plane through $P$ is the one for which the $n$-gon (elliptical) outer face has $P$ as its centroid. (In the elliptical case, one can even draw a convincing analogue to Figure 1.)

Day [**5**] showed that a polyhedron with a given number of faces having minimum volume and containing a given convex solid is tangent to the solid at the centroid of each face. (This involves some work with his Lemma 4.2 and the fact that Day concerns himself with centrally symmetric bodies.) This then gives us a three-dimensional analogue to Theorem 1 in [**2**]. (See [**15**, p. 6].)

Finally, I have learned in the meantime that there is in [**6**] a figure with an element something like what's in my Figure 1, above. In answer to my inquiry Professor Kazarinoff tells me he simply cannot remember where he learned the result, saying that he'd possibly learned it from his father or from Erdös.

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## ON CONTINUOUS FUNCTIONS AND CONVERGENCE OF NETS

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Let $f$ be a function on a topological space $X$ to a topological space $Y$. In terms of convergence of nets, the usual criterion for the continuity of $f$ may be stated as follows (see, for example, [**1**, page 86] or [**2**, page 75]).

THEOREM 1. *A function $f: X \to Y$ is continuous on $X$ if and only if whenever $\{S(n), n \in D\}$ is a net in $X$ converging to $x \in X$, then the net $\{f(S(n)), n \in D\}$ converges to $f(x)$.*

## SOME REMARKS ON THE CAYLEY-HAMILTON THEOREM

DAVID A. KLARNER

Let $k$ denote a positive integer, let $M$ denote a $k \times k$ matrix over a commutative ring $\mathcal{R}$, let $I$ denote the $k \times k$ identity matrix over $\mathcal{R}$, and let $\det(xI - M) = c_M(x)$ denote the *characteristic polynomial* of $M$. Note that this polynomial is monic (the leading coefficient is 1) and has degree $k$. The Cayley-Hamilton Theorem asserts that $M$ satisfies its characteristic polynomial; that is, $c_M(M)$ is the all-zero matrix over $\mathcal{R}$. For example, if $M = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, then $c_M(x) = x^2 - 5x + 7$, so

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^2 - 5\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^1 + 7\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Given a $k \times k$ matrix $M = [m_{ij}]$ over $\mathcal{R}$, for each non-negative integer $h$ let $M^h = [m_{ij}(h)]$, and let $c_M(x) = x^k + a_1 x^{k-1} + \cdots + a_k$. Since $M^h c_M(M)$ is the all-zero matrix for every non-negative integer $h$, it follows that for each pair of integers $i, j$ with $1 \leqq i, j \leqq k$, we have

$$m_{ij}(h+k) + a_1 m_{ij}(h+k-1) + \cdots + a_k m_{ij}(h) = 0, \tag{1}$$

for $h = 0, 1, \cdots$. In other words, the sequence $(m_{ij}(h): h = 0, 1, \cdots)$ satisfies a linear homogeneous difference equation with constant coefficients (called a difference equation for short in the balance of this note), and the auxiliary polynomial $x^k + ax^{k-1} + \cdots + a_k$ of this difference equation is the characteristic polynomial of $M$. Returning to the earlier example, the powers of $M$ are:

| $h$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $M^h$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ | $\begin{bmatrix} 3 & 5 \\ -5 & 8 \end{bmatrix}$ | $\begin{bmatrix} 1 & 18 \\ -18 & 19 \end{bmatrix}$ | $\begin{bmatrix} -16 & 69 \\ -69 & 39 \end{bmatrix}$ | $\begin{bmatrix} -87 & 276 \\ -276 & 62 \end{bmatrix}$ |

The sequence of entries in the $ij$th position of $M^h$ satisfies the difference equation $y(n+2) - 5y(n+1) + 7y(n) = 0$ because the characteristic polynomial of $M$ is $x^2 - 5x + 7$. Thus, the sequence of entries in the upper left positions of this sequence of matrices could be computed by noting that $m_{11}(0) = 1$, $m_{11}(1) = 2$ and $m_{11}(h) = 5m_{11}(h-1) - 7m_{11}(h-2)$ for $h = 2, 3, \cdots$.

If it is shown that (1) holds, then the Cayley-Hamilton theorem is a consequence. Our proof depends on a simple fact from the algebraic theory of sequences, sometimes called the algebra of formal power series; see Niven [3] for a fine survey of this topic. This theory involves forming *Laurent sequences* over the ring $\mathcal{R}$, that is, mappings $f$ which send the integers $\boldsymbol{Z}$ to $\mathcal{R}$ such that there exists a number $\operatorname{ord}(f)$ depending on $f$ such that $f(j) = 0$ for all $j < \operatorname{ord}(f)$. The *sum* $f + g$ of two Laurent sequences $f, g$ is defined by setting $(f+g)(j) = f(j) + g(j)$ for all $j \in \boldsymbol{Z}$, and the *product* $fg$ is defined by setting $fg(j) = \Sigma f(j-i)g(i)$, where the sum extends over all $i$ with $\operatorname{ord}(g) \leqq i \leqq j - \operatorname{ord}(f)$. Laurent sequences together with these operations form a commutative ring. A Laurent sequence is called a *polynomial* if all but a finite number of terms are zero. The result we require is that a Laurent sequence $f$ satisfies a difference equation if and only if there exist polynomials $p, q$ such that $p = qf$.

If we write

$$G_f(x) = \sum_{h \in \boldsymbol{Z}} f(h)x^h,$$

the result just quoted may be given a sharper formulation in terms of formal power series. Let $f$ denote a Laurent sequence with $\operatorname{ord}(f) = 0$, and suppose there exist elements $a_1, \cdots, a_k \in \mathcal{R}$ such that

$$f(h+k) + a_1 f(h+k-1) + \cdots + a_k f(h) = 0, \tag{2}$$

for $h = 0, 1, \cdots$, then there exist elements $b_0, \cdots, b_{k-1} \in \mathcal{R}$ such that

$$\frac{b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}}{1 + a_1 x + \cdots + a_k x^k} = \sum_{h=0}^{\infty} f(h) x^h. \tag{3}$$

Conversely, if (3) holds, then (2) holds. For a proof of this result and a discussion of Laurent sequences and their relation to difference equations see [2].

Now we are ready to prove (1). Put

$$G_{ij}(x) = \sum_{n=0}^{\infty} m_{ij}(h) x^h,$$

for all $i, j$ with $1 \leqq i, j \leqq k$. In order to show that (1) holds for particular pair $i, j$ we have to show $G_{ij} = p_{ij}/q_{ij}$ with $q_{ij} = 1 + a_1 x + \cdots + a_k x^k$ and $p_{ij}$ a polynomial with degree less than $k$. Note that $1 + a_1 x + \cdots + a_k x^k = \det(I - xM) = x^k c_M(x^{-1})$.

Since $M^{h+1} = MM^h$ for $h = 0, 1, \cdots$, we have

$$m_{ij}(h+1) = m_{i1} m_{1j}(h) + \cdots + m_{ik} m_{kj}(h) \tag{4}$$

for $1 \leqq i, j \leqq k$ and $h = 0, 1, \cdots$. Multiplying through this equation with $x^{h+1}$ and summing over $h = 0, 1, \cdots$, we get

$$G_{ij}(x) - m_{ij}(0) = m_{i1} x G_{1j}(x) + \cdots + m_{ik} x G_{kj}(x) \tag{5}$$

for $1 \leqq i, j \leqq k$. Holding $j$ fixed for the moment and letting $i$ vary in (5) we obtain a system of $k$ equations in the $k$ unknowns $G_{ij}, \cdots, G_{kj}$. This system can be solved by Cramer's rule in the ring of Laurent sequences over $\mathcal{R}$. It is easy to check that $G_{ij} = p_{ij}/q_{ij}$ where $p_{ij}$ is a polynomial with degree less than $k$, and

$$q_{ij} = \det(I - xM). \tag{6}$$

This is just what we wanted to prove.

The connection between difference equations and the sequence of powers of a matrix has been noted and exploited many times, for example, in the theory of Markov chains (see Feller [1]). We close with an application which may be new.

Suppose a sequence of integers $(f(h)\colon h = 0, 1, \cdots)$ satisfies the difference equation

$$f(h+k) = a_1 f(h+k-1) + \cdots + a_k f(h)$$

for $h = 0, 1, \cdots$; also, let

$$F_h = [f(h) f(h+1) \cdots f(h+k-1)],$$

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & a_1 \\ 1 & 0 & 0 & \cdots & a_2 \\ 0 & 1 & 0 & \cdots & a_3 \\ & & & & \\ 0 & 0 & 1 & \cdots & a_4 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_k \end{bmatrix},$$

then $F_h = F_{h-1} A$. It follows by a simple induction argument that

$$F_h = F_0 A^h. \tag{7}$$

There is an algorithm to compute $A^h$ in a number of operations proportional to $\ln h$. This is done as follows: Let $l$ denote the greatest integer not greater than $\ln h$ and compute the sequence of matrices $A^1, A^2, A^4, \cdots, A^{2^l}$ (the matrix $A^{2^i}$ is computed by squaring the matrix $A^{2^{i-1}}$). By expressing $h$ in

base 2 digits, we see that $A^h$ can be computed by multiplying together an appropriate subsequence of terms of $A^1, A^2, \cdots, A^{2^l}$. Using (7) we can compute $F_h$ using a number of operations proportional to $\ln h$. I learned through Ronald Rivest that J. Berstel at the University of Paris had also discovered this method for computing the $h$th term of a sequence which satisfies a difference equation. He had also found special methods for sequences satisfying second or third order difference equations.

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## AN INDUCTIVE PROOF OF THE A.M.-G.M. INEQUALITY

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There are a number of inductive proofs for the arithmetic-mean – geometric-mean inequality (see [**1**, pp. 220–224], [**2**, §5, pp. 4–5, and §11, pp. 9–10], [**3**, p. 46], [**4**, Section 2.6, pp. 18–21], [**5**] and [**6**]). This note presents another simple inductive proof.

The inequality in question is

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geqq \sqrt[n]{a_1 a_2 \cdots a_n}, \tag{1}$$

where $a_1, a_2, \cdots, a_n$ are positive numbers, and there is equality if and only if $a_1 = a_2 = \cdots = a_n$.

Let $A$ be the arithmetic mean of the given numbers, i.e., the left hand side of (1). Suppose that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$ and $a_1 < a_n$. Then, clearly, $a_1 < A < a_n$ and so

$$A(a_1 + a_n - A) - a_1 a_n = (a_1 - A)(A - a_n) > 0. \tag{2}$$

Now, (1) is true for $n = 2$. Suppose that the result is true for $n - 1$. Since the arithmetic mean of the numbers $a_2, a_3, \cdots, a_{n-1}$ and $a_1 + a_n - A$ is $A$, we have, by the induction hypothesis,

$$A^{n-1} \geqq a_2 a_3 \cdots a_{n-1}(a_1 + a_n - A)$$

whence it follows from (2) that

$$A^n > a_1 a_2 \cdots a_n.$$

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# MATHEMATICAL EDUCATION

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## PEAKS, RIDGE, PASSES, VALLEY AND PITS
### A Slide Study of $f(x, y) = Ax^2 + By^2$

CLIFF LONG

The advent of computer graphics is making it possible to pay heed to the suggestion, "a picture is worth a thousand words." As students and teachers of mathematics we should become more aware of the variational approach to certain mathematical concepts, and consider presenting these concepts through a sequence of computer generated pictures.

To illustrate this notion, consider the following. In the study of functions of two variables it is usually emphasized that a regular non-planar point on a smooth surface can be classified as one of three distinct types: elliptic, parabolic, hyperbolic [1]. These may be illustrated using the functions $f(x, y) = Ax^2 + By^2$ with the origin being:

(a) elliptic if $A \cdot B > 0$;

(b) parabolic if $A \cdot B = 0$, $A \neq B$ (planar if $A = B = 0$);

(c) hyperbolic if $A \cdot B < 0$.

The slides reproduced here (see page 371) were made at Bowling Green State University from the screen of an Owens-Illinois plasma panel which is an output device for a Data General Nova 800 mini-computer. Many slide sets and super eight movies have been produced by college mathematics teachers under an NSF grant for "Computer Graphics for Learning Mathematics." The institute was held at Carleton College in Northfield, Minnesota, 55057, during the summers of 1973 and 1974. It was under the direction of Dr. Roger B. Kirchner, who, with no small amount of personal effort, has made these slides and movies available at minimum reproduction cost.

It must of course be kept in mind that while one good picture may be worth a thousand words, a thousand poorly chosen ones may be worthless.

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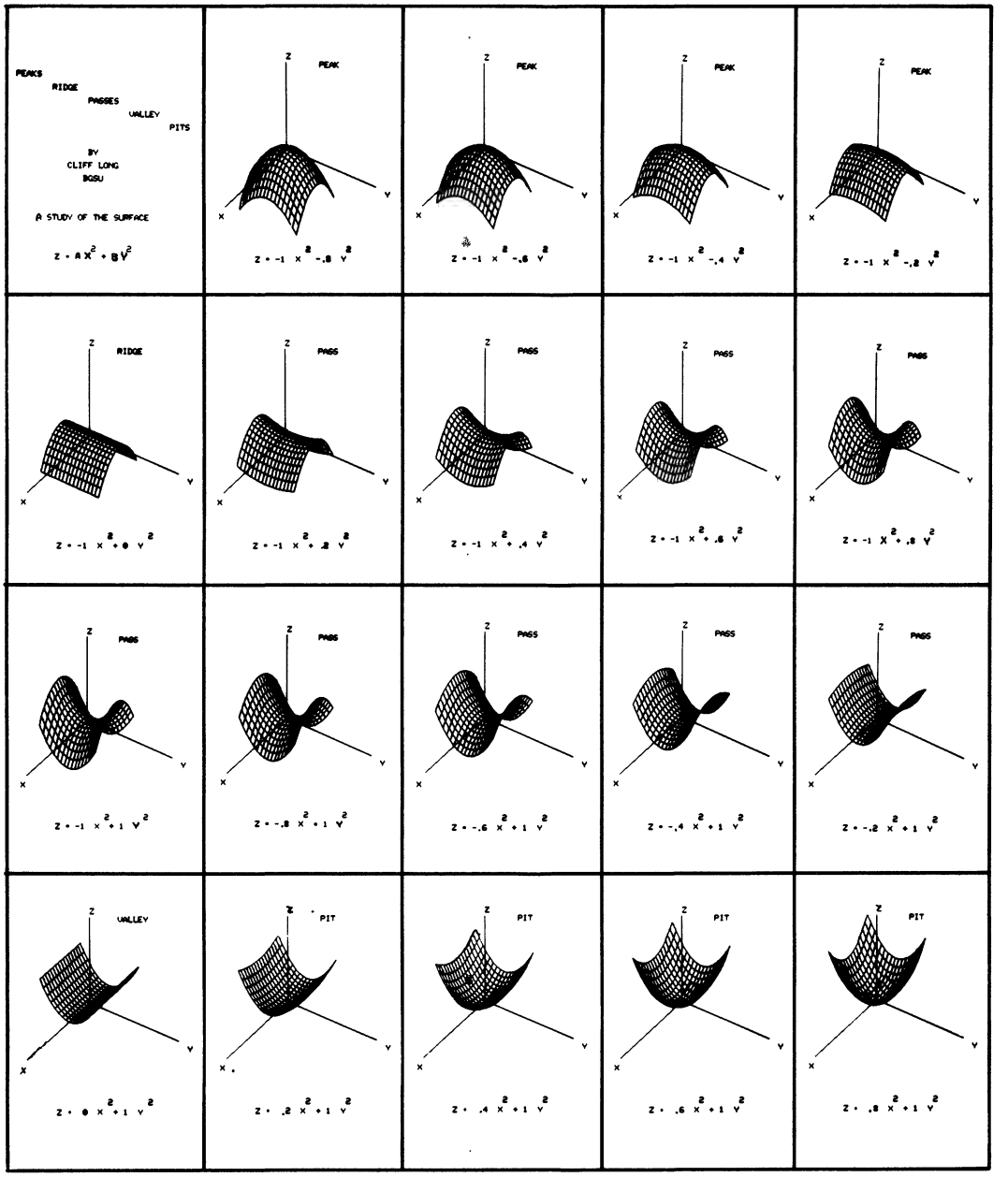
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## THE TEACHING OF ELEMENTARY CALCULUS USING THE NONSTANDARD ANALYSIS APPROACH

KATHLEEN SULLIVAN

In the 1960's a mathematical logician, Abraham Robinson, found a way to make rigorous the intuitively attractive infinitesimal calculus of Newton and Leibniz, beginning a branch of mathematics called nonstandard analysis. When elementary calculus is developed from this nonstandard approach, the definitions of the basic concepts become simpler and the arguments more intuitive (see Robinson [2] or Keisler [1]). For example, the definition of the continuity of a function $f$ at a point $c$ is simply that $x$ infinitely close to $c$ implies that $f(x)$ is infinitely close to $f(c)$.




It would seem that there ought to be considerable pedagogical payoff from this greater simplicity and closeness to intuition. However, anyone considering using this approach will have questions that need to be answered. Will the students "buy" the idea of infinitely small? Will the instructor need to have a background in nonstandard analysis? Will the students acquire the basic calculus skills? Will they really understand the fundamental concepts any differently? How difficult will it be for them to make the transition into standard analysis courses if they want to study more mathematics? Is the nonstandard approach only suitable for gifted mathematics students?

In order to get answers to some of these questions, an experiment was carried out, using an

institutional cyclic design (see Sullivan [3]). Five instructors who were teaching a traditional calculus course in schools located in the Chicago-Milwaukee area during the 1972–73 school year volunteered to use the nonstandard approach during the 1973–74 school year. The text used was H. Jerome Keisler's *Elementary Calculus: An Approach Using Infinitesimals*, published by Prindle, Weber & Schmidt, which is the first text book to adapt the ideas of Robinson to a first year calculus course. It seemed important that the same teachers be involved each year, teaching basically the same student population. Otherwise, differences in attitude and performance might well be written off to differences in instructors or in student populations. Only instructors who had taught calculus several times before were asked to participate in the study so that they would be in a good position to make comparisons.

Four of the five schools participating were small private colleges. The fifth was a public high school with over 2000 students in Glendale, an upper middle-class suburb of Milwaukee. At Saint Xavier College, the control and experimental classes consisted of students planning to major in mathematics; at Lake Forest College, of students in an honors program; at Nicolet High School, of a group of accelerated mathematics students in an advanced placement class. Barat College offers only one calculus course, and the same is true of Mount Mary College.

The assumption that the two groups would be comparable in ability was supported by a check of the SAT mathematics ability scores which were available, i.e., scores for 58 out of the 68 students in the control group and for 55 out of the 68 students in the experimental group. The distribution of scores is shown in the table below.

TABLE 1: *SAT Math Ability Scores*

| | Control Group | Experimental Group |
|---|---|---|
| 700+ | 13 | 13 |
| 600–699 | 29 | 29 |
| 500–599 | 14 | 11 |
| 400–499 | 2 | 2 |

The means chosen to collect data were the following: a calculus test given to both groups of students; interviews with the instructors who taught the control and experimental classes; and a questionnaire filled out by all those who had used Keisler's book within the past five years.

The calculus test was a fifty-minute exam which the instructors did not see until after it had been given to both groups. The purpose was to explore whether or not there did seem to be differences in performance between the two groups. The questions tested the ability of the students to define basic concepts, compute limits, produce proofs, and apply basic concepts.

The single question which brought out the greatest differences between the two groups was question 3:

$$\text{Define } f(x) \text{ by the rule } f(x) = x^2 \quad \text{for } x \neq 2,$$

$$f(x) = 0 \quad \text{for } x = 2.$$

Prove, using the definition of limit, that $\lim_{x \to 2} f(x) = 4$.

The responses of the students are summarized in Table 2.

The fact that there were a greater number of students in the experimental group willing to try to solve the problem was part of a pattern. Table 3 shows the numbers in the two groups who attempted an answer to the corresponding test questions.

There was also a notable difference between the two groups in the way in which they responded to question 6, which called for an explanation as to why a certain integral represented the volume of a given solid. In the control group there were just 9 students who spoke of the integral as a sum, compared with 16 in the experimental group. And of these students, only 3 in the first group, but 10 in the nonstandard group, saw it as a sum of small volumes rather than a sum of small areas.

TABLE 2: *Student Responses to Question* 3

| | Control Group (68 students) | Experimental Group (68 students) |
|---|---|---|
| Did not attempt | 22 | 4 |
| *Standard arguments* | | |
| satisfactory proof | 2 | |
| correct statements falling short of proof (e.g., one is only concerned with $x \neq 2$) | 15 | 14 |
| incorrect arguments | 29 | 23 |
| *Nonstandard arguments* | | |
| satisfactory proof | | 25 |
| incorrect arguments | | 2 |

TABLE 3: *Number of Students Attempting a Solution*

| | Control Group (68 students) | Experimental Group (68 students) |
|---|---|---|
| Defining basic concepts | 48 | 52 |
| Computing limits | 49 | 68 |
| Producing proofs | 18 | 45 |
| Applying basic concepts | 60 | 60 |

One of the instructors mentioned during an interview that formerly his students had found the use of the symbol $dx$ in the integral very mysterious, but that the students in the experimental class had seemed completely satisfied. This observation was supported by the answers to question 6. The first year, only 2 students commented on the significance of $dx$. The second year, 16 students brought it into their explanations.

As for success in giving definitions, there was little difference between the two groups, except that the nonstandard group gave nonstandard definitions. (Since both standard and nonstandard definitions are presented in Keisler's book, a choice was involved.) The two groups also did about equally well in computing limits. The only striking difference was that 23 students in the standard group took derivatives in computing limits without there being any justification for doing so (including some students from each of the 5 schools). Only one student in the experimental group made this mistake.

Seeking to determine whether or not students really do perceive the basic concepts any differently is not simply a matter of tabulating how many students can formulate proper mathematical definitions. Most teachers would probably agree that this would be a very imperfect instrument for measuring understanding in a college freshman. But further light on this and other questions can be sought in the comments of the instructors. The instructor questionnaire was responded to by all of the 12 instructors who had taught a course using the text *Elementary Calculus: An Approach Using Infinitesimals* within the past three years. The responses are tabulated in Table 4, with the numbers indicating how many instructors gave each response.

Note that the group as a whole responded in a way favorable to the experimental method on every item in Part One (agreeing with the advantages cited, disagreeing with the objections proposed). Note also that in Part Two there was almost complete agreement that the proofs of nonstandard calculus are easier to explain and closer to intuition. This seems quite remarkable since the instructors would be much more familiar with standard proofs. The experimental group was also given the edge regarding the ease with which they were able to learn the basic concepts. One instructor commented that, "When my most recent class were presented with the epsilon-delta definition of limit, they were outraged by its obscurity compared to what they had learned."

In the individual interviews with the 5 teachers who had taught the control and experimental classes, 3 of them said they felt that the students in the experimental class had a much better feeling for limits. One of them found the explanation in the fact that a student can move forward in thought (if

TABLE 4: *Instructor Questionnaire: Part One*

(−2 strongly disagree, −1 disagree, but not strongly, 0 neither agree nor disagree, 1 agree, but nor strongly, 2 strongly agree)

The responses below refer to the experimental classes.

| | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| 1. The students had a problem with accepting axioms for the hyperreal numbers. | 3 | 4 | 1 | 4 | |
| 2. The students seemed to find "infinitely small" a natural concept. | | 1 | 2 | 4 | 5 |
| 3. The time that must be used for nonstandard material makes it difficult to cover topics that ought to be included in a 1st year course. | 8 | 2 | 1 | 1 | |
| 4. The course seemed to give the students a better feeling for the historical development of mathematics than a standard course. | 1 | 3 | 3 | 4 | 1 |
| 5. I think that a student who has had two semesters of nonstandard calculus will be at a disadvantage in a standard 3rd semester calculus course. | 6 | 4 | 1 | 1 | |
| 6. I enjoyed teaching calculus using this approach. | | | 2 | 2 | 8 |
| 7. I probably should not have tried to teach the course without a better background in nonstandard analysis. | 9 | 1 | 1 | 1 | |
| 8. I feel that the experience of teaching calculus from the infinitesimal approach will enrich my future teaching of calculus. | | | 1 | 5 | 6 |
| 9. I am afraid that the introduction of infinitesimals left the students confused about the real numbers. | 4 | 6 | | 2 | |
| 10. I would prefer to use the nonstandard approach the next time I teach calculus. | | 2 | 1 | 4 | 5 |

*Instructor Questionnaire: Part Two*

(S Standard, NS Nonstandard, ND No Difference)

The instructors indicated which approach seemed to them to have the advantage or that neither approach seemed to have an advantage over the other.

| | S | NS | ND |
|---|---|---|---|
| 1. The students learn the basic concepts of calculus more easily. | | 8 | 4 |
| 2. The students seem to be more "turned on." | | 5 | 7 |
| 3. The proofs are easier to explain and closer to intuition. | 1 | 10 | 1 |
| 4. The students find it easier to formulate their questions. | | 2 | 9 |
| 5. The students end up with a better understanding of the basic concepts of calculus. | | 5 | 7 |

I pick a point $x$ which is infinitely close to $c$, $f(x)$ will be infinitely close to $L$) rather than backward (if I want $f(x)$ to be within epsilon of $L$, I must pick $x$ to be within delta of $c$). It was also pointed out that the nonstandard approach makes it easier to illustrate concepts like the derivative on a static blackboard since one is dealing with a single number — a representative infinitesimal — and applying operations to it rather than letting something approach zero.

Two instructors expressed the opinion that the nonstandard method of learning calculus has special merit for students planning to major in engineering or physics, fields in which infinitesimals have always been considered a useful tool. On the other hand, some uncertainty was voiced on the question of how well students who want to study more analysis will be able to make the transition from an experimental class to a traditional course. Conversations with students at the University of Wisconsin, who had been in nonstandard calculus classes, suggest that the attitude of the instructor in the standard class may be the crucial factor. The students will have the necessary mathematical background to make the transition, but perhaps they ought to be prepared for the fact that their future instructors may or may not be convinced of this fact. As G. R. Blackley remarked in a letter to Prindle, Weber & Schmidt, concerning *Elementary Calculus: An Approach Using Infinitesimals*, "Such problems as might arise with the book will be political. It is revolutionary. Revolutions are seldom welcomed by the established party, although revolutionaries often are."

Certainly there is no claim to have found an instant solution to the problem of teaching calculus. This is not "Calculus made easy." Hard work and practice will still be required of the student, and a poor background in algebra will still make it difficult to learn calculus.

On the other hand, there does seem to be considerable evidence to support the thesis that this is indeed a viable alternate approach to teaching calculus. Any fears on the part of a would-be experimenter that students who learn calculus by way of infinitesimals will achieve less mastery of basic skills have surely been allayed. And it even appears highly probable that using the infinitesimal approach will make the calculus course a lot more fun both for the teachers and for the students.

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## USE OF CANNED COMPUTER PROGRAMS IN FRESHMAN CALCULUS

MARGARET S. MENZIN

**1. Introduction.** Many articles and newsletters ([1], [2], [3], [4], and their references) have been published on the subject of "computer calculus," which means a calculus course in which the students write and run computer programs. It is the purpose of this article to describe a freshman calculus course that uses canned computer programs, to argue that canned programs are an excellent alternative to user-written programs, and to encourage instructors who have not used any computer programs in freshman calculus to try using canned ones.

**2. General description.** The following description refers to a course of 100 math, science and economics majors taught at Simmons College. Students were enrolled in 4 sections, each of which met 3 hours a week. The text was Thomas' *Calculus and Analytic Geometry* ([5]). Students had periodic access to two teletype terminals hooked into a Hewlett Packard HP–2000C. The terminal usage was a required part of the course; computer exercises were collected, but not graded. Students, working in pairs, ran 6 programs each semester, for a total expenditure of $1\frac{1}{2}$ hours or \$7.50 per student per semester.

**3. The logistics of terminal usage.** One-third of the way through the fall semester, the calculus class was shown how to use the terminal. Students did the actual typing of responses to the first

program (to show them how easy it was). They received one sheet of paper with instructions for signing on and off the computer and helpful hints (most importantly not to confuse the letter *O* and number 0, how to erase at a teletype, and always to press the button which returns control to the computer after they make an entry). Later in the term they received a page on how to write functions in BASIC, enabling them to use a canned program on a function of their own choice.

For each program run the student received a set of "program notes." The notes set forth (1) the aim of the program; (2) what the students needed to know to run the program: the name of the program to "call," and any data they may need to make up or enter *before* beginning to run (e.g., if the program plots a function of their choice; the notes will tell them to prepare one in BASIC notation in advance, if the function must be continuous, etc.); (3) (usually) what they would be asked for in the program; (4) a *minimum* number of examples to try; (5) hints for further explanation, the more explicit the better (e.g., in investigating $\lim p(x)/q(x)$ as $x \to \infty$ we suggested trying polynomials $p(x)$ of degree 4 and $q(x)$ of degree 2); (6) a reference to the relevant pages of the text for theory, and suggestions that they look there for other cases to illustrate; (7) what the student should look for after her first computer run and how to prepare for the second run. (We found that on some problems it took students quite a while to analyse the calculations; for these problems it was best to have students run some examples, think about the answers, and then run more examples.) The notes were designed to be used by the student *before* she approached the terminal, not while using it.

The computer programs were written so as to be self-contained, and to make the student feel as though she were *in control of a routine calculation.* It was felt that this was very important if the terminal was to make the limit process seem *less mysterious.* The programs were written to "talk to" the user (in a conversational mode). Comments were used liberally so that the student could follow the calculation. For example, in the program which computed upper and lower sums for the area under $y = 1/x$, $1 \leqq x \leqq 2$, a comment, before the calculation began, reminded the student that this was a monotonic decreasing function and therefore the maximum and minimum values could be found at the left and right hand endpoints of any subinterval. Then the terminal gave the formula $x(i) = 1 + i/N$ (student chosen $N$) and filled in a table with the headings: $i$, $x(i-1)$, $x(i)$, $y$-min, $y$-max, inscribed area, circumscribed area. Finally the terminal added up the inscribed and circumscribed areas to produce the upper and lower sums, announcing that it was doing exactly that. We did not mind the extra computing done when neglecting that the $i$th $y$-min was the $i+1$st $y$-max. Again, the aim was for the student to feel that each step was a simple arithmetic operation that she was quite capable of doing by hand.

Likewise, elegance of programming was a secondary consideration in writing the programs. Programs were written so that a student who did not know BASIC could follow (e.g., many comments, use of same names for variables as in text, etc.), and students were encouraged to use the LIST command (which types out a copy of the program for the user).

**4. Examples of typical programs.**

| *Program investigates* | *Student chooses* | *Terminal Computes* |
|---|---|---|
| $\lim_{x \to 0} \sin x/x$ | $x$ | $\sin x$, $1/x$, $\sin x/x$ |
| Estimating $f(x_0+h)$ by the tangent at $x_0$ for $f(x)=x^2$, $\sqrt{x}$, $\sin x$ | $x_0$, $x_0+h$ | $f(x_0)$; $f'(x_0)$; $f(x_0+h)$; $f(x_0)+hf'(x_0)$ $\mathrm{Err}(h) = f(x_0+h)-f(x_0)-hf'(x_0)$; $\mathrm{Err}(h)/h$ |
| Riemann Sum | Then $f(x), a, b, n$ $c_i \in [x_{i-1}, x_i]$ | Then $x(i-1), x(i) = a + i(b-a)/n$ $f(c_i), f(c_i)\cdot\Delta x, \Sigma f(c_i)\cdot\Delta x$. |

In addition, students used the extended mean value theorem (truncated Taylor series), the trapezoidal rule, and Simpson's rule to construct trig and log tables, and compared the error estimates of these methods for various numbers of subdivisions. They used the inscribed and circumscribed area under $x^2 + y^2 = 4$, $x \geqq 0$, $y \geqq 0$ to estimate $\pi$ (a highlight of the course for many), used Newton's Method to estimate the root of an equation, and investigated the limiting processes (both convergent and divergent).

**5. Benefits of using canned computer programs.** The most important benefit of computer usage was that it made the course more concrete, and accessible to the student. After going through many examples on the computer, the student believed that perhaps the sequences did converge, that calculus was not a giant hoax, and then she was willing to try to understand the theory. (Here, it was crucial that the student be able to follow the computer's calculations.)

Another, and surprising, benefit was that the terminal usage made students feel more comfortable about approaching but never reaching a limit. It was clear to them that the computer could not for many $f(x)$ calculate $f(x)$ at $x = \sqrt{2}$, but only as $x$ approached $\sqrt{2}$. This made them used to, and then more comfortable with approaching, but never reaching a limit.

A third general benefit, was that the computer usage gave the course a point of view: calculus and approximation. Since most of our students know no physics, the usual applications for calculus did not interest them. On the other hand, they already knew that log and trig tables were important, and were very eager to use calculus to construct them.

Another surprising benefit was that the computer usage made students more aware of the importance of the mean value theorem. We used the "extended mean value theorem" (truncated Taylor series) and bounds on $f^{(n)}(x)$, $a \leqq x \leqq b$ to calculate the maximum error in estimating $f(x) = \log x$ and $f(x) = \sin x$ by Riemann sums, the trapezoidal rule, and Simpson's rule. Our students easily appreciated that there was no point in calculating a Riemann sum to more places than their error estimate, and that they didn't really know what it was unless they knew the error bound on a particular given value. They then appreciated the mean value theorem for enabling them to compute these error bounds.

Finally, the students had fun. Computers have undeniable glamour, and the power to order a calculation is enjoyable. For those students who had had a little calculus in high school, the computer added something new. And all the students felt that in estimating $\pi$ and constructing log and trig tables they were being let in on some things that had always been held secret.

**6. Comparison of canned and student written programs.** There are two aspects of canned programs which are inferior to user-written programs. First, while canned programs allow for some experimentation, and although use of canned programs did motivate some students to learn BASIC, few of our students took advantage of the terminals to try out ideas of their own.

Second, we lacked the reenforcement of understanding of an algorithm which comes from writing a program to compute with that algorithm. In studying differential equations or number theory, for example, this would be a serious loss. In (non-numerical) freshman calculus, however, it is not.

On the other hand, use of canned programs has several important positive aspects. First, and in many cases crucial, it takes much less class time and homework time. It takes 20–30 minutes to teach students to use a terminal, as opposed to a minimum of 4 hours to teach rudimentary BASIC. The savings of time for homework is even more dramatic. (The savings in students' time is accompanied by a saving in computer time. We estimate the cost of canned programs to be 10% of the cost for student-written programs.) For an instructor who is already short of time, the choice may be canned programs or no programs.

Second, if the calculus course is to include student programming, then the instructor would do well to choose a text compatible with that approach, and may even wish to choose a text written specifically for "computer-calculus." If the calculus course uses canned programs, then almost any calculus text

may be used. For me, the two factors of time-saving and flexibility in choice of text make canned programs preferable to user-written programs.

Finally, even if a department is aiming towards a numerical calculus with student-written programs, canned programs may be an excellent first step, especially if the department has limited computing budget, limited computer time available, or members of limited programming experience.

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## PROBLEMS AND SOLUTIONS

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

### ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before August 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 966* [1951, 338]. *Proposed by C. W. Bruce*

Any integer $N$ can be written in the form $2^k(a^2-b^2)$, where $k, a, b$ are nonnegative integers. When are $k, a, b$ unique? When not unique, how many sets of $k, a, b$ are there?

E 2593. *Proposed by Jeanne W. Kerr and John E. Wetzel, University of Illinois*

Three points are given on each of three parallel lines, the three lines not all lying in the same plane.

These points by threes, one on each line, determine 27 triangular plates and these triangular plates could, on the face of it, meet to determine as many as $\binom{27}{3} = 2925$ points, though it is clear that not that many can actually occur. At most how many points can the 27 plates determine?

E 2594.* *Proposed by David P. Robbins, Phillips Exeter Academy, New Hampshire*

Suppose that $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are vectors corresponding to the edges of an oriented regular polygon. Since their sum is 0, an object undergoing displacements by each of these vectors in some order traces out a closed polygon. If this order is chosen at random, what is the probability that the polygon does not intersect itself?

E 2595. *Proposed by Sidney Penner, Bronx Community College*

Consider $(2n+1)^2$ hexagons arranged in a "diamond" pattern, the $k$th column from the left and also from the right consisting of $k$ hexagons, $1 \leqq k \leqq 2n+1$ (Figure 1 illustrates the $5^2$ case). Show that if $n \not\equiv 1 \pmod 3$ and the center hexagon is deleted, then the remaining hexagons can be tiled by trominoes as in Figure 2.

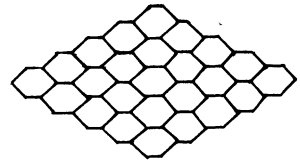


FIG. 1



FIG. 2

E 2596*. *Proposed by Mark A. Spikell, Lesley College, Cambridge, Massachusetts*

Suppose one is supplied with a collection of Cuisenaire rods having dimension $1 \times 1 \times a$, where the length $a$ belongs to a finite set $A$ of positive integers and the number of rods of length $a \in A$ may be supposed to be unlimited. For which $s$ can one build a $1 \times s \times s$ square from one's collection?

(*Editor's comment.* Let g.c.d.$(A) = m$. Then $m \mid s$ by a theorem of de Bruijn, *Filling boxes with bricks*, this MONTHLY 76 (1969), 37–40.)

E 2597. *Proposed by R. W. Farebrother, University of Manchester, England*

Let $j$, $n$ be integers such that $0 \leqq j \leqq n$ and let

$$(1-x)^j(1+x)^{n-j} = \sum_{i=0}^{n} c_{ij}(n)x^i.$$

If $C(n)$ is the matrix $(c_{ij}(n))$ where $i, j = 0, 1, \cdots, n$ show that

$$C(n)^2 = 2^n I, \qquad \det C(n) = (-2)^{n(n+1)/2},$$

$$\operatorname{tr} C(n) = 0 \quad \text{if } n \text{ is odd}$$

$$= 2^{n/2} \quad \text{if } n \text{ is even.}$$

E 2598. *Proposed by Erwin Just, Bronx Community College of the City University of New York*

Does there exist a set of rational points which is dense in the plane such that the distance between each pair of points in the set is irrational?

## SOLUTIONS OF ELEMENTARY PROBLEMS

### The Knight's Distance

E 2392 [1973, 74]. *Proposed by David Singmaster, London, England*

On the $n \times n$ chessboard, for $n \geqq 4$, define the *knight's distance* $D(A, B)$ between two squares $A$

and $B$ to be the minimum number of knight's moves required to go from $A$ to $B$. Define the *knight's diameter* $M(n)$ of the $n \times n$ board to be the maximum knight's distance between any two squares of the board.

(1) Is $M(n)$ monotonic?

(2) Does $M(n)$ always equal the knight's distance between opposite corners of the board?

(3) Prove or disprove: For $n \geq 5$, $M(n) = [2n/3]$.

(4) Determine the knight's distance $D(O, P)$ from the origin to an arbitrary square $P = (a, b)$ on the infinite chessboard.

I. *Solution to* (1), (2) *and* (3) *by Michael Goldberg, Washington, D.C.* In order to compute $M(4)$ consider the following diagrams

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 3 | 2 | | 2 | 3 | 2 | 3 | | 1 | 2 | 1 | 4 |
| 2 | 1 | 4 | 3 | | 1 | 2 | 1 | 4 | | 2 | 3 | 2 | 1 |
| 3 | 4 | 1 | 2 | | 2 | 3 | 2 | 1 | | 3 | 0 | 3 | 2 |
| 0 | 3 | 2 | 5 | | 3 | 0 | 3 | 2 | | 4 | 3 | 2 | 1 |

The numbers indicate the distance from the square marked 0 on the $4 \times 4$ board. It follows that $M(4) = 5$.

In order to compute $M(5)$ one must draw six diagrams which correspond to the six essentially different starting positions of the knight. We leave this to the reader to do and check that $M(5) = 4$.

Since $M(4) > M(5)$ and $M(5) < M(n)$ for large $n$ it follows that the answer to (1) is negative. The example $n = 4$ shows that the answer to (2) is negative. The example $n = 5$ shows that the answer to (3) is negative.

II. *Solution to* (4) *by the editor based on suggestions by Michael Goldberg.* We replace the infinite chessboard by the lattice $\mathbf{Z}^2$ of integral points in $\mathbf{R}^2$. Let $f(m, n)$ denote the knight's distance from the origin $(0, 0)$ to $(m, n) \in \mathbf{Z}^2$. Let $EO$ be the set of all points $(m, n) \in \mathbf{Z}^2$ such that $m$ is even ($E$) and $n$ is odd ($O$). One defines similarly the sets $EE$, $OE$, and $OO$. If the knight starts from a point in $EE$, then it can only go to either $EO$ or $OE$ but not to $OO$. The possible changes of "states" for one move of the knight are indicated in the following diagram

$$\begin{array}{ccc} EE & \leftrightarrow & EO \\ \updownarrow & & \updownarrow \\ OE & \leftrightarrow & OO \end{array}$$

It is clear that if the knight makes a path from one point in $EE$ to another point in $EE$ then the number of moves must be even, etc.

If $a \in \mathbf{R}$ we denote by $[a]$ (resp. $\{a\}$) the largest (resp. smallest) integer $m$ such that $m \leq a$ (resp. $m \geq a$).

We claim that

(i) $f(4k, 0) \leq 2k, \quad k \geq 0;$

(ii) $f(4k+1, 0) \leq 2k+1, \quad k \geq 1;$

(iii) $f(4k+2, 0) \leq 2k+2, \quad k \geq 0;$

(iv) $f(4k+3, 0) \leq 2k+3, \quad k \geq 0.$

The statement (i) is true for $k = 0$. Let $k > 0$ and use induction. Since one can reach $(4k, 0)$ from $(4k-4, 0)$ in two moves we have

$$f(4k, 0) \leq f(4k-4, 0) + 2.$$

By induction hypothesis $f(4k-4,0) \leqq 2k-2$ so that $f(4k,0) \leqq 2k$. Thus (i) is proved. One can similarly prove (ii), (iii), and (iv).

Let $\varepsilon = 0, 1, 2,$ or 3. Since each move may increase the first coordinate by at most 2 we have $f(4k+\varepsilon,0) \geqq \{(4k+\varepsilon)/2\} = 2k + \{\varepsilon/2\}$. On the other hand we have $f(4k+\varepsilon,0) \leqq 2k+\varepsilon$, except when $k=0$, $\varepsilon = 1$. Since the difference

$$(2k+\varepsilon) - (2k + \{\varepsilon/2\}) = [\varepsilon/2] \leqq 1$$

and $f(4k+\varepsilon,0) \equiv 4k+\varepsilon \pmod 2$ it follows that

$$f(4k+\varepsilon,0) = 2k+\varepsilon, \qquad k \geqq 0, \tag{v}$$

where $\varepsilon = 0, 1, 2,$ or 3 and if $k=0$ then $\varepsilon \neq 1$.

By similar arguments (which we leave to the reader) one can prove that

$$f(3k+1+\varepsilon, 3k+1+\varepsilon) = 2k+2, \qquad k \geqq 0, \tag{vi}$$

where $\varepsilon = 0, 1, 2$ and if $k=0$ then $\varepsilon \neq 1$.

In order to compute $f(m,n)$ it suffices to consider the case $m \geqq n \geqq 0$ because $f(\pm m, \pm n)$ is independent of the four choices of $\pm$ signs and $f(m,n) = f(n,m)$.

CASE 1. $m \geqq 2n \geqq 0$. One can reach $(m,n)$ from $(m-2n,0)$ in $n$ moves. Thus $f(m,n) \leqq n + f(m-2n,0)$. Write $m-2n = 4k+\varepsilon$ where $\varepsilon = 0, 1, 2,$ or 3. If $m-2n \neq 1$ we use (v) to get $f(m,n) \leqq 2k+n+\varepsilon$. But $f(m,n) \geqq \{m/2\} = 2k+n+\{\varepsilon/2\}$ and the difference

$$(2k+n+\varepsilon) - (2k+n+\{\varepsilon/2\}) = [\varepsilon/2] \leqq 1.$$

Since $f(m,n) \equiv m+n \pmod 2$ this implies that $f(m,n) = n+2k+\varepsilon$.

If $m-2n=1$ and $n \geqq 1$ then one can reach $(m,n)$ from $(3,1)$ in $n-1$ moves. Hence $f(m,n) \leqq n-1+f(3,1) = n+1$. Since $f(m,n) \geqq \{m/2\} = n+1$ we must have $f(m,n) = n+1$.

Summarizing, we have in this case,

$$f(m,n) = n+2k+\varepsilon,$$

where $\varepsilon = 0, 1, 2,$ or 3, $m-2n = 4k+\varepsilon$ and $(m,n) \neq (1,0)$.

CASE 2. $2n > m \geqq n \geqq 0$. One can reach $(m,n)$ from $(2n-m, 2n-m)$ in $m-n$ moves. Thus $f(m,n) \leqq m-n+f(2n-m, 2n-m)$. Write $2n-m = 3k+1+\varepsilon$ where $\varepsilon = 0, 1,$ or 2. If $2n-m \neq 2$ we use (vi) to get $f(m,n) \leqq m-n+2k+2 = n-k+1-\varepsilon$. But $f(m,n) \geqq \{(m+n)/3\} = n-k-[(1+\varepsilon)/3]$. The difference

$$(n-k+1-\varepsilon) - (n-k-[(1+\varepsilon)/3]) = 1 + [(1+\varepsilon)/3] - \varepsilon \leqq 1.$$

Since $f(m,n) \equiv m+n \pmod 2$ it follows that $f(m,n) = n-k+1-\varepsilon$.

If $2n-m=2$ and $n \geqq 2$ then one can reach $(m,n)$ from $(4,3)$ in $n-3$ moves. Hence $f(m,n) \leqq n-3+f(4,3) = n$. Since $f(m,n) \geqq \{(m+n)/3\} = n$ we must have $f(m,n) = n$.

Summarizing, we must have in this case,

$$f(m,n) = n-k+1-\varepsilon,$$

where $\varepsilon = 0, 1,$ or 2; $2n-m = 3k+1+\varepsilon$ and $(m,n) \neq (2,2)$.

Part (4) also solved by Dragoš Cvetković & Slobodan Simić (Yugoslavia). Partial solutions were submitted by Carolyn MacDonald, and by the proposer.

*Editor's comment.* Goldberg gives the formulas

$$f(m,n) = m-n-2[(m-2n)/4] \qquad \text{(Case 1)}$$

$$f(m,n) = m-n+[(8n-4m+2)/3] - [(4n-2m)/3] \qquad \text{(Case 2)}$$

with $(m,n) \neq (1,0), (2,2)$.

Cvetković and Simić give the formulas

$$f(m,n) = m + n - 2[m/2] - 2[(n - [m/2])/2] \qquad \text{(Case 1)}$$

$$f(m,n) = m + n - 2[(m+n)/3] \qquad \text{(Case 2)}$$

with the same exceptions.

It is easy to verify that these formulas agree with the formulas above.

The proposer originally asked whether $M(n) = \{2n/3\}$ for $n \geqq 5$. This was inadvertently changed to $M(n) = [2n/3]$. The original question seems still to be open.

**Tetrahedron with Three Equiareal Faces**

E 2498 [1974, 1026]. *Proposed by R. E. Smith, Bishop Union High School, Bishop, California*

Given triangle $ABC$, find the locus of all points $R$ (not necessarily in the plane of $ABC$) with the property that the three triangles $RAB$, $RBC$, and $RCA$ have the same area.

*Solution by Carolyn MacDonald, University of Missouri, Kansas City.* We can choose a rectangular coordinate system so that $A = (a,0,0)$, $B = (0,b,0)$, $C = (c,0,0)$. If $R = (x,y,z)$, then the equal area condition can be written in the vector form as follows:

$$|\boldsymbol{RB} \times \boldsymbol{RC}| = |\boldsymbol{RC} \times \boldsymbol{RA}| = |\boldsymbol{RA} \times \boldsymbol{RB}|.$$

After squaring and expressing everything in terms of coordinates we obtain the equations of the required locus:

$$(b^2 + c^2)z^2 + (bx + cy - bc)^2 = (a-c)^2(y^2 + z^2) = (a^2 + b^2)z^2 + (bx + ay - ab)^2.$$

This locus intersects the plane $z = 0$ in the points $A' = (c-a, b, 0)$, $B' = (a+c, -b, 0)$, $C' = (a-c, b, 0)$ and $P = ((a+c)/3, b/3, 0)$. $P$ is the centroid of $ABC$ and $A$, $B$, $C$ are the midpoints of the sides of $A'B'C'$.

As an example, when $a = b = 1$, $c = -1$ the locus consists of the hyperbola $x^2 - 2z^2 = 4$, $y = 1$ and the ellipse $(3y+1)^2 + 6z^2 = 4$, $x = 0$.

Also solved by Thomas Elsner, Leon Gerber, L. Kuipers, Edmondo Morgantini (Italy), and the proposer. Partially solved by Walter Bluger (Canada) and by Michael Goldberg.

**A Polynomial Inequality**

E 2519 [1975, 169]. *Proposed by H. L. Montgomery, University of Michigan*

Let $P$ be a complex polynomial of degree $n$ with $P(1) = 0$ and $P(0) = 1$. Show that

$$\max\{|P(z)| : |z| \leqq 1\} \geqq 1 + \frac{1}{3n}.$$

*Solution by O. P. Lossers, Technological University, Eindhoven, The Netherlands.* Let $z_j = \exp(2\pi ij/(n+1))$, $0 \leqq j \leqq n$, and let $P(z) = \sum_{k=0}^{n} a_k z^k$. Since $P(z_0) = P(1) = 0$ and $\sum_{j=0}^{n} z_j^i = (n+1)\delta_{0,i}$ it follows that

$$\sum_{j=1}^{n} P(z_j) = \sum_{j=0}^{n} P(z_j) = \sum_{k=0}^{n} a_k \sum_{j=0}^{n} z_j^k = a_0(n+1) = n+1.$$

This implies that $|P(z)|$ attains a value $\geqq (n+1)/n = 1 + 1/n$ at one of the $(n+1)$st roots of unity.

Also solved by Paul Bateman, Peter Bundschuh (W. Germany), Dan Eustice, Ron Evans, Irving Gerst, L. E. Mattics, M. R. Modak (India), Q. G. Mohammad (India), Ram Murty & Kumar Murty (Canada), Peter Nicholls, St. Olaf Problem Solving Group, R. A. Rankin (Scotland), Allen Stenger, Selden Trimble, S. G. Udpikar (India), Hartmut Wellstein (W. Germany), and the proposer.

*Editor's comment.* Most solvers use Parseval's Theorem and the Cauchy-Schwartz inequality to show that $\max\{|P(z)|: |z| \leqq 1\} \geqq (1+1/n)^{1/2}$ in the following manner:

$$(\max\{|P(z)|: |z| \leqq 1\})^2 = \frac{1}{2\pi}\int_0^{2\pi} |P(e^{it})|^2\,dt = \sum_{k=0}^{n} |a_k|^2$$

$$\geqq 1+\frac{1}{n}\left|\sum_{k=1}^{n} a_k\right|^2 = 1+\frac{1}{n}.$$

Bateman, the Murtys, Rankin and Trimble give the same improvement in the proposer's inequality as that obtained by Lossers.

### Odd Factorials

E 2520 [1975, 169]. *Proposed by G. B. Huff, University of Georgia*

The non-trivial sequence $a_0, a_1, \cdots$ satisfies the following recursion formula:

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k}^2 (n-2k)!\,a_k^2$$

Find $a_n$.

I. *Solution by Allen Stenger, Student, Pennsylvania State University.* For $n = 0$ we get $a_0 = a_0^2$. In order to avoid the trivial sequence we must take $a_0 = 1$. It will be shown that $a_k$ is the "odd factorial"

$$a_k = (2k-1)!! = 1\cdot 3\cdot 5\cdots(2k-1); \tag{1}$$

where we agree that $(-1)!! = 1$. The proof is by induction. If (1) is valid for $k < n$, where $n$ is a positive integer, then the recursion formula gives

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k}^2 (n-2k)![(2k-1)!!]^2 = \sum_{k=0}^{[n/2]} \frac{(n!)^2 2^{-2k}}{(k!)^2(n-2k)!}.$$

By the trinomial theorem the constant term in the expansion of $(x+1/x+2)^n$ is equal to $2^n a_n/(n!)$. Since this function is equal to $x^{-n}(x+1)^{2n}$ we obtain

$$\frac{2^n a_n}{n!} = \binom{2n}{n}$$

which gives $a_n = (2n-1)!!$.

II. *Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands.* Let $S$ be the set of all sequences formed from the letters $A$, $B$, $C$, $D$ such that each sequence has length $n$ and $A$ occurs the same number of times as $B$ in each sequence. Let $T$ be the set of all 0, 1-sequences such that each sequence has length $2n$ and both 0 and 1 occur $n$ times in each sequence. Clearly

$$|S| = \sum_{k=0}^{[n/2]} \binom{n}{2k}\binom{2k}{k} 2^{n-2k},$$

$$|T| = \binom{2n}{n}$$

where the index $k$ counts the number of occurrences of $A$ in a sequence belonging to $S$.

By replacing $A$, $B$, $C$, $D$ by 11, 00, 10, 01, respectively, in each sequence belonging to $S$ we get a bijection $S \to T$. The equation $|S| = |T|$ can be written in the form

$$\sum_{k=0}^{[n/2]} \binom{n}{2k}^2 (n-2k)![(2k-1)!!]^2 = (2n-1)!!.$$

Thus $a_n = (2n-1)!!$.

Also solved by M. T. Bird, David Bloom, Ben Bowen, Leonard Carlitz, Irving Gerst, Sylvan Greene, M. G. Greening (Australia), Emil Grosswald, Stanton Hales, Jr. & Richard Poppen, Robert Heller, John Kelly & Charles Wexler, Yudell Luke, Ram Murty & Kumar Murty (Canada), Otto Ruehr, Michael Somos, David Zeitlin, and the proposer.

*Editor's comment.* Several contributors remark that the identity $|S| = |T|$ occurs in H. W. Gould, *Combinatorial Identities*, p. 34, formula 3.99 and also in J. Riordan, *Combinatorial Identities*, Ch. 3, Problem 4, pp. 71–72.

### Arithmetic Progressions in Sequences with Bounded Gaps

E 2522 [1975, 300]. *Proposed by Joël Spencer, Massachusetts Institute of Technology*

A theorem due to van der Waerden asserts that if the set $N$ of positive integers is partitioned into a finite number of subsets, then at least one of the subsets contains arbitrarily long arithmetic progressions.

An infinite subset $S = \{s_1, s_2, \cdots\}$ of $N$ $(s_1 < s_2 < \cdots)$ has *bounded gaps* if $(s_{n+1} - s_n)$ is bounded. Show that if $S$ has bounded gaps, then it contains arbitrarily long arithmetic progressions.

(Remark: This result is a special case of a theorem of E. Szemeredi (Acta Arithmetica 27 (1975), 199–245) which states that $S \subseteq N$ contains arbitrarily long arithmetic progressions if it has positive upper density.)

*Solution by William E. Gould, California State College, Dominguez Hills.* With $S$ as above, let $s_0 = 1$ and $b = \max(s_{n+1} - s_n)$ for $n = 0, 1, 2, \cdots$. If $S_k = \{s_1 - k, s_2 - k, \cdots\}$ for $k = 0, 1, 2, \cdots, b-1$, then $S_0, S_1, \cdots, S_{b-1}$ cover $N$ (and partition $N$ if we eliminate repetitions and non-positive integers). By van der Waerden's theorem, some $S_k$ contains arbitrarily long arithmetic progressions, hence $S$ contains arbitrarily long arithmetic progressions which simply translate those of $S_k$ to the right by $k$.

Also solved by Thad Dankel, Paul Edelman, Ram Murty & Kumar Murty (Canada), Ton Odda, Stefan Porubsky (Czechoslovakia), Terry Therneau, and the proposer.

*Editorial note.* Professor N. S. Mendelsohn has brought it to our attention that a solution to this problem appears in one of his papers: *The Golden Ratio and van der Waerden's Theorem*, Proceedings of the 5th South-Eastern Conference on Combinatorics, Graph Theory, and Computing, pp. 93–109.

### A Logarithmic Integral

E 2523 [1975, 300]. *Proposed by Keith P. Kerney, Naval Ship Research and Development Center, Bethesda, Maryland*

Evaluate the following integral exactly:

$$\int_0^1 \log(1+x)\log(1-x)\,dx.$$

(This integral arose in connection with a problem in wing theory.)

I. *Solution by Allen Stenger, Pennsylvania State University.*

$$\int_0^1 \log(1+x)\log(1-x)\,dx$$

$$= \tfrac{1}{2}\int_{-1}^1 \log(1+x)\log(1-x)\,dx \qquad \text{(symmetry)}$$

$$= \int_0^1 \log(2t)\log(2-2t)\,dt \qquad (x = 2t-1)$$

$$= \int_0^1 [\log 2 + \log t][\log 2 + \log(1-t)]\,dt$$

$$= \log^2 2 + 2\log 2 \cdot \int_0^1 \log t\, dt + \int_0^1 \log t \log(1-t)\, dt.$$

The first integral is $-1$, while the second, by the Maclaurin series for $\log(1-t)$, is

$$-\sum_{k=1}^{\infty} \frac{1}{k} \int_0^1 t^k \log t\, dt = \sum_{k=1}^{\infty} \frac{1}{k(k+1)^2} = \sum_{k=1}^{\infty} \left( \frac{1}{k(k+1)} - \frac{1}{(k+1)^2} \right)$$

$$= 1 - (\zeta(2) - 1) = 2 - \pi^2/6.$$

Thus

$$\int_0^1 \log(1+x)\log(1-x)\, dx = \log^2 2 - 2\log 2 + 2 - \frac{\pi^2}{6}.$$

II. *Remark by Otto G. Ruehr, Michigan Technological University.* Integration by parts ($u = \log(1+x)$) followed by the substitution $x = 1 - e^{-t}$ and series expansion reduces the problem to the evaluation of $\Sigma_{n=1}^{\infty} 1/2^n n^2$; thus a direct calculation of the given integral (such as Solution I above) leads to an elementary derivation of the known result that the sum of this series is $\pi^2/12 - \frac{1}{2}(\log 2)^2$. (A similar remark was made by M. L. Glasser & M. S. Klamkin, and by Paul Mielke.)

Also solved by sixty-four others, including the proposer.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before August 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6090. *Proposed by T. Šalát and O. Strauch, Bratislava, Czechoslovakia*

I. J. Schoenberg has defined (this MONTHLY, 66 (1959), 365) $\phi$-convergence of a sequence $\{\gamma_n\}$ of real numbers in this way: $\phi$-lim $\gamma_n = \lambda$ if and only if lim $s_n = \lambda$ where $s_n = n^{-1}\Sigma_{d|n}\phi(d)\gamma_d$. ($\phi$ denotes Euler's function.) Find a sequence which is $\phi$-convergent and is not convergent.

6091. *Proposed by H. S. Shapiro, Royal Institute of Technology, Stockholm, Sweden*

Let $\Gamma$ denote the set of complex numbers of modulus one, and consider for positive integers $m, n$ the map $T: \Gamma^n \to \mathbb{C}^m$ defined by

$$w_1 = z_1 + z_2 + \cdots + z_n,$$

$$w_2 = z_1^2 + z_2^2 + \cdots + z_n^2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$w_m = z_1^m + z_2^m + \cdots + z_n^m$$

where each $z_i$ ranges over $\Gamma$. Prove that for any $m$, and any positive $R$, the range of $T$ contains the ball $\|w\| \leqq R$ for all sufficiently large $n$.

6092. *Proposed by Ignacy Kotlarski, Oklahoma State University*

Let $X_1$ and $X_2$ be two independent Student distributed random variables with 1 and 3 degrees of freedom respectively. Denote

(1) $$Y = \tfrac{1}{2}X_1\sqrt{3} + \tfrac{1}{2}X_2.$$

Show that the probability density functions of $X_1, X_2, Y$ satisfy the relation

$$f_Y(y) = \tfrac{1}{2} f_{X_1 \vee 3}(y) + \tfrac{1}{2} f_{X_2}(y) \tag{2}$$

almost everywhere on $R$.

*Query.* Can this analogy be generalized to

$$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n \tag{1'}$$

where $X_1, \cdots, X_n$ are independent Student distributed random variables with $1, 3, \cdots, 2n-1$ degrees of freedom, and $a_1, a_2, \cdots, a_n$ are constants?

6093. *Proposed by Richard Johnsonbaugh, Chicago State University*

Let $X$ be a completely regular Hausdorff space and let $C(X)$ denote all real-valued continuous functions on $X$ with the topology of uniform convergence on compact sets. Let $F$ be a continuous non-zero linear functional on $C(X)$. Prove that there exists a smallest compact set $K$ with the property that if $f = 0$ on $K$, then $F(f) = 0$.

6094*. *Proposed by Francis Cald, Robertsbridge, England*

A pair of primes, $P$ and $Q$, such as 7 and 17 is said to be acquainted if the set of quadratic residues and the set of quadratic nonresidues of $P$ are, respectively, a subset of the set of residues and the set of nonresidues of $Q$. Is there a positive constant $C$ such that infinitely many pairs of acquainted primes exist for which $Q - P \leqq C$?

6095. *Proposed by Anon, Erewhon-upon-Thames*

Let $P, Q, B$ be $m \times m$, $n \times n$, $n \times m$ resp. complex matrices with $P$ and $Q$ positive definite. Show that $P - B^*Q^{-1}B$ is positive definite if and only if $Q - BP^{-1}B^*$ is positive definite.

## SOLUTIONS OF ADVANCED PROBLEMS

### Zeros in the Fractional Calculus

5988 [1974, 910]. *Proposed by Bertram Ross, University of New Haven*

Show that the fractional integration of order $v$, $0 < v < 1$, of $\log x$ equals zero at $x = e^{\psi(1+v)+\gamma}$, and the fractional derivative of order $1 - v$ of $\log x$ equals zero at $x = e^{\psi(v)+\gamma}$. $\gamma$ is Euler's constant and $\psi$ is the logarithmic derivative of the gamma function.

The Riemann-Liouville definition of fractional integration of order $v$ is

$${}_0D_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)\,dt, \qquad v > 0$$

and fractional differentiation of order $1 - v$, $0 < v < 1$, is

$${}_0D_x^{1-v} f(x) = \frac{d}{dx} [{}_0D_x^{-v} f(x)],$$

where $d/dx$ is ordinary differentiation.

*Solution by Fokko J. de Boer, Saclant ASW Research Centre, La Spezia, Italy.* The fractional integration ${}_0D_x^{-v} f(x)$ is given in Bateman and Erdelyi, *Tables of Integral Transforms*, Vol. II, Chapter 13.1 formula 24, and is

$${}_0D_x^{-v} f(x) = \frac{x^v}{\Gamma(v+1)} (\log x + \psi(1) - \psi(v+1)).$$

This function is zero for $x = \exp(\psi(1+v)+\gamma)$.

The second problem can be solved by the equation

$$\frac{d}{dx}\{{}_0D_x^{-v} f(x)\} = 0,$$

or

$$\frac{vx^{v-1}}{\Gamma(v+1)}\{\log x + \psi(1) - \psi(v+1)\} + \frac{x^{v-1}}{\Gamma(v+1)} = 0.$$

This equation gives $x = \exp(\psi(v)+\gamma)$.

Also solved by F. J. Arena, Paul Bruckman, Mark Holmes, Rhonda Hughes, L. Kuipers, O. P. Lossers (Netherlands), M. E. Muldoon (Canada), Ram Murty & Kumar Murty (Canada), Otto Ruehr, Allen Stenger, R. Tremblay (Canada), and the proposer.

*Editor's comment.* The result differs slightly from the published statement of the problem which contained a misprint.

Using formulas which he derives in his solution (stated in one instance in Oldham and Spanier, *The Fractional Calculus*, New York, 1974) Tremblay shows that

$${}_0D_x^{-v} x^{\alpha} \log x \text{ is zero at } x = \exp(\psi(1+\alpha+v) - \psi(1+\alpha)),$$

$${}_0D_x^{1-v} x^{\alpha} \log x \text{ is zero at } x = \exp(\psi(\alpha+v) - \psi(1+\alpha)).$$

**Substitution Groups**

5990 [1974, 910]. *Proposed by H. Kestelman, University College, London, England*

$G$ is a group of permutations of $1, 2, \cdots, n$. For each $f$ in $G$, $w(f)$ denotes the number of integers $r$ with $f(r) = r$, and $L(f)$ denotes the transformation of $R^n$ mapping $(x_1, x_2, \cdots, x_n)$ onto $(x_{f(1)}, x_{f(2)}, \cdots, x_{f(n)})$. Prove that the arithmetic mean of the $w(f)$ is the dimension of the set of $x$ that are fixed points for all the $L(f)$.

*Solution by George Gearhart, Marblehead, Massachusetts.* Let $i \in N = \{1, 2, \cdots, n\}$. First consider $S =$ the set of all $x$ in $R^n$ which are fixed by all the $L(f)$. Let $Gi =$ the orbit of $i$ under all elements of $G$. Then $x = (x_i)$ is fixed by all $f$ in $G$ if and only if $x_i = x_j$ whenever $j \in Gi$, $i = 1, 2, \cdots, n$. Hence $x = (x_i) \in S$ contains $r$ sets of equal numbers, where $r$ is the number of distinct orbits $Gi$ in $\{1, 2, \cdots, n\}$, and the dimension of the set $S$ is $r$.

Now consider the arithmetic mean of the $w(f)$. The mean equals $(1/|G|)\Sigma_f w(f)$, where $|G| =$ the order of $G$. But

$$\sum_f w(f) = \sum_{i \in N} v(i), \quad \text{where } v(i) = \text{ the number of } f \in G: f(i) = i.$$

$$= \sum_{i \in N} |G_i|, \quad \text{where } G_i = \text{ the subgroup of } f \in G: f(i) = i,$$

$$= \sum_i \frac{|G|}{|Gi|} \quad \text{since } |Gi| \cdot |G_i| = |G|.$$

Therefore the arithmetic mean of the $w(f) = \Sigma_i 1/|Gi|$ which is

$$\sum_{i \in Gs_1} \frac{1}{|Gi|} + \sum_{i \in Gs_2} \frac{1}{|Gi|} + \cdots + \sum_{i \in Gs_r} \frac{1}{|Gi|} = r,$$

because $\Sigma_{i \in Gs_j} 1 / |Gi| = 1$ for all $Gs_j$. Here the $Gs_j$ are the $r$ distinct orbits in $N$.

Also solved by Ethan Bolker, William Dodge, John Dyer, Clark Givens, William Gustafson, Ellen Hertz, Stefen Hui, A. A. Jagers (Netherlands), David Johnson (England), O. P. Lossers (Netherlands), John Moore, Jr., Ram Murty & Kumar Murty (Canada), Allen Schwenk, Robert Snyder, B. F. Woodward, and the proposer.

**Sum of Blocks in a Hermitian Matrix**

5992 [1974, 911]. *Proposed by H. J. Thiebaux, Boulder, Colorado*

Let

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

denote an hermitian, positive definite matrix, where each block of the partitioning is $m \times m$, and define

$$Y = \begin{pmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & & \vdots \\ Y_{n1} & & Y_{nn} \end{pmatrix} = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & & X_{nn} \end{pmatrix}^{-1} = X^{-1}$$

with corresponding partitions. Prove that $(\Sigma_{r,t=1}^{n} Y_{rt})^{-1}$ is hermitian, positive definite.

*Solution by Clark Givens, Michigan Technological University.* If $A = \Sigma_{r,t=1}^{m} Y_{rt}$, then $A = E^* Y E$ where $*$ denotes hermitian conjugate and $E$ is the $mn \times m$ matrix $E = (I_m \mid I_m \mid \cdots \mid I_m)^*$. By a conjugation of $A$ and consideration of $z^*Az$, for $z$ in $C^m$, it follows that $A$ is hermitian, positive definite since $Y$, as $X^{-1}$, possesses these properties and $E^*E = I_m$. Thus, the existence and hermitian positive definite nature of $A^{-1}$ is established.

REMARKS. (1) $A$ is just a special case of the following. If $H$ is an hermitian positive definite $N \times N$ matrix and $F$ is $N \times d$ and of maximal rank, then

$$R(F) = (F^*F)^{-1/2}(F^*HF)(F^*F)^{-1/2}$$

is hermitian positive definite if $(F^*F)^{-1/2}$ is the positive definite root of $F^*F$. In particular, if $d$ is a divisor of $N$, then there are compatible partitionings of $H$ and $F$ so that the $d \times d$ matrix $R(F)$ is given by

$$R(F) = \sum_{i,j=1}^{N/d} (F^*F)^{-1/2} F_i^* H_{ij} F_j.$$

(2) Choose $F^*F = I_d$, so that $R(F) = F^*HF$. Denote the numerical ranges of $R(F)$, $H$ by $\gamma(R(F))$, $\gamma(H)$ respectively. Then for any $z$ in $C^d$,

$$z^*R(F)z = (Fz)^*H(Fz)$$

and we obtain $\gamma(R(F)) \subset \gamma(H)$. Since $\gamma(H)$ is the interval $[\gamma_{\min}, \gamma_{\max}] \subset \{x : x > 0\}$, we again conclude that $R(F)$ is positive definite. This conclusion could also have been drawn without the attendant calculation, for it is known from min-max theory that $R(F)$ estimates the intermediate eigenvalues of $H$.

(3) In the real cases, i.e., $H$ symmetric positive definite, it is suggestive to regard $R(F)$, when restricted to the "sphere" $S = \{F : F \text{ is real } N \times d,\ F^TF = I_d\}$, as a generalized Rayleigh quotient and, by analogy with $\lambda_{\max} = \max_{z} T_{z=1}(z^THz)$, and then to ask whether, with the partial ordering '$A \geqq B$ if $a_{ij} \geqq b_{ij}$ for all $i, j$', it is possible to find a $d \times d$ matrix $\Lambda_{\max}$ for which $\Lambda_{\max} \geqq R(F)$ for all $F$ in $S$.

Also solved by A. J. Bosch (Netherlands), C. G. Cullen, John Dyer, Thomas Foregger, Emilie Haynsworth, A. A. Jaggers (Netherlands), Thomas Markham, Paul Nikolai, G. S. Rogers, F. B. Strauss, Lawrence Thomas, and the proposer.

**Integration by Parts**

5994 [1974, 1033]. *Proposed by J. J. Higgins, University of Missouri, Rolla*

Let $F$ and $G$ be right-continuous distribution functions corresponding to the Lebesgue-Stieltjes measures $\mu_F$ and $\mu_G$ respectively. If $F$ and $G$ have no common points of discontinuity, the well-known integration by parts formula gives

$$\int_{(a,b]} F(x)\mu_G(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} G(x)\mu_F(dx).$$

Give the corresponding integration by parts formula when $F$ and $G$ are allowed to have common points of discontinuity.

*Solution by William S. Griffith, University of Pittsburgh.* Both the above formula and the proposed problem are special cases of a more general integration by parts result which appears, for example, as Theorem 21.67 in Hewitt and Stromberg, *Real and Abstract Analysis.* From that result we know

$$\int_{(a,b]} F(x)\mu_G(dx) = F(b)G(b) - F(a-)G(a-) - \int_{(a,b]} G(x-)\mu_F(dx). \tag{1}$$

Let $A$ be the discontinuity points of $F$ in $(a,b]$ which are continuity points of $G$, $B$ the discontinuity points of $G$ in $(a,b]$ which are continuity points of $F$, and $C$ the common points of discontinuity in $(a,b]$. Since $F$ and $G$ are increasing, we know that each has at most countably many discontinuities. We let $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ be enumerations of the sets $A$, $B$ and $C$ respectively. Further we let $\alpha_i = F(c_i) - F(c_i-)$ and $\beta_i = G(c_i) - G(c_i-)$.

Next we investigate $\int_{(a,b]} G(x-)\mu_F(dx)$. Since $G(x) = G(x-)$ for $x$ in $(a,b]\backslash B \cup C$ and $\mu_F(B) = 0$, we have

$$\int_{(a,b]\backslash B\cup C} G(x-)\mu_F(dx) + \int_B G(x-)\mu_F(dx) = \int_{(a,b]\backslash B\cup C} G(x)\mu_F(dx) + \int_B G(x)\mu_F(dx).$$

Further

$$\begin{aligned}\int_C G(x-)\mu_F(dx) &= \int_C G(x)\mu_F(dx) - \int_C [G(x) - G(x-)]\mu_F(dx)\\ &= \int_C G(x)\mu_F(dx) - \sum \alpha_i\beta_i.\end{aligned}$$

This entails

$$\int_{(a,b]} G(x-)\mu_F(dx) = \int_{(a,b]} G(x)\mu_F(dx) - \sum \alpha_i\beta_i.$$

Putting this together with (1), and simplifying, we end up with

$$\int_{(a,b]} F(x)\mu_G(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} G(x)\mu_F(dx) + \sum \alpha_i\beta_i.$$

Also solved by Paul Nelson, Jr., and by the proposer.

*Editor's comment.* Nelson notes that questions concerning integration by parts for Stieltjes integrals may be found in D. K. Butler, Nuclear Science and Engineering 52 (1973), pp. 492–494; P. Nelson, J. Nuclear Energy 26 (1972) pp. 43–44, and elsewhere.

**Analytic Functions of Bounded Operators**

5995 [1974, 1034]. *Proposed by L. A. Harris, University of Kentucky*

Let $A$ be a bounded linear operator on normed linear space. Suppose $A$ satisfies a quadratic equation with roots $\lambda_1$ and $\lambda_2$ and that $A$ is not a scalar multiple of the identity operator. Show that $\lambda_1$ and $\lambda_2$ are eigenvalues of $A$, $\sigma(A)=\{\lambda_1, \lambda_2\}$, and

$$f(A)=\begin{cases} f(\lambda_1)I+f'(\lambda_1)(A-\lambda_1 I) & \text{for } \lambda_1=\lambda_2, \\ \dfrac{1}{\lambda_1-\lambda_2}[f(\lambda_1)(A-\lambda_2 I)-f(\lambda_2)(A-\lambda_1 I)] & \text{for } \lambda_1\neq\lambda_2, \end{cases}$$

for any function $f$ which is analytic in a disk containing both $\lambda_1$ and $\lambda_2$. (For example, any $2\times 2$ matrix satisfies its characteristic equation, which is quadratic.)

*Solution by M. L. J. Hautus, Technological University, Eindhoven, The Netherlands.* Consider the more general situation where $A$ is a bounded operator on a Banach space (modified below) and $p$ is a minimal polynomial of $A$ (i.e., $p(A)=0$; $\deg q \geqq \deg p$ for every $q$ with $q(A)=0$). We show that $\sigma(A)$ is the set of zeros of $p$ and that each element of $\sigma(A)$ is an eigenvalue of $A$. Furthermore, if $f$ is an analytic function on a neighborhood of $\sigma(A)$, then there exists a polynomial $r$ of degree at most $n-1$ and depending only on $p$ and $f$ such that $f(A)=r(A)$. More specifically,

$$r(\lambda)=\frac{1}{2\pi i}\oint_C \frac{f(z)q(\lambda,z)}{p(z)}\,dz, \tag{1}$$

where $C$ is any contour (or collection of contours) surrounding the points of $\sigma(A)$ and

$$q(\lambda,z)=(p(\lambda)-p(z))/(\lambda-z). \tag{2}$$

If $p$ is a quadratic polynomial, the expression for $f(A)$ given in the problem follows from these results by formal residue calculus.

It follows from the equation

$$(z-\lambda)q(\lambda,z)=p(z)-p(\lambda) \tag{3}$$

that

$$(zI-A)q(A,zI)=p(z)I. \tag{4}$$

We note that $q(a, zI)\neq 0$ for every $z$, since the degree of $q$ with respect to $\lambda$ is at most $n-1$ and $q$ is not zero (see (3)), $p$ is not constant. We consider two cases:

(a) $p(z)=0$. Choose $x$ such that $y=q(A,zI)x\neq 0$. Then according to (4), $y$ is an eigenvector of $A$ and $z$ is an eigenvalue.

(b) $p(z)\neq 0$. Then it follows from (4), that

$$q(A,zI)/p(z)=(zI-A)^{-1} \tag{5}$$

hence $z\notin\sigma(A)$.

This proves the statements about the spectrum and the eigenvalues of $A$. Now, if $f$ is analytic on $\sigma(A)$ we have

$$f(A)=\frac{1}{2\pi i}\oint_C f(z)(zI-A)^{-1}dz \tag{6}$$

and substitution of (5) gives the desired result.

The completeness of the space is used to ensure the existence of the integral in (6). If the space is not complete, we may extend $A$ continuously on the completion of the space. Afterwards it turns out that $f(A)$ is also defined in the original space.

Also solved by Alan Berger, Paul Chauveheid (Belgium), D. K. Cohoon, Amiel Feinstein, Wolfe Snow, and the proposer.

# REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.*

*Women in Mathematics.* By Lynn Osen. M.I.T. Press, Cambridge, Massachusetts, 1974. xii + 185 pp. $4.95 (paper). (Telegraphic Review, June–July 1975.)

*Women in Mathematics* addresses itself to a topic of current interest. One would like to have such a book to recommend to young women mathematics majors who aren't sure women have a place in mathematics and to some professors who feel the same way.

The principal part of the book lies in the eight middle chapters, one each devoted to Hypatia, Maria Agnesi, Emilie de Breteuil, Caroline Herschel, Sophie Germain, Mary Fairfax Somerville, Sonya Kovalevsky, and Emmy Noether. These are flanked by chapters called "History" and "The Golden Age of Mathematics." The book ends with a chapter called "The Feminine Mathique."

The preface begins with the sentence, "*Women in Mathematics* grew out of the desire to trace the impact women have had on the development of mathematical thought, to profile the lives of these women, and to explore the social context within which they worked." Ms. Osen ignores the first desire, devotes 99% of her effort to the second, and superficially considers the last. The major failure of the book is not that we do not learn how these women influenced "the development of mathematical thought." The major failure is that we are not told what mathematics they did. Why should we consider Hypatia a significant mathematician? What theorems are due to Emilie de Breteuil? From the amount of space devoted to them, one might infer that Mary Somerville is more important than Emmy Noether. This is most upsetting since Ms. Osen piously excludes from consideration contemporary women mathematicians because "...whoever attempts to do full justice to the subject of women in mathematics needs the perspective of history to look back with sympathy and understanding on their *work*" (p. 8). One hoped to at least look back on the work of the women included.

Ms. Osen does not seem to know much mathematics or its history. Thus, we have "...Karl Weierstrass, a logician..." (p. 123); "...cross products..." instead of crossed products (p. 147) and a reference to the great Oxford topologist A. N. Whitehead (p. 160). In the introduction she says, "There seems to be general agreement that Newton, Gauss, and Archimedes are the names of the first order; beyond these, the order is equivocal." Therefore, it is "tiresome and counterproductive to argue about the relative merits of male and female mathematicians..." I am not interested in making such comparisons, but Ms. Osen should not base her unwillingness to make them on a silly and false premise. It is not correct, especially in a book addressed to the general reader, to imply that some calculating hack is comparable to Euler or Lagrange.

There is a scholarly problem beyond her insufficient mathematical knowledge. Repeatedly we see the construction: "Some writers have made the claim P(X)," where P(X) is some statement about the value of Madam X. Invariably the reference is the same: Mozens, H. J., *Women in Science*, 1913. Such narrow research is unforgivable. It is not excused by saying this is a book of modest ambition. Slipshod work is not modest, it is merely discouraging.

R. P. INFANTE, Seton Hall University

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
P = professional reading
L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T*(13: 1), L*. *Mathematical Relationships in Business and Economics*. William E. Beatty. Prindle, 1970, xiv + 465 pp, $9.50 (P). An interesting freshman prelude to calculus. In the setting of business applications, linear and nonlinear functions, matrix algebras, linear programming are introduced, along with a brief glimpse of differential calculus. Except for the lack of any trigonometry, this could serve as an attractive format for precalculus courses. TAV

GENERAL, *Mathematics, Models of the Real World*. Peter Lancaster. P-H, 1976, xi + 164 pp, $9.95. Intended as a text for a course in modelling at the precalculus level. Topics: linear programming, population growth, extremal search techniques, population genetics, collision of particles. In place of calculus, difference equations are used extensively. An interesting alternative course for freshmen. TAV

GENERAL, T(13-14: 1, 2), L. *Mathematics, The Man-made Universe: An Introduction to the Spirit of Mathematics, Third Edition*. Sherman K. Stein. Freeman, 1976, xv + 573 pp, $12.50. Good selection of topics; can be used in classes for non-science, education, or math majors; helpful suggestions for lecture planning in teachers manual. Includes a new chapter on probability; chapter 1 of second edition has been omitted. JG

GENERAL, T(13-14: 1, 2), L. *A Mathematical Journey*. Stanley Gudder. McGraw, 1976, xiii + 434 pp, $11.95. A lively text for non-science majors; includes chapters on mathematics and art, computer science, group theory (with a section on communications systems) and graph theory, as well as the more standard topics, number theory and probability. Good bibliography. JG

GENERAL, T(15-16: 1, 2), *Mathematics for Economists*. K. Holden, A.W. Pearson. David & Charles, 1975, 338 pp, $25. Written primarily for students of economics with very little mathematical training, this is a *very* unrigorous introduction to a vast amount of mathematics, including linear and quadratic equations, trigonometry, matrix algebra, calculus, Taylor's series, linear programming, and differential and difference equations. Many exercises, solutions with hints. JD-B

BASIC, T(13), *Plane Trigonometry, Third Edition*. Nathan O. Niles. Wiley, 1976, xiii + 394 pp, $10.95. A nice treatment of trigonometry. Trig functions are defined first for acute angles, using the right triangle, then for any angle in standard position, then as a function of real numbers. The rest of the book is standard fare: identities, equations, solving triangles, and a chapter on logarithms. One good feature is the immediate following of each new concept by a similar problem for the student to do. PJM

BASIC, T(13: 1), *Elementary Algebra with Geometry*. Irving Drooyan, William Wooton. Wiley, 1976, x + 334 pp, $11.50. Similar to fourth edition of *Elementary Algebra for College Students*, but with added chapter on geometry (area, volume formulas, trigonometry ratios). Textual material is brief, with many sample problems. Separate booklet of sample problems and completely worked out solutions to even-numbered problems is available. LCL

BASIC, T(13: 1), *Algebra, A College Course*. William L. Hart. Prindle, 1975, viii + 269 pp, $10.95. Sensible balance, by a long-time teacher and author, between manipulative and theoretical aspects of the algebra needed at the college level. LCL

BASIC, T(13: 1), *Elementary Algebra*. Donald Cohen, Roy Cameron. Cummings, 1976, 436 pp, $8.95 (P). Many detailed examples and problems. Lack of semantic and other distinctions makes some of the material more difficult than it should be. FLW

BASIC, T(13: 1), *Algebra: A Personalized Approach*. George L. Henderson, Walter W. Leffin, Mary I. Van Beck. Prindle, 1976, xiii + 578 pp, $9.95 (P). An attractive and useful text for students who have some algebra. Not a "programmed" text but there are many exercises and diagnostic quizzes. FLW

BASIC, T(13: 1, 2), *Business Mathematics*. Steven C. Lawlor. Canfield Pr, 1976, ix + 469 pp, $11.95. Review of arithmetic, pricing, interest, taxes, depreciation, insurance, descriptive statistics. FLW

BASIC, T(13: 1), *Beginning Algebra: A Modular Approach, Module I-VIII*. Irving Drooyan, William Wooton. Wiley, 1976. *Module I, Natural Numbers*, 56 pp; *Module II, Integers*, 90 pp; *Module III, First-Degree Equations and Inequalities in One Variable*, 83 pp; *Module IV, Products and Factors*, 67 pp; *Module V, Fractions and Fractional Equations*, 129 pp; *Module VI, Graphs and Linear Systems*, 129 pp; *Module VII, Radicals*, 68 pp; *Module VIII, Quadratic Equations*, 92 pp, $27.50 set (P). A sequence of eight programmed modules which cover the topics of ninth grade algebra. Designed so that a conscientious student could work independently. Almost complete reliance on learning by answering questions. CEC

BASIC, T(13: 1), *Basic Mathematics*. Virginia Lee. Canfield Pr, 1976, xi + 384 pp, $8.95 (P). A work-text, covering operations on whole numbers, primes, fractions, decimals, order of operations, and ratio and proportion. An abundance of examples, exercises, tests, and word problems. PJM

BASIC, T(13: 1, 2). *Intermediate Algebra, A Straightforward Approach for College Students.* Martin M. Zuckerman. Norton, 1976, xiii + 666 pp, $11.95; *Instructor's Manual to Accompany Intermediate Algebra,* 117 pp, free (P); *Workbook to Accompany Intermediate Algebra,* xi + 210 pp, $4.95 (P). Topics include factoring, rational expressions, word problems, linear systems, radicals, logarithms, quadratic equations, inequalities, variation and progressions. A direct approach with no frills. Set theoretic notation is avoided. CEC

BASIC, T*(13: 1, 2). *Introductory Algebra.* Ronald J. Harshbarger. Har-Row, 1976, x + 294 pp, $10.95; *Intermediate Algebra,* viii + 407 pp, $10.95. Linear and quadratic equations, exponents and radicals; functional approach to polynomial, exponential, logarithmic equations. Careful presentation plus attractive format. Step by step instructions sandwiched between large numbers of examples and exercises. Semiprogrammed approach allows for individualized instruction. LCL

PRECALCULUS, T(13: 1). *Trigonometry.* Frank C. Denney. Canfield Pr, 1976, ix + 241 pp, $9.95. A look at triangle trigonometry followed by a study of the trigonometric functions, complex numbers, polar coordinates, and some applications. FLW

PRECALCULUS, T*(13: 1, 2), S. *Algebra and Trigonometry Modules, V. I-V.* James E. Hall, Marion B. Smith. Cummings, 1976. *V. I: Fundamentals,* 87 pp; *V. II: Equations,* 120 pp; *V. III: Functions,* 122 pp; *V. IV: Trigonometric Functions: Theory,* 87 pp; *V. V: Trigonometric Functions: Applications,* 100 pp, $2.95 each (P). A very nicely written set of modules which cover the standard topics of college algebra and trigonometry. CEC

EDUCATION. *Mathematics at the School/Industry Interface.* Inst Math Appl, 1975, v + 77 pp, £2.50 (P). Heavy sounding title intended to lend weight to these regional conference proceedings (five papers) expressing grass-roots disenchantment over shortcomings in British elementary mathematics education. LCL

HISTORY, P. *Opera matematica.* Nicolae Cioranescu. Editura Academiei Romania, 1975, 695 pp, Lei 43. Mostly functional analysis in French, with the few post-war papers in Romanian. JAS

FOUNDATIONS, T(18), P. *Proof Theory.* Gaisi Takeuti. Stud. in Logic and Found. of Math., V. 81. North-Holland, 1975, vii + 372 pp, $35.50. Study of Gentzen's approach to proof theory. Much of the material represents the author's own work in the field; minimal overlap with other textbooks. LCL

FOUNDATIONS, T(18), P. *Large Infinitary Languages, Model Theory.* M.A. Dickmann. Stud. in Logic and Found. of Math., V. 83. North-Holland, 1975, xv + 464 pp, $39.95. Development and applications of the model theory of the languages $L_{\kappa\lambda}$, $\infty \geq \kappa \geq \lambda$. $L_{\omega_1\omega}$ is not treated. LCL

FOUNDATIONS, P. *Intensional and Higher-Order Modal Logic with Applications to Montague Semantics.* Daniel Gallin. Math. Stud., V. 19. North-Holland, 1975, ix + 148 pp, $10 (P). The first complete technical account of the highly original work concerning natural language syntax and semantics begun by Richard Montague in 1967 but cut short by his tragic death in 1971. LCL

FOUNDATIONS, P. *Oeuvres Choisies, Tome III.* Waclaw Sierpiński. PWN, 1976, 685 pp. This volume contains more than one hundred papers on set theory selected on the basis of importance and intrinsic beauty. The papers span the period 1930-1966. JAS

NUMBER THEORY, T*(14-15: 2), L*. *Introduction to Number Theory.* William W. Adams, Larry Joel Goldstein. P-H, 1976, xiii + 362 pp, $13.95. The first half of this book is a beginning level introduction to elementary number theory. The second half is an introduction to algebraic number theory which assumes some abstract algebra. This is a well-written text with outstanding exercises. CEC

NUMBER THEORY, S(18), P. *Abstract Analytic Number Theory.* John Knopfmacher. Math. Lib., V. 12. North-Holland, 1975, ix + 322 pp, $29.50. Classical analytic number theory is applied to a wide variety of subjects not usually treated in an arithmetic way. Based on recent research. Includes a thorough bibliography and a selection of unsolved questions. CEC

NUMBER THEORY, P. *The Theory of Numbers.* Ed: S. Iyanaga. Trans: K. Iyanaga. Math. Lib., V. 8. North-Holland, 1975, xi + 541 pp, $51.95. A complete development of class field theory via cohomology of groups. The cohomology is developed first; then, valuation theory and the structure of adele rings and idele groups are described; finally, the main theorems are proved. An appendix on ideal theory (including Artin-Whaples) and a history of class field theory are included. A useful complement to the books of Lang, Weil, Janusz, and Goldstein. SG

NUMBER THEORY, S(15-17), P. *Einführung in Die Zahlentheorie.* Wolfgang Schwarz. Wissenschaftliche Buchgesellschaft, 1975, viii + 180 pp, 13 DM (P). A discussion of concepts, methods, results and problems in number theory, including quite recent work, without proofs or with only sketches of them. JD-B

LINEAR ALGEBRA, T(13-14: 1, 2). *Introductory Linear Algebra with Applications.* Bernard Kolman. Macmillan, 1976, xiii + 426 pp, $11.95. Intended for freshman-sophomore level course, either semester or year long; calculus not a prerequisite. Covers matrices, determinants, vector spaces (over R), linear transformations, eigenvalues, linear programming, applications to quadratic forms, graph theory, game theory, economic models, and numerical linear algebra. Plenty of exercises. SG

LINEAR ALGEBRA, P. *Lecture Notes in Mathematics-489: Etude Géométrique des Espaces Vectoriels, Une Introduction.* Jacques Bair, René Fournea. Springer-Verlag, 1975, vii + 185 pp, $8.20 (P). A study of convex sets and other geometric objects in vector spaces. The geometric objects are defined in terms of algebraic operators (e.g., convex hull for convexity) on sets in vector spaces. The program is then to characterize properties of vector spaces in terms of those operators. PJM

LINEAR ALGEBRA, S(13). *Exploring Matrices and Vectors.* Paul R. Scott. Wiley, 1975, v + 98 pp, $3.95 (P). Uneven collection of discovery exercises grouped into fifteen sections. Credibility undermined by illustrations whose appeal could only be to pre-teens. LCL

ALGEBRA, S(18), P*, L. *Conference on Commutative Algebra-1975.* Ed: A.V. Geramita. Pure and Appl. Math., No. 42. Queen's U, 1975, ii + 268 pp, (P). Swan's exposition of Serre's problem is featured, and includes most of the better known results in addition to the most recent work of Suslin and Vaserstein. Also, texts of invited addresses on other topics in commutative algebra by Bass, Eisenbud, Murthy, Hochster and others. JG

ALGEBRA, T(16-17), S, L. *Homological Methods in Commutative Algebra.* S. Raghavan, Balwant Singh, R. Sridharan. Tata Inst, 1975, ix + 121 pp, $2.25 (P). After preliminaries (functors on module categories, results from commutative algebra, definition of Tor and Ext), the authors discuss dimension theory, a homological characterization of local rings, and unique factorization in regular local rings. Could be used by good senior undergraduates with strong background in algebra. PJM

ALGEBRA, P. *Lie Algebras Generated by Finite-dimensional Ideals.* I. Stewart. Pitman, 1975, 154 pp, $12.90 (P). Generalizes the classical theory on the structure of finite-dimensional Lie algebras over algebraically closed fields of characteristic 0. Contains new results on the Frattini subalgebra. Substantial bibliography. JG

ALGEBRA, T(17-18: 1, 2), S, P, L. *Commutative Algebra, V. 1.* Oscar Zariski, Pierre Samuel. Grad. Texts in Math., V. 28. Springer-Verlag, 1975, xi + 329 pp, $14.80. Reprint of the 1958-1960 Van-Nostrand edition, with correction of some misprints and amplification of Theorem 29 of Chapter V.JG

ALGEBRA, S(17-18), P, *Lie Algebras.* Zhe-Xian Wan. Trans: Che-Young Lee. Pure and Appl. Math., V. 104. Pergamon Pr, 1975, vii + 231 pp, $19.80. An introduction to the classical theory of complex semisimple Lie algebras. Treats structure, automorphisms, representations, real forms. Most of it requires no knowledge of Lie groups. No exercises or bibliography. JD-B

ALGEBRA, T(15-16: 1), *Abstract Algebra, An Active Learning Approach.* Neil Davidson, Frances Gulick. HM, 1976, xv + 336 pp, $12.95; *Instructor's Manual,* v + 75 pp, (P). A gentle introduction to abstract algebra, conventional in coverage but designed for use in classes in which the students work in small groups, assisted by the instructor, who almost never lectures. Definitions and theorems are stated, but students are expected to construct proofs. JD-B

ALGEBRA, T*(16-17: 1, 2), S, P, L. *Presentations of Groups.* D.L. Johnson. London Math. Soc. Lect. Notes, No. 22. Cambridge U Pr, 1976, v + 204 pp, $11.95 (P). Study of groups specified by means of generators and relations--concrete, combinatorial flavor. First half: examples, main ideas and methods; excellent senior seminar material. Second half: development of main tool--cohomology theory! --and applications to theory of minimal presentation. LCL

FINITE MATHEMATICS, T(14-16: 1, 2), *Finite Mathematics with Models.* Gareth Williams. Allyn, 1976, xii + 466 pp, $12.95. Models selected from a variety of disciplines (e.g., economics, sociology, biology) are formulated and applied. Some of the necessary theoretical background is provided in chapters on matrices, computing and functions; the rest is introduced where needed. Heavily computer oriented; uses programming in BASIC. JG

FINITE MATHEMATICS, T(13-14: 1), *Matrix Methods in Finite Mathematics: An Introduction with Applications to Business and Industry.* Steven C. Althoen, Robert J. Bumcrot. Norton, 1976, xii + 451 pp, $11.95. Informal text, with emphasis on linear programming and applications to business. Each chapter includes supplementary and review exercises. JG

FINITE MATHEMATICS, T*(13-14: 1), *Introductory Finite Mathematics with Computing.* William S. Dorn, Daniel D. McCracken. Wiley, 1976, xiv + 449 pp, $12.95. Emphasizes model building in real situations and includes the use of BASIC. Difference equations, probability, matrices, linear equations, and linear programming. FLW

FINITE MATHEMATICS, T(13-14: 1-3), *Finite Mathematics and Calculus with Applications.* G.M. Stratopoulos. Dun-Donnelley, 1976, x + 470 pp, $13.50. Sets, probability, matrices, linear programming, game theory, and a brief introduction to the calculus. FLW

CALCULUS, S(13), L, *Calculator Calculus.* George McCarty. Page Ficklin, 1975, xiv + 254 pp, $8.95 (P). Good as a workbook to accompany a standard calculus text. Uses the pocket calculator to illustrate tedious numerical calculations--roots, approximations to limits of difference quotients, Riemann sums, trapezoidal rule, error bounds, partial sums of series, numerical solutions to differential equations. Appendix on efficient use of a calculator. Reference tables. Bibliography. Index. RJA

CALCULUS, T*(13: 1, 2), L**, *Calculus, A Practical Approach.* Kenneth Kalmanson, Patricia C. Kenschaft. Worth, 1975, xiv + 335 pp, $10.95. If you can teach calculus without any mention of trigonometric or inverse trigonometric functions, this text is worthy of consideration. Built around the concept of modelling, there are more relevant applications in this book than most double its size. Even if the constraints of the "algebra only" prerequisite is too severe, consider this as a source of classroom examples. Nice features: a teacher's guide and preconstructed tests. TAV

CALCULUS, T(13: 1), *Calculus for Business.* Richard D. Anderson, Cecil L. Smith. Dun-Donnelley, 1976, xiii + 465 pp, $12.50. A one-semester text covering basics of differentiation, integration, exponentials and logs, partial derivatives, and first order differential equations. Most applications and many exercises are taken from business and economics. Many exercises. SG

CALCULUS, T(13-14: 1, 2), *Calculus for Business, Biology, and The Social Sciences, Second Edition.* David G. Crowdis, Susanne M. Shelley, Brandon M. Wheeler. Glencoe Pr, 1976, vii + 471 pp, $12.95. An intuitive treatment that emphasizes examples from the social and biological sciences. This new edition has a review of trigonometry and additional exercises. (First edition TR, August 1974.) FLW

REAL ANALYSIS, T(14-15; 1). *Introduction to Analysis, Second Edition.* Edward D. Gaughan. Brooks/Cole, 1975, viii + 228 pp, $13.95. Intended to bridge the gap between intuitive calculus and advanced analysis courses. Carefully written, with a small but carefully chosen collection of exercises. Topics range from sequences to uniform convergence. The author's style and definition-theorem-corollary format make for somewhat dry reading. TAV

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-483: Riemann Surfaces, Theta Functions, and Abelian Automorphisms Groups.* Robert D.M. Accola. Springer-Verlag, 1975, 105 pp, $7.40 (P). Study of vanishing properties of theta functions for surfaces which admit an abelian group of automorphisms, and the extent to which vanishing properties characterize such surfaces. RBK

COMPLEX ANALYSIS, T(15-16). *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering.* E.B. Saff, A.D. Snider. P-H, 1976, xiii + 444 pp, $18.75. An attractive text for an undergraduate course. Concepts are geometrically and physically motivated. Deformation approach to Cauchy's theorem. Readable, mathematically sound. Good examples and problems. Extensive sections on physical applications of conformal maps. RBK

DIFFERENTIAL EQUATIONS, S(18), P*. *The Wave Equation on a Curved Space-Time.* F.G. Friedlander. Cambridge U Pr, 1975, x + 282 pp, $39.50. Uses the language of the theory of distributions and differential geometry. Presents modern treatment of wave equations, e.g., fundamental solutions, existence and uniqueness theorems, application to the Cauchy problem. Ties well the ideas familiar from wave propagation in a flat Lorentzian manifold (also called space-time) to generalizations in curved space-time. I-CH

DIFFERENTIAL EQUATIONS, P. *Improperly Posed Boundary Value Problems.* Ed: A. Carasso, A.P. Stone. Pitman, 1975, vi + 157 pp, $12.90 (P). A collection of papers comprising almost all of the lectures delivered by the participants, but not by the principal speaker, in an NSF-CBMS Regional Conference in May, 1974 at the University of New Mexico. I-CH

DIFFERENTIAL EQUATIONS, P. *Differentialgleichungen im Komplexen.* Horst Herold. Vandenhoeck & Ruprecht, 1975, 190 pp, 22 DM (P). An introduction to the theory of ordinary differential equations in the complex domain. Assumes knowledge of some function theory, but very little about differential equations over the reals. No problems, tiny bibliography. JD-B

NUMERICAL ANALYSIS, S(16-18), P*. *A Handbook of Numerical Matrix Inversion and Solution of Linear Equations.* Joan R. Westlake. Krieger, 1975, viii + 171 pp, $12.50. Reprint of the 1968 original edition, an internal report for Control Data Corp. Intended to be "nearly encyclopedic." Users may find this handbook very helpful, especially the chapter which comments on direct and iterative methods with comparisons. Excellent set of references and appendices. Glossary of matrix terminology, theorems and test matrices. I-CH

NUMERICAL ANALYSIS, T(15-16: 2), S*, L. *Numerical Methods for Engineers and Scientists, A Students' Course Book.* A.C. Bajpai, I.M. Calus, J.A. Fairley. Taylor & Francis, 1975, xii + 380 pp, £6.75 (P). Develops the practical side of numerical analysis in carefully sequenced steps. Three basic units: Equations and Matrices, Finite Differences and their Applications, and Differential Equations. Each unit consists of programs grouped on allied topics. A number of frames divide up each program. A good book for self-instruction with its own limitations (e.g., lack of algorithm testing on computers). I-CH

NUMERICAL ANALYSIS, T(15-16: 1), S*, L. *Scientific Analysis on the Pocket Calculator.* Jon M. Smith. Wiley, 1975, xii + 380 pp, $12.95. Examines the numerical methods and formulas for performing (numerical) analysis on the different types of pocket calculators. Discusses their limitations and capabilities from the standpoint of software. Demonstrates the other important aspects of the pocket calculators (even the simplest four-function ones): as a learning machine and as a research tool. A good book--readable, stimulating and money-stretching. I-CH

FUNCTIONAL ANALYSIS, P. *Extensions of Positive Operators between Banach Lattices.* Donald I. Cartwright. Memoirs No. 164. AMS, 1975, iv + 48 pp, $6 (P). Gives a necessary and sufficient condition for a Banach lattice to be injective in terms of certain intersection properties for such a lattice. References. RJA

FUNCTIONAL ANALYSIS, T(18), P. *Automatic Continuity of Linear Operators.* Allan M. Sinclair. London Math. Soc. Lect. Notes, No. 21. Cambridge U Pr, 1976, 92 pp, $6.95 (P). A linear function from the reals to the reals is continuous. The algebraic condition of linearity forces continuity. If the reals are replaced by a Banach space, linearity alone is no longer sufficient. This monograph investigates what additional algebraic conditions on maps from one Banach space to another force continuity.PJM

FUNCTIONAL ANALYSIS, T(18: 2), S, P. *Vector Measures and Control Systems.* Igor Kluvánek, Greg Knowles. Math. Stud., V. 20. North-Holland, 1976, ix + 180 pp, $13.50 (P). Presents the theory needed for investigating the range of a vector-valued measure. Purposely omits the material on construction of vector measures, Radon-Nikodym theory, representation of linear maps, etc. Studies the so-called control systems (objects slightly more general than vector measures) which serve as suitable models for many control problems. I-CH

FUNCTIONAL ANALYSIS, P. *Approximation of Functions of Several Variables and Imbedding Theorems.* S.M. Nikol'skiĭ. Trans: J.M. Danskin. Grund. math. Wissenschaften, B. 205. Springer-Verlag, 1975, viii + 420 pp, $46.50. Presents both direct and inverse theorems for the theory of approximations and the theory of imbeddings. The inverse theorems of approximations indicate the class to which a function belongs, given estimates of function-approximations. The imbedding theorems are investigated for the fundamental classes W, H, and B of differential functions given on the entire n-dimensional Euclidean space. I-CH

FUNCTIONAL ANALYSIS, S(17-18), P. *Locally Convex Spaces*. Kelly McKennon, Jack M. Robertson. Lect. Notes in Pure and Appl. Math., V. 15. Dekker, 1976, v + 65 pp, $12.50 (P). A comprehensive treatment with examples of the basic theory of locally convex linear topological spaces. Designed for use as a source or reference work. No index but large quantities of tables tracing implications among properties. JAS

FUNCTIONAL ANALYSIS, S(17-18), P. *Integration of Functionals: A Seminar*. K.O. Friedrichs, H.N. Shapiro, *et al*. Courant Inst, 1976, iv + 196 pp, $5 (P). A reprint of the 1957 (unavailable since 1968) original 1955 seminar notes wherein is presented a theory of integration of functionals of sufficient generality to be applicable both to probability theory and quantum mechanics. JAS

OPTIMIZATION, T(16-17: 1, 2), S, P, L. *Variational Methods in Statistics*. Jagdish S. Rustagi. Acad Pr, 1976, xiii + 236 pp, $19.50. Develops "an exposition of calculus of variations and its modern generalizations in order to apply them to statistical problems." Includes an "introduction to Pontryagin's maximum principle as well as Bellman's dynamic programming." FLW

ANALYSIS, P. *Theory and Applications of Differentiable Functions of Several Variables, V*. Ed: S.M. Nikol'skiĭ. Proc. of Steklov Inst. of Math., No. 131. AMS, 1975, iv + 252 pp, $33.60 (P).

ANALYSIS, T(18: 2), P. *Convex Analysis and Variational Problems*. Ivar Ekeland, Roger Temam. Stud. in Math. and its Appl., V. 1. North-Holland, 1976, ix + 402 pp, $29.50. In three sections: a summary of convex analysis, duality for convex variational problems, and the relaxation of non-convex variational problems. The translation is very smooth, and the English edition contains appendices updating the original (1973) work. Extensive bibliography. TAV

ANALYSIS, T(16-17), L. *Unitary Representations and Harmonic Analysis, An Introduction*. Mitsuo Sugiura. Halsted Pr, 1975, xiii + 402 pp, $34. A careful introduction to the subject. Starting with a discussion of Fourier series from a group theoretic viewpoint, the author constructs all the unitary representations for five other Lie groups, concluding with SL(2,R). Harmonic analysis on each group is then developed in terms of the representations. PJM

ANALYSIS, T(16-17: 2), S. *Fourier Series*. Alois Kufner, Jan Kadlec. Trans: G.A. Toombs. Academic, 1971, 358 pp. Deals with the theory of Fourier series in the wider context of Hilbert space, and in greater depth than comparable standard texts. Includes such topics as the Gibbs phenomenon, principle of localization, functions of bounded variation, numerical evaluations, applications, etc. A good text for readers seeking more than just a working knowledge of Fourier series. I=CH

ANALYSIS, P. *A Topological Linearization of Vector Measures*. William H. Graves. U of No Carolina, 1976, 55 pp, $3.50 (P). Presents a universal measure theory for measures on a fixed ring of sets in the sense that every vector measure on the ring is the universal measure followed by a continuous linear map. JAS

ANALYSIS, T*(15-16: 2), L. *Mésure et Intégration*. Roger Jean. Pr U Quebec, 1975, xxii + 305 pp, (P). A nice text, including the Fubini, Riesz representation, and Radon-Nikodym theorems. Lots of exercises, and a well-motivated, carefully written exposition. PJM

GEOMETRY, T*(15-16: 1, 2), L*. *Differential Geometry of Curves and Surfaces*. Manfredo P. do Carmo. P-H, 1976, viii + 503 pp, $22.50. A well-written, attractively presented introduction to the subject. Covers plane and space curves, regular surfaces in $R^3$, curvature via the Gauss map, intrinsic geometry of surface, global geometry of surfaces. Uses elementary linear algebra but machinery is kept to a minimum. Suitable for one semester or a full year course. Deserves serious consideration as a text. SG

GEOMETRY, S(18), P. *Leçons de Géométrie Différentielle, V. IV: Groupes Discrets, Variétés Différentiables*. G. Vranceanu. Editura Academiei Romania, 1975, 359 pp, Lei 26. A French translation of the 1968 Romanian edition. This volume provides a substantial though moderately classical exposition of the theory. No index or bibliography. JAS

GEOMETRY, P. *Lecture Notes in Mathematics-490: The Geometry of Metric and Linear Spaces*. Ed: L.M. Kelly. Springer-Verlag, 1975, x + 244 pp, $10.30 (P). Proceedings of (and elaborations on) a conference in recognition of L.M. Blumenthal held at Michigan State University in East Lansing in June, 1974. Concludes with a list of problems edited by R.C. Guy. JAS

GEOMETRY, T(16-18: 1), S, P, L. *Convex Sets*. Frederick A. Valentine. Krieger, 1976, ix + 238 pp, $14. Reprint of the original 1964 McGraw-Hill edition. Topics include: hyperplanes, separation theorem, Minkowski metric, local convexity, dual cone, convex functions, etc. Basic text in the area. Exercises. Appendices. Bibliography. Index. RJA

PROBABILITY, T(14-15: 1), L. *Introduction to Probability: Theory and Applications*. R.L. Scheaffer, W. Mendenhall. Duxbury Pr, 1975, x + 289 pp, $13.50. A clearly written text for a first course presupposing calculus. The outstanding feature: the exercises refer to actual situations, not the "urn-model" approach. Contains introductory material on topics such as Markov chains and queues. A worthy candidate for a one semester course preceding mathematical statistics or stochastic processes. TAV

PROBABILITY, T**(14: 1), L*. *Elementary Probability Theory with Stochastic Processes, Second Edition*. Kai Lai Chung. Springer-Verlag, 1975, x + 325 pp, $12. Minor editorial changes and corrections distinguish this edition from the earlier one (TR, January 1975). An exceptionally well-conceived text written in a clear, conversational manner. TAV

PROBABILITY, T(14: 1), L. *A First Course in Probability*. Sheldon Ross. Macmillan, 1976, x + 305 pp, $13.95. At the post-calculus level, but well below Feller's classic. Discusses combinatorial methods, random variables, densities and expectations, and assorted limit laws. Could nicely serve as

an introduction to courses in mathematical statistics or stochastic processes. Numerous exercises, both theoretical and applied. TAV

PROBABILITY, P. *Sample Path Properties of Stable Processes.* J.L. Mijnheer. Math. Centre Tracts, No. 59. Math Centrum, 1975, 124 pp, Dfl. 14 (P). The author's thesis. The problem considered is the determination of the probabilistic behavior of large sums of random variables (not necessarily of finite expectation). TAV

PROBABILITY, T(15-16). *Introduction à la Théorie des Probabilités.* Yves Lepage, Marc Moore, Roch Roy. Pr U Quebec, 1975, viii + 551 pp, (P). A rigorous, axiomatic treatment of probability. Prerequisite: a good advanced calculus course. French is easy; lots of exercises; answers in back of book. PJM

STATISTICS, T(14-17: 1, 2), S, P, L. *Nonparametrics: Statistical Methods Based on Ranks.* E.L. Lehmann, H.J.M. D'Abrera. Holden-Day, 1975, xvi + 457 pp, $22.95. A unified treatment of the most important rank tests as well as point and interval estimates and multiple decision procedures related to these tests. Several basic chapters require very little mathematical background and some more theoretical matters are presented in appendices. FLW

STATISTICS, T(15-17: 1), S, P, L. *Sequential Methods in Statistics, Second Edition.* G. Barrie Wetherill. Chapman & Hall, 1975, x + 232 pp, $12. Surveys sequential methods, omitting most proofs, emphasizing practical results, and supplying references to applications. FLW

STATISTICS, T(13-14: 1, 2). *Elementary Statistics, Fourth Edition.* Paul G. Hoel. Wiley, 1976, xi + 361 pp, $12.95. This edition has updated problems and illustrations and a new chapter on multiple regression. The basic structure of this sound, attractive treatment of pre-calculus statistics is unchanged. (Third edition TR, January 1972.) FLW

STATISTICS, T(13: 1), S. *Beginning Statistics.* Gene Zirkel, Robert Rosenfeld. McGraw, 1976, xiii + 318 pp, $9.95 (P). Collection of simple statistical experiments, projects and outside reading suggestions designed to stimulate class discussion and develop intuitive understanding of elementary statistical methods. Light approach written to appeal to poorly motivated student. No algebra, no theory. LCL

STATISTICS, T(13: 1). *Elementary Statistical Concepts.* Ronald E. Walpole. Macmillan, 1976, 231 pp, $10.95. Presupposes no college mathematics. "Illustrative examples have been substituted for mathematical proofs." No Bayesian methods. FLW

STATISTICS, T(13: 1). *Introduction to Statistics.* Robert G.D. Steel, James H. Torrie. McGraw, 1976, xiv + 382 pp, $13.95. Non-calculus version distinguished mainly by inclusion of nonparametric point of view after standard material on inference, analysis of variance and regression. Not enough material for the claimed two semesters. LCL

STATISTICS, T(16-17: 1, 2), S, P, L. *Multivariate Statistical Methods, Second Edition.* Donald F. Morrison. McGraw, 1976, xv + 415 pp, $15.95. Presupposes a course in statistics and some calculus. Includes a review of univariate statistics and a chapter on matrix algebra, and considers analysis of variance, discriminant functions, analysis of covariance, principle components, and factor analysis.FLW

STATISTICS, T(16-17: 2). *Multivariate Statistical Methods in Behavioral Research.* R. Darrell Bock. McGraw, 1975, xiii + 623 pp, $21. An encyclopedic treatment of multivariable methods. Presumes familiarity with estimation, hypothesis testing, multiple regression and analysis of variance, multivariable calculus and linear algebra. The treatment is mathematically informal: few theorems, fewer proofs. A variety of possible techniques is analyzed for given situations. Extensive bibliography.TAV

STATISTICS, T(13: 1). *Introductory Statistics.* John A. Ingram. Cummings, 1974, 436 pp, $11.95. Lots of worked-out examples together with a variety of problem sets written for students of business, social science, and general education. Very little theoretical discussion even for a noncalculus version. Comes with a 10 page mimeographed *Errata.* LCL

STATISTICS, T(13-14: 1). *Statistics, A First Course, Second Edition.* John E. Freund. P-H, 1976, x + 374 pp, $11.25. An attractive text for a pre-calculus course. Includes consideration of subjective probabilities and a brief discussion of decision theory. FLW

STATISTICS, T(13: 1). *Essentials of Statistics.* Robert D. Mason. P-H, 1976, xii + 320 pp, $8.95 (P). Semi-programmed text with frequent questions (and answers) so the reader may check his progress. Presupposes no college mathematics. FLW

COMPUTER SCIENCE, T(13-14: 1), S, L. *Computers and Society: The Technology and Its Social Implications, Second Edition.* Stanley Rothman, Charles Mosmann. SRA, 1976, viii + 423 pp, $11.95. A work on "computer appreciation." Proceeds from computer technology to applications to social impact and the future. All-pervading undercurrent: the computer's relationship to society. No knowledge of programming required. Exercises. References. Glossary. Index. RJA

COMPUTER SCIENCE, T(13: 1). *Computers: Programming and Applications.* Herbert Maisel. SRA, 1976, xiv + 364 pp, $12.95. A general, introductory text. Programming in FORTRAN, PL/I and BASIC--first at an elementary level then at a more advanced level. Also discussions of computers, information representation, algorithms and applications. At times the content is too brief. Good bibliographies. RWN

COMPUTER SCIENCE, T(15-17: 1), S, P, L. *Principles of Program Design.* M.A. Jackson. APIC Stud. in Data Proc., No. 12. Acad Pr, 1975, xii + 299 pp, $22.75. Attempts to present a coherent method and procedure for designing programs that are simple and correct by using the basic ideas of structured programming. Most problem solutions are coded in COBOL. Exercises and discussion questions. Appendix on COBOL. References. RJA

APPLICATIONS (CONTROL THEORY), P. *Control and Dynamic Systems: Advances in Theory and Applications, V. 12.* Ed: C.T. Leondes. Acad Pr, 1976, xx + 627 pp, $28.50. This twelfth annual volume is devoted to contributions on the topic of filtering and stochastic control in dynamic systems. JAS

APPLICATIONS (ECONOMICS), P. *Lecture Notes in Economics and Mathematical Systems-114: Produktionstheorie.* Götz Uebe. Springer-Verlag, 1976, xvii + 301 pp, $11.50 (P). On recent developments in production theory. For specialists. Extensive bibliography. JD-B

APPLICATIONS (ELECTRONICS), T(14-16: 1), S, L. *Problems and Solutions in Logic Design.* D. Zissos. Oxford U Pr, 1976, viii + 146 pp, $4.95 (P); $8.75. Design of logic circuits, including states, unclocked sequential and clocked sequential circuits, counters, and combinational circuits. Appendix on Boolean algebra. Problems with solutions. Index. RJA

APPLICATIONS (ENGINEERING), P. *The Structure of Turbulent Shear Flow, Second Edition.* A.A. Townsend. Cambridge U Pr, 1976, xi + 429 pp, $42.50. Expands many sections of the 1956 edition and adds new ones on convection; includes a sizeable bibliography of recent articles. As before, the aim is to describe a wide variety of flows, and much of the discussion is empirically based. DFA

APPLICATIONS (HEALTH), P. *Proceedings of SIMS Conference on Epidemiology.* Ed: Donald Ludwig, Kenneth L. Cooke. SIAM, 1975, ix + 158 pp, $13.50 (P). 17 papers by mathematicians, statisticians, medical researchers and public health workers brought together in Alta, Utah in July 1974 by the SIAM Institute for Mathematics and Society. LAS

APPLICATIONS (INFORMATION THEORY), P. *On Measures of Information and Their Characterizations.* J. Aczél, Z. Daróczy. Math. in Sci. and Eng., V. 115. Acad Pr, 1975, xii + 234 pp, $24.50. Not intended to be inclusive. Discusses the entropies due to Shannon, Rényi, and others, and their characterizations. Includes preliminaries on functional equations. References. RWN

APPLICATIONS (MANAGEMENT), T(15), L. *Fundamentals of Operations Research for Management: An Introduction to Quantitative Methods.* Shiv K. Gupta, John M. Cozzolino. Holden-Day, 1975, x + 405 pp, $15.95. Attempts to describe operations research techniques with only a calculus background assumed. Matrix notation is not used resulting in clumsy development of linear programming. In spite of the mathematical inelegance, the topics are quite encyclopedic and the exercises well chosen. TAV

APPLICATIONS (OPERATIONS RESEARCH), S, P. *Mathematics in Educational Planning.* Inst Math Appl, 1974, v + 92 pp, £2.00 (P). Six papers included in this proceedings of a symposium organized by the Institute of Mathematics and its Applications held in London on June 11, 1974. Planning problems concerned: budget and staff, class sizes and unit cost, timetabling and student scheduling, setting of priorities in education policy making, etc. Mathematics used: computer simulation, mathematical modeling and matrix methods. I-CH

APPLICATIONS (OPERATIONS RESEARCH), S*(15-17), L. *Operations Research for Immediate Application: A Quick & Dirty Manual.* Robert E.D. Woolsey, Huntington S. Swanson. Har-Row, 1975, xvii + 204 pp, $5.95 (P). Authors attempt to bridge the communication gaps existing between people who write journal articles and those who really need the application of operations research. Manual format: general discussions and brief summaries precede the Quick & Dirtys, each with an example and a reference; a number of "conversational FORTRAN programs" also supplied. Authors' warning: check the actual circumstances of the problem before getting Quick and Dirty. I-CH

APPLICATIONS (PHYSICS), T(13-14), L. *Modern Introduction to Classical Mechanics & Control.* David N. Burghes, Angela M. Downs. Wiley, 1975, 320 pp, $18.50. A sophomore level text on mechanics with a stress on practical problems. Topics: Newtonian mechanics, force fields, conservation laws; space dynamics, rocket flight, oscillation, variational principles, optimal control. A modest number of exercises. SG

APPLICATIONS (PHYSICS), P. *Difference Methods of Solving Problems of Mathematical Physics. II.* Ed: N.N. Janenko. Trans: S. Smith. Proc. of Steklov Inst. of Math., No. 122. AMS, 1975, iii + 99 pp, $21.40 (P). Papers by A.N. Konovalov, V.E. Neuvažev, and V.A. Sučkov on gas dynamics and by Ju. I. Sokin and N.N. Janenko on difference schemes for systems of partial differential equations. Some discuss numerical methods. All appeared in 1973. DFA

APPLICATIONS (PHYSICS), P. *Group Theory and Its Applications, V. III.* Ed: Ernest M. Loebl. Acad Pr, 1975, xvi + 480 pp, $49.50. Continues a series of expository articles leading the reader from basics (in professional-level physics) to the research frontiers. Groups are still without exception groups of symmetries, but the breadth of mathematical tools and sophistication are at a higher level than in most presentations. JAS

APPLICATIONS (PHYSICS), T(13: 1), S. *Theoretical Mechanics.* G.S. Light, J.B. Higham. Longman, 1975, xi + 360 pp, $12.50. A pre-calculus treatment to prepare candidates for British examinations. LAS

APPLICATIONS (SOCIAL SCIENCE), T*(15-17: 1, 2), S, P, L. *Mathematical Methods in the Social and Managerial Sciences.* Patrick Hayes. Wiley, 1975, xvi + 460 pp, $19.95. Differential equations (first and second order, and linear systems), convex analysis, convex and geometric programming. Presupposes calculus and matrix algebra. FLW

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet Carleton; Jennifer Galovich, St. Olaf; Steven Galovich, Carleton; Ih-Ching Hsu, St. Olaf; Roger B. Kirchner, Carleton; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

Professor Marion Wetzel, Denison University, represented the Association at the inauguration of P. H. Jordan, Jr., as President of Kenyon College on October 25, 1975.

*California Institute of Technology*: Messrs. F. K. Dashiell, Jr., UCLA, and J. S. Schlipf, University of Wisconsin, have been appointed Bateman Research Instructors; Professor Marshall Hall, Jr., has been elected a member of the American Academy of Arts and Sciences.

*Le Moyne College*: Assistant Professor J. E. Billotti has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics; Professor J. F. Smith has been appointed Visiting Professor at Gonzaga University for the academic year 1975–76.

*South Dakota School of Mines and Technology*: Associate Professors D. W. Ballew and C. L. Harbison have been promoted to Professors; Assistant Professor Roger Opp has been promoted to Associate Professor; Professor Edgar L. Swanson retired in January 1976 with the title of Professor Emeritus.

*St. Olaf College*: Dr. Richard Allen, Indiana University, has been appointed Assistant Professor; Associate Professor L. A. Steen has been promoted to Professor; Dr. T. A. Vessey has been appointed Chairman of the Mathematics Department.

*SUNY—College at New Paltz*: Dr. Richard Moynihan, University of Massachusetts, has been appointed Assistant Professor; Dr. Edward Eisenberg, Vassar College, has been appointed Visiting Assistant Professor; Assistant Professor Norman Roth has been promoted to Associate Professor.

Associate Professor K. E. Atkinson, University of Iowa, has been promoted to Professor.

Mr. Raphael Phillip Finkelstein, Bowling Green State University, has changed his name to Ray Phillip Steiner.

Mr. D. J. Gittinger, St. Philip's College, has been promoted to Assistant Professor.

Dr. Martin Golumbic, Columbia University, has been appointed Assistant Professor of Computing Science at the Courant Institute for Mathematical Sciences of New York University.

Associate Professor Joseph Teeters, University of Wisconsin-Eau Claire, has been promoted to Professor.

Dr. P. M. Winkler, Yale University, has been appointed Assistant Professor at Stanford University.

Associate Professor Henry S. Cole, West Coast University, died on July 17, 1975, at the age of 59. He was a member of the Association for eight years.

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# MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## 1976 SUMMER MEETING — UNIVERSITY OF TORONTO

The Mathematical Association of America will hold its Fifty-sixth Summer Meeting at the University of Toronto from Thursday, August 26 through Saturday, August 28, 1976. This meeting will be held in conjunction with the American Mathematical Society which will hold sessions during the period Tuesday, August 24 through Friday, August 27, 1976.

Participants should note that the two organizations are meeting in order opposite from the usual schedule for a summer meeting.

## THE FIFTY-NINTH ANNUAL MEETING OF THE ASSOCIATION

The Fifty-Ninth Annual Meeting of the Mathematical Association of America was held at the San Antonio Convention Center, San Antonio, Texas, from Saturday to Monday, January 24–26, 1976, in conjunction with meetings of the American Mathematical Society, and the Association for Women in Mathematics. The program for the Association meeting was centered about the Bicentennial. There were registered 2840 persons, including 1640 members of the Association.

Sessions were held on Saturday morning, Saturday evening, Sunday morning, Monday morning and Monday afternoon in the San Antonio Convention Center Theater. Presiding officer at the lecture by Professor D. J. Struik was Professor J. D. Tarwater; at the lecture by Professor Judith V. Grabiner, Professor M. W. Pownall; at the lecture by Professor Garrett Birkhoff, MAA President-Elect H. L. Alder; at the lecture by Professor P. R. Halmos, Professor Alex Rosenberg; at the lecture by Dr. Mina S. Rees, Professor R. H. Bing; at the lecture by Dr. R. W. Hamming, MAA President H. O. Pollak; at the lecture by Professor P. D. Lax, MAA Second Vice-President, Betty J. Hinman; at the lecture by Professor Mark Kac, Dean Paul Olum.

The Program Committee consisted of Leonard Gillman, Chairman; Betty J. Hinman, Morris Kline, K. O. May, C. V. Newsom, M. W. Pownall and J. D. Tarwater.

## FIRST SESSION OF THE ASSOCIATION

*Mathematics in Colonial America*, by Professor D. J. Struik, Massachusetts Institute of Technology (retired).

Professor Struik pointed out that mathematics in colonial times was mainly appreciated as a tool for astronomy, surveying, and navigation. In the age of Newton, attention was also paid to such subjects as fluxions. Also discussed was the teaching in Mexico, at Harvard, and in Quebec and such figures as Enrico Martinez, Velazquez deLeon, John Winthrop, David Rittenhouse, De Bonnecamps, and Lotbinlere.

*Mathematics in America: The First Hundred Years*, by Professor Judith V. Grabiner, Small College, California State College, Dominguez Hills.

American mathematics in the first hundred years of the Republic was generally undistinguished. Nevertheless, by the end of the 19th century, American mathematics had become quite respectable by international standards. Mathematics was always part of the college curriculum, but the reasons for including it—mental discipline, applications to engineering, research—changed as American society changed. The most interesting questions about American mathematics from 1776 to 1876 are: (1) What mathematics was there (2) How did the mathematics there was relate to American culture and society (3) What brought about the development of a mature American mathematical community by the 1890's.

*Some Leaders in American Mathematics*: 1891–1941, by Professor Garrett Birkhoff, Harvard University.

During the half-century preceding Pearl Harbor, the United States changed from a developing mathematical country to a position of world leadership. Brief sketches were given of some of the leaders, events, and institutions that were most responsible for this rapid transition. These sketches were based in part on personal impressions and recollections.

## SECOND SESSION OF THE ASSOCIATION

*American Mathematics from 1940 to the Day before Yesterday*, by Professor P. R. Halmos, Indiana University.

Professor Halmos presented an account of ten great achievements in American mathematics during the post-war era.

## THIRD SESSION OF THE ASSOCIATION

*Panel Discussion: Two-Year College Mathematics in 1976*

A panel discussion with Professor D. J. Albers, Mission College, Professor P. A. Lindstrom, Genesee Community College and Professor Shelba J. Morman, University of Houston, Victoria Center, moderated by R. D. Larsson, President, Schenectady County Community College.

President Larsson indicated that there are two basic reasons that make it appropriate for the role of two-year colleges to be reviewed as part of the history of mathematics. First, they have made the major contribution in recent history to the enrollments in higher education. Second, the MAA, through the Committee on the Undergraduate Program in Mathematics, has published four reports for two-year colleges since 1969 and it is well to note whether their present concerns imply a need for the development of additional reports of that kind.

Professor Albers spoke on "Trends in Two-Year College Mathematics." He examined the first half of the question: "What is two-year college mathematics and where is it going?," via comparisons of: 1. the two-year

college curriculum with the four-year college curriculum, and 2. the two-year college faculty member with the four-year college faculty member. He hoped that the extensive data used in comparison 1. shatters the (mainly four-year) delusion that a two-year college is simply the first two years of a four-year college. Trends were examined in view of the youth of two-year colleges, the rapid growth of occupational-technical programs, and the very small number of Ph.D's in two-year colleges.

Professor Lindstrom spoke on "Role and Status of Two-Year College Faculty." He noted that for the past ten years, two-year college mathematics has been characterized by the development of many state two-year college organizations, recognition and support from the MAA at the section and national levels and the professional growth of the two-year college mathematics teacher. Much of this is the result of the efforts of a small handful of two-year college people. There are still many who are "good teachers" in the classroom, but have no interest in professionalism and mathematics outside of the classroom. For these teachers, the future is a time of unlimited opportunities for making contributions to all areas of mathematics.

Professor Shelba J. Morman spoke on "The Place of Teacher Education in Two-Year Colleges." She pointed out that the two-year college teacher of mathematics can exert important and unique, although often somewhat subtle, influences on preservice and inservice mathematics teacher education, both elementary and secondary. To effectively maximize this influence the two-year college teacher must be responsive to professional demands perhaps presently unrealized.

*Panel Discussion*: *Mathematics in Our Culture*

A panel discussion with Professor Morris Kline, Brooklyn College (CUNY) and New York University, and Professor R. A. Rosenbaum, Wesleyan University, moderated by Professor R. H. McDowell, Washington University.

Professor McDowell indicated that the panel would examine the failure of mathematicians and educators to make clear the significance of mathematics, especially to educated non-scientists.

Professor Kline attempted to show why the courses usually offered to non-mathematicians, the liberal arts courses, are woefully deficient. The topics usually taught were examined and evaluated on the basis of their purported contribution to a liberal arts education. He contended that these topics are not representative of the value of mathematics itself and fail completely to show the enormous cultural influences of mathematics. Thereby mathematicians forego their best opportunity to present a favorable image of their subject and to win adherents and support for its role in a liberal arts education. Some positive recommendations were described.

Professor Rosenbaum noted that these days (more than ever before?), even people with extensive formal education tend to see a dichotomy between mathematics and the humanities. This situation is not only unfortunate for mathematics and mathematicians, but also dangerous to the health of our culture. An analysis of "how we got this way" suggests ways in which the gap might be bridged and the condition rectified.

## FOURTH SESSION OF THE ASSOCIATION

*Mathematics and the Government*: *The Post-War Years as Augury of the Future*, by Dr. Mina S. Rees, President Emeritus, the Graduate School and University Center, City University of New York.

Federal support of university research in mathematics, begun after World War II by the Office of Naval Research and later expanded to include other mission-oriented agencies as well as the National Science Foundation, had a substantial impact on the development of mathematics and the careers of mathematicians in America, and on collegiate mathematics. The early mathematics programs of ONR were discussed with a consideration of some problems that were present in embryo in the early days and now loom large in government programs, and with mention of Navy-sponsored mathematical research that has influenced undergraduate curricula.

*Panel Discussion*: *The Teaching of Mathematics in College*: *A 1976 Perspective for the Future*

A panel discussion with Professor I. N. Herstein, University of Chicago, and Professor P. J. Hilton, Battelle Memorial Institute and Case Western Reserve University, moderated by Dr. C. V. Newsom, Vice President, Radio Corporation of America (retired).

The introductory statement by Dr. Newsom for the panel discussion entitled, "The Teaching of Mathematics in College: A 1976 Perspective for the Future," treated briefly as background for the presentations of the panelists some of the challenges facing present-day designers of the college program in mathematics. The statement implied that academic mathematicians need to develop a new model as the basis for the design of undergraduate programs in mathematics.

Professor Herstein spoke on "What Will Be Expected of the Good College Mathematics Teacher?"

Professor Hilton emphasized that we must first decide what mathematics the college mathematics teacher will be teaching. This will give us a crude lower bound on the mathematics which must be covered in graduate school. The graduate student must also develop a deep knowledge of some area of mathematics in order to be able to

"keep alive" in an environment in which research and scholarship are not likely to be specifically fostered. Further, he must develop a good feeling for the methodology of applying mathematics and a sympathetic approach to the point of view of the user of mathematics. Finally, he must acquire a genuine respect for the role of the teacher.

*The History of Computing in the United States*, by Dr. R. W. Hamming, Bell Telephone Laboratories.

History of science, like the classic political history, tends to emphasize the individual and the first time something happens. The talk on the history of computing in the U. S. put relatively more emphasis than usual on the general availability of computing. It also discussed the subject under three headings: the hardware available, the software to run the computers that are available, and finally the uses made of the first two, namely, in the area of mathematics.

## FIFTH SESSION OF THE ASSOCIATION

*The Bomb, Sputnik, Computers and European Mathematicians*, by Professor P. D. Lax, New York University.

The pre-war and post-war mathematical milieus were compared and the causes responsible for the changes were explored with special emphases on the contributions of refugee mathematicians and on the challenge of technology to mathematics. This report was based partly on personal experiences.

*Panel Discussion*: *The Role of Applications in the Teaching of Undergraduate Mathematics*

A panel discussion with Professor F. S. Roberts, Rutgers University, and Dr. A. B. Willcox, Executive Director of the MAA, moderated by Professor S. K. Stein, University of California, Davis.

Professor Stein noted that the place of applications in the mathematics curriculum is closely related to the textbooks available. The responsibilities and frustrations facing the textbook author in these times of (perhaps) changing emphasis was discussed.

Dr. Willcox said that many fields depend heavily on mathematics. Mathematics is itself absolutely dependent on the bridges joining it to these other fields. Traffic on these bridges is two-way, carrying indispensable tools for organizing ideas and reaching conclusions in one direction and a disorganized mass of observed data in the other. The tools help society; the mass of data contains hidden patterns that stimulate and direct the creative imagination of the mathematician. Mathematics thus lives in a healthy symbiosis with the other disciplines. If our colleges and universities are to give students *realistic* training in mathematics for today's realities and tomorrow's challenges, this symbiosis must enter the classroom. Students should explore not only the island mathematics but also the bridges linking it to other islands.

If a bridge from mathematics is to influence the student in any significant way, it must touch an island where he or she lives. There are many rapidly developing areas of application of mathematics in widely ranging fields, many bearing closely on problems of great societal importance. These should be represented in the classroom because they command instant attention from the student. However, physical ideas, such as motion, content, speed, are part of the universal experience of living, and therefore applications to the physical sciences will probably always be preeminent in the teaching of most mathematical subjects, especially at the elementary level.

Several examples, including a bridge between elementary calculus and physiology was used to illustrate these ideas.

Professor Roberts indicated that among those problems of society which mathematicians have begun to attack are problems involving energy, transportation, pollution, and health care delivery. Societal problems often involve hard-to-quantify, imprecise relationships, and the simplest types of mathematics are often the most appropriate for describing these relationships qualitatively. Moreover, societal problems tend to involve many variables, and quickly lead to extremely difficult mathematical problems unless major simplifying assumptions are made. These observations have implications for undergraduate mathematics education. To name two such: applied problems, with no "nice" solutions *should* be brought into the classroom; and mathematical problems, simple to state but at the frontier of modern research, *can* be brought into the classroom. These ideas will be illustrated with examples.

*Probability Theory*: *Reflections on the Past and Speculations on the Future*, by Professor Mark Kac, Rockefeller University.

A brief historical discussion of the development of Probability and Statistics in the United States, beginning with J. W. Gibbs and ending with the day of the lecture was given.

The place of Probability Theory in Science was considered and tentative predictions as to the future were made.

## SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Mission Room of the Convention Center on Friday and Sunday at 7:00 P.M. The following films were shown on Friday:

7:00—7:16 P.M. THE DEFINITE INTEGRAL, a film of the MAA CALCULUS FILM SERIES

7:18—7:44 P.M. ISOMETRIES, a film of the COLLEGE GEOMETRY PROJECT
7:46—7:54 P.M. POWERS OF TEN, a film by Charles Eames
7:55— P.M. FILMS OF THE TOPOLOGY FILM PROJECT
7:55—8:09 P.M. REGULAR HOMOTOPIES IN THE PLANE, PART I
8:11—8:30 P.M. REGULAR HOMOTOPIES IN THE PLANE, PART II

The following films were shown on Sunday:

7:00—7:25 P.M. ERRORS THAT DIE, a BBC Broadcast as part of the Open University Foundation Course in Mathematics (b&w)
7:28—7:36 P.M. NEWTON'S EQUAL AREAS, a film produced by Bruce and Catherine Cornwell
7:39—8:00 P.M. SHAPES OF THE FUTURE: SOME UNSOLVED PROBLEMS IN GEOMETRY, PART II, THREE DIMENSIONS with Victor Klee, a film of the MAA INDIVIDUAL LECTURES FILM PROJECT.
8:05—8:17 P.M. INVERSION, a film of the COLLEGE GEOMETRY PROJECT
8:20—9:20 P.M. THE KAKEYA PROBLEM, with A. S. Besicovitch, a film of the MAA MATHEMATICS TODAY SERIES

The AMS and MAA sponsored a recital of music for piano and violoncello performed by Professor Leonard Gillman, University of Texas at Austin, and Professor Louis Rowen, Bar Ilan University, in the Convention Center Theater at 8:45 P.M. on Saturday. The program was as follows:

| | |
|---|---|
| Sonata No. 2, in D Major, BWV 1028 | Bach |
| Sonata No. 1, in E Minor, Op 38 | Brahms |
| Intermission | |
| Sonata No. 3, in A Major, Op 69 | Beethoven |
| Sonata in G Minor, Op 65 | Chopin |

## MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Friday, January 23, at 9:00 A.M. in the La Vista Room of the Hilton Palacio del Rio with 38 members present. Among the items of business transacted were the following:

The Board elected Professor G. B. Price of the University of Kansas, as a member of the Finance Committee for the term 1976–79.

The Board elected Professor K. W. Fogarty, Bronx Community College (CUNY), and Professor George Polya, Stanford University, as Associate Editors of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL for the period 1976–78.

The Board voted to approve the recommendation of the Committee on Earle Raymond Hedrick Lectures that Professor M. D. Davis, Courant Institute of Mathematical Sciences, be invited to deliver the twenty-fourth set of Earle Raymond Hedrick Lectures at the Association's meeting at the University of Toronto, August 26–28, 1976.

Professor W. F. Lucas presented a report of the Committee on Institutes and Workshops in which he reported that an MAA Workshop on Modules in Applied Mathematics will be held at Cornell University from July 20–August 20, 1976 under his direction. This Workshop is funded by a grant of $86,267 from the National Science Foundation. Professor Lucas also described other potential programs under study by the Committee on Institutes and Workshops.

Professors Eileen L. Poiani, St. Peter's College, and D. B. Small reported to the Board on the activities of the Committee on Secondary School Lectures and, in particular, on WAM: Women and Mathematics. This is a secondary school lecture program designed to interest more women in mathematics. This pilot program is being carried out in New York/New Jersey, Greater Chicago, and the San Francisco Bay Area. The program is sponsored by the International Business Machines Corporation.

Professor D. W. Bushaw reported that the Committee on the Undergraduate Program has completed several projects including a revision of the *Basic Library List.*

The Board received the report of Joint (AMS-MAA) Committee on Employment Opportunities in Two-Year Colleges. This report recommended development of programs at universities appropriate to future employment in two-year colleges, provisions for internships at two-year colleges, and enhancing opportunities for professional growth of two-year college faculty.

The Board of Governors approved the operating budget for 1976 and discussed preliminary budgets for 1977 and 1978.

The Executive Director reported that the membership of the Association on December 31, 1975 was 18,774 compared with 18,762 a year earlier. As of the same date, there were 397 academic members of the Association compared with 373 a year earlier.

The Board approved the recommendation of the Joint (AMS-MAA) Committee on Women in Mathematics that SIAM be invited to join in sponsoring this Committee.

The members of the Conference Board of the Mathematical Sciences (CBMS) has both constituent and affiliate members and unanimous consent of the organizations which are constituent members is required for admission of new affiliate members. The MAA is a constituent member of CBMS and the Board of Governors voted to approve the application for affiliate membership of the Association for Women in Mathematics.

The Board approved the following schedule of meetings:

Fifty-seventh Summer Meeting: University of Washington, August 14–16, 1977.

Sixty-first Annual Meeting: Atlanta, Georgia, January 20–22, 1978.

Sixty-second Annual Meeting: Milwaukee, Wisconsin, January 13–15, 1979.

Sixty-fourth Annual Meeting: San Francisco, California, January 10–12, 1981.

The University of Toronto, August 26–28, 1976, and St. Louis, Missouri, January 29–31, 1977, had previously been approved as the sites for the Fifty-sixth Summer Meeting and the Sixtieth Annual Meeting.

The Board of Governors adopted a resolution thanking the City of San Antonio for its reception of this fifty-ninth annual meeting of the MAA. This resolution was forwarded to appropriate city officials by the Secretary.

## ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting of the Association was held on Sunday, January 25, 1976, in the Theater of the San Antonio Convention Center, with President Pollak presiding.

The Association's Fifteenth Award for Distinguished Service was made to Professor Leon W. Cohen of the University of Maryland. The citation (which appears on pages 83–84 of the February issue of the MONTHLY and pages 47–48 of the May issue of the TYCMJ) was prepared by Professor E. J. McShane. This citation was read at the Business Meeting by Professor G. B. Price. Professor Cohen, in accepting the Award, said:

"Let me express my very great appreciation to the Board of Governors for the Distinguished Service Award they have given me. It makes me happy and proud. What occurred to me as I thought about it was how much I owe to my teachers. I recall two high school teachers, Miss Palmer and Mr. Barnwell, the late Professor J. F. Ritt, who set a high standard for me when I was a freshman in 1920, and, particularly, Professor R. L. Wilder, my thesis advisor, whose wisdom and friendship continue to be a joy to me. There were many others. Such service as I may have been privileged to offer the mathematical community is really part payment for the affection offered me by my teachers and colleagues, including my wife, during a rewarding life. My thanks go to all of them."

A specially-bound copy of the citation was presented to Mrs. Cohen in appreciation of her continual help and support of Professor Cohen's many and varied activities.

The Chauvenet prize for 1976 was awarded to Professor Lawrence A. Zalcman of the University of Maryland for his paper "Real Proofs of Complex Theorems (and vice versa)," which appeared in the MONTHLY, 81 (1974), 115–37. Further details concerning this Prize and its recipient appears on pages 84–85 of the February, 1976 issue of this MONTHLY.

At the time of the meeting in San Antonio, Professor Zalcman was in transit from Israel, where he spent a first part of a leave from the University of Maryland at the Weizmann Institute of Science, to Switzerland, where he was to spend a second part of his leave at Eidgenössiche Technische Hochschule in Zürich. Professor Zalcman requested that the Secretary convey his delight and very great honor for having been named the recipient of the 1976 Chauvenet Prize.

The Secretary read a letter from Professor Zalcman, dated September 2, 1976, in which, in addition to expressing his pleasure at having been named the recipient of the Prize, he wrote: "And now, some philosophy. If one believes, as I do, that a major component of mathematics (and perhaps *the* major function of a mathematical proof) is communication, good exposition acquires an almost overriding importance. Thus, it has always seemed to me that an excellent expository article is genuinely more valuable than all but the most seminal research papers. I hope very much that the MONTHLY will continue to publish first-rate expository papers and (what is anterior to that) that mathematicians will find the time to write them.

Personal influences are subtle and indirect, but it remains a fact that I have been lucky enough to sit at the feet of such masters of written and oral exposition as John Kemeny, Kenneth Hoffman, and Max Schiffer. From afar, I have viewed and admired greatly the expositions of Chandrasekharan, Kac, Rudin, and (the master of them all) Courant. It seems to me a very good sign indeed that some of the very best younger mathematicians (Charles Fefferman, for example, or Dennis Hejhal) are superb expositors."

The Secretary presented his report, announcing first the results of the balloting for First Vice-President and Governors in which 1454 ballots were cast. Professor R. C. Buck of the University of Wisconsin was elected First Vice-President for the two year term 1976–77 and Professors Roy Dubisch of the University of Washington and David Gale of the University of California at Berkeley were elected Governors for the three-year term 1976–78. Professor Henry L. Alder of the University of California was elected President-Elect for 1976.

The Secretary reported on some of the actions taken by the Board of Governors, the budgetary situation of the

Association, and the membership figures compared with one year earlier. He emphasized that the success of efforts to increase memberships depends upon individual members asking their colleagues and departments to seek membership. Both individual and academic memberships have recently been made much more attractive by increasing benefits and journal options.

The Secretary called attention to the very fine program at this meeting and the wonderful reception of the program as evinced by the very large attendance at the lectures. In particular, it was reported that nearly 2000 persons had attended the lecture by Professor Halmos. The Secretary singled out Professor Leonard Gillman, MAA Treasurer and Chairman of the Program Committee, as being especially deserving of thanks. The Secretary then called on Professor R. L. Wilder who introduced a motion instructing the Secretary to send a formal letter of appreciation to Professor Gillman and Professor Louis Rowen for the piano and violoncello recital they presented on Saturday evening. This motion was unanimously adopted.

The Secretary also thanked the members of the Committee on Arrangements for so carefully and effectively attending to the many details necessary to arrange this meeting. In particular, the Secretary thanked Professor Peter Terwey, Jr., the Chairman of the Committee on Arrangements, and also called attention to the great success achieved by Professor F. J. Carter and Dean D. W. Lick in publicizing the meeting in the San Antonio media.

It was reported that, at the meeting of the Board of Governors, the members of the Executive and Finance Committees had presented Professor E. F. Beckenbach with a letter and gift in appreciation of his very successful Chairmanship of the Committee on Publications. During Professor Beckenbach's tenure as Chairman of the Committee on Publications, both sales and number of volumes published by the MAA have greatly increased. In particular, the recent addition of the NEW MATHEMATICAL LIBRARY under the editorship of Professor Anneli Lax greatly improves the selection of volumes available to Association members.

The Secretary noted that the TWO-YEAR COLLEGE MATHEMATICS JOURNAL is now an official journal of the Association. On account of appeal of this journal to all two-year college mathematics teachers, the Board of Governors has approved subscription rates for non-members at a rate lower than members' dues. Thus, the Secretary moved an amendment which would make implementation of these rates possible. Specifically, the motion was to delete "but in no case shall an official journal be sold to nonmembers for less than the annual dues of individual members" from Article VII, Section 5. This motion carried.

## MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Thursday, January 22, to Sunday, January 25. The forty-ninth Josiah Willard Gibbs Lecture was presented by Arthur S. Wightman, Department of Physics, Princeton University, at 8:30 P.M. on Thursday, January 22 in the Convention Center Theatre. The title of Professor Wightman's lecture was "Nonlinear functional analysis and some of its applications in quantum field theory."

A set of Colloquium Lectures was delivered by I. M. Singer of Massachusetts Institute of Technology and the Institute for Advanced Study. His subject was "Connections between analysis, geometry and topology." The four lectures in the series were scheduled for 1:00 P.M. on Thursday, Friday, Saturday and Sunday in the Convention Center Theatre.

The Oswald Veblen Prize in Geometry was awarded to James Simon of SUNY at Stony Brook and W. P. Thurston of Princeton University at a session at 3:30 P.M. on Friday, January 23, in the Convention Center Theatre.

The American Mathematical Society presented a two-day short course on Mathematical Aspects of Production and Distribution of Energy on Tuesday and Wednesday, January 20 and 21, in the Fiesta Rooms of the San Antonio Convention Center. The course was designated to acquaint mathematicians with the state of the art in energy-related research in applied mathematics.

The program was under the direction of a committee whose members are George B. Dantzig, Department of Operations Research and Computer Science, Stanford University; Herbert B. Keller, Firestone Laboratory, California Institute of Technology; Peter D. Lax (chairman), Courant Institute of Mathematical Sciences, New York University; and Robert Marr, Brookhaven National Laboratory.

The program consisted of eight one-hour lectures providing concentrated introductions to several different areas of current activity, making it possible for participants to judge whether they would be interested in pursuing any of these areas further. Jean Abadie, Electricité de France spoke on "Large scale nonlinear optimization problems arising in power production"; George B. Dantzig spoke on "A model for assessing physical impact on the economy of a changing energy picture," joint work with S. C. Parikh, Systems Optimization Laboratory, Stanford University; Richard L. Garwin, IBM, Thomas J. Watson Research Center, spoke on "The problem of aggregation in modeling physical and social systems and processes;" Harold Grad, Courant Institute of Mathematical Sciences, New York University, spoke on "Mathematical problems arising in controlled thermonuclear research;" William W. Hogan, Office of Quantitive Methods, Federal Energy Administration, spoke on "Project Independence evaluation system;" Gordon M. Kaufman, Sloan School of Management, Massachusetts

Institute of Technology, spoke on "Estimation of undiscovered oil and gas," joint with work of Eytan Barouch, Sloan School of Technology and Clarkson College of Technology; Milton S. Plesset, Engineering Science Department, California Institute of Technology, spoke on "Nuclear fission energy-problems and promise"; and Frederick D. Tappert, Courant Institute of Mathematical Sciences, New York University, spoke on "Laser fusion."

AMS invited addresses were given as follows, all in the Convention Center Theater:

*Recent Results on Infinite Games*, Professor D. A. Martin, Rockefeller University, Thursday at 9:00 A.M.

*Groups, Logic, and Embeddings*, Professor C. F. Miller, III, Princeton University, Thursday, at 10:30 A.M.

*2-Local Subgroups of Simple Groups*, Professor D. M. Goldschmidt, University of California, Berkeley, Thursday, 2:30 P.M.

*Transformations without Finite Invariant Measure*, Professor A. B. Hajian, Northeastern University, Thursday, 4:00 P.M.

*Concrete Applications and a New Foundation of Abstract Differential Geometry*, Professor Walter Noll, Carnegie Mellon University, Friday, 9:00 A.M.

*Coefficients of Univalent Functions*, Professor P. L. Duren, University of Michigan, Friday, 10:30 A.M.

*Time Evolution of One-Dimensional Systems whose Equilibrium State is a Renewal Process*, Professor F. L. Spitzer, Cornell University, Friday, 2:15 P.M.

*K-Theory of Rings*, Professor S. M. Gersten, University of Utah, Saturday, 2:30 P.M.

*Some Applications of the Frobenius in Characteristic Zero*, Professor Melvin Hochster, Purdue University, Saturday, 4:00 P.M.

*Equivariant Topology as the Study of Actions of Topological Categories*, Professor E. E. Floyd, University of Virginia, Sunday, 2:30 P.M.

*A Survey of Infinite Loop Space Theory*, Professor J. P. May, University of Chicago, Sunday, 4:00 P.M.

Presentation of the University of Texas Mathematics Award was made to Professor W. T. Eaton of the University of Texas following a breakfast Saturday at 7:00 A.M. in the Corte Real Room of the Hilton Palacio del Rio.

The Conference Board of the Mathematical Sciences held a panel discussion on "Issues in school-level mathematical education" at 2:00 P.M. on Saturday, January 24, in the Hilton's Corte Real Room. The moderator was James T. Fey of the University of Maryland. Members of the panel were Shirley Frye, Morris Kline, Shirley Hill and Peter Hilton.

The Association for Women in Mathematics held a panel discussion on "Women mathematicians in business, industry, and government" on Friday, January 23, at 10:30 A.M. Lenore Blum of Mills College was the moderator; members of the panel were Leila D. Bram, F. Jessie MacWilliams, Alice Peters, Lillian H. Rice, Marjorie L. Stein, and Mary F. Wheeler. This panel discussion preceded the AWM business meeting. Both sessions took place in the Corte Real Room of the Hilton. An open meeting of the AWM Executive Committee was held at 7:00 P.M. on Thursday, January 22, in Room #32 of the Convention Center.

The National Association of Mathematicians held its business meeting at 11:00 A.M. on Sunday, January 25, in Room #32 of the Convention Center. At 1:00 P.M. on Sunday, NAM presented a panel discussion on "Graduate programs in mathematics at traditionally black institutions" in Room #32 of the Convention Center. The panel was moderated by Geraldine Darden, Hampton Institute; members of the panel included James Donaldson, Johnny Houston, Frank James, and Theodore Sykes. At 9:00 A.M. on Sunday, Louis Dale of Miles College and the University of Alabama at Birmingham spoke on "The structure of monicfree $K$-ideals in polynomial semirings" in Room #33 of the Convention Center. At 9:30 A.M., Japheth Hall, Jr., of Stillman College spoke on "The geometric structure of groupoids," also in Room #33.

An open meeting of the Mathematics Action Group Steering Committee took place in Room #22 in the Convention Center at 10:00 A.M. on Wednesday, January 21. The MAG business meeting was at 1:30 P.M. on Friday, January 23, in the LaReina Room of the Hilton.

## ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements consisted of Peter Terwey, Jr., Chairman; E. P. Armendariz, P. T. Bateman, F. J. Carter, W. E. Hunt, Constance J. Jones, R. A. Northcutt, D. P. Roselle, G. L. Walker, S. G. Wayment.

Registration headquarters were located in the Banquet Hall of the Convention Center. The Mathematical Sciences Employment Register was maintained from 9:30 A.M. until 4:00 P.M. in the International Ballroom of the Hilton Palacio del Rio. Interviews were held from 9:00 A.M. until 5:40 P.M. on both Saturday and Sunday. Books and educational media were displayed in the Banquet Hall of the Convention Center on Wednesday from 2:00 P.M.-8:00 P.M., on Thursday-Sunday from 8:00 A.M.-4:00 P.M. and on Monday from 8:30 A.M.-2:30 P.M.

DAVID P. ROSELLE, *Secretary*

## NOVEMBER MEETING OF THE OHIO SECTION

The Ohio Section of the MAA held its Fall 1975 meeting at Otterbein College, Westerville, Ohio, November 7 and 8, 1975. One hundred and eighty-four individuals registered for the meeting. Section Chairman, R. G. Laatsch, Miami University, presided.

The theme of the Fall meeting centered upon the teaching of college mathematics, with the invited address, "The Problem of Learning to Teach", presented by Professor E. E. Moise, Queens College, CUNY, New York.

The following contributed papers were presented:

*Present and future use of computer generated graphic teaching aids*, by Bernard Baltz, Ohio State University.
*Teaching applications—some new written materials*, by P. M. Tuchinsky, Ohio Wesleyan University.
*Minimum requirements for a course*, by J. F. Leetch, Bowling Green State University.
*A home-study course in geometry*, by J. A. Murtha, Marietta College.
*Experiment on conjecture in mathematics for college freshmen and sophomores*, by H. C. Nixt, Ohio State University (Marion Branch).
*Finding missing geometric problems*, by L. H. Miller, Ohio State University.
*A broadly based mathematics honors program*, by A. E. Ross, Ohio State University.
*Vector operations of identities in n-space*, by G. L. Szoke, University of Akron.
*Differentials in elementary calculus*, by R. M. McLeod, Kenyon College.
*Providing for continuity in sequential courses—some examples*, by Marilyn R. Studer, Kent State University.
*Mathematics in the humanities—a liberal arts course*, by J. R. C. Leitzel, Ohio State University.
*Beyond the syllabus*, by Joseph Altinger, Youngstown State University.
*Individualized instruction: an alternative?*, by Helen Medley and Marion Walker, Kent State University.

Additional lectures presented, with question/discussion periods, included:

*Pocket calculators in the classrooms*, by Harry Allen, Ohio State University, and Charles Capel, Miami University.
*The mathematics preparation of elementary teachers—what approach?*, by J. E. Schultz, Ohio State University.
*Secondary teacher training and certification in Ohio*, by S. P. Meiring, Ohio Department of Education.

Further, the following 'swap' sessions were also presented:

*The modernized metric system*, led by Louis Ross, University of Akron.
*Searching for mathematical talent*, led by A. E. Ross, Ohio State University.

In conclusion, meetings were scheduled with the Executive Committee of the Section (chaired by Professor Laatsch), and also of ad hoc committees: Committee on Cooperation Among Colleges and Universities (chaired by J. Stein, University of Toledo), Committee on Curriculum (chaired by H. E. Krehbiel, Bluffton College), and Committee on Teacher Training and Certification (chaired by W. A. Kirby, Bowling Green State University). Section Chairman Laatsch was assisted in the meeting by Professor R. S. Varga, Kent State University, Program Committee Chairman, and by Professor S. E. Bohn, Miami University, MAA Sectional Governor.

G. MAVRIGIAN, *Secretary-Treasurer*

## CALENDAR OF FUTURE MEETINGS

Fifty-sixth Summer Meeting, University of Toronto, August 26–28, 1976.

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 wks. bef. mtg.

FLORIDA, early March. Deadline for paper titles 2 wks. bef. mtg.

ILLINOIS, Chicago State University, Chicago, May 14–15, 1976.

INDIANA, Franklin College, Franklin, May 1, 1976.

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 wks. bef. mtg.

LOUISIANA-MISSISSIPPI, University of New Orleans, Louisiana, February 25–26, 1977.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Calvin College, Grand Rapids, May 7–8, 1976.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Montclair State College, Upper Montclair, May 8, 1976.

NORTH CENTRAL, St. Cloud State University, St. Cloud, Minnesota, April 30–May 1, 1976.

NORTHEASTERN, University of New Hampshire, Durham, June 18–19, 1976 (special Spring Meeting).

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Youngstown State University, Youngstown, May 7–8, 1976.

OKLAHOMA-ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 wks. bef. mtg.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Montgomery County Community College, Blue Bell, Pennsylvania, November 20, 1976.

ROCKY MOUNTAIN, Ft. Lewis College, Durango, Colorado, April 30–May 1, 1976.

SEAWAY, College of St. Rose, Albany, April 30–May 1, 1976.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 wks. bef. mtg.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Beloit College, Beloit (Friday), and University of Wisconsin, Rock County Center, Janesville (Saturday), May 7–8, 1976.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 20–26, 1977.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, San Francisco, October 29–30, 1976.

AMERICAN MATHEMATICAL SOCIETY, University of Toronto, August 24–27, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, Chase Park Plaza Hotel, St. Louis, Missouri, January 27–28, 1977.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

INTERNATIONAL PEIRCE CONGRESS, Grand Hotel Krasnapolsky, Amsterdam, June 16–20, 1976.

MU ALPHA THETA, West Chester State College, West Chester, Pennsylvania, August 9–11, 1976.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cincinnati, Ohio, April 20–23, 1977.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami Beach, November 3–5, 1976.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Commodore Perry Hotel, Toledo, Ohio, November 11–13, 1976.

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## Volume 11: STUDIES IN GRAPH THEORY, PART I
## Volume 12: STUDIES IN GRAPH THEORY, PART II

Edited by D. R. Fulkerson, Cornell University

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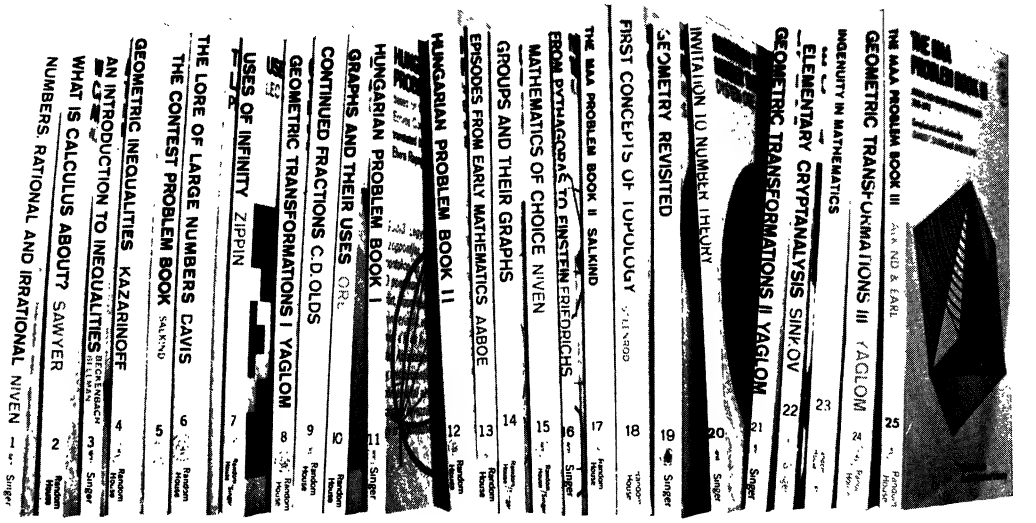
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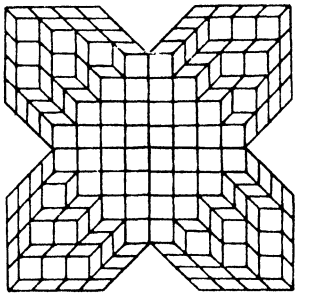
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THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 6

CODEN: AMMYAE

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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

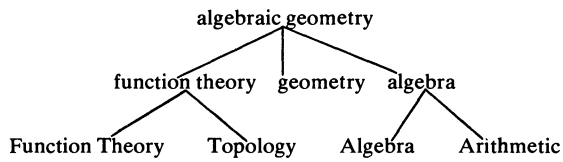
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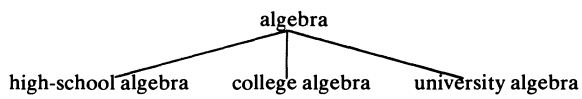
# HISTORICAL RAMBLINGS IN ALGEBRAIC GEOMETRY AND RELATED ALGEBRA

SHREERAM S. ABHYANKAR

**1. Various perspectives.** Algebraic geometry has been approached from various perspectives:




The relevant algebra can be divided into 3 categories:




We associate different catchwords with these divisions:

function theory: function, integral;
geometry: curve, surface, variety;
high-school algebra: polynomial, power series;
college algebra: ring, field, ideal;
university algebra: functor.

We give a chronological tabulation of some of the distinguished proponents of the various divisions, together with one approximate date of work for each proponent; (for most of them, a publication — somewhat randomly chosen — is listed in the References; a large source list can be compiled by looking up the references in the References):

*function theory*: Euler (1748), Abel (1826), Jacobi (1832), RIEMANN (1857), Picard (1897), Poincaré (1910), Lefschetz (1921), Zariski (1929), Hodge (1941), Kodaira (1954), Hirzebruch (1956), Griffiths (1972).

*geometry*: Cremona (1850), M. NOETHER (1870), Bertini (1882), C. Segre (1894), Castelnuovo (1894), Enriques (1894), Zariski (1934).

*high-school algebra*: Bhaskara (1114), Cardano (1530), Ferrari (1540), NEWTON (1680), Tschirnhausen (1683), Euler (1748), Sylvester (1840), Cayley (1870), Kronecker (1882), Mertens (1886), König (1903), Perron (1905), Hurwitz (1913), Macaulay (1916), Zariski (1941), Hironaka (1964).

*college algebra*: DEDEKIND (1882), E. NOETHER (1925), Krull (1930), Zariski (1941), Chevalley (1943), Cohen (1946), Nagata (1960).

*university algebra*: Serre (1955), Cartan (1956), Eilenberg (1956), GROTHENDIECK (1960), Mumford (1965).

**2. Fundamental thesis of this paper** (obviously a partisan claim). The method of high-school algebra is powerful, beautiful, and accessible. So let us not be overwhelmed by the groups-rings-fields or the functorial arrows of the other two algebras and thereby lose sight of the power of the explicit algorithmic processes given to us by Newton, Tschirnhausen, Kronecker, and Sylvester.

Later on in the paper I shall relate four personal experiences to substantiate the above thesis.

---

Expanded version of an invited lecture at the November 30, 1974 Indianapolis meeting of the Indiana Section of the Mathematical Association of America. Extracts were also presented at the Workshop on the Evolution of Modern Mathematics held by the American Academy of Arts and Sciences at Boston on August 8–9, 1974. Excerpts of this paper appeared in the *Proceedings of the Workshop* in "Historia Mathematica," vol. 2, 1975, No. 4.

This work was supported by the National Science Foundation under Grant Number MPS–75–09090 at Purdue University.

## Chapter I. Function Theory

**3. Integral.** After the invention of differential calculus by Newton and Leibnitz, at the end of the seventeenth century, attention was turned to the reverse process: integration. Rational functions of one variable (rational function = polynomial divided by polynomial) and then trigonometric functions were integrated. Then it was found that integrals of the form

$$\int [v(x)+w(x)\sqrt{(x-a)(x-b)(x-c)}]\,dx,$$

where $v(x)$ and $w(x)$ are rational functions and $a$, $b$, $c$ are distinct constants, could not be integrated in terms of "known" functions. These integrals are called elliptic integrals since they first occurred in trying to find the arc length of an ellipse. Euler and others studied elliptic integrals. Then Abel initiated the study of general algebraic integrals of the form

$$\int r(x, y)\,dx,$$

where $r(x, y)$ is a rational function and where $y$ is the $n$-valued function of $x$ defined by

$$f(X, Y) = u_0(X)Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X) = 0$$

with polynomials $u_i(X)$ where $u_0(X) \neq 0$; it is assumed that $f$ is irreducible, i.e., it cannot be factored into two polynomials. In honor of Abel, these integrals became known as abelian integrals.

The differential ( = integrand = integral − integral sign) $r(x, y)\,dx$ is said to be of first kind if it has no pole. The integral $\int r(x, y)\,dx$ is said to be of first kind if the corresponding differential is so. It was shown that, for a fixed $f$, there are at most a finite number of linearly independent differentials of first kind; this number is called the genus of $f$ and is denoted by $g$:

($g_1$) $g$ = number of linearly independent differentials of first kind.

It was also shown that (without restricting to first kind)

($g_2$) $2g - 2$ = number of zeros of a differential − number of its poles.

That the right hand side depends only on $f$, and not on the particular differential, follows from the corresponding property which was proved about rational functions $s(x, y)$:

(1) number of zeros of a function = number of its poles.

Perhaps it may be preferable to regard ($g_2$) as the definition of the genus and ($g_1$) as a theorem. In fact, by and by, we shall write down several genus formulas and, depending on the mood, any one of them can be taken as the definition. At any rate, these formulas do provide a helpful thread running through various aspects of one-dimensional algebraic geometry.

Abel (1826, **[1]**) proved a far-reaching general theorem about sums of integrals of first kind. A little later, Jacobi (1832, **[39]**) inverted these integrals and thereby constructed periodic functions of $g$ complex variables with $2g$ periods.

**4. Riemann.** The above matter needed clarification. Thus, already for elliptic integrals we have to know which of the two square-roots to start with; in the general case we have to know which "branch" of $y$ to take. A picturesque clarification was provided by Riemann by constructing a surface $S$, called the Riemann surface of $f$, on which $y$ "becomes" single-valued. The functions $r(x, y)$ and $s(x, y)$ and the differentials $r(x, y)\,dx$, together with the locations and orders of their zeros and poles, are now to be viewed as happenings on $S$, and so also the abelian integrals $\int r(x, y)\,dx$ are to be integrated along paths on $S$.

Using his surface, Riemann (1857, **[59]**) showed that

($g_3$) $\left\{\begin{array}{l}\textit{number of linearly independent rational functions } s(x, y) \\ \qquad \textit{with prescribed } d \textit{ poles} = d + 1 - g + i,\end{array}\right.$

where $i$ is a nonnegative integer which is zero if $d > 2g - 2$, and later Roch (1865, **[60]**) showed that

(*) $\left\{\begin{array}{l} i = \textit{number of linearly independent differentials of first kind} \\ \qquad\quad \textit{having zeros at the prescribed poles.} \end{array}\right.$

Again, $(g_3)$ may be regarded as giving a definition of the genus. In any case, the *Riemann-Roch theorem* $[(g_3)+(*)]$ shows that the resulting definition is equivalent to $(g_1)$ as well as to $(g_2)$.

Now any (closed) surface $S^*$ is topologically equivalent to (i.e., by stretching and shrinking but without tearing, can be converted into) a sphere with a certain number of handles; say the number of handles is $g^*$. (See Fig. 1.)

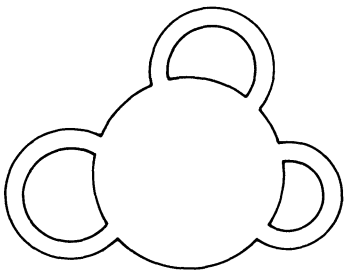
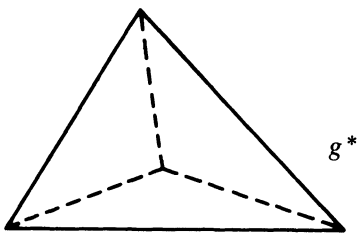


FIG. 1

*Euler's theorem*, which is one of the oldest theorems of topology, gives the following method of computing $g^*$: *On $S^*$ draw any polyhedron $\Delta^*$ (all whose faces are curvilinear triangles); let $\Delta_0^*$, $\Delta_1^*$, and $\Delta_2^*$ be the number of vertices, edges, and faces of $\Delta^*$; then*

$$\Delta_0^* - \Delta_1^* + \Delta_2^* = 2 - 2g^*.$$



$g^* = 0,\ \Delta_0^* = 4,\ \Delta_1^* = 6,\ \Delta_2^* = 4.$

FIG. 2

For verification for a tetrahedron see Fig. 2.

Using formula $(g_2)$ and Euler's theorem, Riemann showed that for the Riemann surface $S$:

$(g_4)$ $\qquad g = \text{number of handles.}$

As a reference to the material described in these two sections, we may cite the excellent book of Stahl (1896, **[68]**).

Let us close our discussion with a philosophical remark:

Jacobi's approach seems to have been algorithmic: write down numerous explicit formulas! As a dividend he could deduce things such as: every integer is a sum of four squares and in exactly so many ways. Jacobi's approach received a big push in the hands of the arithmetician Kronecker.

On the other hand, Riemann's approach appears to have been existential: know that the formulas exist! Many of his existence arguments were based on "Dirichlet's principle," the genesis of which can be expressed by saying something like this: the real and imaginary parts of an analytic function of a

complex variable satisfy the same sort of partial differential equation as a gravitational potential or an electric potential (or a fluid flow or heat conduction...); so the existence of analytic functions or differentials with prescribed boundary behaviour or assigned singularities can be adduced from physical considerations. To carry the analogy further (or backwards), one could say that the mathematical formula "number of zeros of a function = number of its poles" corresponds to the physical fact that there is no gravitational pull inside a hollow shell, or that there is no electric intensity inside a hollow charged conductor. Riemann's approach was furthered by Hilbert (1904, **[34]**) who put the Dirichlet principle on firmer grounds.

**5. Higher dimensions.** A survey of the corresponding development in the hands of Picard, ..., Hirzebruch, can be found in the second edition of Zariski's book (1971, **[80]**). By extending and employing some of these developments, Clemens and Griffiths (1972, **[22]**) have proved the nonrationality of nonsingular hypersurfaces of degree 3 in 4-space, which was a problem of long standing.

## Chapter II: Geometry

**6. Spatial intuition.** In the geometric viewpoint one studies the polynomial equation $f(X, Y) = 0$ by visualizing the (algebraic) plane curve consisting of all points $(a, b)$ in the $(X, Y)$-plane satisfying $f(a, b) = 0$. To take care of "points at infinity" one introduces homogeneous coordinates and then the old point $(a, b)$ is represented by all triples $(ka, kb, k)$ with $k \neq 0$, and the plane curve is given by $F(X, Y, Z) = 0$ where $F$ is the homogeneous polynomial (i.e., a polynomial all of whose terms have the same degree) obtained by "homogenizing" $f$; for example, homogenizing $Y^2 - X^3$ we get $Y^2Z - X^3$ and then dehomogenizing $Y^2Z - X^3$ (i.e., putting $Z = 1$) we get back $Y^2 - X^3$. The degree of $f$ (or of $F$) is called the degree of the plane curve.

More generally, the totality of common solutions of a finite number of polynomial equations in $r$ variables is visualized as a geometric configuration or "an (algebraic) variety" in $r$-dimensional space; again, to take care of "points at infinity" one may use homogeneous equations in $r + 1$ variables. A curve is a variety of "dimension" 1, a surface is a variety of "dimension" 2, ...

**7. Max Noether.** Noether deduced many properties of algebraic curves (including the geometric version of the Riemann-Roch theorem) as consequences of a theorem which acquired the designations "Noether's $AF + B\Phi$ Theorem" or "Noether's Fundamental Theorem." This is indeed one of the most proved theorems; already in Berzolari's 1906 encyclopedia article **[13]** one can find references to proofs by a dozen different authors; Noether's original proof appeared in (1870, **[54]**) and (1873, **[55]**).

*Genesis.* In elementary analytic geometry one may be given the following exercise: Let $F = 0$ and $\Phi = 0$ be two circles. Let it be required to find a third circle $H = 0$ which passes through the points of intersection of $F$ and $\Phi$, and which does something else. The pedantic way would be to first actually find the points of intersection — by solving the simultaneous equations $F = 0$ and $\Phi = 0$ — and then proceed to construct $H$. The clever way, given in good books of analytic geometry (e.g., Salmon (1852), **[62]**), would be thus. For any (constants) $A$ and $B$, the conic

$$AF + B\Phi = 0$$

clearly passes through all the points of intersection of $F$ and $\Phi$. So it is reasonable to surmise that conversely any conic passing through the points of intersection of $F$ and $\Phi$ can be so expressed.

Generalizing the above idea, let $F = 0$ and $\Phi = 0$ be plane curves of any degree. Then again, for any (homogeneous polynomials) $A$ and $B$, the curve

$$AF + B\Phi = 0$$

clearly passes through all the points of intersection of $F$ and $\Phi$.

*Noether's Fundamental Theorem* says that the converse is true. That is, *if* $H = 0$ *is any plane curve such that "*$H$ *passes through all the points of intersections of* $F$ *and* $\Phi$*" then*

$$H = AF + B\Phi \text{ for suitable } A \text{ and } B.$$

Of course, if, say, some point $P$ is a double point of both $F$ and $\Phi$, then $P$ must be a double (or higher) point of every plane curve of the form $AF + B\Phi = 0$.

In other words, Noether's theorem must be qualified by giving a "good" meaning to the phrase "$H$ passes through ...." What meaning to give? ANSWER: That which will make Noether's theorem true.

Indeed the peculiar characteristic wisdom of geometric algebraic geometry is the

**8. Dictum.** Study of simple cases gives rise to a nice succinct statement. Take it as an axiom that the statement is true most generally. Make it true by the provision that we learn to "count properly" the intervening quantities. Or better still, have faith that god (or, if you prefer, nature) has a good meaning in mind, and march on!

**9. Severi.** After Max Noether, geometric algebraic geometry went to live in Italy which produced the great geometers: Bertini, C. Segre, Castelnuovo, Enriques, Severi. To quote from Zariski's address (1950, [**79**]): "The Italian geometers have erected ... a stupendous edifice: the theory of algebraic surfaces." This edifice was surveyed by Zariski (1934, [**80**]).

It is perhaps appropriate to elucidate the *Dictum* by quoting Severi: "Every good theorem must have a good counterexample." PARAPHRASE: don't be deterred if your formula is presently invalid in some cases; it only means that you have not yet completely deciphered god's mind.

At any rate, Severi's *Vorlesungen* (1921, [**66**]) is certainly one of the most readable books on algebraic geometry. Let me here venture to say that

**10. Semple and Roth** (1949, [**64**]) is perhaps the only book on algebraic geometry written after, say, 1940 which covers a lot of territory and is at the same time "readable" like an entertaining novel. However, it may not be the best introduction to the subject for the present-day student trained in contemporary rigor. May I, as a plea, suggest that someone please write a reasonably exhaustive book on algebraic geometry which is rigorous as well as "readable"!

**11. Bryn Mawr.** Before proceeding to give two more examples of the *Dictum*, let us note that a frequently cited proof of Noether's theorem was given by Charlotte Angas Scott of Bryn Mawr (1899, [**63**]). It is worth giving a long quote from her paper: "This (Noether's) theorem, discovered in the course of, and developed for the sake of, purely algebraic researches, is not however tabooed to the geometer ...; but it does not appear that any simple proof depending on geometrical conceptions has yet been given. Cayley (1887, [**18**]) regarded the theorem as intuitive for simple intersections. Zeuthen's proof (1888, [**81**]) depends on an elaborate determination of the number of conditions imposed by the intersections of two curves, when these are simple, the case of multiple intersections being then deduced by the somewhat dangerous process of proceeding to the limit."

**12. Another example of the Dictum.** Two lines meet in a point. A line and a conic meet in two points. Two conics meet in four points. So we are led to the very first theorem of algebraic geometry.

BEZOUT'S THEOREM. *Any two algebraic plane curves of degree* $N$ *and* $M$*, devoid of common components, meet in exactly* $MN$ *points (counted properly!).*

This was first asserted by Maclaurin (1720, [45]). It was discussed by Euler (1748, [26]) and Cramer (1750, [24]), and then more completely by Bezout (1770, [14]).

*Algebraic Proof.* See §21.

*Geometric Proof.* The curves can be degenerated into $N$ lines and $M$ lines. These clearly meet in $MN$ points. Hence so do the original curves.

At any rate, we are led to the geometric characterization of the degree $N$ of a plane curve $C$: $f(X, Y) = 0$ as the number of intersection of $C$ with a line (either count properly or maximize over all lines). Furthermore: a point $P$ of $C$ is said to be a $d$-fold point (or a $d$-ple point, or a point of multiplicity $d$) if most lines through $P$ meet $C$ at $P$ in $d$ coincidental points, or alternatively, if they meet $C$, outside $P$, in $N - d$ points; the finite number of lines through $P$ which meet $C$ at $P$ in more than $d$ coincidental points are called the tangents to $C$ at $P$; $P$ is called a singular point (or a singularity) of $C$ if $d > 1$; $P$ is called a simple point of $C$ if $d = 1$. $C$ is said to be nonsingular if it has no singularities. Algebraically, the multiplicity $d$ of a point $P = (a, b)$ of $C$ at finite distance can be characterized as the smallest degree of a term occurring in the expansion of $f$ around $P$; in other words

$$f(X + a, Y + b) = \zeta_d(X, Y) + \sum_{i+j>d} \lambda_{ij} X^i Y^j,$$

where $\zeta_d(X, Y)$ is a nonzero homogeneous polynomial of degree $d$; moreover, the $d$ factors $\mu_i X + \nu_i Y$ of $\zeta_d(X, Y)$ give the tangents $\mu_i(X - a) + \nu_i(Y - b) = 0$ to $C$ at $P$, which are not necessarily all distinct. So in particular: $P$ is a singularity of $C$ iff $f_Y(a, b) = 0 = f_X(a, b)$, where $f_Y$ and $f_X$ are the $Y$-derivative and $X$-derivative of $f$; moreover, if $P$ is a simple point of $C$ then: $f_Y(a, b) = 0$ iff the tangent to $C$ at $P$ is $X - a = 0$.

A similar situation prevails in higher dimensions.

The problem of resolution of singularities asks whether a given variety can be desingularized, i.e., can be transformed by an almost one-to-one (algebraic) transformation into a nonsingular variety.

For a plane curve $f(X, Y) = 0$, the Riemann surface of $f$ is nothing but the desingularization of $f$. Geometrically (or algebraically) the same thing was achieved by Noether by means of "quadratic transformations." For example, the curve $Y^2 - X^3 = 0$ has a double point at the origin; the quadratic transformation (or substitution)

$$X = X' \text{ and } Y = X'Y'$$

yields

$$0 = Y^2 - X^3 = X'^2 Y'^2 - X'^3 = X'^2(Y'^2 - X')$$

and cancelling out the extraneous factor $X'^2$ we get the nonsingular curve $Y'^2 - X' = 0$. To desingularize the general plane curve $f = 0$ one has to make a finite succession of such transformations. Moreover, Noether also gave an "analysis" of the singularities of $f = 0$ in terms of these resolution steps. A brief exposition of related matter can be found in my Bloomington lecture [7].

**13. Genus formula.** As a final example of the *Dictum*, study of simple cases which we shall describe later, led the geometers from Maclaurin to Noether towards the following formula for the genus $g$ of an irreducible plane curve $C$: $f(X, Y) = 0$ of degree $N$:

$$g = (1/2)(N - 1)(N - 2) - \text{number of double points counted properly}$$

i.e.,

$$(g_5) \qquad g = (1/2)(N - 1)(N - 2) - \Sigma\, \delta(P),$$

where the sum is over all the singular points $P$ of $C$ and where $\delta(P)$ is a certain positive integer which says that "$P$ accounts for $\delta(P)$ ordinary double points."

## CHAPTER III: HIGH-SCHOOL ALGEBRA

**14. Brahmins.** It might be said that (high-school) algebra started with the Brahmins (my ancestors) of India: Aryabhatta (476), Brahmagupta (598), Bhaskara (1114); quadratic equations were solved by completing the square; also rules were given for solving certain types of so-called diophantine equations. Bhaskaracharya (*acharya* = professor) was the director of an observatory at Ujjain; his treatise on astronomy (*Siddhantashiromani*) contains chapters on geometry (*Lilavati* — named after his daughter) and algebra (*Beejaganit*).

(See *Historical Notes* in Chrystal [**21**] and Krull [**44**].)

Now it is true that the history of mathematics should primarily consist of an account of the achievements of great men; but it may be of some interest to also see their impact on an average student like me. With this in mind I may be allowed some

**15. Personal reminiscences.** It is my fond memory that my father initiated me to mathematics — and at the same time to Sanskrit poetry — by teaching me portions of Bhaskara's treatise. After several years, during my last year in high-school, at my father's suggestion I studied Hobson's *Trigonometry* (1891, [**37**]), Hardy's *Pure Mathematics* (1908, [**31**]), and Chrystal's *Algebra* (1886, [**21**]). By the end of first-year college I had also studied Knopp's *Infinite Series* (1928, [**40**]), Burnside-Panton's *Theory of Equations* (1904, [**16**]), and Bôcher's *Higher Algebra* (1907, [**15**]). These books have served me well; they are all (except, perhaps, Hardy's) predominantly high-school-algebraic and algorithmic. After them my ways were set; after them I could not, (and didn't need to?), acquire much new technique. The goodly dose of analysis obtained during my last three years of college was all but forgotten.

As a graduate student at Harvard, I suppose I did acquire some mathematical sophistication. As a result, I always think high-school algebra but write college algebra. At any rate, Zariski's algebraic geometry gave ample scope to my early algorithmic training. However, what initially attracted me to Zariski was Math 103, Projective Geometry: von Staudt's algebra of throws and the resulting identity of geometry with algebra was indeed very exciting.

To continue with our story of high-school algebra:

**16. Arabia and Italy.** Then travelling through Arabia, algebra reached Europe around 1500. During 1500–1600, the Italians — Tartaglia, Cardano, Ferrari, et al — solved cubic and quartic equations.

Fermat's and Descartes' (1637) introduction of coordinates gave a big impetus to algebra.

(See Historical Notes in Chrystal [**21**] and Krull [**44**].)

Then there appeared the majestic genius of Newton (1680, [**53**]):

**17. Newton.** Truly the father of us all. Newton discovered the Binomial Theorem

$$(1+X)^n = \sum_i \frac{n(n-1)(n-2)\cdots(n-i+1)}{1\cdot 2\cdot 3\cdots i} X^i$$

first for positive integral $n$ and then soon after for any fractional $n$. For fractional exponent this can be regarded as solving, for $Y$, the equation

$$Y^n - u(X) = 0, \tag{1}$$

where $u(X)$ is a power series (and where $n$ is the denominator of the $n$ of the Binomial Theorem).

What is a power series? Take the definition of a polynomial

$$a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$$

and from it delete the last term. This gives

$$a_0 + a_1X + a_2X^2 + \cdots$$

as the definition of a power series; likewise for several variables. So a power series is simpler than a polynomial. Like polynomials, power series can be added, subtracted, and multiplied.

Especially in case of one variable, we may even permit a finite number of negative exponents

$$u(X) = a_{-m}X^{-m} + a_{-m+1}X^{-m+1} + \cdots + a_0 + a_1X + a_2X^2 + \cdots$$

and call it a meromorphic series. Unlike a polynomial, a power series or a meromorphic series need not have a highest degree term; but it does have a lowest degree term; the degree of the lowest degree term in $u(X)$ is called the $X$-order of $u(X)$ and is denoted by $\mathrm{ord}_X u(X)$.

When we allow meromorphic series we can divide. Namely if $u(X) \neq 0$ and $q = \mathrm{ord}_X u(X)$, then

$$u(X) = a_q X^q [1 - v(X)] \text{ with } v(0) = 0;$$

hence, because of the geometric series

$$(1 - X)^{-1} = 1 + X + X^2 + \cdots$$

we have

$$u(X)^{-1} = a_q^{-1} X^{-q} [1 + v(X) + v(X)^2 + \cdots].$$

It is presumed that the "coefficients" $a_i$ (or, in case of several variables, $a_{ij\cdots k}$) can be added, subtracted, multiplied, and divided. Thus they may be rational numbers, or real numbers, or complex numbers, or "integers modulo a prime number $p$," etc. In the case of rational numbers, or real numbers, or complex numbers, we are in zero (or infinite) characteristic, i.e., $1 + 1 + \cdots + 1$ is never zero. In the case of "integers modulo a prime number $p$" we are in the nonzero characteristic $p$, i.e.,

$$\underbrace{1 + 1 + \cdots + 1}_{p \text{ times}} \text{ equals } 0;$$

in this case the Binomial Theorem takes the amusing form: $(1 + X)^p = 1 + X^p$. Frequently, as in the case of complex numbers, every polynomial equation in one variable has a root; we shall usually (as in the theorem below) suppose this to be so.

Having solved the simpler equation (1), Newton proceeded to solve the most general equation of that type; to wit:

*Newton's Theorem* (on Puiseux expansion). Every polynomial

$$f(X, Y) = Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X)$$

with meromorphic series $u_i(X)$, where $n!$ is not divisible by the characteristic, can be completely factored if we allow series with fractional exponents. More precisely, we have

$$f(T^{n!}, Y) = \prod_{i=1}^{n} (Y - y_i(T))$$

with meromorphic series $y_i(T)$; the $y_i(T)$ are power series if all the $u_i(X)$ are. Moreover, if $f$ is irreducible (when fractional exponents are not allowed), then $n!$ may be replaced by $n$, i.e., assuming that $n$ is not divisible by the characteristic, we have

$$f(T^n, Y) = \prod_{i=1}^{n} (Y - y(\omega^i T)),$$

where $y(T)$ is a meromorphic series and $\omega$ is a primitive $n$th root of 1.

Not only did Newton prove the existence of the $y_i$, but he actually gave an algorithm for explicitly finding them by successive polynomial approximations. This algorithm is known as Newton's polygon (or, according to older literature, parallelogram). [The spirit of this algorithm is similar to Newton's iterative procedure of approximation of real roots of polynomials with constant (real) coefficients.]

To quote from the *Historical Note* on page 396 of Part II of Chrystal [21]: "... (Newton's) method

was well understood by Newton's followers, Stirling and Taylor; but seems to have been lost sight of... after their time."

Newton's theorem was revived by Puiseux [**58**] in 1850. So it acquired the name "Puiseux expansion" which is a misnomer. What's more is that Puiseux's proof, being based upon Cauchy's integral theorems, applies only to convergent power series with complex coefficients. On the other hand, Newton's proof, being algorithmic, applies equally well to power series, whether they converge or not. Moreover, and that is the main point, Newton's algorithmic proof leads to numerous other existence theorems while Puiseux's existential proof does not do so.

Generally speaking, from Newton to Cauchy (1830), mathematicians used power series without regard to convergence. They were criticised for this and the matter was rectified by the analysts Cauchy and Abel who developed a rigorous theory of convergence. After another hundred years or so we were taught, say by Hensel [**32**], Krull [**43**] and Chevalley [**20**], that it really didn't matter, i.e., we may disregard convergence after all! So the algebraist was freed from the shackles of analysis, or rather (as in Vedanta philosophy) he was told that he always was free but had only forgotten it temporarily.

**18. Tschirnhausen**(1683, [**70**]). This contemporary of Newton tried to solve fifth degree equations. In that he failed. But his success was greater than his failure. He developed many transformations of equations.

If you pick up almost any book on algebra, written before (but none after) 1931, you will find numerous pages devoted to Tschirnhausen transformations. To cite some more meritorious books on algebra, in addition to those already cited in §15, we have: Weber (1894, [**72**]), König (1903, [**41**]), Macaulay (1916, [**46**]), and Perron (1927, [**57**]).

Then in 1931 came van der Waerden [**71**] drawing the line of demarcation between high-school algebra and college algebra; still, being on the boundary, van der Waerden does include some elimination theory.

As a simple sample of Tschirnhausen's transformations we have the trick of killing the coefficient of $Y^{n-1}$ in

$$G(Y) = Y^n + b_1 Y^{n-1} + \cdots + b_n$$

by the substitution

$$Z = Y + \frac{b_1}{n}$$

thereby getting $G(Y) = Z^n + c_2 Z^{n-2} + \cdots + c_n$, because by the Binomial Theorem we have

$$(Y + X)^n = Y^n + nXY^{n-1} + \cdots.$$

For $n = 2$ this is simply the ancient method of solving a quadratic equation by completing the square.

Of course we can divide by $n$ only if $n$ is not divisible by the characteristic.

This killing of $b_1$ is indeed the basic fact underlying Zariski's [**76**], [**77**], [**78**] and Hironaka's [**35**] proofs of resolution of singularities of algebraic varieties of zero characteristic. For an analysis of this matter see my *Moscow lecture* [**6**], where I went on to say that "instead of killing $b_1$, Zariski used differentiation arguments; but then after all, the Binomial Theorem and differentiation are in essence one and the same thing."

In turn, the nonavailability of the above Tschirnhausen transformation is what makes the resolution problem quite different in nonzero characteristic. To put it differently, we have to grasp the divisibility properties of the binomial coefficients, and in other ways, to understand the Binomial Theorem better! In the *Moscow lecture* I also pointed out that as far as the resolution problem in nonzero characteristic is concerned, much critical information is lost by replacing a power series by the totality of all its multiples, i.e., by the "principal ideal generated by it"! This is an instance of the superiority of high-school algebra over college algebra and geometry, and so this brings us to:

**19. Personal experience 1.** In my Harvard dissertation (1956, [2]) I proved resolution of singularities of algebraic surfaces in nonzero characteristic. There I used a mixture of high-school and college algebra. After ten years, I understood the Binomial Theorem a little better and thereby learned how to replace some of the college algebra by high-school algebra; that enabled me to prove resolution for arithmetical surfaces (1965, [4]). Then replacing some more college algebra by high-school algebra enabled me to prove resolution for three-dimensional algebraic varieties in nonzero characteristic (1966, [5]). But still some college algebra has remained.

I am convinced that if one can decipher the mysteries of the Binomial Theorem and learn how to replace the remaining college algebra by high-school algebra, then one should be able to do the general resolution problem. Indeed, I could almost see a ray of light at the end of the tunnel. But this process of unlearning college algebra left me a bit exhausted; so I quit!

**20. Euler** (1748, [26]). Euler dealt with a mixture of algebra and analysis; the two fields started separating when Cauchy (1830) began to be concerned with questions of convergence. As an amusing evidence of Euler's algebraic technique let me quote his "proof" that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Namely, if $p_1, p_2, \cdots$ are the roots of $a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m = 0$, then

$$\sum \frac{1}{p_i} = \frac{-a_1}{a_0}.$$

Now

$$\cos \sqrt{x} = 1 - \frac{x}{2} + \frac{x^2}{24} + \cdots$$

and the roots of $\cos \sqrt{x} = 0$ are

$$\frac{(2n+1)^2 \pi^2}{4} \text{ with } n = 0, 1, 2, \cdots$$

and hence

$$\sum \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

**21. Elimination theory.** This encompasses the explicit algorithmic procedures of solving several simultaneous polynomial equations in several variables. Here some of the prominent names are: Sylvester (1840, [69]), Kronecker (1882, [42]), Mertens (1886, [47]), König (1903, [41]), Hurwitz (1913, [38]), and Macaulay (1916, [46]). It is a vast theory. There used to be a belief, substantially justified, that elimination theory is capable of handling most problems of algebraic geometry in a rigorous and *constructive* manner. This is of course not surprising; after all, what is algebraic geometry but another name for systems of polynomial equations!

What is surprising is that under Bourbaki's influence it somehow became fashionable to bring elimination theory into disrepute. To quote from page 31 of Weil (1946, [74]): "The device that follows, which, it may be hoped, finally eliminates from algebraic geometry the last traces of elimination theory, is borrowed from C. Chevalley's *Princeton lectures.*"

It seems to me, what Bourbaki achieved thereby was trading in constructive proofs for merely existence proofs.

I shall now describe only the first rudiment of elimination theory; namely the resultant of two polynomials in one variable $T$.

To solve two simultaneous equations

$$\xi = a_0T^n + a_1T^{n-1} + \cdots + a_n = 0, \tag{1}$$

$$\eta = b_0T^m + b_1T^{m-1} + \cdots + b_m = 0, \tag{2}$$

or rather to find a condition for them to have a common solution, we eliminate $T$ between them. To wit, say $n \geqq m$; multiply (1) by $b_0$ and (2) by $a_0T^{n-m}$ and then subtract; this cuts down $m + n$; now iterate; ultimately we get an expression dependent only on the $a$'s and $b$'s. Different (but substantially equivalent) ways of systematizing this procedure were given by Euler (1748, [**26**]), Bezout (1770, [**14**]), and Sylvester (1840, [**69**]); it seems (see Berzolari [**13**]) that a procedure was already known to Newton. Following Sylvester we define the $T$-resultant of $\xi$ and $\eta$, denoted by $\mathrm{Res}_T(\xi, \eta)$, to be the $m + n$ by $m + n$ determinant

$$\begin{vmatrix}
a_0 & a_1 & \cdots & a_n & & & \\
 & a_0 & a_1 & \cdots & a_n & & \\
 & & \cdots & \cdots & \cdots & \cdots & \\
 & & & a_0 & a_1 & \cdots & a_n \\
b_0 & b_1 & \cdots & b_m & & & \\
 & b_0 & b_1 & \cdots & b_m & & \\
 & & \cdots & \cdots & \cdots & \cdots & \\
 & & & b_0 & b_1 & \cdots & b_m
\end{vmatrix}$$

with zeros in all blank spaces. We have the

RESULTANT THEOREM. $\mathrm{Res}_T(\xi, \eta) = 0$ *iff either* $a_0 = 0 = b_0$ *or* $\xi$ *and* $\eta$ *have a common solution.*

Now let

$$C_N\colon F(X, Y, Z) = 0 \text{ and } D_M\colon \Phi(X, Y, Z) = 0$$

be plane curves of degree $N$ and $M$ without common components, i.e., $F$ and $\Phi$ are homogeneous polynomials of degree $N$ and $M$ respectively, and $F$ and $\Phi$ do not have a common factor. Let

$$E(X, Z) = \mathrm{Res}_Y(F, \Phi).$$

By the determinantal expression we see that $E(X, Z)$ is a homogeneous polynomial of degree $MN$. By the above theorem we also see that the "roots" of $E(X, Z)$ are the homogeneous $(X, Z)$-coordinates of the points of intersection of $C_N$ and $D_M$. This indicates that $C_N$ and $D_M$ meet in $MN$ points; the indication is turned into

BEZOUT'S THEOREM. $\Sigma\, i(C, D; P) = MN$, where the sum is over all common points $P$ of $C$ and $D$ and where the positive integer $i(C, D; P)$ is the *intersection multiplicity* of $C$ and $D$ at $P$, and it is also defined in terms of a resultant thus: Given $P$, we may choose our coordinates so that at $P$ we have $Z = 1$ and $X = 0 = Y$. By the Preparation Theorem of Weierstrass (1860, [**73**]) (who, although sometimes called the prince of analysis, was indeed a most distinguished high-school algebraist) we can write

$$F(X, Y, 1) = f'(X, Y)f^*(X, Y) \text{ and } \Phi(X, Y, 1) = \phi'(X, Y)\phi^*(X, Y),$$

where $f'(X, Y)$ and $\phi'(X, Y)$ are polynomials in $Y$ with coefficients power series in $X$ such that $f'(0, 0) \neq 0 \neq \phi'(0, 0)$, and where

$$\left.\begin{aligned} f^*(X, Y) &= Y^e + v_1(X)Y^{e-1} + \cdots + v_e(X) \\ \phi^*(X, Y) &= Y^e + w_1(X)Y^{e-1} + \cdots + w_e(X) \end{aligned}\right\} \begin{array}{l}\text{where } v_i(X) \text{ and } w_j(X) \\ \text{are power series with} \\ v_i(0) = 0 = w_j(0).\end{array}$$

Now $\mathrm{Res}_Y(f^*, \phi^*)$ is a power series in $X$ and we define

$$i(C, D; P) = X\text{-order of } \mathrm{Res}_Y(f^*, \phi^*).$$

One can see that then

$$i(C, D; P) = \text{number of power series in } X \text{ and } Y \text{ which are linearly independent over power series in } F(X, Y, 1) \text{ and } \Phi(X, Y, 1).$$

**22. Personal experience 2.** How many equations are needed for defining a curve $C$ in affine 3–space? The history of this question goes back to Kronecker and is described in my *Montreal lectures* (1970, [**9**]); a brief exposition of related matter can also be found in my very elementary *Poona lectures* (1969, [**8**]). In the Montreal lectures I tried to revive elimination theory for studying this question. Among other things, for any nonsingular $C$ I explicitly constructed three equations. Using these three explicit equations, Murthy and Towber (1974, [**49**]) have just now proved that (1) if $C$ is rational or elliptic then two equations suffice, and that (2) every projective module over the polynomial ring in three variables is free (Serre's conjecture).* The funny thing is that Murthy (1972, [**48**]) had also independently proved the existence of three equations, but because he was using university-algebra (the functor Ext and such) his proof was only existential, and that was not enough for proving either (1) or (2).

So high-school algebra has come to the rescue of university algebra!

**23. Discriminant.** The two roots of the quadratic equation $T^2 + bT + c = 0$ coincide iff its discriminant $b^2 - 4c = 0$.

With this experience in mind, consider the plane curve

$$C\colon f(X, Y) = u_0(X)Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X) = 0,$$

where the $u_i(X)$ are polynomials with $u_0(X) \neq 0$. Let $x$ and $y$ be such that $x$ is a variable and $f(x, y) = 0$. Let it be required to locate those values $a$ of $x$ for which at least one of the corresponding values $b$ of $y$ has multiplicity $> 1$, i.e., $b$ is a multiple root of $f(a, Y) = 0$. To this end, we eliminate $Y$ between $f(X, Y)$ and its $Y$-derivative $f_Y(X, Y)$, thereby obtaining a polynomial in $X$ which, following Sylvester, we call the $Y$-discriminant of $f$ and denote it by $\mathrm{Disc}_Y(f)$; i.e.,

$$\mathrm{Disc}_Y(f) = Y\text{-discriminant of } f = \mathrm{Res}_Y(f, f_Y).$$

Now the roots of $\mathrm{Disc}_Y(f)$ are precisely those finite values of $x$ for which there are less than $n$ finite values of $y$, either because one of the finite values of $y$ has multiplicity $> 1$, or because one of the values of $y$ has gone to infinity. All this follows from the Resultant Theorem and the following very simple but

FUNDAMENTAL PRINCIPLE. *The number of roots (or irreducible factors) of a polynomial $\zeta(T)$ in one variable, counted with their multiplicities (resp. degrees and multiplicities), equals the degree of $\zeta(T)$.*

SUPPLEMENT. *Moreover, if $\xi$ is a root (or irreducible factor) of $\zeta(T)$ with multiplicity $e$ then $\xi$ is a root (resp. irreducible factor) of the $T$-derivative of $\zeta(T)$ with multiplicity $\geqq e - 1$, where equality holds iff $e$ is not divisible by the characteristic.*

---

* This conjecture was proved for any number of variables by D. Quillen of MIT in February of 1976. (*Editor*)

Indeed, much algebraic geometry ultimately gets reduced to the Fundamental Principle, plain or supplemented. It is certainly the algebraical key to the various "counting properly."

Now a comment on "finite values." For a finite value of $x$, some value of $y$ goes to infinity iff that value of $x$ is a root of $u_0(X)$. So "no value of $y$ goes to infinity whenever $x$ is finite" iff $u_0(X)$ is a constant. Geometrically speaking, the roots of $u_0(X)$ give the $X$-coordinates of the asymptotes to $C$ parallel to the $Y$-axis. For *Example*, see Fig. 3.

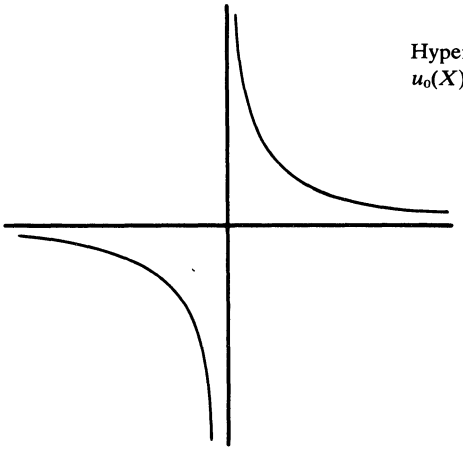



FIG. 3

**24. Personal experience 3.** In a recent paper (1973, [**10**]), Moh and I have made an analysis of: (1) Newton-Puiseux expansion, (2) Tschirnhausen transformations (3) resultants and discriminants, and (4) expansion of a polynomial in terms of another polynomial (analogous to decimal expansion). All these four items can be found in Chrystal [**21**] and Burnside-Panton [**16**], but not in any post-1931 algebra text. As an immediate corollary of this analysis we have proved [**11**] the following Epimorphism Theorem which can be stated in three versions thus:

HIGH-SCHOOL ALGEBRA VERSION. *Let $u(T)$ and $v(T)$ be two polynomials in one variable $T$ such that $T$ can be expressed as a polynomial in $u(T)$ and $v(T)$. Let degree of $u(T) = n > n' =$ degree of $v(T)$, and assume that either $n$ or $n'$ is not divisible by the characteristic. Then $n'$ divides $n$.*

COLLEGE ALGEBRA VERSION. *Any two epimorphisms (whose degrees are nondivisible by the characteristic) of the polynomial ring in two variables onto the polynomial ring in one variable differ from each other by an automorphism of the polynomial ring in two variables. This continues to hold if the polynomial rings are replaced by the free associative (noncommutative) algebras.*

GEOMETRIC VERSION. *Any two embeddings of the affine line into the affine plane differ from each other by a biregular map of the affine plane, (in zero characteristic).*

Most of modern — post 1800 — mathematics seemed powerless to prove this kind of thing. But the 17th century stuff of Newton and Tschirnhausen enabled us to prove certain existence theorems, for instance the existence of the said automorphisms or of the said biregular map. Perhaps the matter is put in a better perspective by saying that what we obtain is a theorem about existence (or better, construction) of curves with prescribed singularities; to further illucidate this point consider:

JUNG-EVYATHAR-NAGATA PENCIL THEOREM (see Nagata [**52**]). *Let $C$ be a nonsingular rational affine plane curve of degree $n$ having only one point $P$ at infinity, and suppose that $C$ is analytically irreducible at $P$. Let $m$ be the multiplicity of $C$ at $P$. Let $P_1, P_2, \cdots, P_k$ be the points of $C$ infinitely near $P$, and let $m_i$*

*be the multiplicity of $C$ at $P_i$. Assume that there exists another nonsingular rational plane curve $D$ ($D \neq C$) of degree $n$ such that: $P$ is the only point of $D$ at infinity, $D$ is analytically irreducible at $P$, $D$ has multiplicity $m$ at $P$, and $D$ has multiplicity $m_i$ at $P_i$ for all $i$. Then $n - m$ divides $n$.*

In effect what we do is to show that, in case of zero characteristic, $D$ automatically exists.

**25. Hamburger.** Hamburger (1871, [**30**]) and Noether (1890, [**56**]) gave an alternative expansion procedure for studying singularities of algebraic curves. In a forthcoming paper I shall present a comparative analysis of the Newton-Puiseux expansion and the Hamburger-Noether expansion. Roughly speaking, the HN expansion is cruder than the NP expansion, and so it works in any characteristic. For instance, upon replacing NP expansion by HN expansion in the last portion of our proof of the Epimorphism Theorem one gets a transparent proof of the above pencil theorem in nonzero characteristic (when the existence of $D$ has to be assumed). But for more delicate questions, such as the automatic existence of $D$ in zero characteristic, one has to use NP expansion.

**26. Personal experience 4.** I have found certain generalizations of Newton's polygon most helpful in obtaining partial results concerning the following problem which is still unproved even for two variables!

*High-School Implicit Function Theorem.* Let $Y_1, Y_2, \cdots, Y_n$ be polynomials in $X_1, X_2, \cdots, X_n$ (with coefficients in a field of zero characteristics) such that

$$\det\left(\frac{\partial Y_i}{\partial X_j}\right)_{\substack{i=1,\cdots,n \\ j=1,\cdots,n}} = \text{a nonzero constant.}$$

Show that then $X_1, X_2, \cdots, X_n$ are polynomials in $Y_1, Y_2, \cdots, Y_n$.

**27. Hironaka.** Newton's polygon has been generalized by Hironaka (1968, [**36**]) to "Newton's polyhedron" which is quite useful in understanding singularities of higher dimensional algebraic varieties.

## Chapter IV: College Algebra

**28. Dictionary.** Telegraphically speaking:

| *college algebra* | *high-school algebra or other suggestive idea* |
|---|---|
| set | collection of objects called its elements |
| group | set in which we can add and subtract |
| ring | also multiply |
| domain | also cancel nonzero common factors from an equation |
| field | also divide |
| algebraically closed field | also solve every one-variable polynomial equation |
| quotient field | from integers to rationals |
| $R$-module | group whose elements can be multiplied by elements of the ring $R$ and still land in the group |
| ideals in $R$ | $R$-module contained in $R$ |
| ideal length | number of linearly independent elements |
| homomorphism | substitute |
| epimorphism | substitute to get everything |
| automorphsim | change of variables. |

**29. Conductor.** Pythagoras (540 B.C.) realized that $\sqrt{2}$ was not a rational number. Gauss (1795) generalized this by showing that if a rational number satisfies an equation of the form

$$T^n + a_1 T^{n-1} + \cdots + a_n = 0$$

with integers $a_i$, then it must be an integer (see exercise 3 on page 7 of Hardy's *Pure Mathematics*

[31]). This gives rise to the following concepts: An element $t$ is said to be integral over (or integrally dependent on) a domain $R$ if $t$ satisfies an equation of the above form with $a_i$ in $R$. For any domain $L$ containing $R$, by the integral closure of $R$ in $L$ we mean the totality of all elements of $L$ which are integral over $R$; the said integral closure is seen to be a domain, i.e., sums and products of integrally dependent elements are themselves integrally dependent. By the normalization of $R$ we mean the integral closure $R^*$ of $R$ in its quotient field. $R$ is said to be normal if $R^* = R$; (Gauss' theorem says that the ring of ordinary integers is normal). An element $r$ of $R$ conducts $R^*$ into $R$ means $rr^*$ is in $R$ for every $r^*$ in $R^*$, i.e., $R$ serves as a common denominator for elements of $R^*$; the conductor of $R$, to be denoted by $\mathrm{Cond}(R)$ is the set of all such $r$. Alternatively, $\mathrm{Cond}(R)$ is the largest ideal in $R$ which remains an ideal in $R^*$. Clearly: $R$ is normal iff $\mathrm{Cond}(R) = R$, iff $\mathrm{Cond}(R) = R^*$. So, as two ways of estimating how far $R$ is from being normal, we introduce the $R$-length of (the factor module) $R/\mathrm{Cond}(R)$ and the $R$-length of (the factor module) $R^*/\mathrm{Cond}(R)$; these lengths are nonnegative integers (or $\infty$) and the vanishing of either of them is a necessary and sufficient condition for $R$ to be normal; the definition of these lengths is thus:

$$R\text{-length of } R/\mathrm{Cond}(R) = \text{the largest length } p \text{ of sequences of } R\text{-modules } \mathrm{Cond}(R) = I_0 \subset I_1 \subset \cdots \subset I_p = R$$

and

$$R\text{-length of } R^*/\mathrm{Cond}(R) = \text{the largest length } q \text{ of sequences of } R\text{-modules } \mathrm{Cond}(R) = I_0 \subset I_1 \subset \cdots \subset I_q = R^*$$

where $I_{j-1} \subset I_j$ means $I_{j-1}$ is contained in but differs from $I_j$.

In addition to the above number-theoretic motivation for the concept of integral dependence, in §23 we have already hinted at the following function-theoretic and geometric motivation. Namely, consider an irreducible plane curve

$$C\colon f(X, Y) = u_0(X)Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X) = 0,$$

where the $u_i(X)$ are polynomials with $u_0(X) \neq 0$. Let $x$ and $y$ be elements such that $x$ is a variable and $f(x, y) = 0$. Then $y$ is integral over $x$, i.e., over the ring of polynomials in $x$, iff $u_0(X)$ is a constant. So, function-theoretically: $y$ is integral over $x$ iff "no value of $y$ goes to infinity whenever $x$ is finite." Geometrically: $y$ is integral over $x$ iff $C$ has no asymptotes parallel to the $Y$-axis.

**30. Dedekind.** Co-inventor (with Kronecker) of ideal theory, Dedekind (1882, [25]) unified algebraic number theory and algebraic curve theory. To give a small sample of his work: Let $C\colon f(X, Y) = 0$ be an irreducible plane curve and let $x$ and $y$ be such that $x$ is a variable and $f(x, y) = 0$. For a point $P$ of $C$ let $R(P)$ denote the local ring of $P$ on $C$; in other words, $R(P)$ is the set of all rational functions on $C$ (i.e., rational functions $r(x, y)$) which do not have a pole at $P$. [If $P = (a, b)$ is at finite distance, i.e., if $a$ and $b$ are both finite, then, more explicitly, $R(P)$ is the set of those rational functions which can be expressed in the form $v(x, y)/w(x, y)$ where $v(X, Y)$ and $w(X, Y)$ are polynomials with $w(a, b) \neq 0$.] Let $R^*(P)$ denote the normalization of $R(P)$. First we take note of

DEDEKIND'S LITTLE THEOREM: *$P$ is a simple point of $C$ iff $R(P)$ is normal.*

So now, as two ways of measuring how singular $C$ is at $P$, in the sense of college algebra, we define

(′) $$\delta(P)_{\mathrm{calg}} = R(P)\text{-length of } R(P)/\mathrm{cond}(R(P))$$

and

(*) $$\delta^*(P)_{\mathrm{calg}} = R(P)\text{-length of } R^*(P)/\mathrm{cond}(R(P)).$$

Then, when $P$ is a singular point of $C$, with $\delta(P)$ as in §13, we can state

DEDEKIND'S CONDUCTOR THEOREM. $\delta^*(P)_{\text{calg}} = 2\delta(P)$.

In other words, upon letting $N$ and $g$ be the degree and genus of $C$, formula $(g_5)$ of §13 acquires the more precise version

$$(g_6) \qquad g = (1/2)(N-1)(N-2) - (1/2)\Sigma\, \delta^*(P)_{\text{calg}}$$

where the sum is over all points $P$ of $C$.

**31. Macaulay.** The pregnant work (1916, [**46**]) of this grand master of high-school algebra has inspired much research in ideal theory. For instance, with notation as in §30, Macaulay's profound analysis of Noether's Fundamental Theorem led another dozen authors, from Gröbner (1934, [**27**]) to Bass (1963, [**12**]), to give various proofs of

*Noether's Fnd Th in new guise*: $\delta^*(P)_{\text{calg}} = 2\delta(P)_{\text{calg}}$

which now converts $(g_6)$ into

$$(g_7) \qquad g = (1/2)(N-1)(N-2) - \Sigma\, \delta(P)_{\text{calg}}.$$

Seeing how this makes two dozen proofs of Noether's Fundamental Theorem, I also got inspired to make the 25th variation which I outlined at the 1973 Calcutta Conference of the Indian Math. Soc. and which will appear in its Proceedings.

**32. Krull.** College algebraist par excellence. Each of Krull's papers, though ring-theoretic in nature, does valuable service to algebraic geometry. For example, he gave us (1938, [**43**]) local rings and their completions; briefly, this formalizes the process by which we "expand a given polynomial as a power series around given values of the variables."

**33. Cohen and Chevalley.** Cohen (1946, [**23**]) and Chevalley (1943, [**19**]) furthered the theory of local rings. In a profoundly beautiful paper (1945, [**20**]) Chevalley, taking the last formula of §21 as his starting point, developed an intersection theory for algebraic and algebroid varieties.

**34. Zariski.** Revolutionized algebraic geometry by harnessing algebra to its service. Since normal varieties have no singularities of codimension 1, it might be thought that Zariski (1939, [**75**]) introduced the normalization process to help out with resolution. The amusing thing is that while normalization has turned out to be quite useful in many aspects of algebraic geometry, for resolution perhaps it was actually a hindrance.

**35. Nagata.** (1962, [**51**]). A fitting successor to Krull.

## CHAPTER V: UNIVERSITY ALGEBRA

**36. Abstraction.** The polynomials of high-school algebra are collected into a ring of college algebra. The rings of college algebra are collected into a category of university algebra, and then the categories are collected into a category of categories, and then..., and then...

**37. Proportion and naturality.** Our forefathers developed numerous techniques. Shall we pick up just a few of them, such as Hilbert's *syzygy*, blow them up out of all proportion, and forget all the remaining ones? When shall we attack those problems which are not amenable to the picked techniques?

What happened to Hilbert's man in the street? Is a theorem natural if it's *statement* requires over five hundred pages? Is a theorem natural if its statement is not readily comprehensible to Kummer?

**38. Line of demarcation.** That was the book of Cartan-Eilenberg (1956, [17]). Then came Grothendieck's *Elements* (1960..., [28]). Then Mumford's *Invariant Theorem* (1965, [50]). Then....

## CHAPTER VI: MORE GEOMETRY

**39. Infinitely near singularities.** Noether's analysis of singularities of plane curves in terms of quadratic transformations, i.e., substitutions of the form

$$X = X' \text{ and } Y = X'Y' \tag{1}$$

alluded to in §13, goes something like this:

The reverse substitution is $X' = X$ and $Y' = Y/X$ and hence, as $X$ and $Y$ both approach zero, $Y'$ takes all possible values. In other words, (1) explodes the point $P$: $X = Y = 0$ into the (projective) line $E$: $X' = 0$; moreover, the points of $E$ corresponds to the tangent directions at $P$. We visualize this by thinking of points of $E$ as the points in the first neighbourhood of $P$. In turn, any point $Q$ of $E$ can be exploded into a line $E(Q)$; we think of the points on the lines $E(Q)$, as $Q$ varies on $E$, as the points in the second neighbourhood of $P$. And so on. Ultimately, we visualize the points in the successive neighbourhoods of $P$ as the points infinitely near $P$. By analogy with optics, we may think of $P$ as the source of a wave with wavefront $E$, every point $Q$ of $E$ as the source of a secondary wavefront $E(Q)$, and so on, thus getting a *Huygens diagram*: (see Fig. 4).

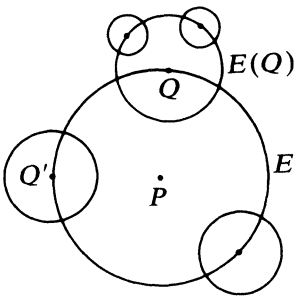



FIG. 4

Given a $d$-fold point $P$ of a plane curve $C$ we choose our coordinates to bring $P$ to the origin and then apply (1). If now $C$: $f(X, Y) = 0$, then the substitution (1) transforms $C$ into the curve $C'$: $f'(X', Y') = 0$ given by

$$f(X', X'Y') = X'^d f'(X', Y').$$

$C'$ will meet $E$ in the points $P^1, \cdots, P^m$ which correspond to the tangents to $C$ at $P$. If $P^i$ is a $d_i$-fold point of $C'$, then we shall have $d_1 + \cdots + d_m \leqq d$. We say that $P^1, \cdots, P^m$ are the points of $C$ in the first neighbourhood of $P$, and the multiplicity of $C$ at $P_i$ is $d_i$. Now iterate this procedure. The points of $C$ infinitely near $P$ can be diagrammed by the *tree of C at P*: (see Fig. 5).

At every fork-point we hang as many fruits as the multiplicity of $C$ at that point which will then be $\geqq$ the number of prongs at that point. It follows that every fork-point higher than a certain level will be unforked. So, although all the shoots of the tree will reach the sky, the number of shoots will eventually stabilize, i.e., the tree is truly infinite vertically but finite horizontally; the stabilized number of shoots may be called the geometric number of branches of $C$ at $P$:

$$(*) \qquad \beta(P)_{\text{geom}} = \text{stabilized number of shoots.}$$

The desingularization theorem says that at every fork-point higher than a certain level, there hangs exactly one fruit; in other words, $C$ has only a finite number of singularities infinitely near $P$.

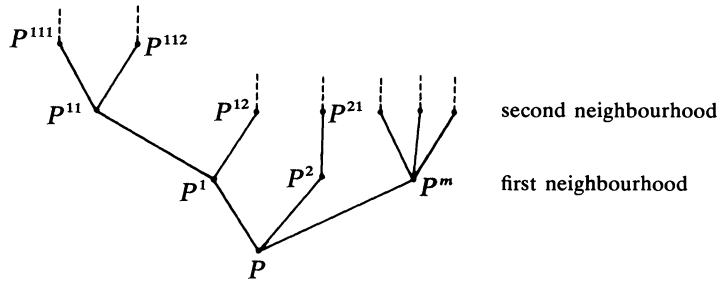



FIG. 5

Following the geometric literature, and only in this Chapter, to distinguish points in the usual sense from infinitely near points we may call the former "distinct" (another suggestive term would be "physically present"). Thus, since $C$ has only finitely many distinct singularities, it follows that in toto $C$ has only a finite number of singular points, distinct as well as infinitely near.

**40. Adjoint.** In most geometric writings on algebraic geometry the use of "adjoints" is pervasive. Here are a few random hints on this concept which might be helpful to a prospective reader of the geometric writings. Firstly, an adjoint to a plane curve $C$ is a plane curve $C^*$ which passes through all the singularities of $C$; secondly, we want this "counting properly." More precisely, if $(Q_j)_{j=1,2,\ldots}$ are all the singularities of $C$, distinct as well as infinitely near, then $C^*$ is adjoint to $C$ means

$$(\text{multiplicity of } C^* \text{ at } Q_j) \geqq (\text{multiplicity of } C \text{ at } Q_j) - 1 \text{ for all } j.$$

Alternatively, if the above condition is satisfied at a distinct singularity $P$ of $C$ and at every singularity $Q$ of $C$ infinitely near $P$, then we say that $C^*$ is adjoint to $C$ at $P$; and now $C^*$ is adjoint to $C$ means $C^*$ is adjoint to $C$ at every distinct singularity of $C$.

This brings us to

*Aphorism* 1. Adjoint is to geometry as derivative is to high-school algebra.

*Commentary.* By the Supplement of the Fundamental Principle, derivative drops the multiplicity of a root (at most) by one.

*More commentary.* But this refers to a one-variable polynomial, whereas $C$ is given by a two-variable polynomial: $f(X, Y) = 0$. So which derivative to take? ANSWER: The derivative in a varying direction. But that is too hard. So we choose say $f_Y$. But now we get something extraneous because we are disregarding the additional condition $f_X = 0$ for singularities. So:

*Aphorism* 2. Adjoint is to geometry as derivative minus "something" is to college algebra.

**41. Bezout and Noether.** In terms of infinitely near points, Bezout's Theorem and Noether's Fundamental Theorem can be refined thus: Let $C_N$: $F = 0$ and $D_M$: $\Phi = 0$ be plane curves of degree $N$ and $M$ without common components. Let $(Q_j)_{j=1,2,\ldots}$ be all the common points, distinct as well as infinitely near, of $C$ and $D$. Let $s_j$ and $r_j$ be the multiplicities of $C$ and $D$ at $Q_j$. Then:

*Bezout's Theorem Refined.* $\Sigma_j r_j s_j = MN$. Or, alternatively,

$$\Sigma\, i(C, D; P)_{\text{geom}} = MN,$$

where the sum is over all distinct common points $P$ of $C$ and $D$, and where the geometric intersection multiplicity of $C$ and $D$ at $P$ is defined by the equation

$$(*) \qquad i(C, D; P)_{\text{geom}} = \Sigma^P r_j s_j$$

where $\Sigma^P$ is the sum over those $j$ for which either $Q_j = P$ or $Q_j$ is infinitely near $P$.

NOETHER'S FUNDAMENTAL THEOREM REFINED. *A plane curve $H = 0$ can be expressed in the form*

*$H = AF + B\Phi$, with plane curves $A = 0$ and $B = 0$ having multiplicities at least $r_j - 1$ and $s_j - 1$ at $Q_j$ for all $j$, if and only if $H$ has multiplicity at least $r_j + s_j - 1$ at $Q_j$ for all $j$.*

*Warning.* This does not make $B$ adjoint to $F$. We can only say something like "$B$ is adjoint to $F$ on $\Phi$, i.e., as far as common points of $F$ with $\Phi$ are concerned." Similarly for $A$ and $\Phi$.

**42. Genus formula refined.** Let us now describe the study of special cases which we spoke of in §13 and which led the geometers to the formula

$$(g_5) \qquad g = (1/2)(N-1)(N-2) - \Sigma\,\delta(P)$$

for the genus $g$ of an irreducible plane curve $C_N$: $f(X, Y) = 0$ of degree $N$, where the sum is over all distinct singularities $P$ of $C$ and where $\delta(P)$ is a certain positive integer which says that "$P$ accounts for $\delta(P)$ ordinary double points."

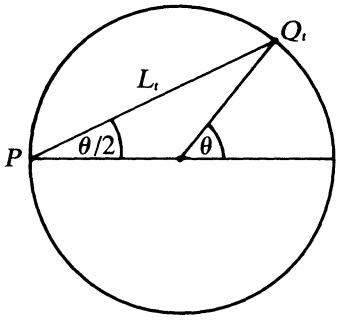



FIG. 6

(1) ($N = 2$). The circle $C_2$: $X^2 + Y^2 = 1$ has the usual parametrization: $X = \cos\theta$ and $Y = \sin\theta$. The familiar substitution $\tan\theta/2 = t$ converts it into the rational parametrization: $X = (1-t^2)/(1+t^2)$ and $Y = 2t/(1+t^2)$. See Fig. 6. Geometrically speaking, fix a point $P$ on the circle, say $P = (-1, 0)$. Let $L_t$ be the line through $P$ of variable slope $t$. Then

$$Q_t = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

is the other point of intersection of $L_t$ with the circle.

This explains why (rational functions of) the trigonometric functions can be integrated.

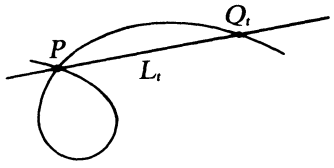



FIG. 7




FIG. 8

(2) ($N = 3$). Fix a point $P$ on $C_3$. If $P$ is a double point of $C_3$, then $L_t$ meets $C_3$ in exactly one more point $Q_t$: (see Fig. 7), and hence the coordinates of $Q_t$ are rational functions of $t$. If $P$ was a simple point of $C_3$ then $L_t$ would meet $C_3$ in two more points: (see Fig. 8), and this "indicates" why elliptic integrals cannot be integrated. If $C_3$ has two double points $P_1$ and $P_2$ then the line $P_1P_2$ would meet $C_3$ in at least $2+2>3$ points contradicting Bezout.

(3) ($N = 4$). Suppose $C_4$ has three double points $P_1$, $P_2$, $P_3$. Fix a simple point $P_4$ of $C_4$. Now a conic through $P_1$, $P_2$, $P_3$, $P_4$ meets $C_4$ in exactly one more point because $2 \cdot 4 - (2+2+2+1) = 1$.

Since 5 points determine a conic, we get a one-to-one correspondence between conics through $P_1$, $P_2$, $P_3$, $P_4$ and points of $C_4$; this gives a rational parametrization of $C_4$. If $C_4$ had another double point $P_5$ then the conic through $P_1, \cdots, P_5$ would meet $C_4$ in at least $2+2+2+1+2>2\cdot 4$ points contradicting Bezout.

(4) In this manner, by passing curves of degree $N-2$ through the double (and some simple) points of $C_N$, we see that

$$\text{number of double points of } C_N \leqq (1/2)(N-1)(N-2),$$

and that if equality holds then $C_N$ is *rational*, i.e., $C_N$ has a rational parametrization, i.e., the points of $C_N$ can be put in an almost one-to-one (algebraic) correspondence with the points of a line.

(5) $(N>1)$. If $C_N$ has a $d$-fold point $P$ and an $e$-fold point $Q$, then the line $PQ$ meets $C_N$ in at least $d+e$ points and hence $d+e \leqq N$ by Bezout. It follows that the multiplicity of any point of $C_N$ is $\leqq N-1$, and if $C_N$ does have an $(N-1)$-fold point then it has no other singularity.

(6) In view of (4) and (5) we surmise that an $(N-1)$-fold point should be counted as $(1/2)(N-1)(N-2)$ double points. From this, and taking infinitely near points into account, we surmise that: If $P=P_1$ is a distinct singular point of $C_N$, if $(P_j)_{j=2,3,\cdots}$ are all the singularities of $C_N$ infinitely near $P$, and if $d_j$ is the multiplicity of $C_N$ at $P_j$ for $j=1,2,\cdots$, then upon defining

$$(*) \qquad \delta(P)_{\text{geom}} = (1/2)\Sigma\, d_j(d_j-1)$$

we have

$$\delta(P) = \delta(P)_{\text{geom}}.$$

(7) In view of (4) and (6) it appears (in effect, by considering adjoints to $C$ of degree $N-2$) that summing over all the distinct singularities $P$ of $C_N$ we have

$$\Sigma\, \delta(P)_{\text{geom}} \leqq (1/2)(N-1)(N-2)$$

with equality if and only if $C_N$ is rational; i.e., in words: $C_N$ has at most $(1/2)(N-1)(N-2)$ double points counted properly, and the maximum is reached iff $C_N$ is rational.

(8) Now the genus $g$ of $f$ is also a nonnegative integer whose vanishing is an iff condition for the Riemann surface of $f$ to be without handles. So, in view of (7), we surmise that:

$$(g_8) \qquad g = (1/2)(N-1)(N-2) - \Sigma\, \delta(P)_{\text{geom}},$$

where the sum is over all the distinct singularities $P$ of $C$. Or, equivalently, if $(Q_j)_{j=1,2,\cdots}$ are the singular points of $C_N$, distinct as well as infinitely near, and if $s_j$ is the multiplicity of $C_N$ at $Q_j$, then

$$(g_9) \qquad g = (1/2)(N-1)(N-2) - (1/2)\sum_j s_j(s_j-1).$$

## Chapter VII: More High-School Algebra

**43. Discriminant locus.** Let

$$C\colon f(X,Y) = u_0(X)Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X) = 0$$

be a plane curve where the $u_i(X)$ are polynomials with $u_0(X) \neq 0$. Let $x$ and $y$ be such that $x$ is a variable and $f(x,y)=0$.

By the *discriminant locus* (for the projection of $C$ on the $X$-axis by lines parallel to the $Y$-axis) we mean the totality of the roots of $\text{Disc}_Y(f)$, each of which we call a *discriminant point.*

First let us remark that the equation $f(X,Y)=0$ is said to be "separable" if $\text{Disc}_Y(f)$ is not identically zero, i.e., if there are only a finite number of discriminant points; we shall usually suppose this to be the case; at any rate, by the Supplemented Fundamental Principle and the Resultant

Theorem one can see that this is automatically so when $f(X, Y)$ is free from multiple factors and the characteristic is zero.

Next let us recall that the discriminant points turn out to be precisely those (finite) values of $x$ for which there are less than $n$ finite values of $y$, either because one of the finite values of $y$ has multiplicity $>1$, or because one of the values of $y$ has gone to infinity; the latter happens exactly when the given value of $x$ is a root of $u_0(X)$; in a sense, the roots of $u_0(X)$ may be regarded as accidental discriminant points.

Disregarding the roots of $u_0(X)$ and restricting to finite values of $x$ and $y$, what causes the discriminant locus?

*Aphorism* 3. Discriminant locus equals branch locus plus projection of the singular locus.

*Aphorism* 4. Derivative locus equals "Different locus" plus singular locus.

NOTE. Count everything properly. Then these Aphorisms provide a guideline for estimating singularities.

*Commentary.* On the one hand, a point $P=(a, b)$ is a singularity of $C$ iff $f(a, b)=f_Y(a, b)=f_X(a, b)=0$; on the other hand, since $\mathrm{Disc}_Y(f)=\mathrm{Res}_Y(f, f_Y)$, by the Resultant Theorem we see that a value $a$ of $x$ is a discriminant point iff for some $b$ we have $f(a, b)=f_Y(a, b)=0$. So in particular, the projection of the singular locus is always part of the discriminant locus. Now when $P=(a, b)$ is a simple point of $C$ then, $f_Y(a, b)=0$ iff the tangent to $C$ at $P$ is parallel to the $Y$-axis; a typical case of this is the parabola $(Y-b)^2-(X-a)=0$; generalizing this we call $P$ a *Different point* if

$$f(X+a, Y+b)=f_0(X, Y)f_1(X, Y),$$

where $f_0(X, Y)$ is a polynomial in $Y$ with coefficients power series in $X$, and where

$$f_1(X, Y)=Y^{e(1)}+v_{11}(X)Y^{e(1)-1}+\cdots+v_{1e(1)}(X)$$

is an irreducible polynomial in $Y$ with $e(1)>1$ and with power series $v_{1j}(X)$ such that $v_{1j}(0)=0$ for all $j$; such an irreducible factor $f_1$, with $e(1)\geqq 1$, is called a *branch* of $C$ centered at $P$. A value $a$ of $x$ is called a *branch point* if it is the projection of a Different point.

Clearly $f(a, b)=f_Y(a, b)=0$ for every Different point $P=(a, b)$. Therefore (1) for a point $P=(a, b)$ we have that: $f(a, b)=f_Y(a, b)=0$ iff $P$ is either a Different point or a singular point. Hence, by the Resultant Theorem, (2) for a value $a$ of $x$ we have that: $a$ is a discriminant point iff it is

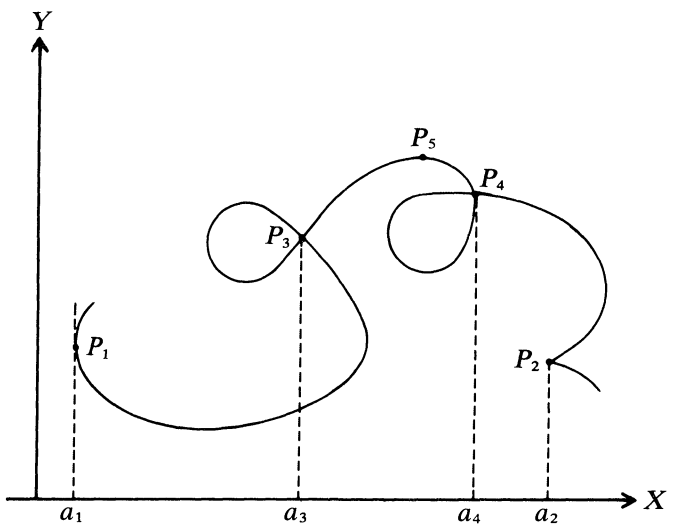



FIG. 9

either a branch point or a point in the projection of the singular locus. By the Fundamental Principle one can also see that: the number of branches of $C$ lying above any given value $a$ of $x$ is always $\leqq n$; this number is $< n$ iff $a$ is a branch point; (recall that we are disregarding the roots of $u_0(X)$).

EXAMPLE: Consider a point $P_i = (a_i, b_i)$. See Fig. 9.

($P_1$) *Simple point with vertical tangent.*

$$f(X + a_1, Y + b_1) = X + \sum_{i+j>1} \lambda_{1ij} X^i Y^j.$$

$P_1$ is a simple point and a Different point. Number of branches centered at $P_1$ is 1; $a_1$ is a branch point but need not be in the projection of the singular locus. Near $P_1$, $C$ could look like the origin of the parabola $Y^2 - X = 0$; (the two roots $y = \pm x^{1/2}$ have the same value at $x = 0$ but they are not power series in $x$; "in the complex case, analytic continuation around $x = 0$ permutes them").

($P_2$) *Cusp.*

$$f(X + a_2, Y + b_2) = f'(X, Y)(Y^2 - X^3) \text{ with } f'(0,0) \neq 0.$$

$P_2$ is a singular point and a Different point. Number of branches centered at $P_2$ is 1; $a_2$ is a branch point and is in the projection of the singular locus. Near $P_2$, $C$ looks like the origin of $Y^2 - X^3 = 0$; (the two roots $y = \pm x^{3/2}$ have the same value at $x = 0$ but they are not power series in $x$; "in the complex case, analytic continuation around $x = 0$ permutes them").

($P_3$) *Node without vertical tangent.*

$$f(X + a_3, Y + b_3) = Y^2 - X^2 + \sum_{i+j>2} \lambda_{3ij} X^i Y^j.$$

$P_3$ is a singular point but not a Different point. Number of branches centered at $P_3$ is 2; $a_3$ need not be a branch point but is in the projection of the singular locus. Near $P_3$, $C$ looks like the origin of $Y^2 - X^2(1 + X) = 0$; (the two roots $y = \pm x(1 + x)^{1/2}$ have the same value at $x = 0$ and they are power series in $x$ (characteristic $\neq 2$); "in the complex case, analytic continuation around $x = 0$ does not permute them").

($P_4$) *Node with a vertical tangent.*

$$f(X + a_4, Y + b_4) = XY + \sum_{i+j>2} \lambda_{4ij} X^i Y^j.$$

$P_4$ is a singular point and Different point; (composite of type ($P_1$) and ($P_3$)). Number of branches centered at $P_4$ is 2; $a_4$ is a branch point and is in the projection of the singular locus. Near $P_4$, $C$ could look like the origin of $Y^3 + X^3 + XY = 0$.

($P_5$) *Simple point with horizontal tangent.*

$$f(X + a_5, Y + b_5) = Y + \sum_{i+j>1} \lambda_{5ij} X^i Y^j.$$

$P_5$ is a simple point but not a Different point; $a_5$ need not be a discriminant point.

NOTE. Let us stress that the singular locus is always where it is, but the "Different locus" (i.e., the totality of Different points) and the "derivative locus $f = f_Y = 0$" vary according to the direction of projection.

For instance, if in the above example we project on the $Y$-axis by lines parallel to the $X$-axis then: $P_2, P_3, P_4$ are still singularities; again $P_2, P_4$ are Different points while $P_3$ is not; $P_1$ is no longer a Different point; $P_5$ is a new Different point;...

**44. Characteristic pairs.** To continue with the Newton-Puiseux expansion, suppose that we are in characteristic zero, and consider

$$f^*(X, Y) = Y^e + v_1(X)^{e-1} + \cdots + v_e(X), \tag{1}$$

where the $v_i(X)$ are power series with $v_i(0) = 0$ for all $i$.

Then, assuming $f^*$ to be irreducible, by Newton's Theorem

$$f^*(T^e, Y) = \prod_{i=1}^{e} [Y - y(\rho^i T)], \tag{2}$$

where $\rho$ is a primitive $e$th root of 1 and

$$y(T) = \Sigma\, \nu_k T^k \tag{3}$$

is a power series. Following Smith (1873, [**67**]) and Halphen (1884, [**29**]) we shall now pick out a finite number of significant exponents from the above expansion which give rise to a finite number of pairs of integers called the characteristic pairs of $f^*$. In terms of the characteristic pairs one can write down explicit formulas for the order of a resultant (and hence for intersection multiplicity) and for the order of a discriminant (and hence for the "measure of a singularity"). The exponents are picked thus:

$m_1 = \operatorname{ord}_T y(T) =$ smallest $k$ such that $\nu_k \neq 0$.

$m_2 =$ smallest $k$ such that $\nu_k \neq 0$ and $\operatorname{GCD}(e, m_1, m_2) < \operatorname{GCD}(e, m_1)$.

. . . . . .

$m_i =$ smallest $k$ such that $\nu_k \neq 0$ and $\operatorname{GCD}(e, m_1, \cdots, m_i) < \operatorname{GCD}(e, m_1, \cdots, m_{i-1})$.

. . . . . .

$m_h =$ smallest $k$ such that $\nu_k \neq 0$ and $1 = \operatorname{GCD}(e, m_1, \cdots, m_h) < \operatorname{GCD}(e, m_1, \cdots, m_{h-1})$.

Here GCD stands for the greatest common divisor. We also put:

| | | |
|---|---|---|
| $q_1 = m_1$ | and | $d_1 = e$ |
| $q_2 = m_2 - m_1$ | | $d_2 = \operatorname{GCD}(e, m_1)$ |
| . . . . . | | . . . . . |
| $q_i = m_i - m_{i-1}$ | | $d_i = \operatorname{GCD}(e, m_1, \cdots, m_{i-1})$ |
| . . . . . | | . . . . . |
| $q_h = m_h - m_{h-1}$ | | $d_h = \operatorname{GCD}(e, m_1, \cdots, m_{h-1})$ |
| | | $d_{h+1} = \operatorname{GCD}(e, m_1, \cdots, m_h) = 1.$ |

The pairs of integers $(q_i, d_i)_{i=1,\cdots,h}$ are called the *characteristic pairs* of $f^*$.

As an illustration of the significance of the characteristic pairs let us write down a formula for the order of a discriminant. So let $\operatorname{Disc}_Y(f^*)$ be the power series in $X$ obtained by eliminating $Y$ between $f^*(X, Y)$ and its $Y$-derivative $f_Y^*(X, Y)$, i.e.,

$$\operatorname{Disc}_Y(f^*) = Y\text{-discriminant of } f^* = \operatorname{Res}_Y(f^*, f_Y^*).$$

Then upon defining

$$\alpha(f^*) = (q_1 - 1)(d_1 - 1) + \sum_{i=2}^{h} q_i(d_i - 1)$$

one has the formula

$$[\operatorname{ord}_X \operatorname{Disc}_Y(f^*)] - e + 1 = \operatorname{ord}_T f_Y^*(T^e, y(T)) - \operatorname{ord}_T \frac{dT^e}{dT} = \alpha(f^*) \tag{4}$$

which one obtains by direct calculation; ($d$ stands for derivative). [To obtain the second equation, substitute (3) in (2) and use the product formula for derivative; to prove the first equation, use the formula

$$\text{(5)}\qquad \mathrm{Res}_Y\Big(\prod_{i=1}^{e}(Y-y_i),\prod_{j=1}^{e}(Y-z_j)\Big)=\prod_{i,j}(y_i-z_j);$$

let us also record the following consequence of (5)

$$\text{(6)}\qquad \mathrm{Disc}_Y(f_1f_2\cdots f_\beta)=\prod_i \mathrm{Disc}_Y(f_i)\prod_{i\neq j}\mathrm{Res}_Y(f_i,f_j),$$

where the $f_i$ are (monic) polynomials in $Y$.]

As another illustration of the significance of the characteristic pairs we shall write a formula for the intersection multiplicity, i.e., for the order of a resultant. So consider another polynomial

$$\phi^*(X,Y)=Y^\varepsilon+w_1(X)Y^{\varepsilon-1}+\cdots+w_\varepsilon(X),$$

where the $w_i(X)$ are power series with $w_i(0)=0$ for all $i$.

Assuming $\phi^*$ to be irreducible, let

$$\text{(2')}\qquad \phi^*(T^\varepsilon,Y)=\prod_{j=1}^{\varepsilon}[(Y-z(\sigma^jT)]$$

be the factorization of $\phi^*$ where $\sigma$ is a primitive $\varepsilon$th root of 1, and

$$\text{(3')}\qquad z(T)=\Sigma\,\nu'_kT^k$$

is a power series. Let $(m'_i,q'_i,d'_i)_{i=1,\cdots,h'}$ be the numbers which correspond to $(m_i,q_i,d_i)_{i=1,\cdots,h}$ when we replace $f^*$ by $\phi^*$. Let $\theta$ measure how close a root of $f^*$ comes to being a root of $\phi^*$, i.e., let

$$\theta=\max_{\substack{1\leqq i\leqq e\\ 1\leqq j\leqq\varepsilon}}\mathrm{ord}_X[y(\rho^iX^{1/e})-z(\sigma^jX^{1/\varepsilon})]$$

and now let $p=$ largest integer such that $m_p/d_1=m'_p/d'_1\leqq\theta$. Then upon defining

$$\alpha(f^*,\phi^*)=\Big(\theta-\frac{m_p}{d_1}\Big)d'_1d_{p+1}+\sum_{i=1}^{p}q_id'_i$$

(and making a direct calculation using (2), (3), (5), (2'), (3')) one gets

$$\text{(7)}\qquad \mathrm{ord}_X\,\mathrm{Res}_Y(f^*,\phi^*)=\alpha(f^*,\phi^*).$$

Note that the above expression for $\alpha(f^*,\phi^*)$ is really symmetric because one can see that

$$\Big(\theta-\frac{m_p}{d_1}\Big)d'_1d_{p+1}=\Big(\theta-\frac{m'_p}{d'_1}\Big)d_1d'_{p+1}\ \text{ and }\ q_id'_i=q'_id_i\ \text{ for }1\leqq i\leqq p.$$

Now drop the assumption of $f^*$ being irreducible, and let

$$f^*(X,Y)=\prod_{i=1}^{\beta}f_i(X,Y)$$

be a factorization of $f^*$ into irreducible polynomials

$$f_i(X,Y)=Y^{e(i)}+\sum_{k=1}^{e(i)}v_{ik}(X)Y^{e(i)-k}\ \text{ with power series }v_{ik}(X).$$

Let $y_i(X^{1/e(i)})$ be a root of $f_i$, i.e., let $y_i(T)$ be a power series with

$$f_i(T^{e(i)},y_i(T))=0.$$

Then upon defining

$$\text{(')}\qquad \alpha(f^*)=\sum_i\alpha(f_i)+\sum_{i\neq j}\alpha(f_i,f_j)$$

[by using (4), (5), (6), (7)] one can see that

$$(8)\qquad \left\{ \begin{aligned} [\mathrm{ord}_X \mathrm{Disc}_Y(f^*)] - e + \beta &= \sum_{i=1}^{\beta} \left[ \mathrm{ord}_T f_Y^*(T^{e(i)}, y_i(T)) - \mathrm{ord}_T \frac{dT^{e(i)}}{dT} \right] \\ &= \alpha(f^*), \end{aligned} \right.$$

where $d$ stands for derivative.

Finally, also drop the assumption of $\phi^*$ being irreducible, and let

$$\phi^*(X, Y) = \prod_{j=1}^{\gamma} \phi_j(X, Y)$$

be a factorization of $\phi^*$ into (monic) irreducible polynomials in $Y$ with coefficients power series in $X$. Then upon defining

$$\alpha(f^*, \phi^*) = \sum_{i,j} \alpha(f_i, \phi_j)$$

[by (5) and (7)] one can see that

$$\mathrm{ord}_X \mathrm{Res}_Y(f^*, \phi^*) = \alpha(f^*, \phi^*).$$

Therefore, with notation as in the end of §21, upon defining the intersection multiplicity of the plane curves $C$ and $D$ at the point $P$, in the sense of high-school algebra, by the formula

$$(\prime\prime)\qquad i(C, D; P)_{\mathrm{halg}} = \alpha(f^*, \phi^*)$$

we get

$$(9)\qquad i(C, D; P)_{\mathrm{halg}} = i(C, D; P).$$

**45. High-school genus formula.** Let $C_N$: $F(X, Y, Z) = 0$ be an irreducible plane curve of degree $N$. Again assume that we are in characteristic zero. Given any point $P$ of $C_N$ we choose coordinates so that $P$ is given by $X = Y = 0$ and $Z = 1$. By Weierstrass' Preparation Theorem (or by Newton's Theorem) we can write

$$F(X, Y, 1) = f'(X, Y) f^*(X, Y)$$

where $f'(X, Y)$ is a polynomial in $Y$ with coefficients power series in $X$ such that $f'(0, 0) \neq 0$, and where

$$f^*(X, Y) = Y^e + v_1(X) Y^{e-1} + \cdots + v_e(X)$$

with positive integer $e$ and power series $v_i(X)$ such that $v_i(0) = 0$ for all $i$. As a measure of how singular $C$ is at $P$ in the sense of high-school algebra, (following the dictate of Aphorism 4 of §43 and the above formula (8)), we define

$$(*)\qquad \delta^*(P)_{\mathrm{halg}} = \alpha(f^*)$$

with $\alpha(f^*)$ as in the above formula ($'$); (we note that then $\delta^*(P)_{\mathrm{halg}}$ is a nonnegative integer which is zero iff $P$ is in a simple point of $C$). By using the chain-rule for differentiation one shows that $\delta^*(P)_{\mathrm{halg}}$ is independent of the coordinate system. Taking formula $(g_2)$ of §3 as the definition of the genus $g$ of $C$, in §47 we shall outline the proof of the following formula:

$$(g_{10})\qquad g = (1/2)(N-1)(N-2) - (1/2)\Sigma\, \delta^*(P)_{\mathrm{halg}}$$

where the sum is over all points $P$ of $C$.

**46. Branches.** Let $C$: $F(X, Y, Z) = 0$ be an irreducible plane curve of degree $N$; so $F$ is an irreducible homogeneous polynomial of degree $N$. Let $f(X, Y) = F(X, Y, 1)$. In other words, "at finite

distance" $C$ is given by

$$f(X, Y) = u_0(X)Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X) = 0,$$

where the $u_i(X)$ are polynomials with $u_0(X) \neq 0$; note that $n \leqq N$. Let $x$ and $y$ be such that $x$ is a variable and $f(x, y) = 0$.

We shall now introduce the concept of branches of $C$ centered at a point $P$ of $C$; the totality of these as $P$ varies over $C$ are called the branches of $C$.

Let $P = (a, b)$ be a point of $C$ at finite distance. By the Weierstrass Preparation Theorem we can write

$$f(X + a, Y + b) = f'(X, Y)f^*(X, Y), \tag{1}$$

where $f'(X, Y)$ is a polynomial in $Y$ with coefficients power series in $X$ such that $f'(0, 0) \neq 0$, and where

$$f^*(X, Y) = Y^e + v_1(X)Y^{e-1} + \cdots + v_e(X) \tag{2}$$

with positive integer $e$ and power series $v_i(X)$ such that $v_i(0) = 0$ for all $i$. Let

$$f^*(X, Y) = \prod_{i=1}^{\beta} f_i(X, Y)$$

be a factorization of $f^*$ into irreducible polynomials

$$f_i(X, Y) = Y^{e(i)} + \sum_{k=1}^{e(i)} v_{ik}(X)Y^{e(i)-k} \text{ with power series } v_{ik}(X).$$

Now we can take power series $x_i(T)$ and $y_i(T)$ such that

$$\operatorname{ord}_T x_i(T) = e(i), \quad \operatorname{ord}_T y_i(T) > 0, \quad \text{and} \quad f_i(x_i(T), y_i(T)) = 0.$$

(In case of characteristic zero we can do all this by Newton's Theorem, and then we may choose $x_i(T)$ to be $T^{e(i)}$.) We define $\beta$ to be the number of branches of $C$ centered at $P$ in the sense of high-school algebra:

$$\beta(P)_{\text{halg}} = \beta. \tag{3*}$$

With each of the "algebroid curves" $f_i(X, Y) = 0$ we associate a "branch" $V_i$ of $C$ centered at $P$. For every rational function $r(x, y)$ we define the *value* $r(V_i)$ of $r(x, y)$ at $V_i$, the *order* $V_i(r(x, y))$ of $r(x, y)$ at $V_i$, and the *degree* $\deg_P(r(x, y))$ of the divisor of $r(x, y)$ at $P$, thus:

$$V_i(r(x, y)) = \operatorname{ord}_T r(a + x_i(T), b + y_i(T))$$

$$\deg_P(r(x, y)) = \sum_{i=1}^{\beta} V_i(r(x, y))$$

and

$$r(V_i) = \begin{cases} \text{coefficient of } T^0 \text{ in } r(a + x_i(T), b + y_i(T)) & \text{if } V_i(r(x, y)) \geqq 0 \\ \infty & \text{if } V_i(r(x, y)) < 0. \end{cases}$$

For any differential $r(x, y)\,dx$ we define its *order* $V_i(r(x, y)\,dx)$ at $V_i$ and the *degree* $\deg_P(r(x, y)\,dx)$ of its divisor at $P$, thus:

$$V_i(r(x, y)\,dx) = \operatorname{ord}_T\left(r(a + x_i(T), b + y_i(T))\frac{dx_i(T)}{dT}\right)$$

and

$$\deg_P(r(x, y)\,dx) = \sum_{i=1}^{\beta} V_i(r(x, y)\,dx).$$

We observe that by using formula (5) of §44 it can be shown that if $D$: $\phi(X, Y) = 0$ is any plane curve without common components with $C$, then

$$i(C, D; P) = \deg_P(\phi(x, y)). \tag{4}$$

As a measure of how singular $C$ is at $P$, (following the dictate of Aphorism 4 of §43), we define

$$\delta^*(P) = \deg_P(f_Y(x, y)) - \deg_P(dx). \tag{5*}$$

(One can see that $\delta^*(P)$ is a nonnegative integer which is zero iff $P$ is a simple point of $C$.) In view of formula (8) of §44 it can be shown that

$$\delta^*(P)_{\text{halg}} = \delta^*(P) \text{ in case of characteristic zero.} \tag{6}$$

It can be seen that the various concepts introduced above are independent of the particular parametrization $(x_i(T), y_i(T))$.

When $P$ is a point of $C$ at infinity, one introduces all the above concepts in a similar fashion after making a suitable change of coordinates. At any rate, let us note that if $P = (a, b)$ is any point of $C$ at finite distance then the branches of $C$ centered at $P$ are exactly those branches $V$ of $C$ for which $x(V) = a$ and $y(V) = b$; moreover, for any branch $V$ of $C$ we have: $x(V) \neq \infty \neq y(V)$ if the center of $V$ is a point of $C$ at finite distance.

For any (nonzero) rational function $s(x, y)$ we define the *degree of its divisor* by the equation

$$\deg(s(x, y)) = \Sigma\, V(s(x, y)) \text{ with sum over all branches } V \text{ of } C$$

and we note that formula (1) of §3 is to be interpreted as the formula

$$\deg(s(x, y)) = 0 \tag{7}$$

a proof of which can be based on the above formula (4) and Bezout's Theorem as stated in §21.

For any (nonzero) differential $r(x, y)\,dx$ we define the *degree of its divisor* by the equation

$$\deg(r(x, y)\,dx) = \Sigma\, V(r(x, y)\,dx) \text{ with sum over all branches } V \text{ of } C$$

and note that the formula $(g_2)$ of §3 for the genus $g$ of $C$ is to be interpreted as saying

$$2g - 2 = \deg(r(x, y)\,dx). \tag{$g_2$)*}$$

Let us regard the above equation as the definition of $g$.

For any branch $V$ of $C$, the *branching degree* or the differential exponent $e(V)$ of $V$ over $x$ is defined by

$$e(V) = \begin{cases} V(x - x(V)) & \text{if } x(V) \neq \infty \\ V(x^{-1}) & \text{if } x(V) = \infty. \end{cases} \tag{8}$$

(In the above situation $e(i) = e(V_i)$.) By the Fundamental Principle it can be shown that

$$\begin{cases} \sum_{x(V)=a} e(V) = n & \text{for every value } a \text{ of } x, \text{ finite or infinite;} \\ & \text{the sum being over all branches } V \text{ of } C \text{ for which } x(V) = a. \end{cases} \tag{9}$$

As a measure of how different $C$ is from the $X$-axis, at a branch $V$ of $C$, we define the *order of the Different* of $C$ over $x$ at $V$ by the equation

$$V(\text{Diff}(C, x)) = \begin{cases} V(dx) & \text{if } x(V) \neq \infty \\ V(dx^{-1}) & \text{if } x(V) = \infty. \end{cases} \tag{10}$$

Again with the same motivation, for any point $P$ of $C$, we define the *degree of the Different* of $C$ over $x$ at $P$ by the equation

$$\deg_P \mathrm{Diff}(C,x) = \Sigma\, V(\mathrm{Diff}(C,x)) \text{ with sum over all branches } V \text{ of } C \text{ centered at } P,$$

and we note then in view of (5*) we have

$$(11^*) \qquad \delta^*(P) = \deg_P(f_Y(x,y)) - \deg_P \mathrm{Diff}(C,x) \text{ if } P \text{ is at finite distance.}$$

Finally, to measure how different $C$ is from the $X$-axis, globally, we define the *degree of the Different* of $C$ over $x$ by the equation

$$(12) \qquad \deg \mathrm{Diff}(C,x) = \Sigma\, V(\mathrm{Diff}(C,x)) \text{ with sum over all branches } V \text{ of } C.$$

Because of the equation $dx^{-1} = -x^{-2}dx$ by definitions (8), (10) and (12) we see that

$$\deg \mathrm{Diff}(C,x) = \deg(dx) + 2 \sum_{x(V)=\infty} e(V)$$

and hence by $(g_2^*)$ and (9) we get

$$(g_{11}) \qquad 2-2g = 2n - \deg \mathrm{Diff}(C,x).$$

In view of definitions (8) and (10), for any branch $V$ of $C$ we have

$$(13) \qquad V(\mathrm{Diff}(C,x)) \geqq e(V)-1 \text{ with equality iff } e(V) \text{ is not divisible by the characteristic,}$$

and hence by $(g_{11})$ we get

$$(g_{12}) \qquad \begin{cases} 2-2g = 2n - \Sigma[e(V)-1] \text{ in case of characteristic zero,} \\ \qquad\qquad \text{with sum over all branches } V \text{ of } C. \end{cases}$$

Formulas $(g_{11})$ and $(g_{12})$ are variously ascribed to Riemann, Zeuthen, or Hurwitz.

A value $a$ of $x$, finite or infinite, is called a *branch point* (for $C$ over $x$, or for the projection of $C$ on the $X$-axis by lines parallel to the $Y$-axis) if $e(V) > 1$ for some branch $V$ of $C$ with $x(V) = a$. Upon letting

$$(14) \qquad \mu(a) = \text{number of branches } V \text{ of } C \text{ with } x(V) = a$$

by (9) we have

$$(15) \qquad \mu(a) = n - \sum_{x(V)=a} [e(V)-1]$$

and hence

$$(16) \qquad a \text{ is a branch point iff } \mu(a) < n.$$

To elaborate (9) in greater detail: Given any finite value $a$ of $x$, let $b_1, \cdots, b_q$ be the finite roots of $f(a, Y) = 0$ and let $p_j$ be the multiplicity of the root $b_j$. Also let $p_0$ be the largest integers such that $u_i(a) = 0$ whenever $0 \leqq i < p_0$, and let $b_0 = \infty$. Then

$$p_0 + p_1 + \cdots + p_q = n$$

and

$$\sum_{x(V)=a,\, y(V)=b_j} e(V) = p_j \text{ for } 0 \leqq j \leqq q.$$

In still greater detail, for $j = 1, \cdots, q$, let

$$f(X+a, Y+b_j) = f_j'(X,Y) f_j^*(X,Y) \text{ with } f_j^*(X,Y) = Y^{e_j} + \cdots$$

be as in (1) and (2) above. Then $e_j = p_j$ for $1 \leqq j \leqq q$ and

$$f(X + a, Y) = f''(X, Y) \prod_{j=1}^{q} f_j^*(X, Y - b_j),$$

where

$$f''(X, Y) = u_0(X + a)Y^{p_0} + \cdots$$

factors into the branches $V$ of $C$ for which $x(V) = a$ and $y(V) = \infty$.

Recalling that the roots of $\mathrm{Disc}_Y(f)$ constitute the discriminant locus, we conclude that the said locus consists of those finite values $a$ of $x$ for which, in the above notation, $q < n$.

**47. Another genus formula.** Again let $C_N$: $F(X, Y, Z) = 0$ be an irreducible plane curve of degree $N$. We can choose coordinates so that the point $(0, 1, 0)$ is not on $C$, i.e., upon letting $f(X, Y) = F(X, Y, 1)$ we may suppose

$$C: f(X, Y) = Y^N + v_1(X)Y^{N-1} + \cdots + v_N(X) = 0.$$

Let $x$ and $y$ be such that $x$ is a variable and $f(x, y) = 0$. By definition (5*) of §46 we have

$$\sum_{x(V) \neq \infty} [V(f_Y(x, y)) - V(dx)] = \sum_{P \neq \infty} \delta^*(P), \tag{1}$$

where the right hand side indicates sum over all points $P$ of $C$ at finite distance. Now $(x^{-1}, yx^{-1})$ serve as coordinates for the points of $C$ at infinity, and the relation between them is $\zeta(x^{-1}, yx^{-1}) = 0$ where

$$\zeta(X, Y) = X^N f(X^{-1}, YX^{-1}). \tag{2}$$

By definition (5*) of §46 we also get

$$\sum_{x(V) = \infty} [V(\zeta_Y(x^{-1}, yx^{-1})) - V(dx^{-1})] = \sum_{P = \infty} \delta^*(P), \tag{3}$$

where the right hand side indicates sum over all points $P$ of $C$ at infinity. Now

$$dx^{-1} = -x^{-2}dx \tag{4}$$

and by (2) we also have

$$\zeta_Y(x^{-1}, yx^{-1}) = x^{1-N} f_Y(x, y). \tag{5}$$

Substituting (4) and (5) in (3), and then adding (1) and (3) we get

$$\deg(f_Y(x, y)) - \deg(dx) + (N - 3) \sum_{x(V) = \infty} V(x^{-1}) = \sum \delta^*(P), \tag{6}$$

where the last sum is over all points $P$ of $C$. Now

$$\deg(f_Y(x, y)) = 0 \qquad \text{by (7) of §46}$$

$$\sum_{x(V) = \infty} V(x^{-1}) = N \qquad \text{by (8) and (9) of §46,}$$

and hence by (6) we get

$$\deg(dx) = N(N - 3) - \Sigma \delta^*(P). \tag{7}$$

By the definition $(g_2)^*$ of §46, the genus $g$ of $C$ is given by

$$\deg(dx) = 2g - 2$$

and hence by (7) we have

$(g_{13})$ $$g = (1/2)(N-1)(N-2) - (1/2)\Sigma\,\delta^*(P),$$

where the sum is over all points $P$ of $C$. In view of formula (6) of §46, this also yields formula $(g_{10})$ of §44.

## Chapter VIII: More College Algebra

**48. Lagrange Interpolation.** From high-school algebra we have the following interpolation formula due to Lagrange (1760). Let

$$f(Y) = \prod_{i=1}^{n} (Y - y_i) \text{ with } y_1, \cdots, y_n \text{ all distinct.}$$

Then for any polynomial $\eta(Y)$ of degree $< n$ we have

$$\eta(Y) = \sum_{i=1}^{n} \frac{\eta(y_i) f(Y)}{f_Y(y_i)(Y - y_i)},$$

because both sides are polynomials of degree $< n$ and their values coincide for each of the $n$ values $y_1, \cdots, y_n$ of $Y$.

**49. Different and Conductor.** Let $C$: $f(X, Y) = 0$ be an irreducible plane curve. Let $x$ and $y$ be elements such that $x$ is a variable and $f(x, y) = 0$. Recall that for any point $P$ of $C$, by $R(P)$ we denote the local ring of $P$ on $C$ and by $R^*(P)$ we denote the normalization of $R(P)$. So far we have introduced various integers $\delta^*(P), \delta^*(P)_{\text{calg}}, \cdots$ to measure how much $C$ is singular at $P$; now we proceed to compare them. To begin with:

*Aphorism* 5. Derivative equals Different times conductor.

*Precise Meaning.* In view of the characterization of $\delta^*(P)$ given in formula (11*) of §46 and the definition of $\delta^*(P)_{\text{calg}}$ given in formula (*) of §30, we interpret this Aphorism to mean that at every point $P$ of $C$ we have

$$\delta^*(P) = \delta^*(P)_{\text{calg}}.$$

Commentary. Since both sides measure how much $C$ is singular at $P$, one may guess them to be equal. Dedekind showed this to be the case. Namely, by Lagrange Interpolation and with the intervention of the concept of "complementary module" invented by him, Dedekind proved that for $P$ at finite distance one has

$$\deg_P(f_Y(x, y)) = \deg_P \text{Diff}(C, x) + R(P)\text{-length of } R^*(P)/\text{Cond}(R(P)).$$

*Aphorism* 6. Adjoint is to geometry as conductor is to college algebra.

*Precise Meaning.* Let $\xi(X, Y)$ be any polynomial and let $P$ be any point of $C$ at finite distance. Then the plane curve $\xi(X, Y) = 0$ is adjoint to $C$ at $P$ iff $\xi(x, y)$ belongs to $\text{Cond}(R(P))$.

Commentary. This definitive form of Aphorism 2 of §40 is indicated by Aphorism 5. For another suggestive reasoning, let us consider, *at a point P of C* at finite distance, "Noether's Fnd Th Refined" (§41) and the "Warning" following it. We regard $F(X, Y, Z)$ as obtained by homogenizing $f(X, Y)$. The Refined Theorem roughly says that, when $H$ satisfies certain conditions at $P$, we can write

(*) $$H(x, y, 1) = B(x, y, 1)\Phi(x, y, 1) \text{ in } R(P),$$

where $B$ will then satisfy certain other conditions at $P$ which, geometrically speaking, (and by taking the "warning" not too seriously) say that $B$ is adjoint to $F$ at $P$; and conversely. Symbolically: conditions on $B$ = conditions on $H/\Phi$ = conditions on $H$ − conditions on $\Phi$. At any rate, we want to show that $H/\Phi$ is in $R(P)$; since $R^*(P)$ is a "nicer" ring, a standard algebraic procedure of doing this would be to first show that $H/\Phi$ is in $R^*(P)$ and then to hope that it is actually in $\text{Cond}(R(P))$ and

hence in $R(P)$; in other words, solve (*) first in $R^*(P)$ and then let the conductor bring it down to $R(P)$; now the conditions put on $H$ seem to guarantee that $H/\Phi$ is in $R^*(P)$ and so we could carry out this procedure if we knew that: geometrically "$H/\Phi(=B)$ is adjoint to $F$ at $P$" corresponds algebraically to "$H/\Phi$ is in Cond$(R(P))$." In any case, this reasoning at least indicates that Aphorism 6 and Noether's Fnd Th Refined are closely related, which bring us to:

MORE COMMENTARY. By induction on the number of quadratic transformations needed to resolve the singularity of $C$ at a given point $P$, and by analyzing the effect of a single quadratic transformation, in one breath one can prove (for instance see my *Calcutta lecture* cited in §31) the following four things:

(1) Aphorism 6.
(2) Noether's Fnd Th Refined (§41).
(3) Noether's Fnd Th in new guise (§31): $2\delta(P)_{\text{calg}} = \delta^*(P)_{\text{calg}}$.
(4) With $\delta(P)_{\text{geom}}$ as defined in (*) of §42: $2\delta(P)_{\text{geom}} = \delta^*(P)_{\text{calg}}$.

CONCLUSION. Thus at every point $P$ of $C$ we have the *equation of concordance*

$$\delta^*(P)_{\text{halg}} = \delta^*(P) = \delta^*(P)_{\text{calg}} = 2\delta(P)_{\text{calg}} = 2\delta(P)_{\text{geom}}$$

where $\delta^*(P)_{\text{halg}}$ was defined in formula (*) of §45, and the first equality was noted in formula (6) of §46. Now let $N$ be the degree of $C$. By regarding

$$(g_2)^* \qquad 2g-2 = \deg(r(x,y)\,dx)$$

as defining the genus $g$ of $C$, in §46 we deduced certain formulas $(g_{11})$ and $(g_{12})$ for $g$ and then in §47 we deduced the formula

$$(g_{13}) \qquad g = (1/2)(N-1)(N-2) - (1/2)\Sigma\,\delta^*(P)$$

where the sum is over all points $P$ of $C$. Now the above equation of concordance converts $(g_{13})$ into the formulas

$$(g_{10}) \qquad g = (1/2)(N-1)(N-2) - (1/2)\Sigma\,g^*(P)_{\text{halg}}$$

$$(g_6) \qquad g = (1/2)(N-1)(N-2) - (1/2)\Sigma\,\delta^*(P)_{\text{calg}}$$

$$(g_7) \qquad g = (1/2)(N-1)(N-2) - \Sigma\,\delta(P)_{\text{calg}}$$

$$(g_8) \qquad g = (1/2)(N-1)(N-2) - \Sigma\,\delta(P)_{\text{geom}}$$

asserted in §45, §30, §31, §42, respectively; these four formulas may also be regarded as refinements of the heuristic formula $(g_5)$ of §13. Finally, formula $(g_9)$ of §42 was only an obvious variation of $(g_8)$. Observe that we have dealt with the equality $\delta^*(P)_{\text{halg}} = \delta^*(P)$ and hence formula $(g_{10})$ only in zero characteristic; however, let it be remarked that, upon replacing Newton-Puiseux expansion by Hamburger-Noether expansion, the characteristic pairs can be suitably reinterpreted so that the equality $\delta^*(P)_{\text{halg}} = \delta^*(P)$ and formula $(g_{10})$ hold also in nonzero characteristic.

This completes an outline of the proof of most of the genus formulas. Needless to say that there are numerous other arrangements of proof, which is hardly surprising: Indeed, ours is an old and venerable subject which has been worked over by many masterminds!

**50. Valuations.** Let again $C$: $f(X, Y) = 0$ be an irreducible plane curve, and let $x$ and $y$ be elements such that $x$ is a variable and $f(x, y) = 0$. By $K$ we denote the "coefficient field" and we note that we are assuming $K$ to be algebraically closed. We put

$$K(x) = \text{field of rational functions } r(x)$$

and

$$K(C) = K(x, y) = \text{function field of } C, \text{ i.e., the field of all rational functions } r(x, y).$$

For any branch $V$ of $C$, the "mapping" which associates $V(r)$ to every element $r$ in $K(C)$ is a *valuation* of $K(C)$, i.e.: as $r$ ranges over $K(C)$, $V(r)$ ranges over all integers together with $\infty$; $V(r)=\infty$ iff $r=0$; and for all $r$ and $s$ in $K(C)$ we have $V(rs)=V(r)+V(s)$ and $V(r+s)\geqq \min(V(r), V(s))$.

It can be shown that in this manner we get all the valuations of $K(C)$. In other words, there is a one-to-one correspondence between all branches of $C$ and all the valuations of $K(C)$. We shall continue to denote a branch of $C$ and the corresponding valuation of $K(C)$ by the same letter.

For any point $P$ of $C$ we define the number of branches of $C$ centered at $P$, in the sense of college algebra, by the equation

$$\beta(P)_{\text{calg}} = \text{number of valuations } V \text{ of } K(C) \text{ such that } V(r)\geqq 0 \text{ for all } r \text{ in } R(P);$$

one can show that these valuations precisely correspond to the branches of $C$ centered at $P$, and hence $\beta(P)_{\text{halg}} = \beta(P)_{\text{calg}}$ where the former was defined in formula (3*) of §46; it can also be shown that

$$R^*(P) = \text{the set of all } s \text{ in } K(C) \text{ for which } V(s)\geqq 0$$
$$\text{for every valuation } V \text{ of } K(C) \text{ such that}$$
$$V(r)\geqq 0 \text{ for all } r \text{ in } R(P).$$

By induction on the number of quadratic transformations needed to resolve the singularity $P$ one can also see that $\beta(P)_{\text{calg}} = \beta(P)_{\text{geom}}$ where the latter was defined in formula (*) of §39. Thus we have the following *equation of concordance*:

$$\beta(P)_{\text{halg}} = \beta(P)_{\text{calg}} = \beta(P)_{\text{geom}}.$$

If $D$: $\phi(X, Y)=0$ is any other plane curve then for any point $P$ of $C$ at finite distance we define the intersection multiplicity of $C$ and $D$ at $P$, in the sense of college algebra, by the equation

$$i(C, D; P)_{\text{calg}} = \Sigma(V(\phi(x, y))) \text{ with sum over all valuations } V \text{ of } K(C)$$
$$\text{such that } V(r)\geqq 0 \text{ for all } r \text{ in } R(P);$$

with suitable modification we define this also when $P$ is a point of $C$ at infinity. In view of formula (4) of §46 and what we have said above, it follows that $i(C, D; P) = i(C, D; P)_{\text{calg}}$. Again, by induction on the number of quadratic transformations needed to resolve the singularity $P$, one can also show that $i(C, D; P)_{\text{calg}} = i(C, D; P)_{\text{geom}}$ where the latter is as defined in formula (*) of §41. Thus we have the following *equation of concordance*:

$$i(C, D; P)_{\text{halg}} = i(C, D; P) = i(C, D; P)_{\text{calg}} = i(C, D; P)_{\text{geom}}$$

where the first equality holds in characteristic zero and was noted in formula (9) of §44.

Let $J$ be a field and let $L$ be a "finite algebraic field extension of $J$," i.e.: $L$ is a field; $L$ contains $J$; and there is a positive integer $n$ such that if $z_1, \cdots, z_m$ are any elements of $L$ which are linearly independent over $J$, then necessarily $m\leqq n$. For a valuation of $W$ of $L$, by the *ramification index* of $W$ over $J$ is meant the smallest positive integer $e$ such that $W(r)=e$ for some $r\in J$. For a valuation $U$ of $J$ by an *extension* of $U$ to $L$ is meant a valuation $W$ of $L$ such that: $W(r)\geqq 0$ for all $r$ in $J$ for which $U(r)\geqq 0$. A valuation $U$ of $J$ is said to be *ramified* in $L$ if the ramification index, over $J$, of some extension of $U$ to $L$ is $>1$.

It can easily be seen that the valuations of $K(x)$ are in one-to-one correspondence with the values of $x$, finite (i.e., in $K$) or infinite. Namely, the valuation $U$ of $K(x)$ and the value $a$ of $x$ correspond if: $U(x-a)=1$ in case $a\neq\infty$, and $U(x^{-1})=1$ if $a=\infty$; it is clear that $a$ is a branch point iff $U$ is ramified in $K(C)$. Also it is clear that for a valuation $V$ of $K(C)$, the ramification index of $V$ over $K(x)$ equals what, in §46, we called the branching degree of the corresponding branch of $C$.

**51. Galois.** The quadratic equation

$$T^2 + bT + c = 0$$

can be solved:

$$t = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Similarly, equations of degree 3 and 4 were solved by radicals; i.e., by extractions of square roots, cube roots, etc.; (in this section we are supposing characteristic to be zero). Then attention was turned to the question whether general equations of degree 5 can be solved by radicals. Abel and Galois (1830) independently proved this to be impossible. Galois' proof went something like this: the general equation of degree 5 cannot be solved by radicals because the "galois group" of the "splitting field" of the said equation is "unsolvable." We shall now briefly explain the concepts of "galois group" and "splitting field" which were invented by Galois.

Let $J$ be a field. By a galois extension of $J$ is meant a finite algebraic field extension $L$ of $J$ such that: whenever a one-variable irreducible polynomial with coefficients in $J$ has one root in $L$, then all its roots are in $L$. For a galois extension $L$ of $J$, by the galois group of $L$ over $J$ is meant the group of all automorphism $\tau$ of $L$ such that $\tau(z) = z$ for every $z$ in $J$; Galois showed that this group is finite and its order (i.e., the numbers of elements in it) equals the (largest) number of elements in $L$ which are linearly independent over $J$. Finally, for a one-variable polynomial $f$ with coefficients in $J$, by the splitting field of $f$ over $J$ is meant the smallest field $L$ containing $J$ such that all the roots of $f$ are in $L$; it is easily seen that $L$ is then a galois extension of $J$.

Let $J$ be a field, let $L$ be a galois extension of $J$, and let $W$ be a valuation of $L$. Following Hilbert (1897, [33]), we define the *inertia group* of $W$ over $J$ to be the subgroup of the galois group of $L$ over $J$ consisting of all those automorphisms $\tau$ such that: for every $z$ in $L$ with $W(z) \geqq 0$, we have $W(z - \tau(z)) > 0$. It turns out that the order of the inertia group equals the ramification index of $W$ over $J$. It can also be shown that the inertia group is cyclic (i.e., is "generated" by a single element) if in addition to assuming $J$ to be of characteristic zero we also assume $W(m) = 0$ for every positive integer $m$ in $J$. In a sense, this theorem can be viewed as half of a college algebra proof of Newton's Theorem on Puiseux expansion.

## Chapter IX: More Function Theory

**52. Riemann surface.** Let now the coefficient field $K$ be the field of complex numbers, and consider the $n$-valued function $y$ of $x$ defined by

$$f(X, Y) = u_0(X)Y^n + u_1(X)Y^{n-1} + \cdots + u_n(X) = 0,$$

where the $u_i(X)$ are polynomials with $u_0(X) \not\equiv 0$, and $f$ is irreducible. Also consider the plane curve $C$: $f(X, Y) = 0$. As said in §4, to clarify the meaning of abelian integrals, Riemann introduced a surface $S$, called the Riemann surface of $f$, on which $y$ "becomes" single valued.

The field $K(C) = K(x, y)$ now becomes the field of all (single-valued meromorphic) functions on $S$. Moreover, for each point $V$ of $S$, there is a function $t$ on $S$, near $V$, such that $t$ has a simple zero at $V$; i.e., near $V$, $S$ looks like a disc around the origin in the complex $t$-plane.

It turns out that the points of $S$ are in a natural one-to-one correspondence with the branches of $C$; so we shall identify them.

Let $S_0$ be the complex $x$-sphere, i.e., the complex $x$-plane together with $\infty$. Let us say that the point $V$ of $S$ lies above the point $x(V)$ of $S_0$. Then for any point $a$ of $S_0$, upon letting

(*) $\mu(a)$ = the number of points $V$ of $S$ which lie above $a$ (i.e., for which $x(V) = a$),

by (15) and (16) of §46 we have

$$(**) \qquad \mu(a) = n - \sum_{x(V)=a} [e(V)-1] \leqq n$$

and:

$$a \text{ is a branch point iff } \mu(a) < n.$$

Now if $a \neq \infty$ is a branch point then it is a discriminant point (but not conversely!); therefore, in all there are only a finite number of branch points. Finally, for any point $V$ of $S$ we have that: if $x(V) = a \neq \infty$ then $(x-a)^{1/e(V)}$ is a uniformizing parameter at $V$; and if $x(V) = \infty$ then $x^{-1/e(V)}$ is a uniformizing parameter at $V$. We summarize all this by saying that *$S$ is an $n$-sheeted branched covering of $S_0$*.

The original construction of $S$ went something like this: On $S_0$ draw an open connected polygon $B_0$ with the discriminant points, together with $\infty$, as vertices. (See Fig. 10.)

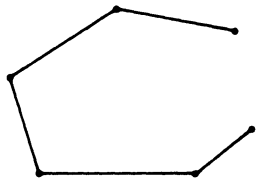


FIG. 10

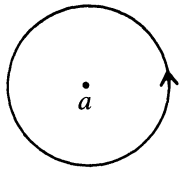



FIG. 11

Slit $S_0$ along $B_0$ to obtain $S_0'$; we suppose that $S_0'$ is connected (i.e., does not fall into pieces) and simply connected (i.e., any closed path on $S_0'$ can be shrunk to a point). Now as $x$ varies in $S_0'$, the $n$ values of $y$ can be sorted out into $n$ analytic functions $y_1(x), \cdots, y_n(x)$; [by the "monodromy theorem" which says that one gets the same analytic continuation along two paths which can be "deformed" into each other]. Take $n$ copies $S_1', \cdots, S_n'$ of $S_0'$ and "put" $y_i(x)$ on $S_i'$. Now $S$ is formed by suitably gluing together $S_1', \cdots, S_n'$ along the edges of $B_1, \cdots, B_n$ to reflect "the manner in which $y_1(x), \cdots, y_n(x)$ permute as we cross $B_0$."

Moreover: Given any point $a$ of $S_0$, as we make analytic continuations along a small circle around $a$, (see Fig. 11), the functions $y_1(x), \cdots, y_n(x)$ will undergo a permutation. We write this permutation as a product of disjoint cycles. [For example

$$\begin{pmatrix} 123456 \\ 135624 \end{pmatrix} = (1)(235)(46);$$

the permutation group being "noncommutative" we write it "multiplicatively" instead of "additively."] It turns out that the number of these cycles is $\mu(a)$ and there is a natural one-to-one correspondence between them and the points $V_1, \cdots, V_{\mu(a)}$ of $S$ lying above $a$, so that for $i = 1, \cdots, \mu(a)$ we have

$$e(V_i) = \text{the length of the corresponding cycle (or, the number of sheets of } S \text{ which come together at } V_i).$$

[In the above example: $\mu(a) = 3$, $e(V_1) = 1$, $e(V_2) = 3$, $e(V_3) = 2$.]

**53. Monodromy group.** On $S_0$ mark a finite number of points $a_1, \cdots, a_m$ so as to include all the branch points. Also fix a point $a$ different from the $a_i$. As we make analytic continuations along various closed paths beginning and ending at $a$, but without hitting any of the points $a_i$, we get various permutations of $(y_1(x), \cdots, y_n(x))$; the totality of these permutations form a group which is called the

*monodromy group* of $f$. The monodromy group is a certain subgroup of the group of all permutations on $n$ symbols; we note that the latter is of order $n!$. The assumption of $f$ being irreducible is reflected in the monodromy group being transitive; i.e., given any $j$ with $1 \leqq j \leqq n$, some member of the monodromy group will carry $y_1(x)$ to $y_j(x)$.

To obtain generators for the monodromy group, we draw a suitable system of loops $A_1, \cdots, A_m$ beginning and ending at $a$, so that $A_i$ makes a small circle around $a_i$. (See Fig. 12.)

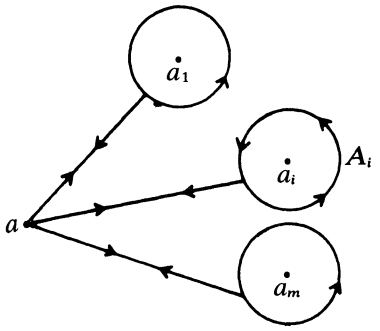



FIG. 12

Let $\tau_i$ be the permutation undergone by $(y_1(x), \cdots, y_n(x))$ as we make analytic continuation along $A_i$. By the monodromy theorem it follows that: (1) $\tau_1 \cdots \tau_m = 1$ (the identity permutation) and $\tau_1, \cdots, \tau_m$ generate the monodromy group.

One of Riemann's existence theorems says that conversely: (2) if any permutations $\tau_1, \cdots, \tau_m$ on $n$ symbols are given such that $\tau_1 \cdots \tau_m = 1$ and the group generated by $\tau_1, \cdots, \tau_m$ is transitive, then there exists $f$, having no branch points outside the $a_i$, so that, for $i = 1, \cdots, m$, the roots of $f$ permute according to $\tau_i$ when analytically continued along $A_i$.

It can be shown that the monodromy group is, in a natural manner, isomorphic to the galois group, over $K(x)$, of the splitting field of $f(x, Y)$ over $K(x)$. This in conjunction with (1) and (2) yields the following

*Theorem about galois theory of algebraic curves.* Let $U_1, \cdots, U_m$ be any valuations of $K(x)$. Then:

(1*) Let $L$ be any galois extension of $K(x)$ such that no valuation of $K(x)$, different from the $U_i$, is ramified in $L$. Then for $i = 1, \cdots, m$, we can pick an extension $W_i$ of $U_i$ to $L$ and also pick a generator $\tau_i$ of the splitting group of $W_i$ over $K(x)$ such that $\tau_i \cdots \tau_m = 1$ and $\tau_1, \cdots, \tau_m$ generate the galois group of $L$ over $K(x)$. Whence, in particular, the galois group of $L$ over $K(x)$ is generated by $m - 1$ generators.

(2*) In this manner, by varying $L$, we can realize every finite group on $m - 1$ generators as the galois group of a galois extension of $K(x)$ in which no valuation other than $U_1, \cdots, U_m$ is ramified.

Now the *statement* of the above theorem is purely algebraic. However, the only available proof is the above sketched function-theoretic and topological argument of Riemann. To obtain a purely algebraic treatment of this theorem, is certainly one of the fundamental problems of algebraic geometry. (For some related matter see [3], and Chapter VIII of the second edition of [80] including the Appendices.)

In a sense, the following may be regarded as a number-theoretic analogue:

*Problem about galois theory of algebraic numbers.* Can any finite group be realized as the galois group of some galois extension of the field of rational numbers?

In a brilliant paper (1954, [61]), Šafarevič has answered this affirmatively for any solvable group. The general case still awaits solution.

While on the subject of analogy, on a simpler level, let us note that the topological simply-connectedness of the sphere and the plane have, respectively, the following algebraic consequences:

(3) $K(x)$ has no unramified extension, [i.e., in every finite algebraic extension of $K(x)$, other than $K(x)$, at least one valuation of $K(x)$ must be ramified].

(4) In every finite algebraic extension of $K(x)$, other than $K(x)$, at least two valuations of $K(x)$ must be ramified.

For this simpler situation a purely algebraic proof does already exist. Namely (4), and hence (3), immediately follow from formulas ($g_{11}$) and (13) of §46 by noting that the genus is always nonnegative. In fact the same formulas show that (3) remains valid also in nonzero characteristic.

This time the number-theoretic analogue is the following beautiful

THEOREM OF KRONECKER. *The field of rational numbers has no unramified extension.*

**54. Sphere with handles.** In §4 we said that $S$ is topologically equivalent to a sphere with a certain number of handles. There we also said that, using formula ($g_2$) and Euler's Theorem, Riemann showed that for $S$:

($g_4$) $$g = \text{number of handles.}$$

Now in §46, from formula ($g_2$) we have deduced formula

($g_{12}$) $$2-2g = 2n - \Sigma[e(V)-1] \text{ with sum over all points } V \text{ of } S.$$

So our explanation of Riemann's deduction will be completed by deducing ($g_4$) from ($g_{12}$) and Euler's Theorem. To do this, draw a sufficiently fine polyhedron $\Delta$ on $S_0$ which includes all the branch points amongst its vertices; let $\Delta_0$, $\Delta_1$, and $\Delta_2$ be the number of vertices, edges, and faces of $\Delta$. Let $\Delta^*$ be the polyhedron on $S$ obtained by "lifting" $\Delta$; let $\Delta_0^*$, $\Delta_1^*$, and $\Delta_2^*$ be the number of vertices, edges, and faces of $\Delta^*$. Let $g^*$ be the number of handles of $S$. In view of (*) and (**) of §52 we see that

$$\Delta_0^* = n\Delta_0 - \Sigma[e(V)-1] \text{ with sum over all points } V \text{ of } S$$

and clearly $\Delta_1^* = n\Delta_1$ and $\Delta_2^* = n\Delta_2$. By Euler's Theorem we have

$$\Delta_0^* - \Delta_1^* + \Delta_2^* = 2-2g^* \text{ and } \Delta_0 - \Delta_1 + \Delta_2 = 2.$$

Combining the above three displayed lines we get

$$2-2g^* = 2n - \Sigma[e(V)-1]$$

and hence by ($g_{12}$) we get $g^* = g$ which proves ($g_4$).

**55. Abel and Jacobi.** Let $\Gamma_1, \cdots, \Gamma_{2g}$ be closed paths on $S$ where $\Gamma_i$ and $\Gamma_{g+i}$ are a latitude and a longitude of the $i$th handle. (See Fig. 13.)

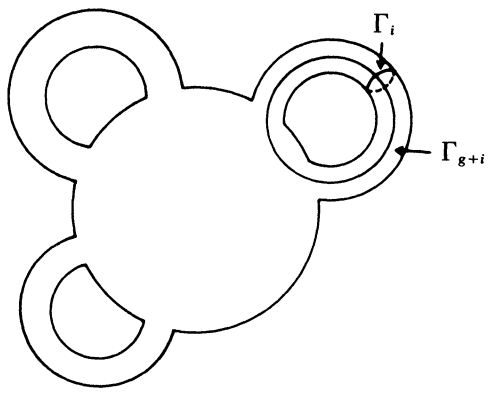



FIG. 13

Now any closed path $\Gamma$ on $S$ can be "deformed into $m_1\Gamma_1 + \cdots + m_{2g}\Gamma_{2g}$ with integers $m_i$" and then for any integral of first kind we have

$$\int_\Gamma r(x, y)\,dx = \sum m_i \int_{\Gamma_i} r(x, y)\,dx;$$

thus, for points $Q$ and $V$ of $S$, the value of the integral

$$\int_Q^V r(x, y)\,dx$$

is only determined up to a linear combination, with integer coefficients, of the $2g$ numbers

$$\int_{\Gamma_1} r(x, y)\,dx, \cdots, \int_{\Gamma_{2g}} r(x, y)\,dx.$$

To state the theorems of Abel and Jacobi, mentioned in §3, let

$$\Omega_1 = r_1(x, y)\,dx, \cdots, \Omega_g = r_g(x, y)\,dx$$

be linearly independent differentials of first kind, where the $r_i(x, y)$ are rational functions, and let

$$\gamma_{ij} = \int_{\Gamma_i} \Omega_j \text{ for } i = 1, \cdots, 2g \text{ and } j = 1, \cdots, g.$$

Also fix any points $Q_1, Q_2, \cdots$ on $S$.

ABEL'S THEOREM. *Let $V_1, \cdots, V_d, V_1', \cdots, V_d'$ be any points on $S$. Then $V_1, \cdots, V_d$ and $V_1', \cdots, V_d'$ are the zeros and poles of some rational function $s(x, y)$ iff for $j = 1, \cdots, g$ we have*

$$\sum_{k=1}^{d} \int_{Q_k}^{V_k} \Omega_j \equiv \sum_{k=1}^{d} \int_{Q_k}^{V_k'} \Omega_j \text{ modulo } ((\gamma_{ij})),$$

*i.e., iff, upon taking any paths $\Lambda_k$ and $\Lambda_k'$ joining $Q_k$ to $V_k$ and $V_k'$, for $j = 1, \cdots, g$ we have*

$$\sum_{k=1}^{d} \int_{\Lambda_k} \Omega_j = \sum_{k=1}^{d} \int_{\Lambda_k'} \Omega_j + \sum_{k,i} n_{ki}\gamma_{ij} \text{ with integers } n_{ki}.$$

JACOBI'S INVERSION THEOREM. *Given any general $g$-tuple $(z_1, \cdots, z_g)$ of complex numbers, there is a unique $g$-tuple of points $(V_1, \cdots, V_g)$ of $S$ such that for $j = 1, \cdots, g$ we have*

$$\sum_{k=1}^{g} \int_{Q_k}^{V_k} \Omega_j \equiv z_j \text{ modulo } ((\gamma_{ij})).$$

*The resulting $2g$ functions*

$$x(V_1(z_1, \cdots, z_g)), y(V_1(z_1, \cdots, z_g)), \cdots, x(V_g(z_1, \cdots, z_g)), y(V_g(z_1, \cdots, z_g))$$

*of the $g$ variables $z_1, \cdots, z_g$ are (single-valued meromorphic) periodic function with the $2g$ periods $(\gamma_{i1}, \cdots, \gamma_{ig})$, $i = 1, \cdots, 2g$.*

These multiply periodic functions are called abelian functions. They can be viewed as functions on the $g$-dimensional (complex dimension $g$ = real dimension $2g$) multi-torus obtained from the space of $g$ complex variables by "dividing" it by the $2g$ periods. The said multi-torus is called the jacobian variety of $f$ (or, of $C$; or, of $K(x, y)$):

($g_{14}$) $g$ = dimension of the jacobian variety.

In Abel's theorems, when we spoke of $V_1, \cdots, V_d$ and $V_1', \cdots, V_d'$ being the zeroes and poles of $s(x, y)$, what we precisely meant was that: given any point $V$ of $S$, upon letting $p$ to be the number of

times $V$ occurs amongst $V_1, \cdots, V_d$ and upon letting $p'$ to be the number of times $V$ occurs amongst $V_1', \cdots, V_d'$ we have $V(s(x, y)) = p - p'$.

In analogy with number theory, the purport of the theorems of Abel and Jacobi is sometimes expressed by saying that the jacobian variety is the "divisor class group" of $f$. Roughly speaking: the various tuples of points $(V_1, \cdots, V_d; V_1', \cdots, V_d')$ of $S$ constitute the "group of divisors of degree zero"; the subgroup of "principal divisors" or "divisors linearly equivalent to zero" consists of those tuples for which $V_1, \cdots, V_d$ and $V_1', \cdots, V_d'$ are the zeroes and poles of some rational function $s(x, y)$; finally, the divisor class group is the factor group of the group of divisors of degree zero modulo the subgroup of principal divisors.

To describe the analogous situation in number theory, let $E$ be the integral closure of the ring of ordinary integers in a finite algebraic field extension $M$ of the field of rational numbers. $E$ is called the ring of (algebraic) integers in $M$. It was found that $E$ need not be a UFD ( = unique factorization domain), i.e., in $E$ one may not have the unique factorization property of ordinary integers. To estimate how far the said property fails in $E$, one can consider the totality of all (nonzero) fractional $E$-ideals, i.e., finitely generated $E$-modules contained in $M$. The fractional $E$-ideals form a group under multiplication; the principal fractional $E$-ideals (i.e., those which consist of all the multiples of a single element) form a subgroup; the factor group of the former modulo the latter is called the divisor class group of $M$, and the order of the divisor class group is called the class number of $M$. Clearly the class number is 1 iff $E$ is a UFD. The divisor class group of $M$ is evidently the number-theoretic analogue of the jacobian variety of $K(x, y)$. Moreover, in a sense, the number-theoretic analogue of the finite dimensionality of the jacobian variety is the

THEOREM OF DEDEKIND AND KRONECKER which says that *the class number is always finite.*

Let us conclude this Chapter with the observation that the Riemann surface (sphere with $g$ handles) and the jacobian variety ($g$-dimensional multi-torus) are two distinct generalizations of the surface of a torus ( = doughnut or bicycle-tube).

## CHAPTER X: MORE UNIVERSITY ALGEBRA

**56. Serre.** To again recall the theorem of Euler (and Riemann):

$$\textit{genus of a curve in the complex case} = \sum_{i=0}^{2} (-1)^i \Delta_i^*,$$

where $\Delta_0^*$, $\Delta_1^*$, and $\Delta_2^*$ are the number of vertices, edges and faces of a polyhedron drawn on the Riemann surface of the curve. Serre (1966, [65]) has made the following far-reaching generalization of this:

$$\textit{arithmetic genus of a variety of any dimension and in any characteristic} = \Sigma(-1)^i h^i,$$

where each of the individual $h^i$ (unlike Euler's $\Delta_i^*$) has an invariant meaning, namely, it is the number of linearly independent elements in the "$i$th cohomology group of the variety."

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# DIOPHANTINE REPRESENTATION OF THE SET OF PRIME NUMBERS

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**1. Introduction.** Martin Davis, Yuri Matijasevič, Hilary Putnam and Julia Robinson [4] [8] have proven that every recursively enumerable set is Diophantine, and hence that the set of prime numbers is Diophantine. From this, and work of Putnam [12], it follows that the set of prime numbers is representable by a polynomial formula. In this article such a prime representing polynomial will be exhibited in explicit form. We prove (in Section 2)

THEOREM 1. *The set of prime numbers is identical with the set of positive values taken on by the polynomial*

$$
\begin{aligned}
(1)\quad & (k+2)\{1-[wz+h+j-q]^2-[(gk+2g+k+1)\cdot(h+j)+h-z]^2-[2n+p+q+z-e]^2\\
& -[16(k+1)^3\cdot(k+2)\cdot(n+1)^2+1-f^2]^2-[e^3\cdot(e+2)(a+1)^2+1-o^2]^2-[(a^2-1)y^2+1-x^2]^2\\
& -[16r^2y^4(a^2-1)+1-u^2]^2-[((a+u^2(u^2-a))^2-1)\cdot(n+4dy)^2+1-(x+cu)^2]^2-[n+l+v-y]^2\\
& -[(a^2-1)l^2+1-m^2]^2-[ai+k+1-l-i]^2-[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m]^2\\
& -[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x]^2-[z+pl(a-p)+t(2ap-p^2-1)-pm]^2\}
\end{aligned}
$$

*as the variables range over the nonnegative integers.*

(1) is a polynomial of degree 25 in 26 variables, $a, b, c, \ldots, z$. When nonnegative integers are substituted for these variables, the positive values of (1) coincide exactly with the set of all prime numbers 2,3,5,.... The polynomial (1) also takes on negative values, e.g., $-76$.

In 1971, Yuri Matijasevič [10] outlined the construction of a prime representing polynomial in 24 variables and degree 37, using the Fibonacci numbers. In the addendum to his paper, an improvement to 21 variables and degree 21 was made. (These polynomials were not written out explicitly.) Our construction here yields a polynomial in 19 variables and degree 29. It also yields a polynomial in 42 variables and degree 5. Thus we might ask what is the smallest possible degree and how few variables are actually necessary to represent primes?

Let us consider first the question of the degree. We know that a prime representing polynomial of degree 5 is possible. All that is necessary to reduce the degree to 5 is the Skolem substitution method (cf. [3], p. 263). However, this procedure increases the number of variables (to 42 when applied to (1)). We do not know whether there is a prime representing polynomial of degree $< 5$.

The question of the minimum number of variables is also open. A simple argument shows that at least 2 variables are necessary. But we do not know the minimum number. The method of proof of Theorem 1 yields a polynomial in 16 variables. To reduce the number of variables below 16 requires an entirely different construction. The best result we were able to obtain is a polynomial in 12 variables. We shall prove

THEOREM 2. *There exists a prime representing polynomial in* 12 *variables.*

This result was reportedly known to Yuri Matijasevič in 1973, although no literature is available concerning this. Our proof uses methods developed by Yuri Matijasevič and Julia Robinson in [11]. The construction is carried out in §3. The polynomial constructed has very large degree.

The proofs of Theorem 1 and 2 are both based on Wilson's Theorem. In each case we show that the set of prime numbers is Diophantine; i.e., that there exists a Diophantine equation solvable only for prime values of a parameter. We construct a polynomial $M(k, x_1, \cdots, x_n)$ with the property that for each nonnegative integer $k$

(3) $\qquad k+2$ is prime $\leftrightarrow M(k, x_1, \cdots, x_n) = 0$ is solvable in nonnegative integers.

It will turn out that $M$ is a sum of squares and hence nonnegative. From such a nonnegative polynomial $M$, satisfying (3), a prime representing polynomial $P$ is constructible by the method of Putnam [12]. We have only to set $P$ equal to

$$(k+2)\{1-M(k,x_1,\cdots,x_n)\}.^{\dagger} \tag{4}$$

The difficulty is of course the construction of $M$. We shall see that this requires nearly all the techniques invented to solve Hilbert's tenth problem. And there is good reason why this should be so. Several years before Hilbert's tenth problem was solved in the negative by Matijasevič [8], Julia Robinson [16] proved that *if the set of prime numbers was Diophantine, then every recursively enumerable set would be Diophantine.* Hence Theorem 1 and 2 actually imply the unsolvability of Hilbert's tenth problem.

As was mentioned previously, our polynomials take on negative values. Hence it cannot be claimed that they represent primes exactly. This is not an accidental feature of the Putnam construction. It is a limitation inherent in algebraic functions. The reader is probably familiar with the theorem that no polynomial can represent only primes. This theorem is proved in Section 4 and the result is also extended to all algebraic functions of several variables, of which the polynomial is only a special case.

To overcome the inexactness of the polynomial representation, it is necessary to use exponential functions or other transcendental functions. Julia Robinson noticed [4] that we may conveniently employ the function $0^x$ for this purpose, provided we define $0^0 = 1$. If we take $M$ as in (3) then we can prove

THEOREM 3. *The set of prime numbers is the exact range of a function of the form*

$$2+k\cdot 0^{M(k,x_1,\cdots,x_n)}$$

*in which $M(k,x_1,\cdots,x_n)$ is a polynomial and $n \leqq 11$.*

Here we used the function $0^x$ to distinguish between zero and positive integers. We may also use the proper subtraction function, absolute value function, remainder function, signum function or integer part function (but no algebraic function). Define $r(y,x)$ to be the remainder after division of $y$ by $x$ (take $r(y,0)=y$). Define $y \dot{-} x$ to be $y-x$ for $y \geqq x$ and 0 for $y<x$. Then $y \dot{-} x = (|y-x|+y-x)/2$. Any one of the following six functions may be used in Theorem 3. (The domain is the nonnegative integers.)

$$0^x = 1 \dot{-} x = \frac{|1-x|+1-x}{2} = 1-r(1,1+x) = 1-\operatorname{sgn}(x) = \left[\frac{1}{1+x}\right].$$

Indeed, using these more general functions it is easy to give a short formula for the $n$th prime, $p_n$. The following formula is derived in [7].

$$p_n = \sum_{i=0}^{n^2}\left(1 \dot{-} \left(\left(\sum_{j=0}^{i} r((j \dot{-} 1)!^2, j)\right) \dot{-} n\right)\right). \tag{5}$$

The $n$th prime function may also be represented by a polynomial, though not of course in one variable. We can prove

THEOREM 4. *There exists a polynomial $P(n,x_1,\cdots,x_k)$ such that for any positive integers $n$ and $m$,*

$$p_n = m \leftrightarrow (\exists x_1,\cdots,x_k)\{P(n,x_1,\cdots,x_k)=m\}.$$

---

† Note the apparent paradox. The polynomial $P$ factors! However, the factors are improper, $P = P\cdot 1$.

*Proof.* The proof is a simple elaboration on Putnam's idea, (due to Yuri Matijasevič [9]). The binary relation $p_n = m$ is recursive and hence Diophantine by [4] [8]. Therefore there exists a polynomial $Q$ such that $p_n = m \leftrightarrow (\exists x_1, \cdots, x_l) Q(n, m, x_1, \cdots, x_l) = 0$.
We have only to put $P = x_{l+1}(1 - Q^2(n, x_{l+1}, x_1, \cdots, x_l))$. (It follows from the central result of [11] that we may take $l = 13$ and hence $k = 14$.)

The Diophantine character of the set of prime numbers has one further consequence which deserves mention. This concerns the problem of *proving* that a number is prime. If $p$ is a composite number, then there is a proof that $p$ is composite consisting of a single multiplication. Julia Robinson remarks in [14] that prior to the solution of Hilbert's tenth problem in 1970, it was not known that there was a similar proof establishing primality in a bounded number of steps. Yet it follows from [4] [8] that this is so. And our construction permits us to compute a bound on this number.

THEOREM 5. *If $p$ is a prime number, then there is a proof that $p$ is prime consisting of only 87 additions and multiplications.*

The number is easily calculated from the equations of Theorem 2.12.

**2. Proof of Theorem 1.** Throughout this paper, all variables are nonnegative integers, unless the contrary is explicitly stated. We shall use the notation $x = \square$ to indicate that $x$ is a perfect square. $r(a, b)$ denotes the remainder of $a$ upon division by $b$. $[x]$ denotes the greatest integer $\leqq x$.

We shall be concerned with the solutions of the Pell equation

$$x^2 - (a^2 - 1)y^2 = 1, \qquad \text{for} \qquad 1 \leqq a.$$

It is well known [3], [16] that the solutions of this equation, $x = \chi_a(n)$, $y = \psi_a(n)$, can be generated via Lucas sequences:

$$\chi_a(0) = 1, \qquad \chi_a(1) = a, \qquad \chi_a(n+2) = 2a\chi_a(n+1) - \chi_a(n),$$
$$\psi_a(0) = 0, \qquad \psi_a(1) = 1, \qquad \psi_a(n+2) = 2a\psi_a(n+1) - \psi_a(n).$$

We shall need the following properties of these sequences.

LEMMA 2.1. $(2a-1)^n \leqq \psi_a(n+1) \leqq (2a)^n$.

LEMMA 2.2. $\psi_a(n) \equiv n \pmod{a-1}$.

These properties are immediate consequences of the definition. Proofs may be found in [3] and [11]. The following lemma will be used to force one unknown to be exponentially larger than another.

LEMMA 2.3. *For $2 \leqq e$, the condition*

$$e^3(e+2)(n+1)^2 + 1 = \square \tag{2.3}$$

*implies that $e - 1 + e^{e-2} \leqq n$. Conversely, for any positive integers $e$ and $t$, it is possible to satisfy 2.3 with $n$ such that $t \mid n+1$.*

*Proof.* Put $a = e + 1$. Then (2.3) becomes a Pell equation in $(a-1)(n+1)$, i.e.,

$$(a^2 - 1)(a-1)^2(n+1)^2 + 1 = \square.$$

If $n$ is any solution of (2.3) then $(a-1)(n+1) = \psi_a(j)$ for some $j$. By Lemma 2.2, $a - 1 \mid j$. Since $0 \neq j$, this implies that $a - 1 \leqq j$. Using Lemma 2.1 we find that

$$(a-2)(a-1) + (a-1)^{a-2} < (2a-1)^{(a-2)} \leqq \psi_a(a-1) \leqq \psi_a(j) = (a-1)(n+1).$$

Hence

$$(a-2) + (a-1)^{a-3} < n + 1,$$

which gives the result. The converse follows easily from the following well-known fact about Pell equations: When $A \neq \square$, the Pell equation $Ay^2 + 1 = x^2$ always has nontrivial solutions, (cf. [11] §2).

LEMMA 2.4. *For any numbers $p, n$ and $a \geqq 1$ we have the congruence*

$$\chi_a(n) \equiv p^n + \psi_a(n)(a-p) \pmod{2ap - p^2 - 1}.$$

*Furthermore, when $0 < p^n < a$, the right side of the congruence is less than or equal to the left side.*

For a proof of the asserted congruence see [16] p. 108 or [3] p. 242. The asserted inequality is not difficult to derive using $\sqrt{a^2-1}\ \psi_a(n) < \chi_a(n)$.

Next we state a Diophantine definition of the sequence $y = \psi_a(n)$. The set of equations is quite economical. It is difficult to assign credit accurately for these equations because they are a synthesis of collective efforts. The equations most resemble those of Julia Robinson (which appear in Theorem 3.1 of Davis [3]). However, they are not identical with these and equation V is due to Yuri Matijasevič.

LEMMA 2.5. *For any numbers $a, n$ and $y$, $(1 \leqq n)$ and $(2 \leqq a)$, in order that $y = \psi_a(n)$ it is necessary and sufficient that there exist numbers $b, c, d, r, s, t, u, v$ and $x$ such that*

(I) $x^2 = (a^2-1)y^2 + 1$,

(II) $u^2 = (a^2-1)v^2 + 1$,

(III) $s^2 = (b^2-1)t^2 + 1$,

(IV) $v = 4ry^2$,

(V) $b = a + u^2(u^2 - a)$,

(VI) $s = x + cu$,

(VII) $t = n + 4dy$,

(VIII) $n \leqq y$.

The proof is virtually identical to that given by Davis in [3] so we shall mention only the differences. Davis used positive integer unknowns in [3], however this is not essential. We need not replace $r$ by $r+1$ in equation IV, (to ensure that $0 < v$), since if $v = 0$ then $u = 1$ by II, $b = 1$ by V, $s = 1$ by III, $x = 1$ by I and VI, and $y = 0$ by I, contradicting VIII. Also, the condition V of [3] was used only to show that $b \equiv 1 \bmod 4y$ and $b \equiv a \bmod u$. However, these conditions are guaranteed by II, IV and V above. In this connection the proof of sufficiency is slightly different from that given in [3]. We need not use the Chinese Remainder Theorem. If we eliminate the unknowns $v, b, s$ and $t$ from I–VIII, by substitution, then we obtain

COROLLARY 2.6. *For any numbers $a, n$ and $y$, $(1 \leqq n)$ and $(2 \leqq a)$, in order that $y = \psi_a(n)$ it is necessary and sufficient that there exist numbers $c, d, r, u$ and $x$ such that*

(I) $x^2 = (a^2-1)y^2 + 1$,

(II) $u^2 = 16(a^2-1)r^2y^4 + 1$,

(III) $(x + cu)^2 = ((a + u^2(u^2 - a))^2 - 1)(n + 4dy)^2 + 1$,

(IV) $n \leqq y$.

We shall also require two basic inequalities:

LEMMA 2.7. *If $0 \leqq \alpha < 1/q$, then $1 - q\alpha \leqq (1-\alpha)^q$.*

LEMMA 2.8 *If $0 \leqq \alpha \leqq \frac{1}{2}$, then $(1-\alpha)^{-1} \leqq 1 + 2\alpha$.*

The fundamental tool in both constructions is Wilson's theorem which characterizes the primes in terms of the factorial function.

LEMMA 2.9. (Wilson's theorem.) *For any number $k \geqq 1$, $k+1$ is prime if and only if*

$$k+1 \mid k! + 1.$$

For a proof see [6], p. 68. The next lemma leads to a Diophantine definition of the factorial function. It is stated in [10], in slightly different form, without proof.

LEMMA 2.10. *For any positive integer $k$, if $(2k)^k \leqq n$ and $n^k < p$ then*

$$k! < \frac{(n+1)^k p^k}{r((p+1)^n, p^{k+1})} < k! + 1.$$

*Proof.* Using the Binomial Theorem we have

$$(p+1)^n = \sum_{i=0}^{k} \binom{n}{i} p^i + p^{k+1} \sum_{i=k+1}^{n} \binom{n}{i} p^{i-k-1}.$$

Now $(np)^{k+1} - 1 = n^k p^k np - 1 \leqq (p-1)p^k np - 1 = pp^k pn - p^k np - 1 < pp^k pn - pp^k = (np-1)pp^k$. Therefore

$$\sum_{i=0}^{k} \binom{n}{i} p^i \leqq \sum_{i=0}^{k} n^i p^i = \frac{(np)^{k+1} - 1}{np - 1} < pp^k$$

so that

(i) $$r((p+1)^n, pp^k) = \sum_{i=0}^{k} \binom{n}{i} p^i \neq 0.$$

First we prove that

(ii) $$k! < \frac{(n+1)^k p^k}{\sum_{i=0}^{k} \binom{n}{i} p^i}.$$

(ii) follows from the inequalities

$$k!\left(\sum_{i=0}^{k} \binom{n}{i} p^i\right) \leqq k!\left(k\binom{n}{k-1} p^{k-1} + \binom{n}{k} p^k\right) \leqq k!\left(k\frac{n^{k-1}}{(k-1)!} p^{k-1} + \frac{n^k}{k!} p^k\right)$$

$$= k!\left(\frac{k^2 n^{k-1}}{k!} p^{k-1} + \frac{n^k}{k!} p^k\right) = k^2 n^{k-1} p^{k-1} + n^k p^k < kn^k p^{k-1} + n^k p^k < kpp^{k-1} + n^k p^k$$

$$= (k + n^k) p^k \leqq (1+n)^k p^k.$$

It remains only to establish

(iii) $$\frac{(n+1)^k p^k}{\sum_{i=0}^{k} \binom{n}{i} p^i} < k! + 1.$$

In consequence of

$$\frac{(n+1)^k p^k}{\sum_{i=0}^{k} \binom{n}{i} p^i} = \frac{(n+1)^k}{\sum_{i=0}^{k} \binom{n}{i} p^{i-k}} < \frac{(n+1)^k}{\binom{n}{k}}$$

we see that (iii) will follow from

(iv) $$\frac{(n+1)^k}{\binom{n}{k}} \leqq k! + 1.$$

To derive (iv) we have only to use Lemmas 2.7 and 2.8, viz.

$$\frac{(n+1)^k}{\binom{n}{k}} < \frac{k!}{\frac{(n+1-k)^k}{(n+1)^k}} = \frac{k!}{\left(1 - \frac{k}{n+1}\right)^k} < \frac{k!}{\left(1 - \frac{k}{n}\right)^k} = k!\left(\left(1 - \frac{k}{n}\right)^k\right)^{-1}$$

$$\leqq k!\left(1 - \frac{k^2}{n}\right)^{-1} \leqq k!\left(1 + \frac{2k^2}{n}\right) \leqq k!\left(1 + \frac{2k^2}{(2k)^k}\right) \leqq k!\left(1 + \frac{1}{k!}\right).$$

Using Lemma 2.10 we may characterize the factorial function in terms of three exponential functions.

LEMMA 2.11. *For any positive integers $k$ and $f$, in order that $f = k!$ it is necessary and sufficient that there exist nonnegative integers $j, h, n, p, q, w$ and $z$ such that*

$$\text{(I)}\quad q = wz + h + j, \qquad \text{(IV)}\quad p = (n+1)^k,$$

$$\text{(II)}\quad z = f(h+j) + h, \qquad \text{(V)}\quad q = (p+1)^n,$$

$$\text{(III)}\quad (2k)^3(2k+2)(n+1)^2 + 1 = \square, \qquad \text{(VI)}\quad z = p^{k+1}.$$

*Proof. Sufficiency.* Suppose $k, f, j, h, n, p, q, w$ and $z$ satisfy conditions I–VI. By II and VI, $0 < h + j \leqq z$. If $h + j = z$ then I implies $z \mid q$ contrary to Lemma 2.10. Hence $h + j < z$ and so I implies $r(q, z) = h + j$. From II and Lemma 2.10 we find

$$f \leqq z/(h+j) \leqq f+1 \quad \text{and} \quad k! < z/(h+j) < k! + 1.$$

Hence $f = k!$ since $f$ and $k!$ are integers.

*Necessity.* Suppose $1 \leqq k$ and $f = k!$ By Lemma 2.3 we may choose $n$ so that $(2k)^k \leqq n$ and III holds. Set $p = (n+1)^k$, $q = (p+1)^n$ and $z = p^{k+1}$. Then IV, V and VI hold. Finally, put $w = (q - r(q, z))/z$, $h = z - fr(q, z)$ and $j = r(q, z) - h$. Then I and II hold. By Lemma 2.10, $h$ and $j$ are nonnegative.

Finally, we are able to give a Diophantine definition of the set of prime numbers.

THEOREM 2.12. *For any number $k \geqq 1$, in order that $k + 1$ be prime it is necessary and sufficient that there exist numbers $a, b, c, d, e, f, g, h, i, j, l, m, n, o, p, q, r, s, t, u, v, w, x, y$ and $z$ such that*

$$\text{(1)}\quad q = wz + h + j,$$

$$\text{(2)}\quad z = (gk + g + k)\cdot(h + j) + h,$$

$$\text{(3)}\quad (2k)^3(2k+2)(n+1)^2 + 1 = f^2,$$

$$\text{(4)}\quad e = p + q + z + 2n,$$

$$\text{(5)}\quad e^3(e+2)(a+1)^2 + 1 = o^2,$$

$$\text{(6)}\quad x^2 = (a^2 - 1)y^2 + 1,$$

$$\text{(7)}\quad u^2 = 16(a^2-1)r^2y^4 + 1,$$

$$\text{(8)}\quad (x + cu)^2 = ((a + u^2(u^2 - a))^2 - 1)\cdot(n + 4dy)^2 + 1,$$

$$\text{(9)}\quad m^2 = (a^2 - 1)l^2 + 1,$$

$$\text{(10)}\quad l = k + i(a - 1),$$

$$\text{(11)}\quad n + l + v = y,$$

$$\text{(12)}\quad m = p + l(a - n - 1) + b(2a(n+1) - (n+1)^2 - 1),$$

$$\text{(13)}\quad x = q + y(a - p - 1) + s(2a(p+1) - (p+1)^2 - 1),$$

$$\text{(14)}\quad pm = z + pl(a - p) + t(2ap - p^2 - 1).$$

*Proof. Sufficiency.* Suppose that numbers $a, b, \cdots, z$ satisfy equations (1)–(14) and that $1 \leqq k$. Equation (3), together with Lemma 2.3, implies that

$$\text{(1')}\qquad 2 \leqq n, \qquad \text{and also (2')} \quad k < n.$$

Equations (4) and (5), together with Lemma 2.3, imply that

$$\text{(3')}\qquad p + q + z + 2n - 1 + (p + q + z + 2n)^{p+q+z+2n-2} \leqq a, \qquad \text{and also (4')} \quad n < a.$$

According to Corollary 2.6, equations (6), (7), (8) and (11) imply that $y = \psi_a(n)$, and hence also that $x = \chi_a(n)$. Equation (9) implies that $m = \chi_a(k')$ and $l = \psi_a(k')$, for some number $k'$. Equation (11) asserts that $l < y$ and hence that $k' < n$. Therefore, by (2') and (4') we have $k' < a - 1$ and also $k < a - 1$. By Lemma 2.2, and equation (10), we have $k \equiv k' \pmod{a-1}$. Therefore $k' = k$ so $m = \chi_a(k)$ and $l = \psi_a(k)$.

From (1'), (2'), and (3') it follows that

$$p < a, \quad (n+1)^k < a \quad \text{and} \quad a < 2a(n+1) - (n+1)^2 - 1.$$

Using Lemma 2.4 and equation (12) we find that $p \equiv (n+1)^k \pmod{2a(n+1)-(n+1)^2-1}$. This together with the above inequalities implies that

$$p = (n+1)^k. \tag{5'}$$

From (1') and (3') it follows that

$$q < a, \quad (p+1)^n < a \quad \text{and} \quad a < 2a(p+1)-(p+1)^2-1.$$

Using Lemma 2.4 and equation (13) we find that $q \equiv (p+1)^n \pmod{2a(p+1)-(p+1)^2-1}$. This, together with the above inequalities, implies that

$$q = (p+1)^n. \tag{6'}$$

From (1'), (2'), (3') and the fact that $p \neq 0$ (which follows from (5')), it follows that

$$z < a, \quad p^{k+1} < a \quad \text{and} \quad a < 2ap - p^2 - 1.$$

Using Lemma 2.4 and equation (14) we find that $z \equiv p^{k+1} \pmod{2ap-p^2-1}$. This, together with the above inequalities, implies that

$$z = p^{k+1}. \tag{7'}$$

According to Lemma 2.11, conditions (1), (2), (3), (5'), (6') and (7') imply that $gk + g + k = k!$ and hence that $k! + 1 = (g+1)(k+1)$. Thus $k+1$ is prime by Lemma 2.9.

*Necessity.* Suppose that $1 \leqq k$ and that $k+1$ is prime. By Lemma 2.9 we may find a number $g$ so that $k! = gk + g + k$. According to Lemma 2.11 numbers $f, h, j, n, p, q$ and $w$ may be chosen to satisfy equations (1), (2), (3) and also the conditions

$$p = (n+1)^k, \quad q = (p+1)^n \quad \text{and} \quad z = p^{k+1}. \tag{8'}$$

Choose $e$ so as to satisfy equation (4). By Lemma 2.3 it is possible to choose numbers $a$ and $o$, $(a \geqq 2)$, satisfying (5). Put $y = \psi_a(n)$. According to Corollary 2.6 we may find numbers $c, d, r, u$ and $x$ satisfying equations (6), (7) and (8). Put $m = \chi_a(k)$ and put $l = \psi_a(k)$. Then (9) holds. By Lemma 2.2 and the fact that $k \leqq \psi_a(k)$ for $2 \leqq a$, we may choose $i$ satisfying equation (10). It is trivial to show (by induction) that for $2 \leqq a$, $n + \psi_a(n-1) \leqq \psi_a(n)$. Using this, and the fact that $k < n$ (which follows from (3)), we find that $n + l \leqq y$. Hence it is possible to choose a number $v$ satisfying (11). Finally, as in the proof of sufficiency, equations (4) and (5) imply that

$$(n+1)^k < a, \quad (p+1)^n < a \quad \text{and} \quad p^k < a.$$

Consequently, by Lemma 2.4 and the equations (8'), numbers $b, s$ and $t$ may be found so that equations (12), (13) and (14) are satisfied. This completes the proof of Theorem 2.12.

Theorem 1 now follows immediately from Theorem 2.12. We have only to replace $k$ by $k+1$ throughout the equations of Theorem 2.12, sum the squares of the equations and employ the device of Putnam [**12**]. This produces the polynomial (1).

Perhaps some industrious reader will construct a shorter prime representing polynomial.

**3. Proof of Theorem 2.** Here we show that primes are Diophantine definable in 11 unknowns. In Section 2 we used what might be called the *congruence method.* In this section we shall use what might be called the *ratio method.* This latter technique, developed by Yuri Matijasevič and Julia Robinson in [**11**], is generally more economical with respect to the number of variables.

LEMMA 3.1. *For* $0 < 2q < \beta$, $\left(\frac{\beta}{\beta-1}\right)^q \leqq 1 + \frac{2q}{\beta}$.

*Proof.* By Lemmas 2.7 and 2.8.

LEMMA 3.2. *For* $0 < n < M$ *and* $0 \leqq x$, $\left(1-\frac{n}{M}\right) < \left(1-\frac{1}{2M(x+1)}\right)^n$.

*Proof.* By Lemma 2.7.

LEMMA 3.3. *If* $x > 8 \cdot 2^n$ *and* $n > k$, *then*

(i) $\frac{(x+1)^n}{x^k} - \left[\frac{(x+1)^n}{x^k}\right] < \frac{1}{8}$, (ii) $\left[\frac{(x+1)^n}{x^k}\right] \equiv \binom{n}{k} \bmod x$, (iii) $\binom{n}{k} < \frac{(x+1)^n}{x^k}$.

*Proof.*

$$\frac{(x+1)^n}{x^k} = \sum_{i=0}^{n} \binom{n}{i} x^{i-k} = \sum_{i=0}^{k-1} \binom{n}{i} x^{i-k} + \sum_{i=k}^{n} \binom{n}{i} x^{i-k},$$

where

$$\sum_{i=0}^{k-1} \binom{n}{i} x^{i-k} \leqq \frac{1}{x} \sum_{i=0}^{k-1} \binom{n}{i} < \frac{2^n}{x} < \frac{1}{8}. \quad \text{Thus} \quad \left[\frac{(x+1)^n}{x^k}\right] = \sum_{i=k}^{n} \binom{n}{i} x^{i-k} \equiv \binom{n}{k} \bmod x.$$

(iii) follows from the assumption $k < n$.

LEMMA 3.4. $n^k \Big/ \binom{n}{k} \leqq k! \left(1 + \frac{2(k-1)^2}{n}\right)$ *for* $n > 2(k-1)^2$.

*Proof.* As in the proof of Lemma 2.10, condition (iv).

LEMMA 3.5. $a\left(1+\frac{1}{10a}\right)^4 < a + \frac{1}{2}$, (*for* $a \geqq 1$).

LEMMA 3.6. $a\left(1-\frac{1}{4a}\right)^2 > a - \frac{1}{2}$, (*for* $a \geqq 1$).

DEFINITION 3.7. $U(x, y) = (x+2)^3(x+4)(y+1)^2 + 1$.

By Lemma 2.3, if $U(x, y) = \square$ then $x^x < y$. Also, for each $x$, arbitrarily large numbers $y$ may be found satisfying $U(x, y) = \square$. (This is the same function $U(x, y)$ defined in [**11**].)

LEMMA 3.8. (Matijasevič-Robinson [**11**]) *Suppose* $A > 1$, $B > 1$ *and* $C > 0$. *Then* $\psi_A(B) = C$ *if and only if the following system of conditions can be satisfied.*

(A1) $DFI = \square$, $F \mid H - C$, $B \leqq C$,
(A2) $D = (A^2 - 1)C^2 + 1$,
(A3) $E = 2(i+1)D(k+1)C^2$,
(A4) $F = (A^2 - 1)E^2 + 1$,
(A5) $G = A + F(F - A)$,
(A6) $H = B + 2(j+1)C$,
(A7) $I = (G^2 - 1)H^2 + 1$.

THEOREM 3.9. *For any positive integer* $k$, *in order that* $k+1$ *be prime, it is necessary and sufficient that the following system of equations has a solution in nonnegative integers*:

(I) $U(2k, n) = \square$,
(II) $U(2n, x) = \square$,
(III) $M = 16nx(w+2) + 1$,
(IV) $A = M(x+1)$,
(V) $B = n + 1$,
(VI) $C = m + B$,
(VII) $DFI = \square$, $F \mid H - C$,
(VIII) $D = (A^2 - 1)C^2 + 1$,
(IX) $E = 2(i+1)DC^2$,
(X) $F = (A^2 - 1)E^2 + 1$,
(XI) $G = A + F(F - A)$,

(XII) $H = B + 2(j+1)C,$

(XIII) $I = (G^2-1)H^2+1,$

(XIV) $\left\{\dfrac{R}{\left(\dfrac{C}{KL}-(w+1)x\right)\left(1-\dfrac{R}{C}\right)^2 L}-(S+1)\right\}^2 < \frac{1}{4},$

(XV) $(M^2-1)K^2+1=\square,$

(XVI) $(M^2x^2-1)L^2+1=\square,$

(XVII) $(M^2n^2x^2-1)R^2+1=\square,$

(XVIII) $K = n-k+1+p(M-1),$

(XIX) $L = k+1+l(Mx-1),$

(XX) $R = k+1+r(Mnx-1),$

(XXI) $S = (z+1)(k+1)-2.$

*Proof of Sufficiency.* Let there be given a solution to the system (I)–(XXI). We must show that $k+1$ is prime. By Wilson's Theorem (Lemma 2.9) it is sufficient to show that $k+1 \mid k!+1$. According to the equation (XXI), $k+1 \mid S+2$. Hence it is sufficient to show that $S+1 = k!$. Define real numbers $\sigma$ and $\beta$ by

$$\sigma = \frac{C}{KL}, \qquad \beta = \frac{R}{(\sigma-(w+1)x)\left(1-\dfrac{R}{C}\right)^2 L}.$$

According to (XIV), $|\beta-(S+1)| < \frac{1}{2}$. Hence we need only prove that

(1) $$\beta - k! | < \tfrac{1}{2}.$$

From (I) we have, by Lemmas 2.3 and 3.7,

(2) $$n > (2k)^{2k} > k, \quad \text{and} \quad n \geqq 5.$$

(II) implies that

(3) $$x > (2n)^{2n} > 8n^k.$$

(III) implies that

(4) $$M \geqq 32nx, \quad \text{and} \quad (5) \quad M > 2n, M > 160 \cdot 10^{10}.$$

From (IV), (V) and (VI), $A > 1$, $B > 1$ and $C \geqq B > 1$. So by Lemma 3.8, (VII)–(XIII) imply that

(6) $$C = \psi_{M(x+1)}(n+1).$$

(XV), (XVIII) and Lemma 2.2 imply

(7) $$K = \psi_M(n-k+1+p'(M-1)),$$

where $p' \geqq 0$, since $M-1 > n-k+1$ by (5). (XVI), (XIX) and Lemma 2.2 imply that

(8) $$L = \psi_{Mx}(k+1+l'(Mx-1)),$$

where $l' \geqq 0$ since $Mx-1 > k+1$ by (4) and (2). (XVII), (XX) and Lemma 2.2 imply that

(9) $$R = \psi_{Mnx}(k+1+r'(Mnx-1)),$$

where $r' \geqq 0$ since $Mnx-1 \geqq k+1$.

We now show by contradiction that $p' = l' = 0$. If $p' > 0$ or $l' > 0$ then

$$\sigma \leqq \frac{(2M(x+1))^n}{(2M-1)^{n+M-k-1}(2Mx-1)^k} \quad \text{or} \quad \sigma \leqq \frac{(2M(x+1))^n}{(2M-1)^{n-k}(2Mx-1)^{k+Mx-1}}.$$

In either case,

$$\sigma < \frac{(2M(x+1))^n}{(2M-1)^M} = \frac{(2M)^n}{(4M(M-1)+1)^n} \cdot \frac{(x+1)^n}{(2M-1)^{M-2n}} < \frac{1}{2}.$$

Hence $\beta < 0$ contradicting (XIV) (which implies that $\frac{1}{2} < \beta$).

(10) Thus $p' = l' = 0$, $K = \psi_M(n-k+1)$ and $L = \psi_{Mx}(k+1)$.

We then have

$$\sigma \leqq \frac{(2M(x+1))^n}{(2M-1)^{n-k}(2Mx-1)^k} < \frac{(x+1)^n}{x^k}\left(\frac{2M}{2M-1}\right)^n \leqq \frac{(x+1)^n}{x^k}\left(1+\frac{n}{M}\right),$$

by Lemmas 2.7 and 3.1. Also,

$$\sigma \geqq \frac{(2M(x+1)-1)^n}{(2M)^{n-k}(2Mx)^k} = \frac{(x+1)^n}{x^k}\left(1-\frac{1}{2M(x+1)}\right)^n > \frac{(x+1)^n}{x^k}\left(1-\frac{n}{M}\right)$$

by Lemma 3.2. Thus,

(11) $$\left|\sigma - \frac{(x+1)^n}{x^k}\right| < \frac{n}{M}\cdot\frac{(x+1)^n}{x^k}, \quad \text{and} \quad \sigma > \frac{1}{2}\frac{(x+1)^n}{x^k}.$$

We next derive bounds for $\sigma - (w+1)x$.

*Case* 1: Suppose $x > \sigma - (w+1)x$. Then $(w+2)x > \sigma$. Put $\epsilon_1 = \left|\sigma - \frac{(x+1)^n}{x^k}\right|$. Then

(12) $$\sigma = \frac{(x+1)^n}{x^k} \pm \varepsilon_1 = \left[\frac{(x+1)^n}{x^k}\right] \pm \varepsilon_1 + \varepsilon_2, \text{ where } 0 < \varepsilon_2 < \frac{1}{8}, \text{ by Lemma 3.3. And}$$

(13) $$\sigma \equiv \binom{n}{k} \pm \varepsilon_1 + \varepsilon_2 \pmod{x}, \quad \text{by Lemma 3.3.}$$

From (11) and (III),

$$0 \leqq \varepsilon_1 < \frac{(x+1)^n}{x^k}\cdot\frac{n}{M} < 2\sigma\cdot\frac{n}{M} < \frac{1}{8}.$$

Since $k+1 \leqq n$, we have

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2} \geqq \frac{n\cdot k(k-1)\cdots 2}{k(k-1)\cdots 2} = n \geqq 5, \quad \text{and} \quad \binom{n}{k} \leqq n^k < \frac{1}{3}x.$$

Then

$$\sigma - (w+1)x \equiv \sigma \equiv \binom{n}{k} \pm \varepsilon_1 + \varepsilon_2 (\bmod x),$$

where $0 < \binom{n}{k} \pm \varepsilon_1 + \varepsilon_2 < \frac{1}{3}x + \frac{1}{4} < \frac{1}{2}x < x$. By (XIV), $0 < \sigma - (w+1)x$. Hence both sides of the congruence are less than the modulus, so we have

(14) $$\sigma - (w+1)x = \binom{n}{k} \pm \varepsilon_1 + \varepsilon_2 < \tfrac{1}{2}x,$$

and $\sigma - (w+1)x = \binom{n}{k} \pm \varepsilon_1 + \varepsilon_2 \geqq \binom{n}{k} - \frac{1}{8} > 4$, i.e.,

(15) $$4 < \sigma - (w+1)x < \tfrac{1}{2}x.$$

From (9), we have $R = \psi_{Mnx}(k+1+r'(Mnx-1))$, where $r' \geqq 0$. Suppose $r' > 0$. Then

$$R \geqq \psi_{Mnx}(k+Mnx) \geqq (2Mnx-1)^{k+Mnx-1} \geqq (2Mnx-1)^{Mnx}.$$

By (6), $C = \psi_{M(x+1)}(n+1) \leqq (2M(x+1))^n < (2Mnx-1)^n$, so $C^3 < R$. Also, $C \geqq (2M(x+1)-1)^n > 9$, so that $C < \frac{1}{3}R^{1/2}$. Thus,

$$R\Big/\left(\frac{R}{C}-1\right)^2 < R/(3R^{1/2}-1)^2 < R/(2R^{1/2})^2 = \tfrac{1}{4}.$$

Since $\frac{1}{2} < \sigma - (w+1)x$ by (15),

$$\beta = \frac{R}{(\sigma-(w+1)x)\left(1-\frac{R}{C}\right)^2 L} < \frac{1}{4(\sigma-(w+1)x)} < \frac{1}{2},$$

contradicting (XIV). Thus

$$r' = 0 \quad \text{and} \quad R = \psi_{Mnx}(k+1). \tag{16}$$

It follows from this, (3) and (4) that

$$\frac{R}{C} = \frac{\psi_{Mnx}(k+1)}{\psi_{M(x+1)}(n+1)} \leqq \frac{(2Mnx)^k}{(2M(x+1)-1)^n} < \frac{(2Mx)^k \cdot n^k}{(2Mx)^n} < \frac{(2Mx)^{k+1}}{(2Mx)^n} \leqq \frac{1}{(2Mx)^2} < \frac{1}{4}. \tag{17}$$

Also,

$$\left(1-\frac{R}{C}\right)^2 > 1-\frac{2R}{C}, \quad \text{so} \quad \frac{1}{\left(1-\frac{R}{C}\right)^2} < \frac{1}{1-\frac{2R}{C}} < 2, \quad \text{by (17).} \tag{18}$$

*Case* 2: Suppose $\sigma-(w+1)x \geqq x$. Then $\sigma-(w+1)x > \frac{1}{2}$, so $r' = 0$, as in Case 1. Also,

$$\beta = \frac{R}{(\sigma-(w+1)x)\left(1-\frac{R}{C}\right)^2 L} < \frac{2(2Mnx)^k}{x(2Mx-1)^k}, \quad \text{using (18), (16) and (10),}$$

$$= \frac{2n^k}{x}\left(\frac{2Mx}{2Mx-1}\right)^k \leqq \frac{2n^k}{x}\left(1+\frac{k}{Mx}\right) \quad \text{by Lemma 3.1, using (3),}$$

$$< \frac{4n^k}{x} < \frac{1}{2}.$$

Thus $\beta < \frac{1}{2}$, contradicting (XIV). Hence, Case 2 does not hold and Case 1 obtains.

Now it is possible to show that $|\beta - k!| < \frac{1}{2}$. We have, from Case 1, that

$$\sigma-(w+1)x = \binom{n}{k} \pm \varepsilon, \quad \text{where} \quad 0 \leqq \varepsilon = |\pm\varepsilon_1 + \varepsilon_2| < \tfrac{1}{4}. \tag{19}$$

Using (10), (16), (17) and (19) we have

$$\beta = \frac{R}{\left(\binom{n}{k} \pm \varepsilon\right)\left(1-\frac{R}{C}\right)^2 L} < \frac{(2Mnx)^k}{\binom{n}{k}\left(1-\left(\varepsilon\Big/\binom{n}{k}\right)\right)\left(1-\frac{1}{(2Mx)^2}\right)^2 (2Mx-1)^k}$$

$$= \left(n^k\Big/\binom{n}{k}\right)\left(\frac{2Mx}{2Mx-1}\right)^k \left(\frac{1}{1-\left(\varepsilon\Big/\binom{n}{k}\right)}\right)\left(\frac{1}{1-\frac{1}{(2Mx)^2}}\right)^2$$

It is easily seen that

$$\frac{1}{1-\left(\varepsilon\Big/\binom{n}{k}\right)} \leqq 1+\frac{2\varepsilon}{\binom{n}{k}}, \quad \left(\frac{1}{1-\frac{1}{(2Mx)^2}}\right)^2 \leqq 1+\frac{1}{Mx}, \quad \text{and} \quad \left(\frac{2Mx}{2Mx-1}\right)^k \leqq 1+\frac{k}{Mx},$$

using Lemmas 2.7, 2.8 and 3.1. By Lemma 3.4,

$$n^k \Big/ \binom{n}{k} \leqq k!\left(1+\frac{2(k-1)^2}{n}\right).$$

Thus

$$\beta < k!\left(1+\frac{2\varepsilon}{\binom{n}{k}}\right)\left(1+\frac{1}{Mx}\right)\left(1+\frac{k}{Mx}\right)\left(1+\frac{2(k-1)^2}{n}\right).$$

We claim

(20) $$\beta < k! + \tfrac{1}{2}.$$

For this, the following inequalities are sufficient, by Lemma 3.5:

(i) $k!\left(2\varepsilon \Big/ \binom{n}{k}\right) < \frac{1}{10}$ (since $10k!k!2\varepsilon < 5(k!)^2 \leqq 5k^{2k} \leqq n < n(n-1)\cdots(n-k+1)$),

(ii) $k!\left(\frac{k}{Mx}\right) < \frac{1}{10}$ (since $10k!\,k \leqq 10k^k < 10n < Mx$),

(iii) $k!\left(\frac{1}{Mx}\right) < \frac{1}{10}$ (by (ii)),

(iv) $k!\left(\frac{2(k-1)^2}{n}\right) < \frac{1}{10}$ (since $20k!(k-1)^2 \leqq 40k^k < (2k)^{2k} < n$, if $k \geqq 2$).

(These inequalities are derived using only (2) and (3).) Thus (20) holds

Also,

$$\beta \geqq \frac{(2Mnx-1)^k}{\binom{n}{k}\left(1+\left(\varepsilon \Big/ \binom{n}{k}\right)\right)(2Mx)^k} \quad \text{by (16) and (10),}$$

$$= \frac{n^k}{\binom{n}{k}}\left(1-\frac{1}{2Mnx}\right)^k\left(1+\frac{\varepsilon}{\binom{n}{k}}\right)^{-1} \geqq \frac{n^k}{\binom{n}{k}}\left(1-\frac{1}{2Mnx}\right)^k\left(1-\frac{\varepsilon}{\binom{n}{k}}\right)$$

$$\geqq k!\left(1-\frac{k}{2Mnx}\right)\left(1-\frac{\varepsilon}{\binom{n}{k}}\right) \quad \text{by Lemmas 2.7 and 2.8.}$$

We claim that

(21) $$\beta > k! - \tfrac{1}{2}.$$

By Lemma 3.6, the following inequalities are sufficient to establish this.

(i) $1-\frac{k}{2Mnx} \geqq 1-\frac{1}{4k!}$ (since $4kk! \leqq 4k^{2k} \leqq 4n < 2Mnx$),

(ii) $1-\frac{\varepsilon}{\binom{n}{k}} \geqq 1-\frac{1}{4k!}$ (since $4\varepsilon k!\,k!(n-k)! < k!\,k!(n-k)! \leqq k^{2k}(n-k)! \leqq n(n-k)! \leqq n!$).

Thus (21) holds. Hence (1) holds.

*Proof of Necessity.* Suppose $k+1$ is a prime number. We must find nonnegative integers satisfying (I)–(XXI). Choose $n$ and $x$ satisfying (I), (II). Let $S = k! - 1$. Then $z$ exists satisfying (XXI), by Wilson's Theorem. Define $w$ by $[(x+1)^n/x^k] = \binom{n}{k} + (w+1)x$. Let $M$ be given by (III), $A$ by (IV), $B$ by (V). Put $C = \psi_A(B)$, and $m = C - B$. Then (VI)–(XIII) may be satisfied by Lemma 3.8. Put $K = \psi_M(n-k+1)$, $L = \psi_{Mx}(k+1)$, $R = \psi_{Mnx}(k+1)$. Then (XV)–(XX) can be satisfied, by Lemma 2.2. It remains only to show that (XIV) holds. Define $\sigma$ and $\beta$ as before. Recall that in the proof of sufficiency it was shown that

$$|\beta - k!| < \tfrac{1}{2}. \tag{22}$$

The only assumptions used in the argument establishing (22) were Lemmas 2.7, 2.8, 3.1–3.7, equations (I)–(XIII), the conditions

(i) $$K = \psi_M(n-k+1), \quad L = \psi_{Mx}(k+1), \quad R = \psi_{Mnx}(k+1),$$

(ii) $$\sigma - (w+1)x = \binom{n}{k} \pm \varepsilon, \quad \text{where} \quad 0 \leqq \varepsilon < \tfrac{1}{4},$$

and the inequalities

(iii) $$0 < \sigma - (w+1)x \quad \text{and} \quad \text{(iv)} \quad \sigma - (w+1)x < x.$$

The condition (XIV) was used only to show (i). Now since $S+1 = k!$, (22) is actually equivalent to (XIV). Hence this same argument may be used to establish (XIV). The conditions (i) have already been satisfied by our choice of $K$, $L$ and $R$. In Case 1 it was shown how to derive condition (ii) using only conditions (iii) and (iv). Therefore we need only derive (iii) and (iv). (iii) is derived as follows using Lemma 3.3 (i) and (3). First

$$M = 16nx(w+2) = 16n\left(\left[\frac{(x+1)^n}{x^k}\right] - \binom{n}{k} + x\right) > 16n\left(\frac{(x+1)^n}{x^k} - \frac{1}{8} - \binom{n}{k} + x\right) > 16n\frac{(x+1)^n}{x^k}. \tag{23}$$

Hence, using (11) and (23), we see that

$$\sigma - (w+1)x = \sigma - \left[\frac{(x+1)^n}{x^k}\right] + \binom{n}{k} > -\frac{n}{M}\frac{(x+1)^n}{x^k} + \binom{n}{k} > -\frac{1}{16} + \binom{n}{k} \geqq -\frac{1}{16} + 5 > 0.$$

Thus (iii) holds. Next we derive (iv). Using Lemma 3.3 (i), (11), (23) and finally (3) we have

$$\sigma - (w+1)x = \sigma - \left[\frac{(x+1)^n}{x^k}\right] + \binom{n}{k} < \sigma - \frac{(x+1)^n}{x^k} + \frac{1}{8} + \binom{n}{k} < \frac{n}{M}\frac{(x+1)^n}{x^k} + \frac{1}{8} + \binom{n}{k}$$

$$< \frac{1}{16} + \frac{1}{8} + \binom{n}{k} < x.$$

Thus (iv) holds. This completes the proof of Theorem 3.9.

The unknowns $M, A, B, C, D, E, F, G, H, I, K, L, R, S$ eliminate from (I)–(XXI) by substitution. This leaves 10 unknowns, $n, x, w, m, z, i, j, p, l, r$, the parameter $k$, six square conditions, one divisibility condition and one inequality. These remaining conditions are definable with one unknown, $y$, by the relation combining theorem of [11]. Thus we obtain a definition $M$, in 11 unknowns. Replacing $k$ by $k+1$ in $M$ we obtain a prime representing polynomial $P = (k+2)(1-M^2)$ in 12 variables.

A direct calculation, based on [11], shows that $M = M_6$ will have degree 148864. However, Yuri Matijasevič has recently worked out a more efficient version of the relation combining theorem. If we suppose that $1 + |\sqrt{A_i}| \leqq V_i$, then in $M_q$ we may replace $W^i$ by $W_i = V_1 \cdot V_2 \cdots V_i$, $(i = 1, 2, \cdots, q)$.

Thus when a square condition arises from a Pell equation, $(\alpha^2-1)\beta^2+1=\square$, we may choose any $V \geqq |\alpha\beta|+|\beta|+2$. Also, if the quantities $B$ and $C$ of [11] are non-negative, as they are here, they need not be squared in $M_q$. These refinements yield a polynomial $M_6$ of degree 13376.

Matijasevič also noticed that our first two square conditions, (I) and (II), may be combined into one square condition

$$U(2k,n)[((2U(2k,n)-1)^2-1)(n+1)^2(x+1)^24+1]=\square. \tag{24}$$

Observe that the first factor of (24) is prime to the second. This gives a polynomial $M_5$ of degree 6848. Hence the degree of the 12 variable polynomial $P$ is 13697. (Recently Yuri Matijasević has announced that he has been able to reduce the number of variables still further, from 12 to 10.)

**4. Functions not formulas for primes.** Many classical theorems are known concerning the impossibility of representing primes with certain sorts of functions. Since these results are negative, they, together with the previous, shed considerable light on the question of the logical complexity of prime representing forms. The oldest result of this type is of course.

THEOREM 4.1. *A polynomial $P(z_1, z_2, \cdots, z_k)$, with complex coefficients, which takes only prime values at nonnegative integers, must be constant.*

*Proof.* It is not difficult to show that the coefficients of an integer valued polynomial must be rational numbers. Let $l$ be any multiple of the denominators of these coefficients of $P$. We are assuming that $P(1,1,\cdots,1)=p$ is a prime. Then, if $n_1, n_2, \cdots, n_k$ are integers, $P(1+n_1lp,\ 1+n_2lp, \cdots, 1+n_klp) \equiv P(1,1,\cdots,1) \bmod p$. Hence, for all $n_1, n_2, \cdots, n_k$, $P(1+n_1lp, 1+n_2lp, \cdots, 1+n_klp)=p$. This implies that $P$ is a polynomial of degree 0. The theorem is proved. (Cf. also [6], p. 18.)

This result was extended to rational functions by R. C. Buck [2]. A rational function is a special case of an *algebraic function*, (cf. [1] for the definition of algebraic function). We now proceed to extend the result to all algebraic functions. We shall need

THEOREM 4.2. *An integer valued algebraic function $W(z_1, z_2, \cdots, z_k)$ is a polynomial.*

*Proof.* We consider first the case of an algebraic function $W=W(z)$ of a single complex variable. Suppose that whenever $z$ is a nonnegative integer, $W(z)$ is also an integer. Let us temporarily restrict $z$ to real values. The point at infinity, $z=\infty$, may or may not be one of the branch points of the function $W$. However, in any case the function has a Puiseux series expansion around the point at infinity [19]. This is the Laurent series expansion of $W=W(t^{-1/h})$, where $h$ is the order of the branch point at infinity and $t=1/z$ is the local parameter. In this Puiseux series

$$W(z)=\sum_{l=0}^{\infty} a_l z^{\alpha-l\delta} \tag{1}$$

$\alpha$ is a fixed rational number and $\delta$ is a fixed positive rational number.

Now, if $W=W(z)$ is integer valued, its first, second, $\cdots$, $r$th differences, defined by $\Delta W(z)=W(z+1)-W(z)$ and $\Delta^{r+1}W(z)=\Delta(\Delta^r W(z))$, are also integer valued algebraic functions of $z$.

It is easy to prove by induction on $r$ that

$$\Delta^r W(n)=\int_0^1\int_0^1\cdots\int_0^1 W^{(r)}(n+x_1+x_2+\cdots+x_r)dx_1dx_2\cdots dx_r \tag{2}$$

where $W^{(r)}(z)$ denotes the $r$th derivative of the function $W=W(z)$. Since $\delta$ is positive, $W^{(r)}(z)\to 0$ as $z\to\infty$, for all sufficiently large $r$. Therefore (2) implies that $\Delta^r W(n)\to 0$ as $n\to\infty$, for all sufficiently large $r$. Since $\Delta^r W(n)$ is integer valued, this implies that $\Delta^r W(n)=0$ for all sufficiently large $n$ and $r$. Since a nonzero algebraic function cannot have infinitely many zeroes [17], it follows that for all $z$ and all sufficiently large $r$, $\Delta^r W(z)=0$. This implies that $W(z)$ is a polynomial in $z$, for all real $z$.

The case of a complex variable $z$ is an immediate consequence of the analytic continuation of the case of the real variable, since an algebraic function is uniquely determined by its values at an infinite number of arguments [17].

The general case, that of an algebraic function of several complex variables, $W(z_1, z_2, \cdots, z_k)$, now follows from the result for one variable. For if we replace all arguments but one with nonnegative integers, we obtain an algebraic function of a single variable

$$W(n_1, \cdots, n_{i-1}, z_i, n_{i+1}, \cdots, n_k). \tag{3}$$

By the preceding proof, (3) is a polynomial in $z_i$. Since $W$ is algebraic, one can find a polynomial $P(z_i)$ (whose degree is independent of the $n_i$'s) such that $|W(n_1, \cdots, n_{i-1}, z_i, n_{i+1}, \cdots, n_k)| < |P(z_i)|$ for all sufficiently large $z_i$. This implies that for some fixed $d_i$

$$\frac{\partial^{d_i} W(n_1, \cdots, n_{i-1}, z_i, n_{i+1}, \cdots, n_k)}{\partial z_i^{d_i}} = 0. \tag{4}$$

Derivatives of algebraic functions are again algebraic. Hence (4) implies that for all $z_1, \cdots, z_k$

$$\frac{\partial^{d_i} W(z_1, \cdots, z_{i-1}, z_i, z_{i+1}, \cdots, z_k)}{\partial z_i^{d_i}} = 0. \tag{5}$$

Now an algebraic function has at most a finite number of branch points. Hence infinitely many $k$-tuples are not branch points of $W$. Consider the $k$ variable Taylor series expansion of $W$ about such a point. Plainly, (5) implies that this series terminates. $W$ is a polynomial of degree $\leqq d_1 + d_2 + \cdots + d_k$. The theorem is proved.

COROLLARY 4.3. *An algebraic function $W(z_1, z_2, \cdots, z_k)$, which takes only prime values at nonnegative integers, is constant.*

This corollary follows from Theorems 4.1 and 4.2. It was proved for $k = 1$ by a different, $p$-adic, method in [18]. Negative results about prime representing *exponential* functions have also been obtained [13] [18] [20]. We shall prove a multivariable result of this type.

THEOREM 4.4. *Suppose $P_i(x_1, \cdots, x_n)$ and $Q_i(x_1, \cdots, x_n) \geqq 0$ are polynomials with integer coefficients and that $a_1, a_2, \cdots, a_m$ are positive integers. Then, if the function*

$$F(x_1, \cdots, x_n) = \sum_{i=1}^{m} P_i(x_1, \cdots, x_n) a_i^{Q_i(x_1, \cdots, x_n)}$$

*takes only prime values at nonnegative integers, it is constant.*

*Proof.* It suffices to prove the theorem for the case of a function of a single variable. (For if $F(x_1, \cdots, x_n)$ is constant in each variable separately, then $F(x_1, \cdots, x_n)$ is constant.) Hence we may suppose $n = 1$. Now if $F(x)$ takes on infinitely many prime values $p$, then we may choose $x_1$ such that $F(x_1) = p$ is relatively prime to each $a_i$ (since the $a_i$'s are nonzero). By Fermat's theorem we then find that $F(x_1 + kp(p-1)) \equiv p \bmod p$, and hence that for every $k$

$$F(x_1 + kp(p-1)) = p. \tag{8}$$

(Cf. Reiner [13] or Hardy and Wright [6], p. 66.) On the other hand, if $F(x)$ takes on only a finite number of prime values, then one of these values is taken on infinitely many times. In either case, for some prime $p$, the equation

$$F(x) = p \tag{9}$$

holds for infinitely many nonnegative integers $x$. It is not difficult to show that equation (9) must then hold identically. This completes the proof of the Theorem.

It is interesting to compare Theorem 4.4 with Theorem 3 stated in the introduction.

*Acknowledgement.* The authors wish to thank Martin Davis, Yuri Matijasevič and Julia Robinson for several helpful suggestions made during the course of this work.

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## MATHEMATICAL NOTES

EDITED BY RICHARD A. BRUALDI

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### AN ANALOG OF SPERNER'S LEMMA

DUANE W. DETEMPLE AND JACK M. ROBERTSON

A basic combinatorial result is

SPERNER'S LEMMA [1]. *Let $T^n = (e_0, \cdots, e_n)$ be an $n$-simplex in $E^n$, with faces $F_i = (e_0, \cdots, \hat{e}_i, \cdots, e_n)$, $i = 0, \cdots, n$, let $c_0, c_1, \cdots, c_n$ be $n+1$ distinct colors and let vertex $e_i$ be assigned color $c_i$. Let $K^n$ be a triangulation of the closure of $T^n$ with vertices of simplices of $K^n$ assigned one of the colors $c_0, \cdots, c_n$ subject to the conditions*:

(i) *$e_i$ is still assigned color $c_i$.*

(ii) *If $V$ lies on face $F_i$, $V$ is not assigned color $c_i$.*

*Then the number of $n$-simplices of $K^n$ with vertices assigned $n+1$ distinct colors is odd.*

The purpose of this note is to give an equivalent result where the attention is on the faces rather than the vertices. The following lemma will be needed.

LEMMA. *Let each of the faces of a simplex $T^n$ be colored with one of the colors $c_0, \cdots, c_n$, and let $N$ be the number of vertices of $T^n$ at which the colors $c_1, \cdots, c_n$ all appear. Then $N$ is even with the single exception that $N = 1$ when each face of $T^n$ has a distinct color.*

*Proof.* If all colors $c_0, c_1, \cdots, c_n$ are used to color $T^n$ then it is clear $N = 1$. If fewer than all $n+1$ colors are used, we may assume these colors are $c_1, \cdots, c_n$ since otherwise $N = 0$.

Now in the case $c_1, \cdots, c_n$ all appear, there is one color, say $c_r$, which appears on two faces, $F_1$ and $F_2$, say. By temporarily recoloring $F_1$ the color $c_0$ we see all colors $c_1, \cdots, c_n$ appear at the vertex $V_1$ opposite $F_1$, even for the original coloration; similarly all colors $c_1, \cdots, c_n$ appear at the vertex $V_2$ opposite $F_2$. If $V_3$ is a third vertex of $T^n$, then $V_3$ is a vertex of both $F_1$ and $F_2$, so color $c_r$ appears more than once at $V_3$. We conclude $N = 2$.

In what follows it will be necessary to examine triangulations $K^n$ of the closure of a simplex $T^n$ by again looking locally at colorings around vertices. For this purpose it will be convenient to assign each vertex $V$ of $K^n$ a neighborhood $U_v$ bounded by a sphere centered at $V$ with radius sufficiently small so that $U_v$ intersects only faces which have $U$ as a vertex. Thus $U_v$ contains disjoint $n$-dimensional sectors of the sphere bounded by (parts of) faces of simplices in $K^n$. If the sector lies inside $T^n$ it will have $n$ bounding faces from a single simplex in $K^n$. Such sectors we shall call *interior* while those outside $T^n$ will be called *exterior.*

THEOREM. *Let $T^n = (e_0, \cdots, e_n)$ be an $n$ simplex in $E^n$, $n \geqq 2$, with faces $F_i = (e_0, \cdots, \hat{e}_i, \cdots, e_n)$, $i = 0, \cdots, n$. Let $F_i$ bear color $c_{i'}$ where $c_{i'}$ is one of the $n+1$ colors $c_0, \cdots, c_n$. Let $K^n$ be a triangulation of the closure of $T^n$ with faces of the simplices of $K^n$ colored one of the $n+1$ colors $c_0, c_1, \cdots, c_n$ subject to the conditions*:

(i) *If a face lies in $F_i$ it bears color $c_{i'}$.*

(ii) *Not all $n+1$ colors appear at any vertex.* (See Comment 2).

*Then*

($a_n$) *Of the $n$-simplices of $K^n$ which share a vertex $V$, an odd number will have colors $c_1, \cdots, c_n$ appearing at $V$ if and only if $V = e_i$ for some $i$ and faces $F_0, F_1, \cdots, \hat{F}_i, \cdots, F_n$ are colored distinctly with colors $c_1, \cdots, c_n$.*

($b_n$) *There are an odd number of simplices of $K^n$ with distinctly colored faces if and only if faces $F_0, \cdots, F_n$ all bear distinct colors.*

*Proof.* We establish ($a_2$) and for $n \geqq 2$ show ($a_n$) implies ($b_n$) while ($a_k$) and ($b_k$) together imply ($a_{k+1}$).

To show ($a_2$) we first note that we need only examine those vertices $V$, where both colors $c_1$ and $c_2$ occur. Counting both interior and exterior sectors, we start on any face and move around $V$ in a fixed direction and add one to the count whenever there is a change of color on successive faces. Because the count is terminated at the same face at which the count was initiated, the number of changes had to be even. The result now follows, because an $e_i$ as described is the only vertex with an exterior sector bounded by both $c_1$ and $c_2$.

To show that ($b_n$) follows from ($a_n$) we count the number $p$ of interior sectors of $K^n$ bounded by colors $c_1, \cdots, c_n$ in two ways. First if $q$ is the number of simplices in $K^n$ with distinctly colored faces, then from the lemma $p = q + \text{even}$. On the other hand, each vertex $V$ of $K^n$ contributes an even number to the count of $p$, except when $V = e_i$ meeting the conditions of ($a_n$) where an odd contribution is made. By the lemma the number of such $e_i$'s is even, unless the faces $F_0, \cdots, F_n$ bear distinct colors.

Assume ($a_k$) and ($b_k$) with $k \geqq 2$. For statement ($a_{k+1}$) we first examine a vertex $V$ lying in the interior of $K^n$ so that all sectors of $U_v$ are interior. If no sectors are bounded by faces colored $c_1, \cdots, c_{k+1}$ the count is zero and we are done. If such a sector exists in $U_v$, its bounding faces intersect the surface of the sphere in a (topological) $k$-simplex whose faces inherit colors $c_1, \cdots, c_{k+1}$. We may thus make our count on the surface of the sphere as follows: count the one assumed sector and remove its corresponding $k$-simplex from the sphere's surface. Statement ($b_k$) may now be applied to the rest of the surface to tally an additional odd number of sectors bounded by faces bearing colors $c_1, \cdots, c_{k+1}$. Thus the total count is even.

Next consider a vertex $V \neq e_0, \cdots, e_{k+1}$ which lies (say) in $\bar{F}_0 \cap \cdots \cap \bar{F}_r$, where $0 \leqq r \leqq k-1$. The procedure of the preceding paragraph can then be applied to the surface of $U_v \cap T^{k+1}$, this surface again being a topological $k$-sphere. To see this we first notice that, just as before, each interior sector at $V$ intersects the surface of $U_v \cap T^{k+1}$ which lies in the interior of $T^{k+1}$ in a $k$-simplex whose faces inherit the colors of the faces of that sector. The remainder of the surface of $U_v \cap T^{k+1}$ is triangulated by $k$-simplices in a natural way by the faces of interior sectors which lie in any face $F_s$, $0 \leqq s \leqq r$. Because there are at most $k$ ($k$-dimensional) faces of $T^{k+1}$ meeting at $V$, we can color the ($k-1$ dimensional) faces of these $k$-simplices on the surface of $U_v \cap T^{k+1}$ subject to the conditions (i) and (ii). Indeed, since these latter simplices involve at most $k$ colors, we conclude, arguing as before, there are an even number of *interior* sectors bearing colors $c_1, \cdots, c_{k+1}$.

The remaining cases $V = e_0, \cdots, e_{k+1}$ are more easily handled. Taking $V = e_i$, for example, we first note that $\partial U_v \cap T^{k+1}$ is a $k$-simplex $T^k$ whose faces we can suppose inherit the colors of $F_0, \cdots, \hat{F}_i, \cdots, F_{k+1}$, namely $c_{0'}, \cdots, \hat{c}_{i'}, \cdots, c_{k+1'}$. Once again the interior sectors at $V$ triangulate the $k$-simplex $T^k$ and the inherited coloration satisfies the conditions (i) and (ii). Thus by ($a_k$) and ($b_k$) there are an odd number of simplices with colors $c_1, \cdots, c_{k+1}$ iff $F_0, \cdots, \hat{F}_i, \cdots, F_{k+1}$ bear the distinct colors $c_1, \cdots, c_{k+1}$, which completes the verification of ($a_{k+1}$).

COMMENT 1. From ($b_n$), if in an $n$-simplex $T^n$ face $F_i$ bears color $c_i$, $i = 0, \cdots, n$, and if faces of simplices of the triangulation $K^n$ are colored according to (i) and (ii), then an odd number of simplices of $K_n$ bear all $n+1$ colors on faces. This statement is actually equivalent to Sperner's Lemma.

To obtain this from Sperner's Lemma assume faces in $K^n$ have been colored as required by (i) and (ii). Now assign each vertex $V$ of $K^n$ a color not appearing on any face at $V$. The hypothesis of Sperner's Lemma is then satisfied. Hence an odd number of simplices in $K^n$ have vertices assigned distinct colors. Our assignment of colors, however, was made so that a simplex of $K^n$ has vertices assigned distinct colors if and only if its faces bear distinct colors.

Conversely, if conditions in the hypothesis of Sperner's Lemma are met, color faces of simplices in $K_n$ falling in $F_i$ color $c_i$ and for all other faces color each a color not assigned to any of its vertices. Such a coloring meets the conditions of the statement above so that an odd number of simplices have faces colored with distinct colors. But again, a simplex in $K^n$ has distinct colors if and only if the same is true of its faces.

COMMENT 2. The following procedure may be followed to produce colorings as required by the theorem. Color all faces lying in $F_i$ color $c_i$, then color faces having $e_0$ as a vertex one of the colors $c_{1'}, \cdots, c_{n'}$. Assume the coloring has been completed at $k$ vertices and proceed to a $(k+1)$st vertex $V'$. If not all $n+1$ colors appear on the faces already colored at $V'$, the coloring is extended to new faces at $V'$ to avoid a missing color there. On the other hand, if all $n+1$ colors already appear at $V'$, we may eliminate a color, say $c_j$, without destroying conditions already met at the previous $k$ vertices as follows. For any face $F$ bearing color $c_j$ at $V'$ examine each of its other $n-1$ vertices which may or may not be one of the $k$ vertices previously handled. At each of these $n-1$ vertices of $F$ only one color was prohibited so that $F$ could bear two colors without violating the coloring conditions at these $n-1$ vertices. Thus coloring $c_j$ may be changed to that second color on face $F$.

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## PRIMES IN CERTAIN ARITHMETIC PROGRESSIONS

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For any integer $m \geqq 2$ we give a simple proof that there are infinitely many primes in the arithmetic progression $m+1$, $2m+1$, $3m+1, \cdots$. The background material needed for the proof is contained in the first two or three chapters of most books on number theory. For other proofs of this result see Dickson [1], Estermann [2], Nagell [5], Rotkiewicz [6] and Sierpinski [7]. Nagell [5, pp. 170–173] also gives an elementary proof that there are infinitely many primes $\equiv -1 \pmod{m}$.

The result that we prove is of course a special case of Dirichlet's theorem that if g.c.d. $(a, m) = 1$, there are infinitely many primes $\equiv a \pmod{m}$. However, Dirichlet's result is much more difficult to prove; even the "elementary" proofs are quite complicated, c.f. Le Veque [4]. Thus the purpose in offering the argument of this paper is that it makes available an easy proof covering not just one or two arithmetic progressions, but infinitely many.

A very thorough discussion of Dirichlet's theorem, including several special cases, is given by Hasse [3].

To prove that there are infinitely many primes $p \equiv 1 \pmod{m}$ it suffices to prove that there is one such prime for any $m > 1$, because it would then follow that there is a prime $p_2 \equiv 1 \pmod{mp}$ and then a prime $p_3 \equiv 1 \pmod{mpp_2}$, and so on, by induction. Also of course we have $p \equiv p_2 \equiv p_3 \equiv \cdots \equiv 1 \pmod{m}$.

Suppose that there is no prime $p \equiv 1 \pmod{m}$. In the first part of our discussion $t$ will denote an arbitrarily chosen positive integer, and $q$ a fixed prime divisor of $(tm)^m - 1$. Note that $q \nmid tm$. Letting $h$ denote the least positive integer such that $q \mid (tm)^h - 1$, we know from elementary number theory (or group theory) that $h \mid m$, and we prove that $h < m$. For if $h = m$ then since $q \mid (tm)^{q-1} - 1$ by Fermat's theorem, we get $m \mid q - 1$ or $q \equiv 1 \pmod{m}$, contrary to the assumption that there is no such prime. Thus $h < m$, and we write $m = hc$ with $c > 1$.

LEMMA. *Let $q^r$ be the highest power of the prime $q$ that divides $(tm)^m - 1$, i.e., $q^r \| (tm)^m - 1$ in the usual notation. If $s$ is any positive integer such that $h \mid s$ and $s \mid m$ then $q^r \| (tm)^s - 1$.*

First we prove that $q^r \mid (tm)^h - 1$. If this were not so, then $r > 1$ and the second factor on the right side of

$$(tm)^m - 1 = [(tm)^h - 1][(tm)^{ch-h} + (tm)^{ch-2h} + \cdots + (tm)^h + 1] \tag{1}$$

is divisible by $q$. But $(tm)^h \equiv 1 \pmod q$, so this second factor is congruent to $1+1+\cdots+1+1 \equiv c \pmod q$. Hence $q \mid c$ and so $q \mid m$, whereas $q \nmid m$ as already noted. Second, given any $s$ as in the lemma, we note that each of $(tm)^h - 1$, $(tm)^s - 1$, $(tm)^m - 1$ divides the following one because of factorings like (1). Thus $q^r \mid (tm)^h - 1$ follows from $q^r \mid (tm)^h - 1$ and $q^r \| (tm)^m - 1$, and the lemma is proved.

Let the canonical factoring of $m$ into powers of primes be

$$m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}.$$

For $i = 1, 2, \cdots, k$ let $S_i$ be the set of coefficients of $x^i$ in the expansion of $m(1+x/p_1)\cdot(1+x/p_2)\cdots(1+x/p_k)$ so that

$$m\prod_{i=1}^{k}\left(1+\frac{x}{p_i}\right) = m + x\sum_{s\in S_1} s + x^2\sum_{s\in S_2} s + x^3\sum_{s\in S_3} s + \cdots + x^k\sum_{s\in S_k} s. \tag{2}$$

For example we see that

$$S_1 = \{m/p_1, m/p_2, \cdots, m/p_k\}, \quad S_2 = \{m/p_1p_2, m/p_1p_3, \cdots\},$$

and $S_i$ has $\binom{k}{i}$ elements. Then define

$$T = S_1 \cup S_3 \cup S_5 \cup \cdots, \qquad V = S_2 \cup S_4 \cup S_6 \cup \cdots,$$

$$B_t = \prod_{s\in T}[(tm)^s - 1] \Big/ \prod_{s\in V}[(tm)^s - 1]. \tag{3}$$

Now the prime $q$ divides $(tm)^s - 1$ if and only if $h \mid s$, and since $h < m$ we see that $h$ divides at least one of the elements of $S_1$. Let $n$ be the precise number of elements of $S_1$ divisible by $h$, so that $1 \leqq n \leqq k$.

We now prove that for $i = 1, 2, 3, 4, \cdots, k$ exactly $\binom{n}{i}$ of the elements of $S_i$ are divisible by $h$, where $\binom{n}{i}$ is interpreted as zero for values of $i > n$. To prove this we note first that the elements of $S_i$ are precisely the greatest common divisors of the elements of the $i$-subsets of $S_1$. For example, $m/p_1p_2p_3$ is the g.c.d. of $m/p_1$, $m/p_2$ and $m/p_3$. Without loss of generality we may presume, by interchanging subscripts, that the $n$ elements of $S_1$ divisible by $h$ are $m/p_1$, $m/p_2, \cdots, m/p_n$. The greatest common divisors of the elements of the $i$-subsets of these $n$ elements are precisely those elements of $S_i$ that are divisible by $h$, in number $\binom{n}{i}$ as claimed. By the lemma we know that $q^r \| (tm)^s - 1$ if and only if $h \mid s$, and hence the exact power of $q$ dividing $B_t$ is

$$r\left\{\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots\right\} - r\left\{\binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots\right\}.$$

This equals $r$ because it amounts to the binomial expansion of $r - r\,(1-1)^n$, and so $q^r \| B_t$. But also $q^r \| (tm)^m - 1$, and since any prime factor of any expression $(tm)^s - 1$ in $B_t$ in (3) is also a prime factor of $(tm)^m - 1$, we have proved that $(tm)^m - 1$ and $B_t$ are divisible by the same prime powers. Hence $(tm)^m - 1 = B_t$, or

$$[(tm)^m - 1]\prod_{s\in V}[(tm)^s - 1] = \prod_{s\in T}[(tm)^s - 1].$$

This holds for all positive integers $t$, and so the polynomial equation

$$(x^m - 1)\prod_{s\in V}(x^s - 1) = \prod_{s\in T}(x^s - 1) \tag{4}$$

has infinitely many solutions $x = m, 2m, 3m, \cdots$. This is impossible unless the polynomials on the two sides of (4) are identically equal, which is not so because the polynomial on the left side is of higher degree. To see that this is the case, we look at the difference of the two degrees,

$$m + \sum_{s\in V} s - \sum_{s\in T} s = m\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right) > 0,$$

where the equality here comes by setting $x = -1$ in (2).

The above proof establishes a little more than the existence of a prime $q \equiv 1 \pmod m$ for any $m > 2$, namely that there is such a prime satisfying $q < m^{3m}$. This can be seen as follows. To get the contradiction from equation (4) we need to know, not that it has infinitely many solutions, but only that it has more distinct solutions than its degree. The degree of equation (4) is the sum of a subset of the positive divisors of $m$, and so is at most the sum of the natural numbers from 1 to $m$, which is exceeded by $m^2$. Hence we get a contradiction from (4) by knowing that it has solutions $x = tm$ for $t = 1, 2, \cdots, m^2$. Thus we have proved that there is a prime $q \equiv 1 \pmod m$ among the factors of at least one of the integers $\{(tm)^m - 1;\ t = 1, 2, \cdots, m^2\}$ and consequently $q < m^{3m}$.

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## AN INEQUALITY IN TWO MONOTONIC FUNCTIONS

S. K. Stein

G. H. Hardy, J. E. Littlewood, and G. Polya ([1], p. 163) note that, "velocity averaged by time is less than velocity averaged by distance." More formally, they are asserting that the following inequality holds,

$$\frac{\int_a^b v(t)\,dt}{\int_a^b dt} \leqq \frac{\int_{s_0}^{s_1} v(s)\,ds}{\int_{s_0}^{s_1} ds},$$

where $s$ refers to distance and $t$ to time. This inequality may be rewritten as follows:

$$\frac{\int_a^b v(t)\,dt}{\int_a^b dt} \leqq \frac{\int_a^b v^2(t)\,dt}{\int_a^b v(t)dt}.$$

Clearing denominators results in the claim that $(\int_a^b v(t)\,dt)^2 \leqq \int_a^b v^2(t)\,dt \int_a^b dt$. As they point out, this is a special case of Schwarz's inequality,

$$\left[\int_a^b f(t)g(t)\,dt\right]^2 \leqq \int_a^b f^2(t)\,dt \int_a^b g^2(t)\,dt,$$

where $f(t) = v(t)$ and $g(t) \equiv 1$.

Moreover, on purely physical grounds, the assertion is plausible. After all, a high speed held for a short time would be weighted less in a 'velocity averaged by time' than in 'velocity averaged by distance'. This observation suggests that "*velocity squared* averaged by time is less that *velocity squared* averaged by distance." This turns out to be true, but not as a consequence of Schwarz's inequality. Both assertions are special cases of the following inequality that concerns two monotonic functions.

THEOREM. *Let $f$ and $g$ be monotonically increasing nonnegative functions defined on $[0,\infty)$. Let $v$ and $h$ be nonnegative measureable functions. Let $a < b$ be real numbers. Then*

$$\int_a^b f(v(t))g(v(t))h(t)\,dt \cdot \int_a^b h(t)\,dt \geqq \int_a^b f(v(t))h(t)\,dt \cdot \int_a^b g(v(t))h(t)\,dt. \tag{1}$$

*Proof.* (The function $h$ simply puts a measure on $[a, b]$, $dm = h(t)dt$.) Observe that (1) is linear in $f$. Since $f$ is monotonically increasing it can be approximated by a sum of a finite number of step functions that have a single jump. Thus it suffices to establish (1) in the special case that $f(x) = 0$, if $x \leqq v_1$ and $f(x) = 1$ if $x > v_1$. Let

$$A = \{t \mid v(t) \leqq v_1 \text{ and } a \leqq t \leqq b\}$$

$$B = \{t \mid v(t) > v_1 \text{ and } a \leqq t \leqq b\}.$$

Then (1) becomes for the $f$ in question

$$\int_B g(v(t))dm \int_a^b dm \geqq \int_B dm \int_a^b g(v(t))dm. \tag{2}$$

Similarly, to establish (2) it is enough to consider $g$ to be a single-jump function: $g(x) = 0$ if $x \leqq v_2$ and $g(x) = 1$ if $x > v_2$. Let

$$C = \{t \mid v(t) \leqq v_2 \text{ and } a \leqq t \leqq b\}$$

$$D = \{t \mid v(t) > v_2 \text{ and } a \leqq t \leqq b\}.$$

Then (2) becomes an inequality expressible in terms of the sets $B$, $D$, and $[a, b]$.

There are two cases to consider, $v_2 \leqq v_1$ and $v_1 \leqq v_2$. In the first case (2) becomes

$$\int_B dm \int_a^b dm \geqq \int_B dm \int_D dm, \tag{3}$$

which is clearly valid. The second case follows from the first by interchanging $f$ and $g$. This establishes the theorem.

In particular, when $f(x) = x^2$, $g(x) = x$, and $h(x) \equiv 1$, (1) reduces, after division, to

$$\frac{\int_a^b v^2(t)v(t)dt}{\int_a^b v(t)dt} \geqq \frac{\int_a^b v^2(t)dt}{\int_a^b dt},$$

which implies that "velocity squared averaged by distance is greater than or equal to velocity squared averaged by time." The function $x^2$ may be replaced by any nonnegative monotonically increasing function.

Moreover, as the referee pointed out, Schwarz's inequality can be derived from (1) by setting $f(x) = x = g(x)$, $h = G^2$, and $v = F/G$. With this substitution, (1) becomes

$$\int_a^b F^2\,dt \int_a^b G^2\,dt \geqq \int_a^b FG\,dt \int_a^b FG\,dt,$$

which is Schwarz's inequality.

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## RESEARCH PROBLEMS

EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)*

### FIXED-POINT PROBLEMS FOR DISK-LIKE CONTINUA

CHARLES L. HAGOPIAN

Does every disk-like continuum have the fixed-point property? This open question [**12**, p. 253] is more general than two well-known outstanding problems discussed in R. H. Bing's expository article [**2**, Questions 2 and 3]. An affirmative answer to this question would imply that all tree-like continua and all nonseparating plane continua have the fixed-point property. Here we present a theorem that may be helpful to those who are looking for a disk-like continuum that does not have the fixed-point property.

We call a nondegenerate metric space that is both compact and connected a **continuum.** A continuum $M$ has the **fixed-point property** if for each continuous function $f$ of $M$ into itself, there is a point $p$ of $M$ such that $f(p) = p$. A continuous function $f$ of a continuum $M$ is called an **$\varepsilon$-map** if for each point $q$ of $f[M]$, the diameter of $f^{-1}(q)$ is less than $\varepsilon$. A continuum $M$ is **disk-like** if for each positive number $\varepsilon$, there exists an $\varepsilon$-map of $M$ onto a disk.

In this article $E^n$ is Euclidean $n$-space and $S^2$ is the 2-sphere

$$\{(x, y, z) \in E^3 : x^2 + y^2 + z^2 = 1\}.$$

The boundary of a given set $Z$ is denoted by $\operatorname{Bd} Z$. We define $T$ to be the continuum in $E^3$ that is

the union of the disk

$$D = \{(x, y, z) \in E^3 : x^2 + y^2 \leqq 1 \text{ and } z = 0\}$$

and the arc $A = \{(x, y, z) \in E^3 : x = y = 0 \text{ and } 0 \leqq z \leqq 1\}$.

THEOREM. *If a continuum $M$ has a subcontinuum that is homeomorphic to $T$, then $M$ is not disk-like.*

*Proof.* Showing that $T$ cannot be 1/2-mapped into $E^2$ will suffice. To accomplish this, we assume the contrary. Let $f$ be a continuous function of $T$ into $E^2$ such that for each point $q$ of $f[T]$, the diameter of $f^{-1}(q)$ is less than 1/2.

The complementary domain $G$ of $f[\mathrm{Bd}\, D]$ in $E^2$ that contains $f(0,0,0)$ is a subset of $f[D]$. To see this, let $k$ be a continuous function of $D$ onto $S^2$ such that

(1) $k[\mathrm{Bd}\, D]$ is the south pole,

(2) $k$ restricted to the interior of $D$ is a homeomorphism,

(3) the image under $k$ of the disk $B = \{(x, y, z) \in E^3 : x^2 + y^2 \leqq 1/4 \text{ and } z = 0\}$ is the closed upper hemisphere of $S^2$, and

(4) for each pair $p, -p$ of antipodal points of $S^2$, the distance between $k^{-1}(p)$ and $k^{-1}(-p)$ in $D$ is at least 1/2. Note that if $p$ and $-p$ are antipodal points of $S^2$, then $k[B] \cap \{p, -p\} \neq \emptyset$.

Now suppose a point $w$ of $G$ does not belong to $f[D]$. Let $d$ be the projection map of $E^2$ onto the quotient space $E^2/R$ where $R$ relates distinct points $z$ and $z'$ of $E^2$ if and only if $\{z, z'\} \subset E^2 - G$. Since $E^2/R$ is either a 2-sphere or a plane **[13]**, we can assume $E^2/R - \{w\}$ is lying in $E^2$. Note that $h = dfk^{-1}$ is a continuous function of $S^2$ into $E^2$. Also note that since $B$ contains $(0,0,0)$ and $f[B]$ is a connected set that does not meet $f[\mathrm{Bd}\, D]$, the domain $G$ contains $f[B]$. According to the Borsuk-Ulam Theorem **[11,** p. 477], there exist antipodal points $p$ and $-p$ of $S^2$ such that $h(p) = h(-p)$. But this contradicts the assumption that $f$ is a 1/2-map. Hence $G$ lies in $f[D]$.

Since $f[A]$ is a connected set in $E^2$ that misses $f[\mathrm{Bd}\, D]$, the point $f(0,0,1)$ belongs to $G$. But this implies that $f^{-1}(f(0,0,1))$ meets $D$, which is impossible. This contradiction completes our proof.

In Theorems 19, 20, and 21 of **[2],** Bing referred to three cube-like continua that do not have the fixed-point property. Each of these continua contains a copy of $T$ and therefore fails to be disk-like.

In **[16],** G. S. Young defined for each positive integer $n$, a continuum $T_n$ that is the union of an $n$-cell $C$ and an arc $I$ such that $C \cap I$ is a point that is an endpoint of $I$ and an interior point of $C$. Young **[16]** proved that for each $n$, the space $E^{n+1}$ does not contain uncountably many mutually exclusive homeomorphs of $T_n$. This result generalizes R. L. Moore's theorem **[14]** that the plane does not contain uncountably many mutually exclusive simple triods. It follows from Young's result that $E^3$ does not contain uncountably many mutually exclusive copies of $T$.

O. H. Hamilton **[9]** proved that every arc-like continuum has the fixed-point property. In **[4],** E. Dyer extended Hamilton's theorem to all topological products of arc-like continua. Note that the product of two continua is disk-like when each factor is arc-like. Perhaps Dyer's result can be applied to the following unsolved problem.

*Question* 1. If the topological product of two continua is disk-like, then must it have the fixed-point property? The answer is "yes" when each factor is $\lambda$ connected (i.e., every pair of points in the factor can be joined by a hereditarily decomposable continuum in the factor) **[7].**

*Question* 2. Does every disk-like arcwise-connected continuum have the fixed-point property? The fixed-point property has been established for every tree-like arcwise-connected continuum **[3]** and every nonseparating arcwise-connected plane continuum **[5].** Other related results are given in **[1], [6], [8], [10], [15],** and **[17].**

**Added in proof:** A different proof of the theorem above involving the disk-with-a-sticker $T$ is given in Ralph Bennett's paper, *Locally connected 2-cell and 2-sphere-like continua*, Proc. Amer. Math. Soc. 17 (1966), 674–681. In the same article, Bennett showed that every locally connected disk-like continuum is planar and therefore has the fixed-point property.

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## WHAT IF $2^c$ AND $3^c$ ARE INTEGERS?

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*Let $c$ be a real number such that $2^c$ and $3^c$ are integers; must $c$ be an integer?*

Results and comments.

1. In [2] it is proved that if $n^c$ is an integer for $n = 1, 2, 3, \cdots$, then $c$ is an integer.

2. If $2^c$, $3^c$ and $5^c$ are integers, $c$ must be an integer. This is a special case of [1, p. 9, Corollary 1].

Now assume that $c$ is not an integer.

3. $c > 1$. For if $0 < c < 1$, $1 < 2^c < 2$.

4. $c > 2$. For if $1 < c < 2$, $2 < 2^c < 4$ hence $2^c = 3$. A calculation shows that $3^c$ is not an integer.

5. Similar computations showing that $c$ must be somewhat larger depend only on the reader's perseverance.

6. $c$ must be transcendental. This follows from the Gelfond-Schneider theorem, see [1] p. 19.

*Editorial note*: I have seen this problem attributed to Gleason; no doubt it could easily occur to people independently. At one time it might have been dismissed as hopeless, but since the work of Baker, this may not now be a fair appraisal—R. K. G.

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## CLASSROOM NOTES

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### A SIMPLE PROOF OF MEAN CONVERGENCE IN THE LAW OF LARGE NUMBERS

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Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables and let $S_n = (X_1 + \cdots + X_n)$. Khintchine's form of the weak law of large numbers asserts that

$$E(X_1) = 0 \Rightarrow (S_n/n) \to 0 \text{ in probability.} \tag{1}$$

In this note we present a rather simple proof of a stronger result, namely,

$$E(X_1) = 0 \Rightarrow E(|S_n|) = o(n). \tag{2}$$

The Markov inequality

$$P(|Y| \geq c) \leq E(|Y|)/c$$

shows that (2) is stronger than (1). An elementary proof of (1) can be found, for example, on pp. 246–248 of Feller [1]. Our proof uses a different 'truncation' and treats the 'tails' in a simple manner. We use the standard fact that $(x_1 + \cdots + x_n) = o(n)$ whenever $x_n \to 0$.

*Proof of* (2). Let $F$ denote the common distribution function of the $X_n$'s. Define the random variables $Y_n$ and $Z_n$ as follows:

$$Y_n = X_n \quad \text{and} \quad Z_n = 0 \quad \text{if} \quad |X_n| \leq \sqrt{n},$$

$$Y_n = 0 \quad \text{and} \quad Z_n = X_n \quad \text{if} \quad |X_n| > \sqrt{n}.$$

Then $X_n = Y_n + Z_n$ and $S_n = T_n + U_n$, where $T_n = \Sigma_1^n Y_k$ and $U_n = \Sigma_1^n Z_k$. Therefore (2) will be proved as soon as we show that $E(|T_n|) = o(n)$ and $E(|U_n|) = o(n)$.

Since $E(|X_1|) < \infty$, it follows that

$$E(|Z_n|) = \int_{|x| > \sqrt{n}} |x| \, dF(x) \to 0.$$

Therefore $E(|U_n|) \leq \Sigma_1^n E(|Z_k|) = o(n)$.

If we use the relation

$$(E|T_n|)^2 \leq E(T_n^2) = \mathrm{Var}(T_n) + [E(T_n)]^2$$

we could assert that $E(|T_n|) = o(n)$ as soon as we are able to prove that

$$E(T_n) = o(n) \quad \text{and} \quad \mathrm{Var}(T_n) = o(n^2). \tag{3}$$

We now proceed to prove (3). Observe that $E(Y_n) \to E(X_1) = 0$. Therefore $E(T_n) = \Sigma_1^n E(Y_k) = o(n)$.

Let $A$ denote $E(|X_1|)$. Then

$$\mathrm{Var}(Y_n) \leq E(Y_n^2) = \int_{|x| \leq \sqrt{n}} x^2 \, dF(x) \leq \sqrt{n} \int_{|x| \leq \sqrt{n}} |x| \, dF(x) \leq A\sqrt{n}.$$

Therefore $\mathrm{Var}(T_n) = \Sigma_1^n \mathrm{Var}(Y_k) = O(n^{3/2}) = o(n^2)$. Thus both the assertions of (3) are established and the proof is complete.

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## A CHARACTERIZATION OF THE $n$-DIMENSIONAL PARALLELOTOPE

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The Brunn-Minkowski theorem [3, p. 187] is one of the beautiful theorems that can be included in an introductory geometry course. Unfortunately, applications of this theorem are rarely encountered by the undergraduate student of mathematics. In this paper we apply this powerful theorem to obtain a characterization of the $n$-dimensional parallelotope. Characterizations of the parallelotope are of interest similar to characterizations of the sphere, cf. [1, 2, 4, 5].

The setting for this paper is Euclidean $n$-space $R^n$. Compact convex sets with nonempty interiors will be called convex bodies. A non-empty intersection of a compact convex set $K$ with an $i$-dimensional flat in $R^n$ will be called an $i$-dimensional slice (or $i$-slice) of $K$. A direction in $R^n$ is simply a unit vector in $R^n$ and an $(n-1)$-slice $k$ is said to be in a direction $u$ if all the vectors in $k$ are perpendicular to $u$.

We shall prove the following.

THEOREM. *A convex body $K$ is an $n$-dimensional parallelotope if and only if there exist $n-1$ linearly independent directions for each of which all the $(n-1)$-slices of $K$ have the same $(n-1)$-dimensional volume.*

As an immediate corollary to this theorem we have:

COROLLARY. *If a convex body has $n-1$ linearly independent directions in each of which all the $(n-1)$-slices have the same $(n-1)$-dimensional volume, then there exists another direction, linearly independent from the original $n-1$ directions, in which all the $(n-1)$-slices have equal $(n-1)$-dimensional volumes.*

By way of notation, we shall use $V_i$ to denote the volume in Euclidean $i$-dimensional space. If $g$ is either a vector or a set of vectors in $R^n$ and $E$ is an $i$-dimensional hyperplane in $R^n$, then $g|E$ will be used to denote the projection of $g$ onto $E$. If $A$ and $B$ are subsets of $R^n$, then the smallest compact convex set containing $A$ and $B$ will be called the convex hull of $A$ and $B$ and will be denoted by $\mathrm{conv}(A, B)$.

The $n$-dimensional parallelotopes can be defined recursively as follows. The 1-dimensional parallelotopes are defined to be the convex bodies in $R^1$. After defining the $(n-1)$-dimensional parallelotopes, the $n$-dimensional parallelotope is defined to be a convex body in $R^n$ which is the convex hull of an $(n-1)$-dimensional parallelotope and one of its translates.

The Brunn-Minkowski theorem [3, p. 187] states that if $A$ and $B$ are compact convex sets in $R^n$ with $V_n(A)$ and $V_n(B)$ positive, then

$$V_n(\lambda A + (1-\lambda)B)^{1/n} \geqq \lambda V_n(A)^{1/n} + (1-\lambda)V_n(B)^{1/n} \qquad [0 \leqq \lambda \leqq 1]$$

with equality for all $\lambda$ between 0 and 1 if and only if $A$ and $B$ are homothetic.

The following version of the Brunn-Minkowski theorem is a direct consequence of the previously stated version.

LEMMA 1. *If $E$ and $E'$ are parallel $(n-1)$-dimensional hyperplanes in $R^n$ and $K$ and $K'$ are compact convex sets such that $K \subset E$ and $K' \subset E'$ with $V_{n-1}(K)$ and $V_{n-1}(K')$ positive, then*

$$V_{n-1}(\lambda K + (1-\lambda)K')^{1/(n-1)} \geqq \lambda V_{n-1}(K)^{1/(n-1)} + (1-\lambda)V_{n-1}(K')^{1/(n-1)} \qquad [0 \leqq \lambda \leqq 1]$$

*with equality for all $\lambda$ between 0 and 1 if and only if $K$ and $K'$ are homothetic.*

LEMMA 2. *If $K$ is a convex body in $R^n$ such that all the $(n-1)$-slices of $K$ in some direction $u$ have the same $(n-1)$-dimensional volume then all these $(n-1)$-slices of $K$ are translates of each other. In addition, if $H$ and $H'$ are the supporting hyperplanes of $K$ perpendicular to $u$, then*

$$K = \operatorname{conv}(K \cap H, K \cap H').$$

*Proof.* Since $K$ is a convex body in $R^n$ (and, thus, has positive $n$-dimensional volume) it follows from Fubini's theorem [3, p. 165] that the $(n-1)$-dimensional volume of the slices in question must be positive.

Let $E$ and $E'$ be $(n-1)$-dimensional hyperplanes perpendicular to $u$ such that $K$ has points in common with both $E$ and $E'$. Let $k = E \cap K$ and $k' = E' \cap K$. Since $V_{n-1}(k) = V_{n-1}(k')$, we can use Lemma 1 to obtain

$$V_{n-1}(\lambda k + (1-\lambda)k') \geqq V_{n-1}(k) \qquad [0 \leqq \lambda \leqq 1] \tag{1}$$

with equality for all $\lambda$ between 0 and 1 if and only if $k$ and $k'$ are homothetic. From the convexity of $K$ it follows that

$$\lambda k + (1-\lambda)k' \subset (\lambda E + (1-\lambda)E') \cap K. \tag{2}$$

Since $(\lambda E + (1-\lambda)E') \cap K$ is an $(n-1)$-dimensional slice of $K$ perpendicular to $u$ it follows that

$$V_{n-1}((\lambda E + (1-\lambda)E') \cap K) = V_{n-1}(k) \qquad [0 \leqq \lambda \leqq 1]. \tag{3}$$

We combine (2) and (3) to obtain

$$V_{n-1}(k) \geqq V_{n-1}(\lambda k + (1-\lambda)k') \qquad [0 \leqq \lambda \leqq 1].$$

Hence, there is equality in (1) for all $\lambda$ between 0 and 1. It follows that $k$ and $k'$ are homothetic and since they have the same $(n-1)$-dimensional volume, they must be translates of each other.

Let $h = H \cap K$ and $h' = H' \cap K$. Consider an arbitrary $x \in K$. Since $x$ must lie between $H$ and $H'$, there exists an $\alpha$ such that $0 \leqq \alpha \leqq 1$ and $x$ is in $\alpha H + (1-\alpha)H'$. From the convexity of $K$ it follows that

$$\alpha h + (1-\alpha)h' \subset (\alpha H + (1-\alpha)H') \cap K. \tag{4}$$

Since $V_{n-1}(h) = V_{n-1}(h')$, we can use Lemma 1 to obtain

$$V_{n-1}(\alpha h + (1-\alpha)h') \geqq V_{n-1}(h). \tag{5}$$

Since $V_{n-1}(h) = V_{n-1}((\alpha H + (1-\alpha)H') \cap K)$, we can rewrite (5) as

$$V_{n-1}(\alpha h + (1-\alpha)h') \geqq V_{n-1}((\alpha H + (1-\alpha)H') \cap K). \tag{6}$$

Since $\alpha h + (1-\alpha)h'$ and $(\alpha H + (1-\alpha)H') \cap K$ are convex bodies in the $(n-1)$-dimensional hyperplane $\alpha H + (1-\alpha)H'$, we can combine (4) and (6) to yield

$$\alpha h + (1-\alpha)h' = (\alpha H + (1-\alpha)H') \cap K.$$

Thus, we conclude that $x$ is in $\alpha h + (1-\alpha)h'$ and, hence, that $K \subset \operatorname{conv}(h, h')$.

LEMMA 3. *If $K$ is a plane convex body such that for some direction $u$ all the 1-slices of $K$ perpendicular to $u$ have the same length, then $K$ is a parallelogram.*

*Proof.* Let $H$ and $H'$ be the lines of support of $K$ perpendicular to $u$. Hence, $H \cap K$ and $H' \cap K$ are parallel lines of the same length. From Lemma 2 we have $K = \text{conv}(H \cap K, H' \cap K)$ which proves the lemma.

We now proceed to establish the theorem stated in the introduction.

THEOREM. *A convex body $K$ is an $n$-dimensional parallelotope if and only if there exist $n-1$ linearly independent directions $u_1, \cdots, u_{n-1}$ and $n-1$ constants $c_1, \cdots, c_{n-1}$ such that all the $(n-1)$-dimensional slices of $K$ perpendicular to $u_i$ have $(n-1)$-dimensional volume $c_i$.*

*Proof.* The necessity of the conditions follows easily from the definition of the $n$-dimensional parallelotope. We prove sufficiency by induction on the dimension $n$ of the Euclidean space in which the convex body $K$ lies. Lemma 3 proves sufficiency for $n = 2$. We, thus, assume sufficiency in Euclidean $(n-1)$-space and shall prove it in Euclidean $n$-space.

We first note that since $K$ is a convex body and, hence, has positive $n$-dimensional volume, it follows from Fubini's theorem that all the $c_i$ must be positive.

Let $H_i$ and $H_i'$ denote the supporting hyperplanes of $K$ in the direction $u_i$. Let $K_i = H_i \cap K$ and $K_i' = H_i' \cap K$. From Lemma 2 we know that $K_1'$ is a translate of $K_1$ and that $K = \text{conv}(K_1, K_1')$.

Suppose $i$ is such that $1 < i \leqq n-1$. From Lemma 2 we know that there exists an $a_i \in R^n$ such that $K_i' = a_i + K_i$ and

$$K = \cup\{\lambda K_i + (1-\lambda)K_i' \mid 0 \leqq \lambda \leqq 1\}. \tag{1}$$

If $K_i$ lies in the interior of the half space determined by $H_1$, then we must have $K \cap H_1 \subset K_i' \cap H_1$. This follows from (1) using the fact [**6**, p. 9] that if one of two points in a convex set lies in the interior of the set, then, except for (possibly) one of the end points, the line segment joining the points must lie in the interior of the set. However, $K_i' \cap H_1$ lies in the $(n-2)$-dimensional hyperplane $H_i' \cap H_1$ and, hence, cannot contain $K \cap H_1$ which has positive $(n-1)$-dimensional volume $c_1$. Thus, $K_i$ must intersect $H_1$. Similarly, $K_i'$ must intersect $H_1$. The same argument shows that both $K_i$ and $K_i'$ intersect $H_1'$.

This shows that both $K_i$ and $K_i'$ have $H_1$ and $H_1'$ as supporting hyperplanes in the direction $u_1$. Hence, the vector $a_i$ can have no component in the direction $u_1$ and, hence, lies entirely in $H_1$. It follows that $K_i \cap H_1$ and $K_i' \cap H_1$ are translates of each other in $H_1$ and thus

$$V_{n-2}(K_1 \cap H_i) = V_{n-2}(K_1 \cap H_i').$$

Suppose that $E_i$ is an $(n-1)$-dimensional hyperplane that lies between $H_i$ and $H_i'$ and is perpendicular to $u_i$. It follows that there exists an $\alpha$ between 0 and 1 such that

$$E_i = \alpha H_i + (1-\alpha)H_i'.$$

From (1) we have

$$E_i \cap K = \alpha K_i + (1-\alpha)K_i'.$$

Since both $K_i$ and $K_i'$ have points in common with both $H_1$ and $H_1'$, then $E_i \cap K$ also has points in common with both $H_1$ and $H_1'$. From Lemma 2 we know that $E_i \cap K$ is a translate of $K_i$. Now, arguing as before, we can show that $E_i \cap K$ is a translate of $K_i$ by a vector lying entirely in $H_1$. Thus, $K_i \cap H_1$ is a translate of $E_i \cap K \cap H_1$ in $H_1$ from which it follows that

$$V_{n-2}(K_1 \cap H_i) = V_{n-2}(K_1 \cap E_i).$$

This shows that all the $(n-2)$-dimensional slices of $K_1$ (in $H_1$) perpendicular to $u_i \mid H_1$ have the same $(n-2)$-dimensional volume $V_{n-2}(K_1 \cap H_i)$.

Since $u_1, \cdots, u_{n-1}$ are linearly independent and $H_1$ is an $(n-1)$-dimensional hyperplane perpendicular to $u_1$, it follows that $u_2|H_1, \cdots, u_{n-1}|H_1$ are linearly independent. Since $V_{n-1}(K_1) = c_1 > 0$, $K_1$ is a convex body in the $(n-1)$-dimensional Euclidean space $H_1$. Applying the inductive hypothesis to the convex body $K_1$ in $H_1$ shows that $K_1$ is an $(n-1)$-dimensional parallelotope.

Since the convex body $K$ is the convex hull of $K_1$ and $K_1'$ where $K_1'$ is a translation of the $(n-1)$-dimensional parallelotope $K_1$, we conclude that $K$ is an $n$-dimensional parallelotope.

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## CORRECTIONS TO "ELEMENTARY APPROXIMATIONS TO THE AREA OF $N$-DIMENSIONAL ELLIPSOIDS"

(This MONTHLY, 78 (1971) 280–283)

M. S. KLAMKIN

On p. 282, the region of integration for the first double integral for $S$ should be $x^2/a^2 + y^2/b^2 \leqq 1$, $x, y \geqq 0$; the lower inner limit 0 in the next three double integrals should be $\pi/2 - \alpha$; and finally the $a^2$ should be deleted from the last double integral.

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## MATHEMATICAL EDUCATION

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Your attention is drawn to the existence of the new quarterly "Journal of Personalized Instruction," 29 Loyola Hall, Georgetown University, Washington, D.C. 20057. Our continuing backlog prompts us to ask potential contributors of articles on personalized systems of instruction to keep the above journal in mind as a possible publishing alternative. — *The Editors.*

## AN ALGORITHM FOR SCORING BY ATTRIBUTES

MARGARET W. MAXFIELD

**Scoring by clustering.** Departments of mathematics and statistics commonly use the same traditional scoring algorithm used by instructors less skilled in criticizing models:

1. Each item $i$ on the examination is given a point value $v(i)$. The value system is usually described on the examination, to guide students in allocating their time.

2. During grading, each item $i$ on each submitted paper $e$ is assigned a numerical score $p(e,i)v(i)$, where $p(e,i)$, in the interval $[0,1]$, is the relative adequacy of the response. In an effort to achieve fairness, readers often read one item at a time throughout all papers. Different readers may choose $p(e,i)$ according to different criteria: portion attempted, portion free of mistakes, relative success compared with other students' responses, and so on.

3. The sum $\Sigma p(e,i)v(i)$ over all items $i$ is assigned to paper $e$ as its numerical score $n(e)$.

4. A linear scatter diagram of the scores $n(e)$ is prepared, and letter grades are assigned to clusters in the scatter.

**Relative grades.** Scoring by clustering represents a defensive strategy. Anticipating complaints in case papers of comparable merit receive different letter grades, graders assign cut-off points for A's, B's, and so forth, only after seeing the configuration as a whole.

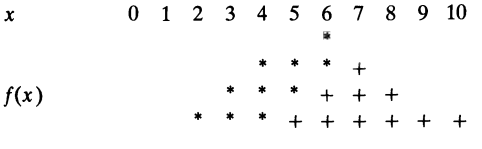
Indeed, dependence upon comparisons, rather than individual performances, is inevitable, given such a low fidelity criterion (the short alphabet of possible letter grades) that an elementary-length topology is necessary. The elementary length in this case is the minimum discrimination possible between letters of the available alphabet. It might be quantified as "1", for grades A, B, $\cdots$, without +'s, in case the letters are mapped to integers for computing grade point averages (A$\rightarrow$4, B$\rightarrow$3, $\cdots$). S. P. Gudder points out [2] that the comparable topology on the space of comparisons for particles in physics reflects the fact that "a single isolated particle is unobservable in physics; a second particle must be introduced to gain information about the first one."

Yet, are we satisfied to have letter grades responsive not to an individual's performance alone, but to accidents in the whole configuration of scores, that is, to clustering?

**Resonances.** Do the clusters themselves give us valuable information? On the contrary, they may actually obscure information, because of resonances and anti-resonances. If there are many items on the test, and if the proportion correct, $p(e,i)$, is chosen from a large set of fractions, the numerical scores will seem to allow fine discriminations. Similarly, in physics, *statistical* observations are possible of elementary particles of diameter less than a supposed elementary length. (It is this same statistical effect that can produce an average family with an indiscrete number 2.3 of children.)

A reinforcing of overlapping numerical scores may yield an artificial *a posteriori* "observation," or grade cluster, that might better be considered a mere "resonance" (a term preferred by some to describe physical objects of ephemeral sub-particle existence time [1]).

EXAMPLE. Suppose that on a 10-exercise quiz the frequency $f(x)$ of $x$-correct among twenty papers is given by



There seems to be a mode at $x = 6$. However, suppose that half the class of 20 students did not understand, or even attempt, the last four exercises. For these ten students, suppose the frequency of $x$ correct answers is $g(x)$; for the ten students who tried all the problems, suppose the frequency is $h(x)$:

| $x$ | $g(x)$ | $h(x)$ |
|---|---|---|
| 0 | | |
| 1 | | |
| 2 | * | |
| 3 | ** | |
| 4 | *** | |
| 5 | ** | + |
| 6 | ** | + + |
| 7 | | + + + |
| 8 | | + + |
| 9 | | + |
| 10 | | + |

When the twenty papers are analyzed in separate sets of 10, according to the last four problems, they reveal two modes, a mode at $x = 4$ for students who omitted the problems, and a mode at $x = 7$ for those who attempted the problems.

Notice that, just as we may seem to observe resonances where reinforcement occurs, we may encounter "anti-resonances" where cancellation occurs. This happens, for instance, when two of our bases for grading have opposite effects for some papers. A familiar example is grading that is sensitive both to understanding concepts and to accuracy in applying the concepts. A natural mode at a high level of understanding may be blurred by the distribution of inaccuracies.

**The numerical model.** Scoring by clustering inherits other deficiencies from the intermediate numerical scores. What does the score $n(e)$ represent? For repeated test items about the same skill, the numerical grade reports the student's mean performance (by reporting the total). Are we interested in such a mean? Do we postulate a model in which a student's performances are subject to a frequency distribution, and if so, have we drawn a random sample from it? Are we content to say, "This calculus student differentiates 74%"?

In case the items $i$ on the examination require different skills, totaling over $i$ seems unsatisfactory. Only a highly artificial model could justify adding a student's score on differentiating a product to his score on stating the mean value theorem.

**Scoring by attributes.** Probably the most satisfactory system for record-keeping throughout the term is a marking system retaining as detailed a labelling as possible [3]. As each student demonstrates acceptable competency with one of the skills covered, a code mark keyed to that course objective is entered by his name. Note that feedback to the student and to the instructor is then about content, not about grades. A test that covers several skills permits a student to earn several code marks. If several items test the same skill, the grader decides what overall performance is acceptable. (However, once he has rejected the statistical model, the instructor is unlikely to set repetitive test items.)

What replaces averaging (totalling) in determining a letter grade for each student? Does grading, deprived of computation, become subjective? No. The grader decides which competencies merit which grades. Often these will be ordered by set inclusion; a B paper demonstrates all the skills necessary for a C, together with additional skills. The grader may decide, though, that alternative skills can be substituted in some cases, so that there are overlapping sets of skills that earn the same grade. There is no intermediate numerical stage of grading, so grades can be geared directly to their functions, such as reportage to students, advisors, employers, and the like.

Under scoring by attributes, examinations are deemphasized, since the student ordinarily has

several opportunities to demonstrate proficiency in each skill. This lowers anxiety and pressure levels. Also, tests can often be shorter, since repetitive test items can be dropped.

**An algorithm.** This algorithm for assigning letter grades to papers, to a term's work, or projects adapts well for department-wide examinations graded cooperatively. Assume that each item is to be read throughout all papers by one grader.

First, decide which responses merit an A that are not required for a B. Read first the test items that call for these A-specific responses. Papers that show acceptable performance for A-specific skills are then placed in the "A" pile, others in the "less than A" pile. If the A-specific items test lesser skills implicitly, graders may elect not to read the rest of the "A" papers. Next, read any B-specific items, dividing the "less than A" pile into "B" and "less than B." If the A-specific items have not tested sufficiently for other skills, then re-read the A pile, deciding whether the B responses are poor enough to demote each paper, and how far—"B" or "less than B."

Once all passing alphabet letters have been sorted, the grade assignment should be checked: Rank the papers within each pile. Compare the worst in each pile with the best in the next lower pile, noting whether they are ranked in the right order and whether there is substantial difference in quality between then. Especially if there seems to be a narrow difference between a pair of neighbors, think further about the ultimate effects of the various letter grades as input for a future process. How separated in effect are a B and a C on the record of a premed student? a general education student? How would another instructor interpret the grade in advising the student about a subsequent course?

This kind of threshold grading, with the question "Is it acceptable?" replacing "How much should I take off?" is easier for the grader than numerical grading. He does have to make decisions as to what is acceptable, but he must do that implicitly in establishing numerical grades, also. If he wants to favor perception of concepts over accuracy in application, he makes his decisions accordingly, rather than "taking off just 1 for a computational error."

It has been found that most students learn best when the objectives for the course are clearly delineated. Increasingly, textbooks state objectives explicitly, often for individual chapters. Scoring by attributes continues this. It encourages instructors to set tests that reveal progress toward the objectives. The student is encouraged to measure himself against these goals, rather than against classmates. Also, with attention consistently on content, there is less danger that the student will see the instructor as an adversary, to be wheedled out of marks given reluctantly, but in the power of gift.

The mechanical computation of statistics, means and standard deviations, learned at some pains by teachers, are ordinarily inappropriate, and when a statistical model is called for, many would now recommend nonparametric procedures based on ranking. The algorithm explained here is consistent with these modern recommendations.

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1. K. Ford, The World of Elementary Particles, Blaisdell, Waltham, Mass., 1963.
2. Stanley P. Gudder, Elementary length topologies in Physics, SIAM J. Appl. Math., 16 (1968) 1011–1019.
3. M. W. Maxfield, Innovative evaluation, The Two-Year College Math. J., 5, No. 1 (Winter 1974), 47–52.

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# PROBLEMS AND SOLUTIONS

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

## ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before September 30, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 2599. *Proposed by Bernardo Recamán, S., Bogotá, Colombia*

Are there arbitrarily large positive integers $N$, such that for all $n \geqq N$ we have $\phi(n) \geqq \phi(N)$ while $\phi(n) \leqq \phi(N)$ when $n \leqq N$? ($\phi$ denotes Euler's totient function.)

E 2600. *Proposed by Ron Evans, University of Wisconsin*

Fix $r \geqq 2$ and suppose that $z_1$, $z_2$, $z_3$, $z_4$ are complex numbers of modulus $\geqq r$. Find the point at which

$$2 - (z_1 + z_2)(z_3 + z_4) + z_1 z_2 z_3 z_4$$

attains its minimum modulus.

E 2601. *Proposed by Robert Weinstock, Oberlin College*

Prove that

$$\sum_{k=0}^{[n/2]} \frac{\binom{2n-k}{k}}{\binom{2n-k}{n}} \frac{2n-4k-1}{2n-2k+1} 2^{n-2k} = 1.$$

E 2602. *Proposed by C. L. Mallows, Bell Laboratories, Murray Hill, New Jersey*

Prove that

$$\sum_{i=0}^{a-1} \binom{b+i-1}{b-1} \binom{2n-b-i}{n-b} = \sum_{i=b}^{n} \binom{a+i-1}{a-1} \binom{2n-a-i}{n-a}.$$

E 2603. *Proposed by M. S. Klamkin, University of Waterloo, Ontario*

Let $x_i > 0$ $(1 \leqq i \leqq n)$. Prove that

$$r \cdot \sum \frac{x_1 x_2 \cdots x_r}{x_1 + x_2 + \cdots + x_r} \leqq \binom{n}{r} \left( \frac{x_1 + \cdots + x_n}{n} \right)^{r-1}$$

and that equality holds iff $x_1 = x_2 = \cdots = x_n$.

(The "symmetric" sum above consists of $\binom{n}{r}$ terms.)

E 2604*. *Proposed by E.T.H. Wang, Wilfred Laurier University, Waterloo, Ontario*

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and let $A : \mathbb{N} \to \mathbb{N}$ be defined by $A(n) = [2n/3]$. For $n \in \mathbb{N}$ let $k \in \mathbb{N}$ be the smallest integer such that $A^k(n) = 0$ and define $f(n) = k$. Find a formula, as simple as possible, for the function $f$.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### $\det(I + A\bar{A}) \geqq 0$

E 2525 [1975, 300]. *Proposed by D. Ž. Djoković, University of Waterloo*

Let $A$ be a complex $n \times n$ matrix, let $\bar{A}$ be its complex conjugate, and let $I$ be the $n \times n$ identity matrix. Prove that $\det(I + A\bar{A})$ is real and nonnegative.

*Solution by O. P. Lossers, Technological University, Eindhoven, The Netherlands.* If $A$ is invertible then it follows from $A^{-1}(A\bar{A} - zI)A = \bar{A}A - zI$ that

$$\det(A\bar{A} - zI) = \det(\bar{A}A - zI).$$

By an obvious continuity argument, the last equality is valid also when $A$ is singular. Hence $\det(A\bar{A} - zI)$ is real for real $z$ (use $\overline{\det Z} = \det \bar{Z}$). To complete the proof it suffices to show that every negative eigenvalue of $A\bar{A}$ has even algebraic multiplicity.

Suppose that $\lambda$ is a negative eigenvalue of $A\bar{A}$ and that $a$ is a corresponding eigenvector. Let $b = A\bar{a}$. Then

$$A\bar{A}b = A\bar{A}A\bar{a} = \lambda A\bar{a} = \lambda b.$$

We assert that $a$ and $b$ are linearly independent over $\mathbb{C}$. For suppose that $b = \mu a$. Then

$$\lambda a = A\bar{A}a = A\bar{b} = \bar{\mu}A\bar{a} = \bar{\mu}b = |\mu|^2 a,$$

which is impossible since $\lambda < 0$.

Now let $e_1, \cdots, e_n$ be the standard basis of $\mathbb{C}^n$ and choose a new basis $a_1, \cdots, a_n$ such that $a_1 = a$, $a_2 = b$. Let $S$ be the $n \times n$ matrix such that $a_i = Se_i$, $1 \leqq i \leqq n$, and let $B = S^{-1}A\bar{S}$. Then $B\bar{B} = S^{-1}A\bar{A}S$ and hence $\det(A\bar{A} - zI) = \det(B\bar{B} - zI)$. Since

$$Be_1 = S^{-1}A\bar{S}e_1 = S^{-1}A\bar{a} = S^{-1}b = e_2,$$

$$Be_2 = S^{-1}A\bar{S}e_2 = S^{-1}A\bar{b} = S^{-1}A\bar{A}a = \lambda S^{-1}a = \lambda e_1,$$

$B$ has the form

$$B = \left[\begin{array}{cc:c} 0 & \lambda & \\ & & C \\ 1 & 0 & \\ \hdashline & 0 & D \end{array}\right]. \qquad \text{Thus} \qquad B\bar{B} = \left[\begin{array}{cc:c} \lambda & 0 & \\ & & E \\ 0 & \lambda & \\ \hdashline & 0 & D\bar{D} \end{array}\right],$$

so that $\det(B\bar{B} - zI) = (z - \lambda)^2 \det(D\bar{D} - zI)$ and the assertion follows by induction on $n$.

Also solved by Thomas Foregger & Jeff Lagarias, Chuck Horowitz & Ron Evans, and the proposer.

*Editor's comment.* After submitting the problem the proposer discovered the following reference: J. Haantjes, *Klassifikation der antilinearen Transformationen*, Math. Ann. 112 (1935), 98–106. The Corollary on p. 101 of this paper is equivalent to the assertion of the problem.

The proposer has shown that every matrix of the form $A\bar{A}$ is similar to the square of a real matrix. The proof will appear in a paper to be published in Linear and Multilinear Algebra. Note that the problem is an immediate corollary of that theorem.

**Sum-Distinct Sets**

E 2526 [1975, 300]. *Proposed by Paul Smith, University of Victoria*

Call a set $\{a_1, \cdots, a_n\}$ of positive integers *sum-distinct* if the $2^n$ possible sums $\Sigma_i \varepsilon_i a_i$ (with $\varepsilon_i = 0$ or 1) are all distinct.

Obviously for any $n$, the set $\{1, 2, 4, \cdots, 2^{n-1}\}$ is an $n$-element sum-distinct set. Do $n$-element sum-distinct sets exist with $a_i < 2^{n-1}$ for every $i$? (For example $\{3, 5, 6, 7\}$ is a 4-element sum-distinct set with this property.)

Cf. P. Erdös, Problem 220, Canad. Math. Bull. 16 (1973), p. 463.

I. *Solution by Max Beresin and Joseph O'Sullivan (jointly), D. M. Bloom, M. G. Greening (Australia), Barry Hirschfeld (Israel), O. P. Lossers (Netherlands), L. E. Mattics, Charles Nicol, Tom Odda, and Bernardo Recamán (England) (independently).* For $n \leqq 3$, direct inspection shows that no such set exists. For $n \geqq 4$ the following construction shows that the answer is affirmative. Let $A_4 = \{3, 5, 6, 7\}$ and $A_{n+1} = \{2k : k \in A_n\} \cup \{t\}$ where $t$ is any odd integer $< 2^n$. Then clearly $A_{n+1}$ is an $(n+1)$-element sum-distinct set all of whose elements are $< 2^n$.

II. *Solution by Terry Therneau, St. Olaf College.* For $n \geqq 4$, let $A = \{1, 2, 4, \cdots, 2^{n-2}\}$, $k = 2^{n-1} - 1$, and $B = \{k\} \cup \{k - a_i : a_i \in A\}$. Clearly $B$ is an $n$-element set, all of whose elements are $< 2^{n-1}$. We claim that it is sum-distinct. Any sum of $m$ elements of $B$ will be of the form $mk - \Sigma \varepsilon_i a_i$ with $a_i \in A$ and some set of $\varepsilon_i$. Since $A$ is sum-distinct, all sums for a fixed $m$ are clearly distinct. It remains to show that any sum of $m+1$ terms is greater than the sum of $m$ terms. If $m = 1$, the greatest sum of $m$ terms is $k$, while the least sum of $m+1$ terms is $2k - 2^{n-2} - 2^{n-3}$ which is greater than $k$ if $n > 3$. If $m > 1$, the greatest sum of $m$ terms is less than $mk$, while the least sum of $m+1$ terms is greater than or equal to $(m+1)k - \Sigma a_i = (m+1)k - k = mk$.

Solutions similar to Solution II were submitted by Carl Hurd, Dale Klamer, S. C. Locke (Canada), Roy Olson, Donald Shell, and J. R. Slack.

Various other solutions were submitted by David Anderson, A. Bialostocki (Israel), Robert Brigham, Lorraine Foster, Fred Galvin, Marcel Herzog (Australia), Bob Ihnot, Virender Kumar (India), L. F. Meyers, A. Oppenheim (Nigeria), Jai Ram (India), Richard Rice, Allen Schwenk, Temple University Problem Solving Group, and Roger Weitzenkamp. Partial solution by J. L. Brenner.

*Editor's comments.* Mattics raises the question: Is there for any $n$, a sum-distinct set whose elements are pairwise coprime? Nicol remarks that a computer search shows that $\{3, 5, 6, 7\}$ is the only 4-element sum-distinct set all of whose elements are less than 8. He also shows that the number of $n$-element sum-distinct sets is at least 2 raised to the exponent $(\frac{1}{2}(n-1)(n-3)) - 3$. Odda remarks that it is an open question of Erdös whether or not for arbitrarily large $k$, there is an $n$-element sum-distinct set all of whose elements are less than $2^{n-k}$. Recamán and others suggest the problem of determining for each $n$, the value $f(n)$ defined to be the smallest integer which can be the largest element of an $n$-element sum-distinct set. Solution I clearly implies that $f(n) \leqq 7 \cdot 2^{n-4}$ for all $n \geqq 4$ and hence $\lim_{n\to\infty} f(n)/2^{n-1} \leqq 7/8$. Hirschfeld constructs a sum-distinct set with largest element $3 \cdot 2^{n-3}$ and thus sharpens the bound to 3/4. He also demonstrates that this limit can be further reduced arbitrarily close to 2/3. Schwenk gives an example for which the limit is 0.63336834. Anderson conjectures that $f(n) \geqq a_{n+1}$ where the sequence $\{a_n\}$ is defined recursively as $a_1 = 0$, $a_2 = 1$, and $a_{n+1} = 2a_n - a_{[n/2]}$.

**The Four-Color Theorem for Touching Pennies**

E 2527 [1975, 301]. *Proposed by F. David Hammer, Stockton State College (New Jersey)*

(a) A finite number of pennies are placed flat on the plane. Prove that these (non-overlapping) pennies can be painted with at most four colors so that touching pennies bear different colors.

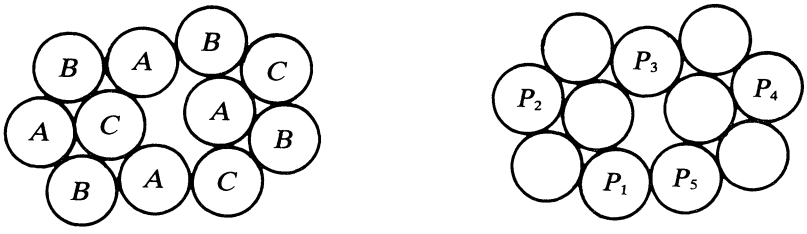
(b) Prove the same result for an infinite collection in the plane.

(c) What is the minimum number of pennies which require four colors?

I. *Solution to* (a) *by the Temple University Problem Solving Group.* One of the pennies, $P$ say, touches at most three others (take one whose center is a vertex of the convex hull of all the centers in the array). Arguing by induction, we may suppose that the remaining pennies can be colored in no more than four colors. Then one of the four colors is available for $P$.

II. *Solution to* (b) *by I. Chaff, The Corn Institute, Mariposa, Ontario.* (a) implies (b) by virtue of König's Lemma, the Compactness Theorem of logic, or the de Bruijn-Erdös Theorem on coloring infinite graphs (see C. St. J. A. Nash-Williams, *Infinite graphs — a survey*, J. Combinatorial Theory 3 (1967), 268–301).

III. *Solution to* (c) *by Allen J. Schwenk, Michigan State University.* The minimum number of pennies which require 4 colors is 11 and there is a unique array of 11 pennies which is not 3-colorable. Let $A$ be a non-3-colorable array. Then $A$ is connected and remains connected on removal of any penny $P$ or pair of touching pennies $PQ$; if not, we could 3-color the components of the resulting array with $P$ (resp. $PQ$) reattached to each and thereby obtain a 3-coloring of $A$, permuting colors to give agreement at $P$ (resp. $PQ$) if necessary. Moreover, each penny $P$ in $A$ must touch at least 3 others, for otherwise a 3-coloring of $A - P$ would give rise to one of $A$. It follows that $A$ consists of a cycle of outside pennies with no touching between them except when they are neighbors in the cycle, together with some inside pennies, every outside penny touching at least one inside penny. If $A$ had only one inside penny, it would consist of this penny surrounded by 6 outside pennies, a 3-colorable array. If $A$ had just 2 inside pennies and these touched, it would consist of these 2 pennies surrounded by 8 outside pennies; there are essentially only two such arrays, both 3-colorable again. Just 2 inside pennies, not touching, necessitates at least 11 pennies altogether and if there are exactly 11 pennies we have one of the two arrays




shown in the figure. A 3-coloring is given for the first (in which, incidentally, a certain amount of sliding can occur, just as with one of the two ten-penny arrays above), but any attempt to 3-color the second requires $P_1$, $P_2$, $P_3$, $P_4$, and $P_5$ to have the same color, so that a fourth color is needed and we have the sought-after array (if there are 3 or more inside pennies then there must be more than 11 pennies in all).

(a) and (b) were solved by Carl Hurd, Rhys Jones (England), S. C. Locke (Canada), O. P. Lossers (Netherlands), William Markel, Allen Schwenk, The Temple University Problem Solving Group, and the proposer. (c) was solved by Hurd, Jones, Lossers, and Schwenk.

*Editor's comment.* This problem occurs as an exercise in G. Ringel, *Map Color Theorems* (Springer-Verlag, 1974), and the minimal array for (c) was exhibited in Martin Gardner's column for the *Scientific American*, April 1975, p. 133. The problem of describing *all* minimal non-3-colorable arrays remains open. A number of

solvers expressed the problem in terms of graphs, the graphs arising being those embedded in the plane in which the distance between any two vertices is $\geqq 1$, with equality if and only if the two vertices are joined by an edge. It is not known what is the minimum number of colors required for graphs in the plane with every edge of unit length, there being no further restriction on the distances between the vertices, though it is known that 7 colors are sufficient.

**Ideals of a Matrix Ring**

E 2528 [1975, 400]. *Proposed by Louis W. Shapiro, Howard University*

Let $R$ denote the ring of $n \times n$ real matrices with the property that every element not in the first row or on the main diagonal is 0. How many two-sided ideals does $R$ have?

I. *Solution submitted unsigned.* Let $A$ and $B$ be rings with identity and ${}_AM_B$ a bimodule. The abelian group $R = A \oplus M \oplus B$ becomes a ring with identity by defining the multiplication via matrices

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} a' & m' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} aa' & am' + mb' \\ 0 & bb' \end{pmatrix}$$

where we identify $(a, m, b) \in R$ with the matrix $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$. Let $R'$ be an ideal of $R$ and let $A'$, $B'$, $M'$ be its projection in $A$, $B$, $M$, respectively. Since $\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \in R'$ implies that

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R', \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \in R'$$

we have that $R' = A' \oplus M' \oplus B'$. It is clear that $A'$ (resp. $B'$) is an ideal of $A$ (resp. $B$) and that $A'M \subset M'$, $MB' \subset M'$. Conversely, if $A'$, $M'$, $B'$ satisfy these conditions then $A' \oplus M' \oplus B'$ is an ideal of $R$.

In our example $A = \mathbb{R}$, $B$ is the ring of diagonal real matrices of order $n-1$ and $M$ is the space of real row-vectors of dimension $n-1$ considered as $(A, B)$-bimodule in the obvious way.

There are $2^{n-1}$ ideals $R' = A' \oplus M' \oplus B'$ such that $A' = A$ and consequently $M' = M$.

$M_B$ has precisely $2^{n-1}$ submodules; these are the $\mathbb{R}$-subspaces of $M$ which are spanned by a subset of the standard basis $\{e_1, \cdots, e_{n-1}\}$ of $M$. If $\dim M' = k$ then there are $2^k$ ideals $B'$ of $B$ such that $MB' \subset M$. Hence the number of ideals $R'$ such that $A' = 0$ is

$$\sum_{k=0}^{n-1} \binom{n-1}{k} 2^k = 3^{n-1}.$$

Altogether there are $2^{n-1} + 3^{n-1}$ ideals in $R$.

II. *Solution by M. Ram Murty and V. Kumar Murty, Undergraduates, Carleton University, Ottawa, Ontario.* $R$ is the incidence algebra $I(P)$ of the poset $P = \{1, 2, \cdots, n\}$, where $1 < i$ $(i = 2, 3, \cdots, n)$ and there are no other strict inequalities valid in $P$. Let $S(P)$ be the poset of segments of $P$ ordered by inclusion. It is known that the lattice of ideals of $I(P)$ is anti-isomorphic with the lattice of order ideals of the poset $S(P)$. (See Theorem 3.1, p. 272 in P. Doubilet, G.-C. Rota and R. Stanley, *On the Foundations of Combinatorial Theory* (VI): *The Idea of Generating Function*, Proc. Sixth Berkeley Symp. Math. Stat. and Prob. 1972, vol. 2.) There are $2^n$ order ideals of $S(P)$ which contain no segment $[1, i]$ $(i > 1)$ of $P$. The number of order ideals containing $k$ segments of the form $[1, i]$ $(k \geqq 1,\ i > 1)$ is

$$\binom{n-1}{k} 2^{n-k-1}.$$

Hence the total number of ideals in $R$ is

$$2^n + \sum_{k=1}^{n-1} \binom{n-1}{k} 2^{n-k-1} = 2^{n-1} + 3^{n-1}.$$

Also solved by David Bloom, Peter de Buda (Canada), Robert Feinberg, Thomas Foregger, P. K. Garlick, Robert Gilmer, Bennette Harris, Melvin Henriksen, G. A. Heuer, Carl Hurd, Colonel Johnson, Jr., Thomas Kezlan, Joel Levy, O. P. Lossers (Netherlands), Robert Patenaude, Martin Schechter (Canada), Jón Stefánsson (Iceland), Temple University Problem Solving Group, E. T. Wong, and the proposer.

*Editor's comment.* Many solvers have noticed that $\mathbb{R}$ can be replaced by any field without affecting the result. Let $R_k$ $(1 \leqq k \leqq n)$ be the ring of $n \times n$ real matrices with the property that the $(i, j)$-th entries for $i > k$ and $i \neq j$ are zero. Patenaude shows that $R_k$ has $2^{n-k} + 3^{n-k}$ ideals.

### An Application of $\psi(x)$

E 2529 [1975, 400]. *Proposed by S. W. Golomb, University of Southern California*

Let $N_k(n)$ denote the number of "digits" in the $k$-ary representation of the natural number $n$. Show that if

$$S_k = \sum_{n=1}^{\infty} \frac{1}{n(N_k(n))^2}$$

then $S_k \sim A \log k$ for some constant $A$. Find $A$ and estimate the error term.

*Solution by Robert E. Shafer, Berkeley, California (revised by the editor).* If we put $N_k(n) = p$, then

$$S_k = \sum_{n=1}^{\infty} \frac{1}{p^2} T(p, k),$$

where

$$T(p, k) = \sum_{n=k^{p-1}}^{k^p - 1} \frac{1}{n}.$$

Using the function

$$\psi(x) = -\gamma + \sum_{m=0}^{\infty} \left( \frac{1}{m+1} - \frac{1}{m+x} \right),$$

where $\gamma$ is the Euler constant, we can write

$$T(p, k) = \psi(k^p) - \psi(k^{p-1}), \qquad p = 2, 3, \cdots.$$

We have for $x > 0$

$$\psi(x) = \log x - \frac{1}{2x} + R(x),$$

where

$$-\frac{1}{12x^2} < R(x) < 0. \tag{1}$$

(See N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, 1924, pp. 106–107.) Thus

$$T(1, k) = \gamma + \psi(k) = \gamma + \log k - \frac{1}{2k} + R(k)$$

and for $p \geqq 2$,

$$T(p, k) = T(1, k^p) - T(1, k^{p-1}) = \log k + \frac{1}{2}\left(1 - \frac{1}{k}\right) \frac{1}{k^{p-1}} + R(k^p) - R(k^{p-1}).$$

Therefore

$$S_k = \gamma + \left( \sum_{p=1}^{\infty} \frac{1}{p^2} \right) \log k - \frac{3}{8k} + R_k,$$

where

$$R_k = -\frac{1}{8k^2}+\frac{1}{2}\left(1-\frac{1}{k}\right)\sum_{p=3}^{\infty}\frac{1}{p^2k^{p-1}}+\sum_{p=1}^{\infty}\left(\frac{1}{p^2}-\frac{1}{(p+1)^2}\right)R(k^p). \tag{2}$$

Thus

$$A=\sum_{p=1}^{\infty}\frac{1}{p^2}=\frac{\pi^2}{6} \quad \text{and} \quad S_k - A\log k = \gamma - \frac{3}{8k}+R_k.$$

Using (1) and (2) it is easy to get the estimate $|R_k|<1/k^2$.

Also solved by Paul Bruckman, Leonard Carlitz, Peter de Buda (Canada), Roger Eggleton, M. G. Greening (Australia), J. D. Hiscocks (Canada), Dennis Jesperson, Jeff Lagarias, Joel Levy, Graham Lord (Canada), O. P. Lossers (Netherlands), L. E. Mattics, Ram Murty & Kumar Murty (Canada), T. Šalát (Czechoslovakia), Michael Skalsky, and the proposer.

*Editor's comment.* Many solvers were satisfied to give an asymptotic formula for $S_k - A \log k$ without computing an upper bound for its absolute value. For instance, Lossers gives $R_k = O(1/k^2)$.

### A Hexagon Theorem

E 2531 [1975, 400]. *Proposed by V. F. Ivanoff, San Carlos, California*

Given points $A, B, C, D, E, F$ in the plane, let $ABC$ denote the *directed* area of triangle $ABC$, etc. Prove that

$$AEF\cdot DBC + BEF\cdot DCA + CEF\cdot DAB = DEF\cdot ABC.$$

(Remark: The special case $D = F$ was shown by W. L. Williams, *A pentagon theorem*, this MONTHLY 60 (1953) 616–617.)

*Solution by Albert Nijenhuis, University of Pennsylvania.* If $A=(a_1, a_2)$ etc., then

$$ABC=\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

etc. The left hand side of the proposed equality is equal to one quarter of the determinant of the matrix

$$M=\begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ a_1 & b_1 & c_1 & -d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & -d_2 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ a_1 & b_1 & c_1 & 0 & e_1 & f_1 \\ a_2 & b_2 & c_2 & 0 & e_2 & f_2 \end{pmatrix}$$

This can be seen by using the Laplace expansion of $\det(M)$ with respect to the first three rows. On the other hand, by subtracting row $i$ from row $3+i$ $(i=1,2,3)$ we see that $\frac{1}{4}\det(M)$ is also equal to the right hand side of the proposed equality.

Also solved by Alex Ferrer (Mexico), M. G. Greening (Australia), J. D. Hiscocks (Canada), P. L. Hon (Hong

Kong), L. Kuipers (Switzerland), Detlef Laugwitz (Germany), O. P. Lossers (Netherlands), J. G. Mauldon, R. D. Nelson (England), C. L. Sabharwal, Anand Tamhankar, William Wernick, Southern University Primer for Research Group. Brief outlines of the solution by M. S. Klamkin (Canada), Lew Kowarski, Gerson Robison, and the proposer.

*Editor's comment.* Nijenhuis mentions that the result can be generalized easily to higher dimensions by considering the analogs of the matrix *M.* H. Guggenheimer notes that E 2531 occurs as Problem 24, Section 5.1 of his book, *Plane Geometry and its Groups.* He also refers to Crelle's Journal, 26 (1843), p. 26 and to J. van Ijzeren, *Moderne vlakke meetkunde* (Zutphen, W. J. Thieme & Cie., 1941) for an extensive study of consequences of the Möbius Theorem.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before September 30, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6096. *Proposed by Jan Mycielski, University of Colorado*

A set of cells of a chessboard is called connected if a rook can visit the whole set without moving over cells which are not in the set. Put $s = a_n n^2$ and let $2^s$ be the number of connected subsets for a chessboard of size $n$ by $n$. Prove that the sequence $a_1, a_2, \cdots$, converges and estimate its limit.

6097*. *Proposed by Glen E. Bredon, Rutgers University*

Consider the polynomial

$$P(t) = 2^{-n}(1 + t^{a_1})(1 + t^{a_2}) \cdots (1 + t^{a_n}).$$

The first $k$ derivatives of $P(t)$ evaluated at $t = 1$, that is

$$q_1 = P'(1), q_2 = P''(1), \cdots, q_k = P^{(k)}(1),$$

are symmetric functions of $a_1, a_2, \cdots, a_n$. Show that in the polynomial algebra generated by these $k$ symmetric functions coincides with that generated by $\sigma_1$ and the $\sigma_{2j}$ for $2 \leqq 2j \leqq k$. Here $\sigma_i$ is the sum of $i$th powers, $\sigma_i = a_1^i + a_2^1 + \cdots + a_n^i$. (Note that it follows, in particular, that $q_3$ is a polynomial in $q_1$ and $q_2$. In fact, $q_3 = 3q_1q_2 - 2q_1^3 - 3q_2 + 3q_1^2 - q_1$. Similarly, $q_5$ is a polynomial in $q_1$, $q_2$ and $q_4$; and so on.)

6098. *Proposed by Peter L. Renz, San Francisco*

Let $A$ be the group of affine bijections from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any subset $S$ of $\mathbb{R}^n$ define $A(S)$ to be the subgroup of $A$ which takes $S$ onto itself. A convex body is a compact convex set with non-empty interior. We say a convex body $K$ in $\mathbb{R}^n$ is *maximally symmetric* if $A(K)$ is not properly contained in $A(L)$ for any convex body $L$ in $\mathbb{R}^n$. Characterize the maximally symmetric convex bodies in $\mathbb{R}^n$.

6099. *Proposed by Jerome Minkus, Berkeley, California*

For $n \geqq 3$ let $G_n$ denote the group generated by $a_1, a_2, \cdots, a_n$ subject to the relations

$$a_1a_2^{-1}a_3 = a_2a_3^{-1}a_4 = \cdots = a_{n-2}a_{n-1}^{-1}a_n = a_{n-1}a_n^{-1}a_1 = a_na_1^{-1}a_2 = 1.$$

(Compare Problem 5327 [1967, 91].) Show that

(i) $G_5$ is isomorphic to the binary dodecahedral group

$$\{a, u: a^5 = u^3 = (au)^2\}.$$

(ii) $G_n$ is nonabelian for all $n \geqq 3$.

6100. *Proposed by Eric Chandler, North Carolina State University*

For fixed integer $n > 1$ find a bijection $T$ on the real numbers such that $T^m$ is a contraction if and only if $m = kn$ for $k = 1, 2, \cdots$. Can $T$ be continuous?

6101. *Proposed by Michael Slater, University of Bristol, England*

Suppose $F$ is an ordered field in which Rolle's theorem holds for polynomials. (That is, if $f(x) \in F[x]$, $f(a) = 0 = f(b)$ with $a < b$, then there exists $c$ with $a < c < b$ such that $f'(c) = 0$, where $f'$ is the formal derivative.) Show that any sum of squares in $F$ is a square in $F$.

## SOLUTIONS OF ADVANCED PROBLEMS

### A Γ-function Inequality

5997 [1974, 1034]. *Proposed by M. S. Klamkin, University of Waterloo, Canada*

Prove that

$$|x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p| \leqq \{\Gamma(1+n)\}^p$$

where $n$ is a positive integer, $p$ is real and $\geqq 1$, and $0 \leqq x \leqq n$. (This inequality has been given by A. Ostrowski for integral $p$, $n$. See Mitrinović and Vasić, *Analytic Inequalities*, Springer-Verlag, 1970, p. 198.)

*Solution by Thomas Foregger, Bell Laboratories, Murray Hill, New Jersey.* Let

$$f_n(x) = |x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p|.$$

We show that

$$f_n(x) \leqq \left(\frac{p-1}{2p-1}\right)^{p-1}\left(\frac{p}{2p-1}\right)^p \{\Gamma(1+n)\}^p \quad \text{if } 0 \leqq x < 1,$$

$$f_n(x) \leqq \frac{1}{n}\{\Gamma(1+n)\}^p \quad \text{if } 1 \leqq x \leqq n.$$

First, suppose that $0 \leqq x \leqq 1$. Then, using $\Gamma(n+1) = n!$ and the relation $|(x-k)^p| = (k-x)^p \leqq k^p$ for $0 \leqq x < 1$ and $k = 1, \cdots, n$, we have that

$$\frac{f_n(x)}{(n!)^p} = x^{p-1}\frac{(1-x)^p}{1^p}\cdot\frac{(2-x)^p}{2^p}\cdot \ldots \cdot\frac{(n-x)^p}{n^p} \leqq x^{p-1}(1-x)^p.$$

Elementary calculus shows that the right hand side has a maximum value of

$$\left(\frac{p-1}{2p-1}\right)^{p-1}\left(\frac{p}{2p-1}\right)^p.$$

Next, suppose that $1 \leqq x \leqq n$ and let $m = [x]$. Then

$$|(x-k)^p| = \begin{cases} (x-k)^p & \text{if } 1 \leqq k \leqq m, \\ (k-x)^p & \text{if } m+1 \leqq k \leqq n, \end{cases}$$

so that

$$\frac{f_n(x)}{(n!)^p} = \frac{x^{p-1}}{n^{p-1}}\prod_{k=1}^{m-1}\frac{(x-k)^p}{(n-k)^p}\cdot\frac{(x-m)^p}{n}\cdot\prod_{k=m+1}^{n}\frac{(k-x)^p}{(k-m)^p}.$$

Clearly, $0 \leqq x-k \leqq n-k$ if $1 \leqq k \leqq m-1$ and $0 \leqq k-x \leqq k-m$ if $m+1 \leqq k \leqq n$. Also $x \leqq n$. Thus

$$\frac{f_n(x)}{(n!)^p} \leqq \frac{(x-m)^p}{n} \leqq \frac{1}{n}.$$

Also solved by Emil Grosswald, L. Kuipers, Robert Shafer, and the proposer.

*Editor's note.* Grosswald observes that the inequality written as $|x(x-1)\cdots(x-n)|x^{-1/p} \leqq n!$ is an easy consequence of Ostrowski's theorem.

**Covering a Set of Integers**

5998 [1974, 1034]. *Proposed by D. E. Daykin, Reading University, England*

Let $n$ be a positive integer and let $N$ be the set $\{1, 2, \cdots, n^2+n+1\}$. Suppose $F$ is a family of distinct subsets of $N$ such that (i) each member of $F$ contains more than $n^2$ integers of $N$, and (ii) each integer of $N$ is in more than $n^2$ members of $F$. Prove that two members of $F$ have union $N$.

*Solution by John V. Erhart, Northeast Missouri State University.* The conjecture is false for $n \geqq 3$. For $n = 3$, construct a collection of subsets of $N$ as follows: Let $S = \{n+1, \cdots, n^2+n+1\}$. $S$ contains $n^2+1$ elements so there are $n^2+1$ distinct subsets of $S$ each of which contains exactly $n^2$ elements. Call these subsets $A_1, A_2, \cdots, A_{n^2+1}$. For $j = 1, 2, \cdots, n$, let $A_{i,j} = A_i \cup \{j\}$. The collection $F = \{A_{i,j} \mid i = 1, 2, \cdots, n^2+1;\ j = 1, 2, \cdots, n\}$ satisfies conditions (i) and (ii) above. We see that at least $n$ members of $F$ are needed to cover $N$.

Moreover, given any collection $F$ of subsets of $N$ satisfying (i) and (ii) above, there exists a subcollection of $F$ which contains at most $n$ members whose union is $N$.

*Proof*: The conclusion follows immediately if $N \in F$. Assume $N \notin F$ and that there exists a member $A$ of $F$ that contains more than $n^2+1$ elements. Let $N \backslash A = \{x_1, \cdots, x_k\}$ where $k < n$. For $i = 1, \cdots, k$ we can find $A_i \in F$ such that $x_i \in A_i$, so that $N = A \cup A_1 \cup \cdots \cup A_k$.

Now suppose that no member of $F$ contains more than $n^2+1$ elements. Pick $A \in F$ and consider $N \backslash A = \{x_1, \cdots, x_n\}$. If there is a member $B$ of $F$ that contains more than one element of $N \backslash A$, then consider $N \backslash (A \cup B) = \{y_1, \cdots, y_k\}$ where $k \leqq n-2$. As before, we can find $k$ sets $A_1, \cdots, A_k$ such that $y_i \in A_i \in F$ for $i = 1, \cdots, k$, so that $N = A \cup B \cup A_1 \cup \cdots \cup A_k$.

Finally, if there is no member of $F$ that contains more than one element of $N \backslash A$, we begin by picking $A_1 \in F$ such that $x_1 \in A_1$. $A_1$ must contain $n^2$ elements of $A$. Let $\{x_0\} = A \backslash A_1$. There exists a member of $F$, say $A_2$, such that $\{x_0, x_2\} \subset A_2$ (since there is more than one set containing $x_2$, each of which contains $n^2$ elements of $A$). Now continue picking sets $A_i \in F$ (if $n \geqq 3$) for $i = 3, \cdots, n$ such that $x_i \in A_i$, so that $N = A_1 \cup A_2 \cup \cdots \cup A_n$. The proof is complete.

Also solved by Charles Blair, David Bloom, Peter Borwein (Canada), Robert Breusch (New Zealand), Kevin Brown, Grady Cantrell, Philip Castevens, Phillip Chase, Paul Comba, Daniel Deignan, Paul Erdös (Israel), David Farnsworth, Thomas Foregger, Michael Garey & David Johnson, Richard Gisselquist, Michael Hager & Martin Schaefer (Germany), J. R. Henderson, C. V. Heuer, William Margolis, L. E. Mattics, E. C. Milner (Canada), Roy Olson, C. B. A. Peek, Bernardo Recamán S. (England), Per Salberger (Sweden), David Sanders, E. F. Schmeichel, Robert Spira, Oto Strauch (Czechoslovakia), Phil Tracy, J. B. van Rongen (Netherlands), Eric Verheiden, Michael Webb, and Ira Wolf.

*Editor's notes.* (1) Foregger proves the following modification of the original statement: Let $a$, $b$ be natural numbers and let $N$ be the set $\{1, 2, \cdots, a+b\}$. Suppose $F$ is a family of distinct subsets of $N$ such that (i) each member of $F$ contains more than $a-1$ integers of $N$, and (ii) each integer of $N$ is in more than $a-1$ members of $F$. Then $b$ members of $F$ have union $N$.

(2) Strauch shows that if besides (i), (ii), $F$ has no more than $n^2+n+1$ sets, then $N$ can be covered by two sets in $F$.

**A Measure in $\mathbb{R}^n$**

5999 [1974, 1034]. *Proposed by R. D. McKelvey, Carnegie-Mellon University and University of Rochester*

Let $\mu$ be a finite measure on the $\sigma$-algebra of sets in $\mathbb{R}^n$ generated by the half spaces defined by hyperplanes through the origin. I.e., for any $\alpha \in \mathbb{R}^n$, let $H_\alpha = \{x \in \mathbb{R}^n \mid x\alpha > 0\}$ and let $H = \{H_\alpha \mid \alpha \in \mathbb{R}^n\}$. Further, let $B$ be the $\sigma$-algebra generated by $H$, and $\mu: B \to R$ be a finite positive measure on $B$. Prove or disprove the conjecture: If $\mu(A) = \mu(-A)$ for all $A \in H$, then $\mu(A) = \mu(-A)$ for all $A \in B$. (Here, of course, $-A = \{x \in \mathbb{R}^n \mid -x \in A\}$.)

*Solution by S. Michael Webb, East Texas State University.* Let $\mu$ be any positive finite measure on a $\sigma$-algebra $B$. Then the set $D = \{A \in B \mid \mu(A) = \mu(-A)\}$ is a $\sigma$-algebra of sets. We have first, since $\varnothing = -\varnothing$, that $\mu(\varnothing) = \mu(-\varnothing)$. Since $\boldsymbol{C}(-A)$, the complement of $-A$, is $-\boldsymbol{C}A$, we have

$$\mu(-\boldsymbol{C}A) = \mu(\boldsymbol{C}(-A)) = \mu(\boldsymbol{R}^n) - \mu(-A) = \mu(\boldsymbol{R}^n) - \mu(A) = \mu(\boldsymbol{C}A)$$

for any $A \in D$. All that remains is closure for countable unions. Let $\{A_n \mid n \in \boldsymbol{N}\}$ be a disjoint sequence in $D$. Then the sequence $\{-A_n\}$ is a disjoint sequence in $D$ and

$$\mu\Big(-\bigcup_n A_n\Big) = \mu\Big(\bigcup_n -A_n\Big) = \sum_n \mu(-A_n) = \sum_n \mu(A_n) = \mu\Big(\bigcup_n A_n\Big).$$

For the given measure, $H \subseteq D$, and $B \supseteq H$ is the smallest $\sigma$-algebra containing $H$. Therefore, $D = B$.

*Note.* This result extends to the extension of $\mu$ obtained from the outer measure

$$\mu^*(E) = \inf\Big\{\sum_n \mu(A_n) \mid E \subseteq \bigcup_n A_n \quad \text{and} \quad \{A_n \mid n \in \boldsymbol{N}\} \subseteq B\Big\}.$$

That $\mu^*$ inherits the required property is easily obtained by observing that $E \subseteq \bigcup_n A_n$ if and only if $-E \subseteq \bigcup_n -A_n$ and that $-(-X) = X$ for any $X \subseteq \boldsymbol{R}^n$.

Also solved by D. A. Overdijk & F. H. Simons & F. W. Steutel, (Netherlands), and by Peter Ungar & the proposer.

**A Continuous Functional**

6000 [1974, 1120]. *Proposed by Siemion Fajtlowicz and Jan Mycielski, University of Colorado*

Let $F$ be the space of all complex-valued functions $f: [0,1] \to \{z : |z| \leq 1\}$ such that $f(0) \neq 0$, $|f(t_2) - f(t_1)| \leq t_2 - t_1$ for all $0 \leq t_1 < t_2 \leq 1$, with the distance $\operatorname{dist}(f_1, f_2) = \max\{|f_1(t) - f_2(t)| : 0 \leq t \leq 1\}$. We define a function $\phi: F \to \{z : |z| = 1\}$ putting $f^*(t) = f(t)/|f(t)|$ and $\phi(f) = f^*(\min\{t : |f(t)| = t\})$. Prove that $\phi$ is continuous.

Is the result true when the space of complex numbers is replaced by a Banach space so that $f$ maps $[0,1]$ into a ball?

*Solution by O. P. Lossers, Technological University, Eindhoven, The Netherlands.* Let $H$ be an inner product space, with norm $|a| = \sqrt{(a,a)}$. If we define $a^* = a/|a|$ for $a \neq 0$, we can prove the following result:

*Let $F$ be the set of functions $f: [0,1] \to \{a \in H : |a| \leq 1\}$, satisfying $f(0) \neq 0$ and $|f(t) - f(s)| \leq |t - s|$ for $s, t \in [0,1]$. On $F$ the distance $\operatorname{dist}(f,g) = \max_{0 \leq t \leq 1} |f(t) - g(t)|$ is used. If we define*

$$\tau_f = \min\{t \in [0,1] : |f(t)| = t\}, \qquad \text{en} \phi(f) = f^*(\tau_f),$$

*then the map $\phi: F \to H$ is continuous.*

*Proof.* We use the equality

$$|a^* - b^*|^2 = \{|a - b|^2 - ||a| - |b||^2\}/|a| \cdot |b|.$$

If $f \in F$ and $f_n \in F$ is a sequence such that $\varepsilon_n = \operatorname{dist}(f, f_n) \to 0$ as $n \to \infty$, then we have for large $n$ $\tau_n \geq \frac{1}{2}\tau$, (where we have put $\tau_n$ for $\tau_{f_n}$ and $\tau$ for $\tau_f$). Consequently,

$$\begin{aligned}
|\phi(f) - \phi(f_n)|^2 &= |f^*(\tau) - f_n^*(\tau_n)|^2 \\
&= \{|f(\tau) - f_n(\tau_n)|^2 - ||f(\tau)| - |f_n(\tau_n)||^2\}/|f(\tau)|\,|f_n(\tau_n)| \\
&\leq 2\{(|f(\tau) - f(\tau_n)| + |f(\tau_n) - f_n(\tau_n)|)^2 - |\tau - \tau_n|^2\}/\tau^2 \\
&\leq 2\{(|\tau - \tau_n| + \varepsilon_n)^2 - |\tau - \tau_n|^2\}/\tau^2 \to 0 \qquad (n \to \infty).
\end{aligned}$$

The result is not true in a general Banach space: Consider the space $\mathbb{R}^2$ with norm $|x| = \max(|x^1|, |x^2|)$. If

$$f_n^1(t) = \begin{cases} \frac{1}{4}+\frac{1}{n} & \left(0<t<\frac{1}{4}\right) \\ t+\frac{1}{n} & \left(\frac{1}{4}<t<\frac{3}{4}\right) \\ \frac{3}{4}+\frac{1}{n} & \left(\frac{3}{4}<t<1\right) \end{cases} \qquad f_n^2(t) = \begin{cases} t & \left(0<t<\frac{1}{4}\right) \\ \frac{1}{4} & \left(\frac{1}{4}<t<1\right) \end{cases}$$

$n = 4, 5, \cdots, f(t) = \lim_{n\to\infty} f_n(t)$, then

$$f_n^*(t) = (1, f_n^2(t)/f_n^1(t)), \qquad f^*(t) = (1, f^2(t)/f^1(t)),$$

$\tau_n = \frac{3}{4} + 1/n$, $\tau = \frac{1}{4}$ and hence

$$\phi(f^*) = (1,1) \neq \lim_{n\to\infty} \phi(f_n^*) = \lim_{n\to\infty} \left(1, \frac{1}{3+4/n}\right) = \left(1, \frac{1}{3}\right).$$

Also solved by J. M. Brown & D. A. Voss, Joel Davis, and the proposers.

### The Remainder Term in Maclaurin's Expansion

6001 [1974, 1121]. *Proposed by J. A. Eidswick, University of Nebraska-Lincoln*

When Taylor's theorem is applied to a power series $f$ and a point $x$ in its interval of convergence, a sequence $\{t_n\}$ is obtained such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(t_n)}{n!} x^n$$

for every $n = 1, 2, \cdots$. Can $\{t_n\}$ always be chosen so that $\liminf |t_n| = 0$?

*Editor's notes.* This problem remains unsolved. The following theorems with proofs were contributed as partial solutions:

(1) *Bryce Parry, Stanford University.* If $x$ is interior to the circle of convergence, then $\liminf |t_n| = 0$.

(2) *M. J. Pelling, University of Benin, Nigeria.* If $x$ is interior to the interval of convergence, then there is a monotonic increasing function $g(x)$ on $(0,1)$ such that if the radius of convergence of $f$ is $\geqq R$, then $\{t_n\}$ can be chosen so that

$$0 < |t_n| < \frac{R \cdot g(|x|/R)}{n+1}$$

for infinitely many $n$.

$$\int_0^x (x-t)^{p-1} \ln \Gamma(t)\, dt$$

6002 [1974, 1121]. *Proposed by Bertram Ross, University of New Haven*

Examine for convergence and, if convergent, evaluate

$$I = \int_0^x (x-t)^{p-1} \ln \Gamma(t)\, dt,$$

where $\Gamma(t)$ is the gamma function and $0 < p < 1$.

I. *Solution by L. Kuipers, Mollens, Switzerland.* Suppose $x>0$. We have

$$I=\int_0^x (x-t)^{p-1}\ln(t\Gamma(t))\,dt-\int_0^x (x-t)^{p-1}\ln t\,dt$$

$$=\int_0^x (x-t)^{p-1}\ln\Gamma(t+1)\,dt-\int_0^x (x-t)^{p-1}\ln t\,dt$$

$$=I_1-I_2, \text{ say.}$$

At $t=x$ the integral $I_1$ is an improper integral but converges since $0<p<1$. At $t=0$ the integrand of $I_1$ is continuous.

At $t=x$ the integral $I_2$ converges for the same reason as does $I_1$. At $t=0$ the integrand is unbounded because of $\ln t$, but since $\int_0^x \ln t\,dt$ converges at $t=0$, so does $I_2$. Hence the integral $I$ is convergent.

II. *Solution by Emil Grosswald, Temple University.* Starting with the formula (Abramowitz and Segun, *Handbook of Mathematical Functions*, Dover, New York, p. 256)

$$\ln\Gamma(1+z)=-\ln(1+z)+z(1-\gamma)+\sum_{n=2}^{\infty}(-1)^n n^{-1}(\zeta(n)-1)z^n$$

valid for $|z|<2$, we find

$$I=\frac{x^p}{p}\left\{\frac{1-\gamma}{p+1}x+S_p(x)+\ln x-\frac{p+1}{p}-\frac{\Gamma'}{\Gamma}(p)\right\},$$

where

$$S_p(x)=1-\gamma-\frac{x}{1+p}+\sum_{r=2}^{\infty}(-1)^r\frac{(r-1)!\,\zeta(r)}{(p+1)(p+2)\cdots(p+r)}x^r.$$

The representation depends on the convergence of $S_p(x)-|x|<1$. Thus the value for $I$ is valid for $0<x<1$ and for all values of $x>0$ to which $S(x)$ may be continued. It would, of course, be valuable to have a closed expression for $S_p(x)$, but this was not achievable.

III. *Solution by Nathaniel Grossman, University of California, Los Angeles.* In my paper, *Polygamma functions of arbitrary order*, contributed to the SIAM Journal of Mathematical Analysis, the following asymptotic expansion is derived, valid for $|\arg x|\leqq\pi-\delta$,

$$\frac{I}{\Gamma(p)}\sim\frac{x^p}{\Gamma(p+1)}\left\{\log\frac{1}{x}+\gamma+\frac{\Gamma'(p+1)}{\Gamma(p+1)}\right\}$$

$$+x^p\sin\pi p\left\{\sum_{k=1}^{\infty}(-1)^k\frac{B_k c_{2k-1}}{2k}\frac{\Gamma(2k-p-1)}{x^{2k}}+\sum_{k=0}^{\infty}\frac{B_k}{2k}\frac{\Gamma(2k-p)}{(2\pi x)^{2k+1}}\right\},$$

where $c_r$ is determined by

$$\Gamma(s)=\frac{(-1)^r}{r!}\frac{1}{s+r}+c_r+O(s+r)$$

near the pole at $s=-r$.

*Editor's comment.* The proposer remarks that the result will enable one to interpolate between integral orders of the psi function.

# REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.*

## FILMS

*Maurits Escher: Painter of Fantasies.* A Coronet Film, produced by Document Associates, Inc. Color only and sound, 16mm, 26½ minutes, copyright 1970. (American Film Festival Honors Award.)

This film is a delight to admirers of Escher's work, and should add new members to that group. It offers the artist in informal conversation about his work, pleasant guitar music, and, best of all, a close view of many of the Escher prints. The details of *Metamorphose, Relativity, Day and Night* and other works are presented very effectively by panning, although in some places the camera is unsteady; in *Reptiles*, the cycle from two to three and back to two dimensions is not quite completed.

Escher mentions mathematics several times during his discussions. He constructs a Möbius strip, and refers to correspondence with an English mathematician who advised showing an ant on one. However, there is no mention of the fact that Escher set himself the problem of representing the concepts of infinity and unboundedness, and that he attempted to do so with regular tessellations of surfaces, cycles of sorts, and with visualizations of sequences converging to a limit.

Although the film does not make the point, several of the Escher works can serve as starting points for mathematical discussions: the Möbius strips for topology, the tessellations of the plane for a discussion of two-dimensional isometries and symmetry groups, and many works for a discussion of continuous deformations. The "spirit of mathematics" is represented by works such as *Belvedere* which show that the artist was willing to experiment with new systems of axioms. It is unfortunate from an analyst's point of view that the *Circle Limits* and *Smaller and Smaller* were not included in the film.

Showing this movie could help build bridges between disciplines and might allow the mathematician to offer some comments to an audience he could not otherwise reach. At the same time, the film could help broaden the mathematics student's conception of his subject. However, there is still a need for a film about Escher made by and for mathematicians.

ROBERT L. RAYMOND, University of Minnesota, Morris

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S**, P**, L***. *Adventures of a Mathematician*. S.M. Ulam. Scribner's, 1976, xi + 317 pp, $14.95. A witty, intellectual, yet very human autobiography; it provides unique insight into the thought and lives of those few Los Alamos theoreticians who altered both the intellectual and political landscape of the world. "It is still an unending source of surprise for me to see how a few scribbles on a blackboard or on a sheet of paper could change the course of human affairs." LAS

GENERAL, S(13-16), L. *The Conquest of Will: Information Processing in Human Affairs*. Abbe Mowshowitz. A-W, 1976, xvi + 365 pp, $8.95 (P). Around the digital computer there has developed a great deal of information processing technology. This book studies the extent to which information technology facilitates the centralization of power in the hands of a few and the consequences thereof. Bibliography. Chapter references. Index. RJA

GENERAL, S(13-16), P, L**. *Ingenuity in Mathematics*. Ross Honsberger. New Math. Libr., No. 23. MAA, 1975, 204 pp, $4 (P). Reprint of the original edition first published in 1970 by Random House. Excellent exposition at an elementary level. LAS

GENERAL, S. *Games With the Pocket Calculator*. Sivasailam Thiagarajan, Harold D. Stolovitch. Dymax, 1976, 47 pp, $2 (P). Two dozen arithmetic-based games designed for four-function calculators. The games, which require considerable mental manipulation, should help make arithmetic fun--at least for those that get hooked on them. LAS

BASIC, T(13: 1). *Elementary Algebra, Without Trumpets or Drums*. Martin M. Zuckerman. Allyn, 1976, x + 437 pp, $10.95. The title is appropriate. This book is primarily long lists of worked out examples, followed by long lists of problems. Set theoretic notation is avoided. CEC

BASIC, T(13: 1), S. *Beginning Algebra, Second Edition*. John H. Minnick, Raymond C. Strauss. P-H, 1976, xiii + 335 pp, $12.50. Suitable as a review of high school algebra without material on logarithms. Numerous exercises. Selected answers. Glossary. Index. RJA

BASIC, T(13: 1). *Business Mathematics: A Consumer Approach*. Robert P. Webber. HM, 1976, xi + 435 pp, $11.50. Emphasis on practical matters of concern to consumers--installment buying, interest, mortgages, taxes, insurance, stocks and bonds. Also chapters on statistics and computers. LCL

BASIC, T?. *Arithmetic Fundamentals*. Jack G. Pease, Robert G. Russell. Merrill, 1975, xi + 225 pp, $7.95 (P). The material covered really belongs to high schools (9th grade, maybe). Students with modest sophistication should be able to wonder why in this book 5 + 7 = 12 is assumed yet 35 + 47 = 82 is proved (pp. 36-37). The reviewer finds the cartoons enjoyable, but the Chinese numerals (hand-written for the text) crooked, therefore, disappointing. I-CH

BASIC, T(13: 1), S. *Trigonometry for Today*. Robert E. Mosher. Har-Row, 1976, xii + 313 pp, $11.95; *Algebra and Trigonometry for Today*, xv + 537 pp, $12.95. These cognate texts have an appealing format and are suitable for students with little background in high school mathematics. Numerous exercises and selected answers. Appendix on logarithmic methods. Tables. Index. RJA

BASIC, S?. *How to Solve Word Problems in Algebra: A Solved Problem Approach*. Mildred Johnson. McGraw, 1976, 166 pp, $3.95 (P). Emphasis is on mechanics of word-problem solving since the author believes students "can learn mechanics even if they are unable to reason... ." What a sad point of view. CEC

BASIC, T(13: 1). *Applied Mathematics, Fourth Edition, Teacher's Edition*. James F. Johnson, Peter Andris, John G. Bradley. Bruce, 1975, 310 pp, $7.95. Practical mathematics for secondary schools, or vocational courses in technical schools or community colleges. Fundamentals such as fractions, powers and roots, formulas, measurement, etc., precede chapters on the mathematics of buying and selling, construction trades, printing, automobiles and machine shop. Helpful teacher's edition. PSJ

EDUCATION, L*. *The Teaching of Algebra at the Pre-College Level*. Ed: Peter Braunfeld, W.E. Deskins. CEMREL, 1975, xi + 471 pp, $10 (P). A compilation of papers presented by a distinguished group of mathematicians at the Third CSMP International Conference. CEC

EDUCATION, T(15-16: 1). *Helping Children Learn Mathematics: A Competency-Based Laboratory Approach*. Gregory R. Baur, Linda Olsen George. Cummings, 1976, 406 pp, $10.95. Methods text for elementary school mathematics. Considers development of a philosophy, formulation of strategies, teaching techniques, and evaluation. PSJ

EDUCATION, S(16). *Laboratory Activities for Teachers of Secondary Mathematics*. Gerald Kulm. Prindle, 1976, 124 pp, $6.50 (P). Discusses construction and teaching of lab lessons. Contains 17 laboratory activities to be completed by the prospective secondary mathematics teacher. Good bibliography. Appears excellent for use in methods class. PSJ

HISTORY, S(15-17), P, L**. *On the History of Statistics and Probability.* Ed: D.B. Owen. Statistics, V. 17. Dekker, 1976, xiv + 466 pp, $29.50. 21 expository and anecdotal papers, including featured papers by W.G. Cochran, H.O. Hartley and J. Neyman, from the May 1974 Bicentennial Symposium at Southern Methodist University. A unique collection of reflection and history by many of the principals involved in the growth of probability and statistics during the last half century. LAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-499: Logic Conference.* Ed: G.H. Müller, A. Oberschelp, K. Potthoff. Springer-Verlag, 1975, v + 651 pp, $20.10 (P). Proceedings of the International Summer Institute and Logic Colloquium held in Kiel, Germany, from July 17 to August 3, 1974. JAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-492: Infinitary Logic: In Memoriam Carol Karp.* Ed: D.W. Kueker. Springer-Verlag, 1975, 206 pp, $9.50 (P). Carol Karp viewed infinitely long formulas as providing the most natural means for generalizing recursion theory. Unfortunately, her planned monograph on this topic was cut short in 1972 by a three-year losing battle against cancer. Her ideas are expressed in this collection of papers by her colleagues and students. LCL

FOUNDATIONS, T(14-16: 1), S, L. *Elementary Set Theory.* Kam-Tim Leung, Doris Lai-Chue Chen. Hong Kong U Pr, 1975, x + 135 pp, (P). Judicious combination of axiomatic and informal elements written specifically for undergraduate use at the University of Hong Kong. Part I: sets, relations, mappings; Part II: integers via Peano postulates, well-ordering principle and equivalences, ordinals and cardinals. LCL

FOUNDATIONS, P. *Lecture Notes in Mathematics-498: Model Theory and Algebra, A Memorial Tribute to Abraham Robinson.* Ed: D.H. Saracino, V.B. Weispfenning. Springer-Verlag, 1975, 463 pp, $15.20 (P). Sixteen contributed papers (none on nonstandard analysis), preceded by a brief biography of Robinson, a tentative bibliography of his works, and a version of his last paper. LAS

COMBINATORICS, S(13-16), P, L**. *Graphs and Their Uses.* Oystein Ore. New Math. Libr., No. 10. MAA, 1975, viii + 131 pp, $4 (P). Reprint of the original edition first published in 1963 by Random House. Excellent exposition at an elementary level. LAS

NUMBER THEORY, P*, L. *Sieve Methods.* H. Halberstam, H.-E. Richert. Acad Pr, 1974, xiv + 364 pp, $26. An engaging introduction to analytic number theory. Focuses on generalizations of the Twin Prime Conjecture and the Goldbach Conjecture. Highlights include: Brun's Sieve, Selberg's upper bound method, weighted sieves, and Chen's Theorem (every sufficiently large even number is the sum of a prime and a number with at most two prime factors). No exercises. SG

NUMBER THEORY, S(13-16), P, L**. *The Lore of Large Numbers.* Philip J. Davis. New Math. Libr., No. 6. MAA, 1975, x + 165 pp, $4 (P). Reprint of the original edition first published in 1961 by Random House. Excellent exposition at an elementary level. LAS

LINEAR ALGEBRA, T*(14-16: 1, 2), S*, L*. *Linear Algebra and its Applications.* Gilbert Strang. Acad Pr, 1976, xi + 374 pp, $11.95. Written with students at all levels in mind, this text explains rather than deduces. Gaussian elimination heads off the book yet change of basis hides itself in the appendix. In between are exhibits of the surprising power of linear algebra (its applications to statistics, numerical analysis, linear programming and duality) and the necessary need for abstraction (the construction of proofs and precision of mathematical arguments). A teachable and readable text containing more attractions than distractions even for beginning students. An enlightening book, not for MIT students only. I-CH

LINEAR ALGEBRA, T(16-17), S*, L. *Linear Algebra and Multidimensional Geometry.* N.V. Efimov, E.R. Rozendorn. Trans: George Yankovsky. MIR, 1975, 495 pp. A solid classical treatment of linear algebra. Includes tensors, groups of matrices, and applications to analytic and projective geometry. The style is very formal and quite sophisticated and there are no exercises. However, the material is not otherwise readily available in this form. JAS

LINEAR ALGEBRA, T(13-14: 1). *Basic Concepts of Linear Algebra.* S. Isaak, M.N. Manougian. Norton, 1976, x + 391 pp, $11.95. A very basic text with lots of easy exercises. Proofs are given in the text but essentially none are assigned in the exercises. The emphasis is on examples in $R^n$ rather than on development of theory. Nice historical comments and marginal notes. Suitable for use in high school since only one part of the last chapter uses any calculus. JAS

ALGEBRA, T(18), P. *Topics in Finite Groups.* Terence M. Gagen. London Math. Soc. Lect. Notes, No. 16. Cambridge U Pr, 1976, viii + 86 pp, $6.95 (P). The main purpose of these lecture notes is to make more accessible the highly original and important results of Bender on the classification of finite groups. LCL

ALGEBRA, P. *Lecture Notes in Mathematics-495: Representations of Permutation Groups II.* Adalbert Kerber. Springer-Verlag, 1975, 175 pp, $8.20 (P). A continuation of Part I on representation theory of wreath products--particularly the ordinary irreducible characters. Applications to the symmetrization of representations and the theory of enumeration under group action (currently an essential part of combinatorics). LCL

ALGEBRA, S(17-18), P, L. *Commutative Algebra, V. II.* Oscar Zariski, Pierre Samuel. Grad. Texts in Math., V. 29. Springer-Verlag, 1960, x + 414 pp, $14.80. Reprint of the 1958-60 Van-Nostrand edition. JG

FINITE MATHEMATICS, T(13-14: 1, 2), L. *Topics in Mathematics: Models, Logic, Number Theory, Graph Theory, Probability, Statistics, Computer Programming.* Edward P. Merkes. Intext, 1975, xiv + 320 pp, $10.50. The chapters on number theory include the infinitude of primes, the Euclidean algorithm (no proof), linear Diophantine equations, and simple Nim; those on probability and statistics include

linear programming and hypothesis testing; those on computer programming give the rudiments of FORTRAN, with applications to business and continuous functions (finding zeros and extrema). JG

CALCULUS, T(13: 1, 2), *Calculus: A Modeling Approach*. Marvin L. Bittinger. A-W, 1976, xv + 531 pp, $10.95 (P). "Brief introduction to calculus as applied to business, the behavioral sciences, the social sciences, and biology." Topics: differentiation, exponential and logarithmic functions, integration, differential equations, functions of several variables. Very intuitive. Has drill exercises, chapter tests, study guides, and many problems with an applied flavor. A calculator can be helpful.DFA

CALCULUS, T(13: 1), *A Brief Course in Calculus with Applications, Second Edition*. Peter Frank, David A. Sprecher, Adil Yaqub. Har-Row, 1976, xi + 322 pp, $12.95. To the first edition (TR, April 1972) have been added: a chapter on improper integrals, two on functions of several variables, over 600 exercises, many applications. The material on statistics no longer appears. Intuitive, with many illustrations. Assumes and uses no trigonometry. DFA

CALCULUS, T(13-14), *Calculus I*. Brian Knight, Roger Adams. Allen & Unwin, 1975, 118 pp, £1.50 (P); *Calculus II*, 97 pp, £1.50 (P); *Complex Numbers and Differential Equations*, 119 pp, £1.50 (P); *Matrices and Vectors*, 96 pp, £1.50 (P); *Lecturer's Notes*, 69 pp, £3.00 (P). Each chapter in each of the first four books is designed as the basis of a single lecture: it contains pre-lecture material, class discussion exercises, and problems. There are 59 chapters in all. The books are very technique-oriented (few proofs appear), and contain many examples and problems. Because of this, perhaps their best use is in a program of self-study to accompany regular courses. DFA

CALCULUS, T(13: 1), S, *Calculus with Applications to Business and Life Sciences*. Abe Mizrahi, Michael Sullivan. Wiley, 1976, xiii + 414 pp, $12.95. Emphasis is on mathematical models from business, economics, and biology. A problem is presented and then the requisite notions from calculus needed to solve it are presented. Excellent illustrations and diagrams. Many exercises and selected answers. Chapter bibliographies. Tables. Index. RJA

CALCULUS, T(13: 1), *Calculus, A Short Course with Applications to Business, Economics, and the Social Sciences*. Gerald Freilich, Frederick P. Greenleaf. Freeman, 1976, xii + 395 pp, $11.95. Quick, intuitive introduction to differentiation, integration and applications. Trigonometry, exponential and logarithmic functions used sparingly in an inessential way. Includes introduction to calculus of several variables, Lagrange multipliers, linear programming, separable differential equations. LCL

CALCULUS, T(13-14: 1-4), *The Calculus with Analytic Geometry, Third Edition*. Louis Leithold. Har-Row, 1976, xvi + 1102 pp, $16.95. A not-very-extensive revision of the second edition (TR, June-July 1972). Changes include a more detailed treatment of the real numbers, a considerable rearrangement of the material on limits, continuity and the derivative, and the addition of two sections on line integrals. JD-B

CALCULUS, T(15-16: 2-4), *Advanced Calculus*. Leopold Flatto. Williams & Wilkins, 1976, xii + 498 pp, $15.95. The first seven chapters treat functions of one variable, the last seven those of several variables. Includes an appendix on linear algebra. Very rigorous; lots of exercises. Includes a good deal of motivation and introduces many problem-solving techniques. DFA

CALCULUS, S(13-16), P, L**, *What is Calculus About?* W.W. Sawyer. New Math. Libr., No. 2. MAA, 1975, 118 pp, $4 (P). Reprint of the original edition published in 1961 by Random House. Excellent exposition at an elementary level. LAS

REAL ANALYSIS, L, *La Mesure des Grandeurs*. Henri Lebesgue. Blanchard, 1975, 184 pp, 30 F (P). Reprint of the original edition, first published in 1931-35 in *l'Enseignement Mathematique*. An English translation is available (Holden-Day, 1966). LAS

REAL ANALYSIS, T(15-17: 2), *Principles of Mathematical Analysis, Third Edition*. Walter Rudin. McGraw, 1976, x + 342 pp, $14.95. Major changes from earlier editions: deferral of Dedekind cuts to appendix, new material on functions of several variables and integration of differential forms. Many new problems reinforce a substantial text to make this an exceptionally strong option for an introductory analysis course. LAS

DIFFERENTIAL EQUATIONS, T(15-16), L, *Partial Differential Equations, Theory and Technique*. George F. Carrier, Carl E. Pearson. Acad Pr, 1976, xi + 320 pp, $16.50. An attractive introduction to the subject. Presumes some knowledge of ordinary differential equations and physical science. Topics include diffusion, wave, potential equations, Laplace transform methods, first and second order equations, perturbations, Green's functions, variational methods, eigenvalue problems. Many exercises. SG

NUMERICAL ANALYSIS, P, *Topics in Numerical Analysis II*. Ed: John J.H. Miller. Acad Pr, 1975, xiv + 266 pp, $22.75. Proceedings of the second of three planned conferences (First Conference--*Topics in Numerical Analysis I*, TR, June-July 1974), which took place in Dublin in the summer of 1974. JAS

NUMERICAL ANALYSIS, P, *Theory of Approximation with Applications*. Ed: Alan G. Law, Badri N. Sahney. Acad Pr, 1976, xviii + 408 pp, $17.50. Invited and refereed papers from a conference at the University of Calgary in August 1975. LAS

OPTIMIZATION, S(15-18), P, L*, *Studies in Graph Theory*. Ed: D.R. Fulkerson. Stud. in Math., V. 11 and 12. MAA, 1975. *Part I*, xi + 218 pp, $10; *Part II*, xi + 231 pp, $10. Twelve papers, six new and six slightly modified reprints, divided equally between the two parts. Topics range from Hamiltonian circuits and coloring problems to network analysis. Authors include C. Berge, G. Dantzig, D. Fulkerson, R. Gomory, and W. Tutte. LAS

OPTIMIZATION, T(14-16: 1-3), *Operations Research, Second Edition*. Frederick S. Hillier, Gerald J. Lieberman. Holden-Day, 1974, xi + 800 pp, $22.50. "The only big change [from the first edition] is that there are innumerable small improvements." The number of problems has been doubled, and the number

of realistic examples has been increased. It remains a popular, comprehensive, flexible, relatively elementary survey of contemporary operations research techniques. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-497: Analyse Harmonique sur les Groupes de Lie.* Ed: P. Eymard, *et al.* Springer-Verlag, 1975, yi + 710 pp, $21.40 (P). Otherwise unpublished presentations from the joint seminar of the Universities in Nancy and Strasbourg during the academic years 1973-74 and 1974-75. JAS

GEOMETRY, S(13-16), P, L**, *Geometric Inequalities.* Nicholas D. Kazarinoff. New Math. Libr., No. 4. MAA, 1975, 132 pp, $4 (P). Reprint of the original edition first published in 1961 by Random House. Excellent exposition at an elementary level. LAS

TOPOLOGY, T(14-16: 1), L, *Basic Topology: A Developmental Course for Beginners.* Dan E. Christie. Macmillan, 1976, xi + 256 pp, $12.95. A gentle, modified Moore course ranging from elementary set theory through compactness to metric spaces. Many results are left for the student to prove. Begins with a survey of general uses and aspects of topology to motivate the detailed text which follows. LAS

TOPOLOGY, P, *Lecture Notes in Mathematics-496: Topics in K-Theory.* L. Hodgkin, V. Snaith. Springer-Verlag, 1975, 294 pp, $11.50 (P). Two related but independent papers in topological K-theory. The first is a much expanded version of a University of Warwick preprint on the equivariant Kunneth Theorem in K-theory. The second is a detailed study of Dyer-Lashof operators in K-theory. JAS

TOPOLOGY, T?(16-17: 1, 2), S, P, *Topologia.* M. Garcia Marrero, *et al.* Editorial Alhambra, 1975, 591 pp, (P). An unusual topology book in Spanish. Essentially no discussion of compactness or separation properties. Instead there is extensive study of initial and final topologies and uniformities. This discussion is based on a very thorough treatment of continuity. JAS

TOPOLOGY, S(18), P, *A Selective Survey of Axiom-Sensitive Results in General Topology.* H.R. Bennett, T.G. McLaughlin. Math. Ser., No. 12. Texas Tech U, iv + 114 pp, $7 (P). A research-level exposition of matters relating to Martin's axiom, the Ponomarev problem, Souslin's conjecture, the Blumberg problem, and Moore spaces. LAS

TOPOLOGY, T(17-18: 1), P, *A Geometric Approach to Homology Theory.* S. Buoncristiano, C.P. Rourke, B.J. Sanderson. London Math. Soc. Lect. Notes, No. 18. Cambridge U Pr, 1976, 149 pp, $10.95 (P). This book answers the question "How would a geometric (=piecewise-linear) topologist do algebraic topology?" One advantage to this approach: relations between geometry of a space and homology groups become more transparent. Suitable for a second course in topology at the graduate level. PJM

STATISTICS, S(13), *Statistics Made Relevant: A Casebook of Real Life Examples.* Paul Baum, Ernest M. Scheuer. Wiley, 1976, xxii + 217 pp, $5.95 (P). Supplement to statistics textbook. Forty-eight articles, advertisements, etc., from the news media, each followed by leading questions on perforated worksheets. No answers. LCL

STATISTICS, T(16-17: 1), S, P, L, *The Analysis of Time Series: Theory and Practice.* C. Chatfield. Chapman & Hall, 1975, xiv + 263 pp, $19.75. Aims to "bridge the gap between theory and practice." Estimation in the time domain, forecasting, spectral analysis, bivariate processes, and linear systems are considered after an introduction to the basic ideas. FLW

COMPUTER SCIENCE, S, P**, L**, *Computer Power and Human Reason, From Judgment to Calculation.* Joseph Weizenbaum. Freeman, 1976, xii + 300 pp, $9.95. A thoughtful blend of insight, expertise, anecdote and passion that will stand for a long time as a definitive integration of technological and humanistic thought. Ranging widely over the nature of computers and the motivation of computer programmers, Weizenbaum argues forcefully against the common reduction of man's problems and responsibilities to mechanistic computer programs. LAS

COMPUTER SCIENCE, T(16-18: 1), S, P, L, *Foundations of Microprogramming: Architecture, Software, and Applications.* Ashok K. Agrawala, Tomlinson G. Rauscher. Acad Pr, 1976, xvii + 416 pp, $15.50. Presents the architectural, software, and applications aspects of microprogramming. Contains material on microprogramming languages and microprogrammable computer systems commercially available. Concludes with survey chapter on past, present, and future trends. Annotated bibliography for each chapter. RJA

COMPUTER SCIENCE, P, *Computer Science and the Education of Computer Scientists.* Inst Math Appl, 1975, xi + 58 pp, £1.50 (P). Proceedings of a symposium on computer science education in British universities. Discusses the balance between theory and practice and the relationships to computer engineering, mathematics and numerical analysis. RWN

COMPUTER SCIENCE, T(14-15: 1), S, L, *Fundamental Concepts of Programming Systems.* Jeffrey D. Ullman. A-W, 1976, ix + 328 pp, $15.95. An ambitious and exciting second course in computing which touches on many different topics: structured programming, computer organization, data structures, assemblers, loaders, link editors, context-free grammars, compilers, etc. Exercises. Bibliography. Appendix. Index. RJA

COMPUTER SCIENCE, S(13-15), *The Compleat Computer.* Dennie L. van Tassel. SRA, 1976, viii + 216 pp, $5.95 (P). A compendium of about 100 nontechnical articles and excerpts--ranging from newspaper features to science fiction. An entertaining and informative supplement to a first course on computing. LAS

COMPUTER SCIENCE, T(14-15: 1), S, L, *Computer Systems Organization and Programming.* Harry Katzan, Jr. SRA, 1976, 459 pp, $14.95. First four chapters provide material for a computer organization and machine language course. Remaining five chapters provide possibility for choices from among: computer software, programming techniques, data management, and organization of software systems. Each chapter contains a vocabulary list, a questions section, an exercise section, and references. Index.RJA

COMPUTER SCIENCE, T(13: 1), S, L. *Foundation Course in Computer Science.* Bryan Higman. Am Elsev, 1975, xi + 99 pp, $5. The "initial indoctrination" for an introductory computer science course. Contents: algorithms, digital manipulation, computing engines, systems and data structures. Style: interesting lecture notes. As a text, it would need to be supplemented. RWN

COMPUTER SCIENCE, S(13-15). *My Computer Likes Me*--when I speak in BASIC.* Bob Albrecht. Dymax, 1972, 63 pp, $2 (P). Reprint of zany, popular BASIC handbook. See TR, February 1974. LAS

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Systems Programming, Concepts of Operating and Data Base Systems.* David K. Hsiao. A-W, 1975, xxii + 328 pp, $16.95. Includes assembly systems, I/O control systems, batch processing systems, multiprogramming, virtual memory systems, and data base systems. Chapter references. Exercises. Index. RJA

COMPUTER SCIENCE, T(13: 1), S. *Computer Programming: An Individualized Course in FORTRAN IV.* Carl A. Grame, Daniel J. O'Donnell. Westinghouse Learning Pr, 1975. *Learning Guide,* 182 pp, $6 (P); *Scriptbook,* 70 pp, $6 (P); *Instructor's Guide,* 41 pp, $6 (P); *Testbook,* $2 (P). A self-paced course in FORTRAN IV. Suitable for use in a classroom with an instructor or on an individual basis. Cassettes available as learning aids. Material is clearly and attractively organized. Complete set of materials is expensive. Answers to exercises. Appendix: summary of various FORTRAN features. Index. RJA

COMPUTER SCIENCE, T(13: 1), S. *BASIC Programming and Applications.* C. Joseph Sass. Allyn, 1976, xi + 354 pp, $8.95 (P). An introductory programming text centered around BASIC. Valuable at any level, it could also be used in a self-paced way. Excellent chapter formats. Good illustrations and diagrams. Questions and exercises with selected answers. Several appendices. References. Index. RJA

COMPUTER SCIENCE, S. *What To Do After You Hit Return, or P.C.C.'s First Book of Computer Games.* Bob Kahn. People's Computer, 1975, vii + 157 pp, $6.95 (P). Several dozen games and simulations together with BASIC code for perhaps half of the games. An outlandish (11" x 15") book in the distinctive, eclectic style of other PCC publications. LAS

SYSTEMS THEORY, S. *Simulation in Human Systems: Decision-Making in Psychotherapy.* Richard Bellman, Charlene Paule Smith. Wiley, 1973, xviii + 205 pp, $15.75. Illustrates how mathematical ideas such as "system" and "multistage decision process" can be used to study the structure of the communication and decision processes of an initial psychotherapy interview. The computer simulation of the initial psychotherapy interview is disucssed in general terms. There is neither a single mathematical formula nor a computer program in the entire book. I-CH

SYSTEMS THEORY, T(15-17: 1), S, L. *Systems Analysis: Definition, Process, and Design.* Philip C. Semprevivo. SRA, 1976, xv + 350 pp, $12.95. Begins by defining a system and systems analysis. Proceeds to describe the various tools of the systems analyst: flowcharts, decision tables, questionnaires. Provides extensive discussion of system components. Relies heavily on practical problems to illustrate ideas. Each chapter contains a vocabulary list and problems. Appendices. Index. RJA

SYSTEMS THEORY, T(16-17: 1, 2), P. *Transform Techniques for Probability Modeling.* Walter C. Giffin. Acad Pr, 1975, xiii + 233 pp, $24.50. Application of Laplace, Fourier, etc., transforms to probability, queueing theory, and linear systems analysis. PJM

SYSTEMS THEORY, P. *Entscheidungstheorie.* Kuno Egle. Birkhauser, 1975, 232 pp, Sfr. 32 (P). A general theoretical treatment of decision theory which subsumes various optimization theories. The mathematical tools are: pre-ordered algebraic structures, topology, and category theory. JAS

APPLICATIONS (BIOLOGY), P. *Population Genetics and Ecology.* Ed: Samuel Karlin, Eviatar Nevo. Acad Pr, 1976, xiv + 832 pp, $25.50. Proceedings of a conference-workshop in Israel, March 1975. In three parts: field and laboratory studies, models and evidence, theoretical studies. LAS

APPLICATIONS (BIOLOGY), T(16-17), S, P, L. *Stochastic Models in Biology.* Narendra S. Goel, Nira Richter-Dyn. Acad Pr, 1974, x + 269 pp, $19.50. Mathematical facts about stochastic processes are summarized and then applied to problems of population size, population genetics, neuron firing, chemical kinetics, and photosynthesis. FLW

APPLICATIONS (BIOLOGY), S, P, L. *Perspectives in Biometrics, V. 1.* Ed: Robert M. Elashoff. Acad Pr, 1975, xii + 200 pp, $21. The first in a series of critical surveys and discussions of methodological aspects of data analysis. Articles on interactive statistical computation, adaptive sampling, computer-aided prognosis, multivariate methods in classification, the jackknife, and estimation of the inverse cumulative function. FLW

APPLICATIONS (POLITICAL SCIENCES), T(15-17), S*, L*. *Paradoxes in Politics: An Introduction to the Nonobvious in Political Science.* Steven J. Brams. Free Pr, 1976, xvii + 231 pp, $6.95 (P). An entirely nontechnical exposition of voting and game-theoretic paradoxes requiring of the reader only "sustained logical argument." A marvelous body of evidence to motivate mathematical modelling in political science. LAS

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# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

Assistant Professor Michael Olinick, Middlebury College, has been promoted to Associate Professor.

Dr. Dirk J. Struik, Professor Emeritus in the Department of Mathematics at MIT and a member of the Technology Studies Program, has been awarded a gold medal of achievement by the National University of Mexico for his service to the teaching and development of mathematics in Mexico over the years.

Assistant Professor Sidelle Wertheimer, Kean College, has been named Coordinator of Developmental Programs to concert the activities of the various segments of the college involved in developmental studies.

Colonel Benjamin Stern, USA (retired), Assistant Professor Emeritus, University of Nebraska at Omaha, died on November 12, 1975. He was a member of the Association for seventeen years.

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# MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twenty-first annual meeting of the Northeastern Section of the MAA was held at Simmons College, Boston, Massachusetts, on November 29, 1975; there were 127 people in attendance. The section chairperson, Anne F. O'Neill, presided.

The morning meeting was devoted to a panel discussion:

*New priorities for undergraduate education in the mathematical sciences*, Moderator: P. J. Davis, Brown University. Panel: Garrett Birkhoff, Harvard University; B. E. Meserve, University of Vermont; H. O. Pollak, President of MAA, Bell Laboratories.

At the afternoon business meeting, the following officers were elected for the coming year: Chairperson, G. P. Murphy, University of Maine; Vice-Chairperson, R. S. Pieters, Groton School; Secretary-Treasurer, G. W. Best, Phillips Academy. Chairperson Anne F. O'Neill announced that the fall 1976 meeting of the Section would be held at Rhode Island College, Providence, Rhode Island. The business meeting concluded with Henry Pollak describing new MAA programs.

The following talks completed the program:

*Deciding some undecidable topological statements*, by W. W. Comfort, Wesleyan University.

*Some aspects of topological graph theory*, by Joan P. Hutchinson, Tufts University.

G. W. BEST, *Secretary-Treasurer*

## CALENDAR OF FUTURE MEETINGS

Fifty-sixth Summer Meeting, University of Toronto, August 26–28, 1976.

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 wks. bef. mtg.

FLORIDA, University of South Florida, Tampa, March 4–5, 1977.

ILLINOIS, second Friday/Saturday in May.

INDIANA

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, Tabor College, Hillsboro, March or April 1977.

KENTUCKY, early April. Deadline for papers 6 wks. bef. mtg.

LOUISIANA-MISSISSIPPI, University of New Orleans, Louisiana, February 25–26, 1977.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 wks. bef. mtg.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, early November and early May.

NORTH CENTRAL, South Dakota State University, Brookings, October 22–23, 1976.

NORTHEASTERN, University of New Hampshire, Durham, June 18–19, 1976 (Special Spring Meeting).

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO

OKLAHOMA-ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 wks. bef. mtg.

PACIFIC NORTHWEST, Portland State University, Portland, Oregon, June 18–19, 1976.

PHILADELPHIA, Montgomery County Community College, Blue Bell, Pennsylvania, November 20, 1976.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers 8 wks. bef. mtg.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 wks. bef. mtg.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, Loyola Marymount University, Los Angeles, March 12, 1977.

SOUTHWESTERN, usually in April. Deadline for papers 2 wks. bef. mtg.

TEXAS, Baylor University, Waco, April 1–2, 1977.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 wks. bef. mtg.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 20–26, 1977.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, San Francisco, October 29–30, 1976.

AMERICAN MATHEMATICAL SOCIETY, University of Toronto, August 24–27, 1976.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of Tennessee, Knoxville, June 14–17, 1976.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, Chase Park Plaza Hotel, St. Louis, Missouri, January 27–28, 1977.

ASSOCIATION FOR WOMEN IN MATHEMATICS

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

INTERNATIONAL PEIRCE CONGRESS, Grand Hotel Krasnopolsky, Amsterdam, June 16–20, 1976.

MU ALPHA THETA, West Chester State College, West Chester, Pennsylvania, August 9–11, 1976.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cincinnati, Ohio, April 20–23, 1977.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami Beach, November 3–5, 1976.

PI MU EPSILON, University of Toronto, August 26–28, 1976.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Commodore Perry Hotel, Toledo, Ohio, November 11–13, 1976.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Georgia Institute of Technology, Atlanta, October 18–20, 1976.

# APPLICATIONS OF UNDERGRADUATE MATHEMATICS IN ENGINEERING

written and edited by Ben Noble
Mathematics Research Center, U. S. Army, University of Wisconsin

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The author wishes to announce to the readers of this MONTHLY

**The Foundations of Geometry and the Non-Euclidean Plane**

by George E. Martin, SUNY at Albany

was published in August of 1975. This text is in the IEP Series in Mathematics published by Dun-Donnelley Publishing Corp., 666 Fifth Avenue, New York, NY 10019.

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(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 7

CODEN: AMMYAE

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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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## AMERICAN MATHEMATICS FROM 1940 TO THE DAY BEFORE YESTERDAY*

J. H. EWING, W. H. GUSTAFSON, P. R. HALMOS,
S. H. MOOLGAVKAR, W. H. WHEELER, AND W. P. ZIEMER

**Preface.** What is the best way to present the small fragment of history described by the title above? Should this report occupy itself mainly with the statistics of the growth of Mathematical Reviews?, with the lives of mathematicians?, with lists of books and papers?, or with retracing the influences and implications that led from the bridges of Königsberg first to *analysis situs* and then to homological algebra? We decided to do none of these things, but, instead, to tell as much as possible about mathematics, the live mathematics of today. To do so within prescribed boundaries of time and space, we present the subject in the traditional "battles and kings" style of history. We try to describe some major victories of American mathematics since 1940, and mention the names of the winners, with, we hope, enough explanation (but just) to show who the enemy was. The descriptions usually get as far as statements only. We omit all proofs, but we sometimes give a brief sketch of how a proof might go. A sketch can be one sentence, or two or three paragraphs; its purpose is more to illuminate than to convince.

Progress in mathematics means the discovery of new concepts, new examples, new methods, or new facts. Schwartz's concept of distribution, Milnor's example of an exotic sphere, Cohen's method of forcing, and the Feit–Thompson theorem about simple groups are surely major by any standards. It was no trouble to find such victories to include in our list; the difficulty was to decide what to exclude. We formulated some rough rules (e.g., theorems, not theories); since at least some aspects of applied mathematics were covered by other presentations, we restricted our attention to pure mathematics; we excluded work that had neither root, nor branch, nor flower in the U.S.; and, in deciding which of two candidates to keep, we leaned toward the one of greater general interest. ("Of general interest" is not quite the same as "famous", but it's close.)

We ended up with ten "battles and kings", and we think that they draw a fair picture of what's been happening. We do not say that our ten are greater than any others, nor that they are necessarily maximal in the mathematical sense of not being lesser than any others. We do say that they would all appear, and would be discussed with respect, in any responsible history of our place and time. The total number of such "non-omittable" victories is certainly greater than ten; it may be twenty or even forty. The choice of our ten was influenced by the limits of our competence and by our personal preferences; that could not be helped. Anyone else would very likely have selected a different set of

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* An expurgated version of this paper was delivered (by P. R. H.) as an invited address at the San Antonio meeting of the Mathematical Association of America on January 24, 1976.The authors are grateful to W. Ambrose, G. Bennett, J. L. Doob, L. K. Durst, I. Kaplansky, R. Narasimhan, I. Reiner, and F. Treves for help, including advice, references, and, especially, encouragement.

The preparation of this paper was supported in part by a grant from the National Science Foundation.

ten. We hope and think, however, that everyone's list would have a large overlap with ours, and that the local differences would not essentially alter the global picture.

In history, every moment influences its successors; to restrict attention to a time interval may be often necessary, and sometimes possible, but it is rarely natural. In the same way, every place influences all others. Since the topology of the surface of our globe is much more intricate than that of the time line, to restrict attention to one country is almost impossible. The history of mathematics is no exception: trying to describe what happened *here*, we frequently yield to the pressure of distant influences and discuss what happened *there*. We were able to stay reasonably close to our original charge just the same; if fractional credits are assigned, something like 8.25 of the ten accomplishments described below can be called American. It might be of interest to observe also that over half of the original papers we refer to appeared in the *Annals of Mathematics*.

The order of the presentations might have been based on any of several principles (e.g., what actually happened first?, what is a prerequisite for what?). We decided to arrange them in order of complexity of the underlying category, or, in other words, very roughly speaking, in order of distance from the foundations of mathematics. At the end of each section there is a small list of pertinent references. The list is intentionally incomplete. All it contains is one (or, if necessary, two or three) of the earliest papers in which the discovery appears, and a more recent exposition of the discovery whenever we could find one.

**Continuum Hypothesis.** All mathematics is derived from set theory (or, in any event, many of us believe it is) and the manipulation of sets is a simple, natural exercise (or, in any event, students have very little trouble catching on to it). Everything that any working mathematician ever needs to know about sets (and a few extra things that he never thought he needed to know) could be summarized on one printed page (or three or four printed pages, if motivation is wanted along with the formalism). Such a page would state the basic ways of making new sets out of old (e.g., the formation of sets consisting of specified elements, the formation of unions of sets of sets, and the formation of the power set, i.e., the set of all subsets, of a set); it would describe the basic properties of sets (e.g., that two sets are equal if and only if each is a subset of the other, and that no set has elements that are themselves sets that have elements continued on downwards *ad infinitum*); and it would state (as an assumption or as a conclusion, but in either case as a description of the universe that sets live in) that infinite sets exist. These basic set-theoretic statements might be regarded either as obvious factual observations or as an axiomatic description of the ZF (Zermelo–Fraenkel) structure. In either case it would be a simple matter to code them in the language of a suitable (not very complicated) computer. Such a machine could easily be taught all the rules of inference that mathematicians ever use. If, in addition, its basic data were increased by two more statements, it could, in principle, easily print out all known mathematics (and a lot that is not yet known).

The two statements that history has subjected to extra scrutiny are AC (the axiom of choice) and GCH (the generalized continuum hypothesis). AC says that, for each set $X$, there is a function $f$ from the power set of $X$ into $X$ itself such that $f(A) \in A$ for each non-empty subset $A$ of $X$; GCH says that each subset of the power set of an infinite set $X$ is in one-to-one correspondence either with some subset of $X$ or with the entire power set — there is nothing in between.

Is AC true? The question has often been likened to a similar one about Euclid's parallel postulate. In both cases there is a more or less pleasant axiom system and a less pleasant, more complicated, non-obvious additional axiom. If the extra axiom is a consequence of the basic ones, it is true, and all is well; if its negation is a consequence of the basic ones, it is false, and, for better or for worse, the question is definitively answered. The same question can, of course, be asked about GCH. It has long been known that GCH implies AC; in view of this there is an obvious connection between the two answers.

The answers are subtle and profound intellectual achievements. Gödel proved (1940) that AC and GCH are not false (i.e., that they are consistent with the axioms of ZF), and Paul Cohen proved (1964) that they are not true (i.e., that they are independent of ZF).

Gödel argued by the construction of a suitable model. If, he said, ZF is consistent, so that there is a universe $V$ of sets satisfying the basic axioms of ZF, then, he proved, there is a "sub-universe" that also satisfies them, and in which, moreover, both AC and GCH are true. The sub-universe Gödel constructed was the class $L$ of "constructible" sets. (The word is given a very liberal but completely precise meaning; roughly speaking, the constructible sets are the ones that can be obtained from the empty set by a transfinite sequence of elementary set-theoretic constructions.) The class $L$ is a substructure of $V$ in the familiar mathematical sense of that word: the objects of $L$ are some of the objects of $V$, and the relation $\in$ among them is the restriction of the set-theoretic $\in$ in $V$ to the objects of $L$. The existence of a model such as $L$ (constructed out of a hypothetically consistent model $V$) proves the consistency of AC and GCH the same way as the existence of the Euclidean plane proves the consistency of the parallel postulate.

Cohen's argument was similar but harder. It is reminiscent of Felix Klein's construction of a Lobachevskian plane by endowing a Euclidean disk with a new metric. Cohen started with a suitable model of ZF and adjoined new objects to it. The new objects are "classes" (but not sets) in the old model. The adjunctions proceed by a new method called "forcing", which, once it was discovered, was found to be applicable in many parts of set theory. Cohen's proof constructs an infinite sequence of better and better finite approximations to the new objects. Roughly speaking, each property of the new model is "forced" by properties of the old model and one of the approximations. Depending on how the details are adjusted, the end result can be a model of ZF in which AC is false, or a model of ZF in which AC is true but even the classical un-generalized continuum hypothesis CH is false. (CH is GCH for a countably infinite set.) Conclusion: AC and CH are independent of ZF.

REFERENCES. [1] P. J. Cohen, The independence of the continuum hypothesis, Proc. N. A. S., 50(1963) 1143–1148 and 51(1964) 105–110. [2] P. J. Cohen, Set theory and the continuum hypothesis, Benjamin, New York, 1966(MR 38 # 999). [3] J. B. Rosser, Simplified independence proofs, Academic Press, New York, 1969(MR 40 # 2536). [4] T. J. Jech, Lectures in set theory, with particular emphasis on the method of forcing, Springer, Berlin, 1971(MR 48 # 105).

**Diophantine Equations.** The continuum hypothesis was the subject of Hilbert's first problem (in the famous list of 23 problems that he proposed in 1900); Hilbert's tenth problem concerned the solvability of Diophantine equations. The problem was to design an algorithm, a computational procedure, for determining whether an arbitrarily prescribed polynomial equation with integer coefficients has integer solutions. It is in some respects more natural and sometimes technically easier to discuss the *positive* integer solutions (solutions in $\mathbb{Z}_+$) of polynomial equations with *positive* integer coefficients. Caution: that does not mean equations such as $p(x) = 0$ only. The problem includes the search for $x$'s such that $p(x) = q(x)$; more generally, it includes the search for $n$-tuples $(x_1, \ldots, x_n)$ such that $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$; and, in complete generality, it means the search for $n$-tuples $(x_1, \ldots, x_n)$ for which there exist $m$-tuples $(y_1, \ldots, y_m)$ such that

$$p(x_1, \ldots, x_n, y_1, \ldots, y_m) = q(x_1, \ldots, x_n, y_1, \ldots, y_m).$$

For each $p$ and $q$ (in $n + m$ variables) the solution set, in the latter sense, is called a "Diophantine set" in $\mathbb{Z}_+^n$.

What does it mean to say that there is an algorithm for deciding solvability? A reasonable way to answer the question is to offer a definition of computability for sets and functions, and then to define an algorithm in terms of computability.

When does a function from $\mathbb{Z}_+$ to $\mathbb{Z}_+$, or, more generally, a function from $\mathbb{Z}_+^n$ to $\mathbb{Z}_+$ deserve to be called "computable"? There is general agreement on the definition nowadays: computable functions (also called "recursive" functions) are the ones obtained from certain easy functions (constant, successor, coordinate) by three procedures (composition, minimalization, primitive recursion). The details do not matter here (they won't be used anyway); it might be comforting to know, however, that they are not at all difficult. A set (in $\mathbb{Z}_+$, or, more generally, in $\mathbb{Z}_+^n$) will be called computable in case its

characteristic function is computable. Consequence: a set (in $\mathbb{Z}_+^n$) is computable if and only if its complement is computable.

Consider now all polynomial equations (in the sense described above), and let $\{E_1, E_2, E_3, \ldots\}$ be an enumeration of them. (In order for what follows to be in accord with the intuitive concept of an algorithm, the enumeration should be "effective" in some sense. That can be done, and it is relatively easy.) The indices $k$ for which $E_k$ has a solution (in the sense described above) form a subset $S$ of $\mathbb{Z}_+$. The Hilbert problem (is there an algorithm?) can be expressed as follows: is $S$ a computable set? The answer is no. The answer was a long time coming; it is the result of the cumulative efforts of J. Robinson (1952), M. Davis (1953), H. Putnam (1961), and Y. Matijasevič (1970).

The central concept in the proof is that of a Diophantine set, and the major step proves that every computable set is Diophantine. The techniques make ingenious use of elementary number theory (e.g., the Chinese remainder theorem, and a part of the theory of Fibonacci numbers, or, alternatively, of Pell's equation). The proof exhibits some interesting Diophantine sets whose Diophantine character is not at all obvious (e.g., the powers of 2, the factorials, and the primes).

One way to prove that $S$ (the index set of the solvable equations) is not computable is by contradiction. If $S$ were computable, then it would follow (by a slight bit of additional argument) that each particular Diophantine set (i.e., the solution set of each particular equation) is computable, and hence (by the "major step" of the preceding paragraph) that the complement of every Diophantine set is Diophantine. The contradiction is derived by exhibiting a Diophantine set whose complement is not Diophantine.

This last step uses a version of the familiar Cantor diagonal argument. The idea is "effectively" to enumerate all Diophantine subsets of $\mathbb{Z}_+$, as $\{D_1, D_2, D_3, \ldots\}$, say, prove that the set $D^* = \{n: n \in D_n\}$ is Diophantine (that takes some argument), and, finally, to prove that the complement $\mathbb{Z}_+ - D^* = \{n: n \notin D_n\}$ is not Diophantine (that's where Cantor comes in).

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**Simple Groups.** So much for the foundations. The next subject up the ladder is algebra; in the present instance, group theory.

Every group $G$ has two obvious normal subgroups, namely $G$ itself and, at the other extreme, the subgroup 1. A group is called "simple" if these are all the normal subgroups it has.

Simple groups are like prime numbers in two ways: they have no proper parts, and every finite group can be constructed out of them. (By general agreement the trivial positive integer 1 is not called a prime, but the trivial group 1 is called simple. Too bad, but that's how it is.)

Suppose, indeed, that $G$ is finite, and let $G_1$ be a maximal normal subgroup of $G$. (To say that $G_1$ is maximal means that $G_1$ is a proper normal subgroup of $G$ that is not included in any other proper normal subgroup of $G$.) If $G$ is simple, then $G_1 = 1$; in any event, the maximality of $G_1$ implies that the quotient group $G/G_1$ is simple. The relation between $G$, $G_1$, and $G/G_1$ (group, normal subgroup, quotient group) is sometimes expressed by saying that $G$ is an extension of $G/G_1$ by $G_1$. In this language, every finite group (except the trivial group 1) is an extension of a simple group by a group of strictly smaller order. The statement is a group-theoretic analogue of the number-theoretic one that says that every positive integer (except 1) is the product of a prime by a strictly smaller positive integer.

If $G_1$ is not trivial, the preceding paragraph can be applied to it; the result is a maximal normal subgroup $G_2$ in $G_1$, such that $G_1$ is an extension of the simple group $G_1/G_2$ by $G_2$. The procedure can

be repeated so long as it produces non-trivial subgroups; the end-product is a chain

$$G = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n = 1$$

(a "composition series") with the property that each $G_i/G_{i+1}$ is simple ($i = 0, \ldots, n-1$). A great part of the problem of getting to know all finite groups reduces in this way to the determination of all finite simple groups. (The celebrated Jordan-Hölder-Schreier theorem is the comforting reassurance that, to within isomorphism, the composition factors $G_i/G_{i+1}$ are uniquely determined by $G$, except for the order in which they occur.)

The abelian ones among the finite simple groups are easy to determine: they are just the cyclic groups of prime order. That's easy. What's hard is to find all non-abelian ones. Some examples of simple groups are easy to come by; among permutation groups, for instance, the most famous ones are the alternating groups of degree 5 or more. The known simple groups did not exhibit any pattern, and even the simplest questions about them resisted attack. Burnside conjectured, for instance, that every non-abelian simple group has even order, but that conjecture stood as an open problem for more than 50 years.

In a spectacular display of group-theoretic power, Feit and Thompson (1963) settled Burnside's conjecture (it is true). The proof occupies an entire issue (over 250 pages) of the *Pacific Journal.* It is technical group theory and character theory. Some reductions in it have been made since it appeared, but no short or easy proof has been discovered. The result has many consequences, and the methods also have been used to attack many other problems in the theory of finite groups; a subject that was once pronounced dead by many has shown itself capable of a vigorous new life.

REFERENCE. W. Feit and J. G. Thompson, Solvability of groups of odd order, Pac. J. Math., 13(1963) 775–1029 (MR29 # 3538).

**Resolution of Singularities.** Algebra becomes richer, and harder, when it is mixed with and applied to geometry; one of the richest mixtures is the old but very vigorous subject known as algebraic geometry. This section reports the solution of an old and famous problem in that subject.

Let $k$ be an algebraically closed field, and let $k^n$ be, as usual, the $n$-dimensional coordinate space over $k$. (The heart of the matter in what follows will be visible to those who insist on sticking to the field of complex numbers in the role of $k$.) An "affine algebraic variety" $V$ in $k^n$ is the locus of common zeros of a collection of polynomials in $n$ variables with coefficients in $k$. Since only the zeros matter, the collection itself is not important; it can be replaced by any other collection that yields the same locus. Thus, if $R$ is the ring of *all* polynomials in $n$ variables with coefficients in $k$, and if $I$ is the ideal in $R$ generated by the prescribed collection, then $I$ will define the same variety; there is, therefore, no loss of generality in assuming that the collection was an ideal to begin with.

The objects of interest on varieties are their "singular points". Intuitively, these are points where the "tangent vectors" are not as they should be. Consider, for example, the curves defined by

$$y^2 = x^3 + x^2 \quad \text{and} \quad y^2 = x^3.$$

(Since the ground field was restricted to be algebraically closed, the *real* planar curves with these equations are not the right things to look at, but they are more lookable at than the complex curves, which lie in the complex plane. Warning: the complex plane has four real dimensions. To the algebraic geometer, the familiar "complex plane" of analysis is the complex *line.*) The first of these comes in to the origin from the first quadrant with slope 2, has a loop in the left half plane, and goes out from the origin to the fourth quadrant with slope $-2$; it has the origin as a double point. The other one comes in to the origin from the first quadrant with slope 0, and goes out the same way to the fourth quadrant; it has the origin as a cusp.

The effective way to deal with singular points begins by giving a purely algebraic description of

them. Consider, for this purpose, the ring $R_V$ of polynomial functions on $V$ (i.e., the restrictions of the polynomials in $R$ to $V$). If $N_V$ is the ideal of $R$ consisting of the polynomials that vanish on $V$, then, clearly, $R_V = R/N_V$. Each point $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $V$ induces a maximal ideal $N_\alpha$ in $R$ (consisting of the set of polynomials that vanish at $\alpha$); clearly $N_V \subset N_\alpha$.

The next step (in the program of defining singular points algebraically) is to form a new ring that studies the local behavior of functions near $\alpha$. The idea is (very roughly) this. (i) Consider pairs $(U, f)$, where $U$ is a "neighborhood" of $\alpha$ and $f$ is a rational function with no poles in $U$. (ii) Define an equivalence relation for pairs by writing $(U, f) \sim (U', f')$ exactly when there is a neighborhood $U''$ of $\alpha$, included in $U \cap U'$, such that $f = f'$ on $U''$. (iii) The equivalence classes ("germs") form a ring (with, for example,

$$[(U, f)] + [(U', f')] = [(U \cap U', f + f')]),$$

called the "local ring" of $V$ at $\alpha$.

From the algebraic point of view, the preceding topological considerations are just heuristic; they will now be replaced by an algebraic construction. The process is, appropriately, called "localization". (i) Consider pairs $(f, g)$, where $f$ and $g$ are in $R$ and $g \notin N_\alpha$. (ii) Define an equivalence relation for pairs by writing $(f, g) \sim (f', g')$ exactly when there is an $h$ not in $N_\alpha$ such that $h \cdot (fg' - gf') = 0$. (iii) Write $f/g$ for the equivalence class of $(f, g)$. The equivalence classes form a ring $R_\alpha$ (with the usual rules of operations for fractions). The ring $R_\alpha$ is indeed a "local ring" in the customary algebraic sense: it has a *unique* maximal ideal, namely the one formed by the elements of $R_\alpha$ that vanish at $\alpha$.

To motivate the next step, pretend, again, that the subject is not algebraic geometry, but analytic geometry. In that case $R_\alpha$ would consist of Taylor series at $\alpha$ convergent near $\alpha$, and the ideal $N_\alpha$ of germs vanishing at $\alpha$ would consist of the Taylor series at $\alpha$ with vanishing constant term. The linear terms of a Taylor series are, in some sense, first order differentials. One way to capture just those terms is to "ignore" higher order terms. More precisely: consider the ideal $N_\alpha^2$, which, in the analytic case, consists of the Taylor series with vanishing constant term and vanishing linear term, and form $N_\alpha/N_\alpha^2$.

The definition is now easy to formulate. The "dimension" $d$ of $V$ is, by definition, the minimum of the dimensions (over the field $k$, of course) of all the quotient spaces $N_\alpha/N_\alpha^2$; a point $\alpha$ is "singular" when $\dim(N_\alpha/N_\alpha^2) > d$. It is not difficult to see that for the two curves mentioned as examples above, the origin is indeed a singular point in the sense of this definition.

One of the main problems of algebraic geometry is to "get rid of" singular points. For this purpose the discussion is restricted to "irreducible" varieties, i.e., to the ones for which $R_V$ is an integral domain, or, equivalently, $N_V$ is a prime ideal. In that case, form the field of fractions $F_V$ of $R_V$. Two varieties $V$ and $W$ are "birationally equivalent" if $F_V$ and $F_W$ are isomorphic. This means roughly that $V$ and $W$ parametrize one another by rational mappings at all but finitely many places. The problem of "resolution of singularities" is that of finding a non-singular variety birationally equivalent to $V$.

The subject has a long history. Curves were handled by Max Noether in the 19th century. Surfaces were the subject of much geometric discussion by the Italian school; a rigorous proof was found by R. J. Walker (1935). For varieties of arbitrary dimension, over fields of characteristic 0, the final victory was inspired by Zariski's work; it was won by Hironaka (1964).

REFERENCE. H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. Math., 79(1964) 109–326 (MR 33 # 7333).

**Weil Conjectures.** The mathematician's work is often most difficult (and most rewarding) when he reasons by analogy, when he guesses that *this* situation ought to be just like *that* one: In 1949 A. Weil, reasoning in this way, proposed three conjectures that have profoundly influenced the development of algebraic geometry over the past 25 years.

The conjectures appeared in a paper entitled "Numbers of solutions of equations in finite fields",

which was ostensibly a survey of previous work. Counting the number of solutions of a polynomial equation in several variables over a finite field was a classical problem, investigated by Gauss, Jacobi, Legendre, and others, but Weil took a new point of view. To understand his approach, consider the special case of the homogeneous equation

$$(\dagger) \qquad a_0x_0^n + a_1x_1^n + \cdots + a_rx_r^n = 0,$$

where the coefficients $a_i$ are in the prime field $F$ of $p$ elements. The basic problem is to count the number of solutions in $F$, but, to number theorists, it is just as important to count the number of solutions in any finite extension field of $F$. Recall that, for every positive integer $k$, there is a unique extension field $F_k$ of $F$ with $p^k$ elements. What Weil did was to count the number of solutions of (†) in each field $F_k$, and then code that information in a generating function.

To do this economically, examine the solution set of an equation such as (†). There is, of course, always the trivial solution, where all the $x_i$ are zero; that one is justly regarded as trivial. If $(x_0, x_1, \ldots, x_r)$ is a non-trivial solution, and if $0 \neq c \in F_k$, then $(cx_0, cx_1, \ldots, cx_r)$ is also a non-trivial solution. Each non-trivial solution generates in this way $p^k - 1$ others, and there is no virtue in counting them all separately. It is natural, therefore, to consider the $r$-dimensional "projective space" $P^r(F_k)$, i.e., the set of non-trivial ordered $(r+1)$-tuples of elements of $F_k$, where two are identified if one is a scalar multiple of the other. (This is exactly analogous to the familiar real and complex projective spaces.) The problem in these terms is to count the number of "points" in $P^r(F_k)$ that are "solutions" of (†).

That is precisely what Weil did. He let $N_k$ be the number of solutions of (†) in $P^r(F_k)$, considered the generating function $G$,

$$G(u) = \sum_{k=1}^{\infty} N_k u^{k-1},$$

and proved a remarkable statement: $G$ is the logarithmic derivative of a *rational* function. That is: there exists a rational function $Z$ such that

$$\sum_{k=1}^{\infty} N_k u^{k-1} = \frac{d}{du} \log Z(u),$$

or, in other words, if

$$Z(u) = \exp\left( \sum_{k=1}^{\infty} \frac{N_k}{k} u^k \right),$$

then $Z$ is rational. The function $Z$ satisfies a functional equation analogous to the one satisfied by the Riemann zeta function, and it is appropriate to refer to $Z$ as the zeta function associated with the equation (†). Motivated by classical problems that the Riemann zeta function gave rise to, Weil studied and was able to determine many properties of the zeros and the poles of $Z$.

Here is where Weil's paper reaches its climax. Weil wanted to extend the results about (†) to algebraic varieties in $P^r(F_k)$, i.e., to the solution sets of *systems* of homogeneous equations in $r$ variables. The notion of a zeta function, originally defined by Riemann, was extended by Dedekind to algebraic number fields, by Artin to function fields, and now, by Weil, to algebraic varieties. (The varieties to be considered should be non-singular. It doesn't matter here what the general definition of that condition is; for most fields it can be defined as usual by requiring that the Jacobian of the system of equations have maximal rank at every point.) Given a system of equations, with coefficients in $F$, let $N_k$ be, as before, the number of solutions in $P^r(F_k)$. Weil advanced the following conjectures. One: the function $Z$, defined as before by

$$Z(u) = \exp\left( \sum_{k=1}^{\infty} \frac{N_k}{k} u^k \right),$$

is rational. Two: $Z$ satisfies a particular functional equation, which, as before, bears a striking resemblance to the one satisfied by the Riemann zeta function. Three: the reciprocals of the zeros and the poles of $Z$ are algebraic integers and their absolute values are powers of $\sqrt{p}$. (This is called the generalized Riemann hypothesis.)

All this might seem far removed from what is normally thought of as geometry, and, although several examples were known, it might seem that Weil made his conjectures on strikingly little evidence. What was really behind the conjectures? The answer is contained in the last paragraph of Weil's paper, where he suggests that there is an analogy between the behavior of these varieties (for fields of characteristic $p$) and that of the classical varieties (for the field of complex numbers).

In 1960 Dwork established the rationality conjecture (without the condition of non-singularity). The final triumph came in 1974: using twenty years' of results of the Grothendieck school, Deligne established all the Weil conjectures, and, perhaps more importantly, proved that there is a beautiful connection between the theory of varieties over fields of characteristic $p$ and classical algebraic geometry. "God ever geometrizes", said Plato, and "God ever arithmetizes", said Jacobi; the Weil conjectures show, better than anything else, how He can do both at once.

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**Lie Groups.** So much for algebra, with or without geometry. The next subject points toward some of the later analytic ones by mixing algebra with topology. The result, like a few other outstanding results of mathematics, seems to get something for nothing, or, at the very least, to get quite a lot for an astonishingly low price. One of the most famous results of this kind occurs in the early part of courses on complex function theory: it asserts that a differentiable function on an open subset of the complex plane is necessarily analytic.

Hilbert's fifth problem asked for such a something-for-nothing result. The context is the theory of topological groups. A topological group is a set that is both a Hausdorff space and a group, in such a way that the group operations

$$(x, y) \mapsto xy \quad \text{and} \quad x \mapsto x^{-1}$$

are continuous. A typical example is the set of all $2 \times 2$ real matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ with $x > 0$; the topological structure is that of the right half plane (all $(x, y)$ with $x > 0$), and the multiplicative structure is the usual one associated with matrices. Equivalently: define multiplication in the right half plane by

$$(x, y) \cdot (x', y') = (xx', xy' + y);$$

since

$$(x, y)^{-1} = \left(\frac{1}{x}, \frac{-y}{x}\right),$$

it is clear that both multiplication and inversion are continuous.

This example has an important special property: it is "locally Euclidean" in the sense that every point has a neighborhood that is homeomorphic to an open ball in (2-dimensional) Euclidean space. (Equivalently: every point has a "local coordinate system".) An even more important special property of the example is that the group operations, regarded as functions on the appropriate Euclidean space, are not only continuous but even analytic. If a group is locally Euclidean, i.e., if it can be "coordinatized" at all, then there are many ways of coordinatizing it; if at least one of them is such

that the group operations are analytic, the group is called a "Lie group". Hilbert's fifth problem was this: is every locally Euclidean group a Lie group?

The analogy of this problem with the one in complex function theory is quite close. It is relatively elementary that a twice-differentiable function is analytic; it has been known for a long time that if a topological group has sufficiently differentiable coordinates, then it has analytic ones.

Immediately after the discovery of Haar measure, von Neumann (1933) applied it to prove that the answer to Hilbert's question is yes for compact groups. A little later Pontrjagin (1939) solved the abelian case, and Chevalley (1941) solved the solvable case. (Sorry about that, but "solvable" is a technical word here and its use is unavoidable.)

The general case was solved in 1952 by Gleason and, jointly, by Montgomery and Zippin; the answer to Hilbert's question is yes. What Gleason did was to characterize Lie groups. (Definition: a topological group "has no small subgroups" if it has a neighborhood of the identity that includes no subgroups of order greater than 1. Characterization: a finite-dimensional locally compact group with no small subgroups is a Lie group.) Montgomery and Zippin used geometric-topological tools (and Gleason's theorem) to reach the desired conclusion.

Warning: the subject cannot be considered closed. The question can be generalized in ways that are both theoretically and practically valuable. Groups can be replaced by "local groups", and abstract groups can be replaced by groups of transformations acting on manifolds. The best kind of victory is the kind that indicates where to look for new worlds to conquer, and the one over Hilbert's fifth problem was that kind.

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**Poincaré Conjecture.** A "manifold" is a topological space (a separable Hausdorff space to be exact) that is locally Euclidean. Manifolds have been the central subject of topology for many years, and still are. Hilbert's fifth problem was about group manifolds; the Poincaré conjecture is about the connectedness properties of smooth manifolds. A "differential manifold" is a manifold endowed with local coordinate systems such that the change of coordinates from one coordinate neighborhood to an overlapping one is smooth. "Smooth" in this context is a generally accepted abbreviation for $C^\infty$, i.e., for infinitely differentiable.

The axioms of Euclidean plane geometry characterize the plane. This kind of activity (find the central core of a subject, abstract it, and use the result as an axiomatic characterization) is frequent and useful in mathematics. Since spheres are the principal concept of a large part of topology, it is natural to try to subject them too to the axiomatic approach. The attempt has been made, and, to a large extent, it was successful.

The 1-sphere, for instance (i.e., the circle), is a compact, connected 1-manifold (i.e., a manifold of dimension 1), and that's all it is: to within a homeomorphism every compact connected 1-manifold is a 1-sphere.

For the 2-sphere, the facts are more complicated: both the 2-sphere $S^2$ and the torus $T^2$ ($= S^1 \times S^1$) are compact connected 2-manifolds, and they are not homeomorphic to each other. To distinguish $S^2$ from $T^2$ and, more generally, from a sphere with many handles, it is necessary to observe that, although both $S^2$ and $T^2$ are connected, $S^2$ is more connected. In the appropriate technical language, $S^2$ is "simply connected" and $T^2$ is not. The relevant definitions go as follows. Suppose that $X$ and $Y$ are topological spaces and that $f$ and $g$ are continuous functions from $X$ to $Y$; write $I$ for the unit interval $[0, 1]$. The functions $f$ and $g$ are "homotopic" if there exists a continuous function $h$ from $X \times I$ to $Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x$. (Intuitively: $f$ can be continuously deformed to $g$.) The space $Y$ is simply connected if every continuous function from $S^1$ to $Y$ is homotopic to a constant. (Intuitively: every closed curve can be shrunk to a point.) Once this

concept is at hand, the characterization of the 2-sphere becomes easy to state: to within a homeomorphism, every compact, connected, simply connected 2-manifold is a 2-sphere.

The discussion of dimensions 1 and 2 does not yet provide a firm basis for guessing the general case, but it does at least make the following concept plausible. There is a way of defining "$k$-connected" that generalizes "connected" ($k = 0$) and "simply connected" ($k = 1$): just replace $S^1$ in the definition of simple connectivity by $S^j$, $j = 0, 1, \ldots, k$. Thus: a space $Y$ is $k$-connected if, for each $j$ between 0 and $k$ inclusive, every continuous function from $S^j$ to $Y$ is homotopic to a constant.

The general Poincaré conjecture is that if a smooth compact $n$-manifold is $(n-1)$-connected, then it is homeomorphic to $S^n$. For $n = 1$ and $n = 2$ the result has been known for a long time; the big recent step was the proof of the assertion for all $n \geqq 5$. The proof was obtained by Smale (1960). Shortly thereafter, having heard of Smale's success, Stallings gave another proof for $n \geqq 7$ (1960) and Zeeman extended it to $n = 5$ and $n = 6$ (1961). For $n = 3$ (the original Poincaré conjecture) and for $n = 4$ the facts are not yet known.

Actually Smale proved a much stronger result. He showed how certain manifolds could be obtained by gluing disks together. His results provide a starting point for a classification of simply connected manifolds.

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**Exotic Spheres.** A "diffeomorphism" between two differential manifolds is a homeomorphism such that both it and its inverse are smooth. Homeomorphism is an equivalence relation between manifolds; the equivalence classes (homeomorphism classes) consist of manifolds with the same topological properties. Similarly, diffeomorphism is an equivalence relation between differential manifolds, and the equivalence classes (diffeomorphism classes) consist of manifolds with the same differential properties. Are these concepts really different? Is diffeomorphism really more stringent than homeomorphism? The answer is yes, even for topologically very well-behaved manifolds, but that is far from obvious. An example constructed by Milnor in 1956 came as a surprise, and, according to Hassler Whitney, that single, isolated example led to the modern flowering of differential topology.

Milnor's example is the 7-sphere. For every positive integer $n$, the $n$-sphere $S^n$ is embedded in Euclidean $(n+1)$-space in a natural way, and thus has a natural differential structure. Milnor showed that there exists a differential manifold that is homeomorphic but not diffeomorphic to $S^7$; such a manifold has come to be called an "exotic" 7-sphere.

To prove the assertion, there are three problems to solve: (1) find a candidate, (2) prove that it is homeomorphic to $S^7$, and (3) prove that it is not diffeomorphic to $S^7$. The first problem was easy (with hindsight); the candidate was a space (a 3-sphere bundle over the 4-sphere) that had been familiar to topologists for a number of years. Milnor solved the second problem using Morse theory. A Morse function on a differential manifold is a real-valued smooth function with only non-degenerate critical points. The $n$-sphere has a Morse function with exactly two critical points (project onto the last coordinate and consider two poles). A theorem of G. Reeb's goes in the other direction: if a differential manifold has a Morse function with exactly two critical points, then it is homeomorphic to a sphere. Milnor showed that his candidate had such a Morse function. The third problem was the hardest. Here Milnor used two facts: first, that $S^7$ is the boundary of the unit ball in $\mathbb{R}^8$, and, second, that his candidate was presented as the boundary of an 8-dimensional manifold $W$. If the candidate were diffeomorphic to $S^7$, then, using the diffeomorphism, one could glue the unit ball onto $W$ and obtain an 8-dimensional manifold that (as Milnor showed) cannot exist.

Once it was known that exotic 7-spheres could exist, it was natural to ask how many there were, i.e., how many diffeomorphism classes there were. Milnor and Kervaire showed that there are 28.

What about the other spheres? Again Milnor and Kervaire showed that the set of differential $n$-spheres (modulo diffeomorphism) could be made into a finite abelian group, with the "natural" sphere as the zero element; the group operation is the "connected sum", which is the natural gluing together of manifolds. The group is trivial for $n < 7$; it has order 28 for $n = 7$, order 2 for $n = 8$, order 8 for $n = 9$, order 6 for $n = 10$, and order 992 for $n = 11$. For $n = 31$, there are over sixteen million (diffeomorphism classes of) exotic spheres.

There are two systematic ways of constructing exotic spheres. The first is Milnor's "plumbing" construction (joining holes by tubes), which presents exotic spheres as boundaries of manifolds assembled by cutting and pasting. The other method (due to Brieskorn, Pham, and others) gives preassembled examples. For each finite sequence $(a_1, \ldots, a_n)$ of positive integers, let $\Sigma(a_1, \ldots, a_n)$ be the set of those zeros of the polynomial $z_1^{a_1} + \cdots + z_n^{a_n}$ that lie on the unit sphere in complex $n$-space. Milnor gave precise criteria on the $n$-tuple that ensure that this manifold is homeomorphic to a sphere of the appropriate dimension (which is $2n - 3$, by the way). For example, as $k$ runs from 1 to 28, the manifolds $\Sigma(3, 6k - 1, 2, 2, 2)$ provide representatives for the 28 different diffeomorphism classes of 7-spheres.

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**Differential Equations.** Differential concepts play an important role everywhere, including pure algebra and, as above, topology. Differential equations are what make the world go around, and anyone who wants to predict and perhaps partly to change how the world goes around must know about differential equations and their solutions.

Differential equations are classified in a curiously primitive manner according to the number of independent variables that are involved in differentiation, and the way in which the unknown functions enter. The classification is "one" and "many" in the one case, and "good" and "not-so-good" in the other, or, in terms of the corresponding adjectives that apply to the equations, "ordinary" and "partial" in the one case, and "linear" and "non-linear" in the other. This report is concerned with linear equations only, and partial ones at that; ordinary ones make just a brief appearance at the beginning, to set the stage.

The beginnings of the theory of ordinary linear differential equations are simple and satisfactory; they can be found in elementary textbooks. If $p$ is a polynomial

$$p(\xi) = \sum_{j=0}^{k} a_j \xi^j,$$

and if $D = d/dx$, then $P = p(D)$ is a differential operator, and $Pu = g$ (for given $g$ and unknown $u$) is the typical linear O.D.E. with constant coefficients. If $g$ is continuous (a reasonable, useful, but much too special assumption), then the equation always has a solution. The conclusion remains true even for variable coefficients (i.e., in case the $a_j$'s themselves are functions of $x$), provided they are subjected to some mild restrictions. It is, for instance, sufficient that the $a_j$'s be continuous and that the "principal" coefficient $a_k$ have no zeros.

For partial differential equations even the beginnings are non-trivial and new, and, for instance, even the theory for constant coefficients belongs to the most recent period of research. The formulation of the problem is easy enough: consider a polynomial in several variables $\xi_1, \ldots, \xi_n$, and obtain a differential operator $P$ by replacing $\xi_j$ by $\partial/\partial x_j$; the problem is to solve $Pu = g$ for $u$.

To avoid some not especially enlightening and not especially useful epsilontic hairsplitting, it has become customary to take $g$ (and to seek $u$) in either the most or the least restrictive class of objects in sight. The most restrictive class consists of the smooth (infinitely differentiable) functions on whatever domain is under consideration ($\mathbb{R}^n$, an open set in $\mathbb{R}^n$, a manifold); the other extreme is represented by

Laurent Schwartz's distributions. (The motivation of distribution theory is that functions $f$ induce linear functionals $\phi \mapsto \int \phi(x)f(x)dx$ on $C^\infty$. A "distribution" is a suitably continuous linear functional, not necessarily one induced by a function. The analogy between the generalization and its source suggests an appropriate definition of differentiation for distributions, and with that definition the theory of partial differential equations is off and running.)

Partial differential equations is an old subject and a widely applied one, and it is astonishing that the basic theorem is as recent as it is; it seems only the day before yesterday that Ehrenpreis (1954) and Malgrange (1955) proved that every linear P.D.E. with constant coefficients is solvable. If the right hand side is smooth, there is a smooth solution; even if the right hand side is allowed to be an arbitrary distribution, there is a distribution solution. The subject is exhaustively treated in Ehrenpreis's book (1962) and can be regarded as closed.

So far so good; the proofs are harder than for O.D.E.'s, but the facts are pleasant. The theory for variable (i.e., function) coefficients is much harder, much less known, and nowhere near finished. Two exciting contributions to it in the late 1950's showed that old guesses and old methods were woefully inadequate.

As for old guesses: Hans Lewy produced (1957) an inspired and amazingly simple example of a P.D.E. with variable (but *very* smooth) coefficients that has no solutions at all. Lewy's polynomial is of degree 1,

$$p(x, \xi) = a_1\xi_1 + a_2\xi_2 + a_3\xi_3,$$

where the coefficients $a_1$, $a_2$, $a_3$ are functions of three variables $x_1$, $x_2$, $x_3$, and, in fact, the first two are constants:

$$a_1 = -i, \quad a_2 = 1, \quad a_3 = -2(x_1 + ix_2).$$

The corresponding differential operator is, of course,

$$P = -i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - 2(x_1 + ix_2)\frac{\partial}{\partial x_3}.$$

What Lewy proved is that for almost every $g$ in $C^\infty$ (in the sense of Baire category) the equation $Pu = g$ is satisfied by no distribution whatever.

At about the same time (1958) Calderón studied the uniqueness of the solution of certain important partial differential equations (under suitable initial conditions). He showed, in effect, that if $Pu = 0$, with $u = 0$ for $t \leqq 0$ (intuitively, "$t$" here is time), then, locally $u$ remains 0 for some positive time. Calderón's methods were transplanted from harmonic analysis; they introduced singular integrals into the subject, whence, a little later, came pseudo-differential operators and Fourier integral operators. These ideas have dominated the subject ever since.

Hörmander analyzed and generalized Lewy's example (1960). What makes it work, he pointed out, was that the coefficients are complex; what is fundamental is the behavior of the commutator of $P$ and $\bar{P}$. The operator $\bar{P}$ here is obtained simply by replacing each coefficient by its complex conjugate. (In operator language: $\bar{P}u = \overline{(P\bar{u})}$.) More precisely: consider, for each polynomial in $(\xi_1, \ldots, \xi_n)$ its "principal part", i.e., the part that involves the terms of highest degree only. (For Lewy's example there is no other part.) If $p(x, \xi)$ is the principal part, write $b(x, \xi)$ for the "Poisson bracket",

$$b(x, \xi) = \sum_j \left(\frac{\partial p}{\partial \xi_j}\frac{\partial \bar{p}}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial \bar{p}}{\partial \xi_j}\right).$$

Assertion: if, for some $(x^0, \xi^0)$, the principal part $p(x^0, \xi^0)$ vanishes but the Poisson bracket $b(x^0, \xi^0)$ does not, then $p$ is, in the sense of Lewy, not solvable in any open set containing $x^0$. It is easy to see that the Lewy example is covered by the Hörmander umbrella. Indeed: since

$$p = -i\xi_1 + \xi_2 - 2(x_1 + ix_2)\xi_3,$$

$$\bar{p} = i\xi_1 + \xi_2 - 2(x_1 - ix_2)\xi_3,$$

elementary computation yields

$$b = 8i\xi_3,$$

and it becomes clear that for every $x = (x_1, x_2, x_3)$ there is a $\xi = (\xi_1, \xi_2, \xi_3)$ such that $p(x, \xi) = 0$ and $b(x, \xi) \neq 0$.

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**Index Theorem.** The Atiyah–Singer index theorem (1963) spans two areas of mathematics, topology and analysis, and that's not an accident of technique but in the nature of the subject: the span is what it's all about. Theorems with such a broad perspective are usually the ones that are the most useful and the most elegant, and the index theorem is no exception. The very breadth of the theorem requires, however, that an expository sketch of it proceed obliquely. In what follows we describe, first and mainly, a historical and conceptual precursor, the Riemann–Roch theorem, and then indicate, briefly, how the Atiyah–Singer theorem generalizes it.

The classical Riemann–Roch theorem deals with the dual nature (topological and analytic) of a Riemann surface. Every compact Riemann surface is homeomorphic to a (two-dimensional) sphere with handles. The number of handles, the "genus", completely determines the topological character of the surface; that part is easy. The analytic structure is more complicated. It consists of a covering by a finite number of open sets and of explicit homeomorphisms from the complex plane $\mathbf{C}$ to each open set, which define holomorphic functions on the overlaps. (It is convenient and harmless to use the homeomorphisms to identify each open set in the covering with an open set in $\mathbf{C}$; that is tacitly done below.) If, for example, the surface is the sphere (with no handles), think of $\mathbf{C}$ as slicing through the equator, and use stereographic projections (toward the north and south poles) as the homeomorphisms. There are two open sets here, the complement of the north pole and the complement of the south pole; the holomorphic function on the overlap is given by $w(z) = 1/z$.

A smooth function on a Riemann surface can be viewed as a set of functions on, say, the open unit disk in $\mathbf{C}$ (one for each of the open sets of the covering) that are smooth ($C^\infty$) and transform into one another under the changes of variables induced by the overlaps. (If, that is, $f$ and $g$ are two of these functions, and $w$ is the transformation on the disk induced by going via the appropriate homeomorphism to the open set corresponding to $f$ and coming back from the overlap with the open set corresponding to $g$, then $f(z) = g(w(z))$.) The function on the Riemann surface is called holomorphic (or meromorphic) if each of these functions on the disk is holomorphic (or meromorphic). Another necessary concept for the analytic study of a Riemann surface is that of a smooth differential: that is an expression of the form $p(x, y)dx + q(x, y)dy$, where $p$ and $q$ are complex-valued smooth functions that, on the overlaps, satisfy the chain rule for change of variables. A holomorphic differential is one of the form $f(z)dz$, where $f$ is holomorphic and $dz = dx + idy$. (In the notation used above, the overlap relation for these differentials becomes $f(z)dz = g(w)dw = g(w(z))w'(z)dz$; the functions $f$ and $g$ no longer merely transform into one another, but are altered by the contribution of the differentials as well.)

The analytic properties of a Riemann surface are the properties of the holomorphic (and

meromorphic) functions and differentials that it possesses. A well-known result is that the only holomorphic functions on a compact Riemann surface are constants: that is essentially what Liouville's theorem says. The Riemann–Roch theorem says much more. In its simplest form it deals with a compact Riemann surface $S$ of genus $g$, and $n$ points $z_1, \ldots, z_n$ on $S$. Let $F$ be the vector space of meromorphic functions on $S$ with poles of order not greater than 1 at each $z_i$ (and nowhere else); let $D$ be the vector space of holomorphic differentials with zeros of order not less than 1 at each $z_i$ (and possibly elsewhere). Conclusion:

$$\dim F - \dim D = 1 + n - g.$$

(In the special case of the classical Liouville theorem, $g = 0$, $n = 0$, and $\dim D = 0$.) The important aspect of the conclusion is that a quantity described completely in *analytic* terms can be computed from nothing but *topological* data.

In the special case $n = 0$, $F$ is the vector space of holomorphic functions on $S$ (so that $\dim F = 1$) and $D$ is the space of all holomorphic differentials. There is a linear map, conventionally denoted by $\bar{\partial}$, from the vector space of all smooth functions on $S$ to the vector space of all smooth differentials: write

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}}\, d\bar{z}$$

in each of the open sets of the prescribed covering. The map $\bar{\partial}$ is an example of a differential operator. The kernel of $\bar{\partial}$ consists precisely of the functions satisfying the Cauchy–Riemann equations; in other words

$$\ker \bar{\partial} = F.$$

The cokernel of $\bar{\partial}$ (the quotient space of the space of all smooth differentials modulo the image of $\bar{\partial}$) is similarly identifiable with $D$. The conclusion of the Riemann–Roch theorem takes, in this case, the form

$$\dim \ker \bar{\partial} - \dim \operatorname{coker} \bar{\partial} = 1 - g.$$

The Atiyah–Singer theorem is a generalization of the Riemann–Roch theorem in that it too states that a certain analytically defined number (the "analytic index") can be computed in terms of topological data. Which aspects are generalized? All. To begin with, the Riemann surface is replaced by an arbitrary compact smooth manifold $M$ of arbitrary dimension. The vector spaces of smooth functions and smooth differentials are replaced by vector spaces of smooth sections of complex vector bundles over $M$ (in fact, complexes of vector bundles). The map $\bar{\partial}$, finally, is replaced by a differential operator $\Delta$, which satisfies a certain invertibility condition (called ellipticity). It follows that both $\ker \Delta$ and $\operatorname{coker} \Delta$ are finite-dimensional; the difference of the two dimensions is the analytic index. The conclusion is that the analytic index can be computed in terms of topological invariants (the "topological index"), which are very sophisticated generalizations of the genus.

Even in its relatively short life the Atiyah–Singer index theorem has had important and interesting consequences, and has been proved in at least three enlighteningly different ways. A recent one depends on the study of the heat equation on a manifold.

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**Epilogue.** Concepts, examples, methods, and facts continue to be discovered; problems get reformulated, placed in new contexts, better understood, and solved every day. We hope that the ten examples above have communicated at least a part of the breadth, depth, excitement, and power of the mathematics of our time. Mathematics is alive, and it's here to stay.

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# ON RATIONAL TRIANGLES

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A rational triangle is one that has (positive) rational sides and area. In Problem 5499 [this MONTHLY, May 1967, pp. 599–600] Daykin and Biggerstaff raised the following interesting questions:

(1) Are there positive rationals $a$, $b$ that are not the sides of any rational triangle?

(2) Is every positive rational $k$ the area of some rational triangle?

We shall answer both of these questions in the affirmative by the following theorems:

THEOREM 1. *Let $q$ be a prime not of the form $8t \pm 1$, $\omega$ a positive odd integer such that no prime divisor of $\Delta = q^{2\omega} - 1$ is of the form $4l + 1$. Then $a = q^{\omega} + 1$ and $b = q^{\omega} - 1$ are not the sides of any rational triangle.*

THEOREM 2. *For every positive rational $k$, there is a rational triangle of area $k$.*

*Proof of Theorem* 1. Suppose there is such a triangle, with $c = 2n_1/n_2$ and area $k$. By Heron's formula

$$k^2 = \left(q^{2\omega} - \frac{n_1^2}{n_2^2}\right)\left(\frac{n_1^2}{n_2^2} - 1\right).$$

Hence the positive integer $w = kn_2^2$ satisfies

$$w^2 = (q^{2\omega}n_2^2 - n_1^2)(n_1^2 - n_2^2). \tag{1}$$

We shall show, by descent on $w$, that there are no solutions in positive integers $n_1, n_2, g_1, g_2$ and $w$ of the system

$$w^2 = (g_1^2 n_2^2 - n_1^2)(g_2^2 n_1^2 - n_2^2), \tag{2}$$

$$g_1 g_2 = q^{\omega}. \tag{3}$$

If $(n_1, n_2) > 1$, we can divide (2) through by $(n_1, n_2)^4$ to get a smaller value of $w$. Hence we may assume that $(n_1, n_2) = 1$. If $q \mid (g_1, n_1)$, then (2) may be written as

$$\left(\frac{w}{q}\right)^2 = (g_1'^2 n_2^2 - n_1'^2)(g_2'^2 n_1'^2 - n_2^2)$$

with $g_1' = g_1/q$, $g_2' = g_2 q$, $n_1' = n_1/q$, so that $g_1' g_2' = q^{\omega}$ and a reduction in $w$ has been achieved. Hence we may assume that $(g_1, n_1) = 1$ and similarly $(g_2, n_2) = 1$.

Now defining

$$e = (g_1^2 n_2^2 - n_1^2, g_2^2 n_1^2 - n_2^2), \tag{4}$$

we have

$$\begin{aligned} g_2^2 n_1^2 - n_2^2 &= eu_1^2, \\ -n_1^2 + g_1^2 n_2^2 &= eu_2^2, \\ w &= eu_1 u_2, \end{aligned} \tag{5}$$

where $u_1, u_2 > 0$, $(u_1, u_2) = 1$. Hence (noting (3) and the definition of $\Delta$)

$$\begin{aligned} \Delta n_1^2 &= e(g_1^2 u_1^2 + u_2^2), \\ \Delta n_2^2 &= e(u_1^2 + g_2^2 u_2^2). \end{aligned} \tag{6}$$

From (6), $(\Delta n_1^2, \Delta n_2^2) = \Delta$ is divisible by $e$. Hence $\Delta = ee'$, and

$$\begin{aligned} e'n_1^2 &= g_1^2u_1^2 + u_2^2, \\ e'n_2^2 &= u_1^2 + g_2^2u_2^2. \end{aligned} \tag{7}$$

Now, by hypothesis, if $p$ is an odd prime divisor of $e'$, it is of the form $4l + 3$, so it divides both terms on the right of both equations in (7), hence is a common divisor of $u_1$ and $u_2$. This contradiction shows that $e' = 2^\theta$. Now if $q = 2$, $\Delta$ is odd, so $e' = 1$. If $q$ is odd, then $q = 8t \pm 3$, so $\Delta = q^{2\omega} - 1 \equiv 8 \pmod{16}$. If either $n_1$ or $n_2$ is even, then one of the right sides in (7) is divisible by 4, and this implies that both terms are even and $2 \mid (u_1, u_2)$. This contradiction shows that $n_1$ and $n_2$ are odd. Since $g_1$ and $g_2$ are odd, (4) shows that $8 \mid e$. Hence $e' = \Delta/e$ is odd and again $e' = 1$, $e = \Delta$. Therefore

$$\begin{aligned} n_1^2 &= g_1^2u_1^2 + u_2^2, \\ n_2^2 &= u_1^2 + g_2^2u_2^2, \\ w &= \Delta u_1u_2. \end{aligned} \tag{8}$$

Since $(g_1, u_2) \mid n_1$ and $(g_1, n_1) = 1$, we have $(g_1u_1, u_2) = 1$. Similarly $(u_1, g_2u_2) = 1$, so the first two equations in (8) are primitive Pythagorean. Since $g_1g_2 = q^\omega$, $\omega$ odd, one of the factors is a square. By symmetry, we may assume that $g_1$ is a square.

First, suppose that $q$ is odd. Then $u_1$ and $u_2$ are of opposite parity.

CASE 1. $u_1$ odd, $u_2$ even. Then from (8) and the known form of primitive Pythagorean triples, we have

$$\begin{aligned} g_1u_1 &= \alpha_1^2 - \alpha_2^2, & u_1 &= \beta_1^2 - \beta_2^2, \\ u_2 &= 2\alpha_1\alpha_2, & g_2u_2 &= 2\beta_1\beta_2, \end{aligned} \tag{9}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are non-negative integers such that $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = 1$. Since $u_2 > 0$, they are all positive. From (9), we get

$$\beta_1\beta_2 = g_2\alpha_1\alpha_2, \qquad g_1\beta_1^2 - \alpha_1^2 = g_1\beta_2^2 - \alpha_2^2. \tag{10}$$

Therefore

$$\begin{aligned} \beta_2^2(g_1\beta_1^2 - \alpha_1^2)^2 &= (g_1(\beta_1\beta_2)^2 - \beta_2^2\alpha_1^2)(g_1\beta_2^2 - \alpha_2^2) \\ &= \alpha_1^2(g_1g_2^2\alpha_2^2 - \beta_2^2)(g_1\beta_2^2 - \alpha_2^2). \end{aligned}$$

Hence, if we define $w' = \beta_2(g_1\beta_1^2 - \alpha_1^2)/\alpha_1$, $g_1' = g_2\sqrt{g_1}$, $g_2' = \sqrt{g_1}$, $n_2' = \alpha_2$, $n_1' = \beta_2$, we have

$$w'^2 = (g_1'^2n_2'^2 - n_1'^2)(g_2'^2n_1'^2 - n_2'^2), \tag{11}$$

with $g_1'g_2' = g_1g_2 = q^\omega$. This is of the same form as (2). Thus, to prove descent, it remains to prove that $0 < w' < w$. If $w' = 0$, then $\alpha_1^2 = g_1\beta_1^2$, so from (10), $\alpha_2^2 = g_1\beta_2^2$ and

$$(\beta_1\beta_2)^2 = g_2^2(g_1\beta_1^2)(g_1\beta_2^2),$$

a contradiction. Hence $w' \neq 0$. From the form of (11), each factor on the right is positive, so $w' > 0$, and furthermore, $w' < g_1'g_2'n_1'n_2' = q^\omega\alpha_2\beta_2$. On the other hand, from (9),

$$\begin{aligned} u_2 &= 2\alpha_1\alpha_2 > \alpha_2, \\ u_1 &= \beta_1^2 - \beta_2^2 = (\beta_1 + \beta_2)(\beta_1 - \beta_2) > \beta_2. \end{aligned}$$

Thus, using the third equation in (8), we get

$$w = (q^{2\omega} - 1)u_1u_2 > (q^{2\omega} - 1)\alpha_2\beta_2 > q^\omega\alpha_2\beta_2 > w'.$$

This completes the descent for Case 1.

CASE 2. $u_1$ even, $u_2$ odd. From (8),

$$\begin{aligned} g_1u_1 &= 2\gamma_1 + \gamma_2, & u_1 &= 2\delta_1\delta_2 \\ u_2 &= \gamma_1^2 - \gamma_2^2, & g_2u_2 &= \delta_1^2 - \delta_2^2, \end{aligned} \tag{12}$$

where, as before, $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$ are positive. Observe that $\gamma_1$, $\gamma_2$ are of opposite parity, as are $\delta_1$, $\delta_2$. Define the positive odd integers

$$\begin{aligned} \alpha_1 &= \gamma_1\gamma_2 & \beta_1 &= \delta_1 + \delta_2, \\ \alpha_2 &= \gamma_1 - \gamma_2, & \beta_2 &= \delta_1 - \delta_2, \end{aligned} \tag{13}$$

and note that $\alpha_2 < u_2$, $\beta_2 < u_1$. From (13) and (12), we get (10) and then (11), with the same definitions of $w'$ etc. in terms of $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $g_1$, $g_2$. As before, $w' > 0$ and

$$w' < q^{\omega}\alpha_2\beta_2 < q^{\omega}u_1u_2 < (q^{2\omega} - 1)u_1u_2 < w.$$

This completes the proof for $q$ odd.

Now suppose that $q = 2$. In this case, since $g_1$ is a square, $g_2$ is even, and from the primitivity of the second equation in (8), $u_1$ is odd. Again we have a case distinction, according to the parity of $u_2$.

CASE 3. $u_2$ even. Then from (8), we get (9), and the proof proceeds exactly as in Case 1.

CASE 4. $u_2$ odd. Then $g_1$ and $g_2$ are both even, and from (8),

$$\begin{aligned} g_1u_1 &= 2\beta_1\beta_2, & g_2u_2 &= 2\gamma_1\gamma_2, \\ u_2 &= \beta_1^2 - \beta_2^2, & u_1 &= \gamma_1^2 - \gamma_2^2, \end{aligned} \tag{14}$$

with $\beta_1$, $\beta_2$, $\gamma_1$, $\gamma_2$ positive and $\gamma_1$, $\gamma_2$ of opposite parity. Define the positive odd integers

$$\begin{aligned} \alpha_1 &= \gamma_1 + \gamma_2, \\ \alpha_2 &= \gamma_1 - \gamma_2. \end{aligned} \tag{15}$$

Writing $\bar{g}_1 = 2g_2$, $\bar{g}_2 = g_1/2$, we derive from (15) and (14) the relations

$$\begin{aligned} \beta_1\beta_2 &= \bar{g}_2\alpha_1\alpha_2, \\ \bar{g}_1\beta_1^2 - \alpha_1^2 &= \bar{g}_1\beta_2^2 - \alpha_2^2, \end{aligned} \tag{16}$$

similar to (10). Note that $\bar{g}_1$ is a square, and that $\bar{g}_1\bar{g}_2 = g_1g_2 = 2^{\omega}$. Again we get (11), except that now $g_1' = \bar{g}_2\sqrt{\bar{g}_1}$, $g_2' = \sqrt{\bar{g}_1}$, $w' = \beta_2(\bar{g}_1\beta_1^2 - \alpha_1^2)/\alpha_1$. As before, $w' > 0$, and $w' < 2^{\omega}\alpha_2\beta_2$. Also,

$$u_1 = \gamma_1^2 - \gamma_2^2 = \alpha_1\alpha_2 \geqq \alpha_2,$$

$$u_2 = \beta_1^2 - \beta_2^2 = (\beta_1 + \beta_2)(\beta_1 - \beta_2) > \beta_2.$$

Finally $w = (4^{\omega} - 1)u_1u_2 > 2^{\omega}\alpha_2\beta_2 > w'$. This completes the proof of Theorem 1.

Some examples of pairs $a, b$ that satisfy the conditions of Theorem 1 are given by $q = 2$, $\omega = 1, 3, 5, 7, 13, 15, 17, 19, 21$. The case $q = 3$, $\omega = 1$ yields the pair 4, 2 or equivalently, 2, 1. Thus no side of a rational triangle can be twice another. There are undoubtedly many pairs not covered by Theorem 1, for example 2, 5. It would be of interest to characterize all pairs.

*Proof of Theorem* 2. If $r$ is a rational greater than 1, then the triangle with sides $r + r^{-1}$, $r - r^{-1}$, 2 is a right triangle with rational sides and area $r - r^{-1}$. Joining two such triangles along the side of length 2, we get the triangle $\Delta$ with sides $r + r^{-1}$, $s + s^{-1}$, $r - r^{-1} + s - s^{-1}$, and area

$$K = r - \frac{1}{r} + s - \frac{1}{s}. \tag{17}$$

If we multiply each side of $\Delta$ by the rational $\lambda$, we get a rational triangle with area $\lambda^2 K$. Hence every $k$ of this form is the area of a rational triangle. Actually the restriction $r > 1$ can be removed, for if

$0<r<1$, we can form the triangle with sides $r+r^{-1}$, $r^{-1}-r$, 2 and subtract it from the $s$ triangle, provided that $K>0$. Even $r<0$ can be interpreted as subtracting or adding the triangle with sides $|r|+|r|^{-1}$, $||r|-|r|^{-1}|$, 2 from the $s$ triangle. The upshot is that every positive $\lambda^2 K$, where $K$ has the algebraic form (17), is the area of a rational triangle.

In order to solve this Diophantine problem, we try

$$r = kx, \qquad s = kx^2, \tag{18}$$

with $x$ rational $\neq 0$. Putting these values in (17) we get

$$kKx^2 = k^2x^4 + k^2x^3 - x - 1. \tag{19}$$

Thus it is sufficient to make the right side of (19) a rational square, say $y^2$. To solve the equation

$$y^2 = k^2x^4 + k^2x^3 - x - 1, \tag{20}$$

we put

$$y = kx^2 + \frac{k}{2}x - \alpha, \tag{21}$$

where $\alpha$ is to be determined so that the coefficient of $x^2$ in the expansion of $y^2$ is 0. This coefficient is $k^2/4 - 2k\alpha$, so $\alpha = k/8$. Hence

$$y = k\left(x^2 + \frac{x}{2} - \frac{1}{8}\right). \tag{22}$$

With this choice of $y$, equation (20) becomes linear in $x$. Its solution is

$$x = \frac{k^2+64}{8(k^2-8)}. \tag{23}$$

With this choice of $x$, the values $r, s$ given by (18) provide a solution. Explicitly, the sides of the $r, s$ triangle are

$$\frac{k^2x^2+1}{k|x|}, \qquad \frac{k^2x^4+1}{kx^2}, K$$

and the area is $K = y^2/(kx^2)$. Hence, if we multiply the sides by $|kx/y|$, we get a triangle

$$\frac{k^2x^2+1}{|y|}, \frac{k^2x^4+1}{|xy|}, \left|\frac{y}{x}\right| \tag{24}$$

with area $k$.

As an example, let $k = 1$. Then $x = -65/56$, so $r = -65/56$, $s = (65/56)^2$. This leads to the triangle with sides

$$\frac{52}{33}, \frac{641}{330}, \frac{33}{10}.$$

The solution is not unique, and the one given above is not the simplest by any means. S. and P. Chowla have found by ingenious trial and error the solution

$$r = \frac{2k}{k-2}, \; s = \frac{k-2}{4}, \tag{25}$$

which works for all $k \neq 2$. (For $k = 2$, use $k = 8$ instead and scale down.) For $k > 2$, this leads to the triangle

$$\frac{5k^2-4k+4}{k^2-4}, \frac{k(k^2-4k+20)}{2(k^2-4)}, \frac{k+2}{2} \tag{26}$$

with area $k$. For $k = 1$, their solution leads to the triangle (5/3, 17/6, 3/2), which is much simpler than the one given above. Other solutions for $k = 1$, suggested by the referee, are (1/2, 13/3, 25/6) and (17/30, 4, 113/30).

We can introduce a second parameter in any solution

$$a = f(k),\ b = g(k),\ c = h(k)$$

by replacing $k$ by $km^2$, where $m$ is any positive rational, and then dividing by $m$. Thus we get

$$a' = \frac{1}{m} f(km^2), b' = \frac{1}{m} g(km^2), c' = \frac{1}{m} h(km^2).$$

For example, (26) above leads to

$$\frac{5k^2m^4 - 4km^2 + 4}{m(k^2m^4 - 4)},\ \frac{km(k^2m^4 - 4km^2 + 20)}{2(k^2m^4 - 4)},\ \frac{km^2 + 2}{2m}. \tag{27}$$

Of course, this device can be used only once.

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## THE FIRST DIGIT PROBLEM

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**1. Introduction and notation.** It has been known for a long time that if an extensive collection of numerical data expressed in decimal form is classified according to first significant digit, without regard to position of decimal point, the nine resulting classes are not usually of equal size. Indeed, while a truly random table should show a frequency of 1/9 for the occurrence of a given first digit $p$ ($p = 1, 2, \ldots, 9$), many observed tables give a frequency approximately equal to $\log_{10}(p + 1)/p$. Thus the initial digit 1 appears about .301 of the time, 2 somewhat less and so on, with 9 occurring as a first digit less than 5 percent of the time. (We do not admit 0 as a possible first digit.)

This particular logarithmic distribution of first digits, while not universal, is so common and yet so surprising at first glance that it has given rise to a varied literature, among the authors of which are mathematicians, statisticians, economists, engineers, physicists and amateurs. The present memoir includes a bibliography as nearly complete as I could collect, deliberately omitting only those references to the problem which make no attempt to add to its understanding. My purpose is to review all the proposed explanations in some rational (but not chronological) order, making plain the hypotheses and results in each case but often suppressing details of proof.

The main bibliography is arranged and numbered chronologically; every simply numbered item is directly and avowedly concerned with the first digit phenomenon. The supplementary bibliography, numbered 1B, 2B, etc., does not refer explicitly to the problem.

A few notations will persist throughout: $R$ stands for the real number system, $R^+$ is the non-negative part of $R$, $N$ stands for the set of positive integers. Intervals in $R$ are given as usual, e.g. $[a, b) = \{x \in R: a \leqq x < b\}$. $D_p$ denotes the set of all members of $R^+$ whose standard decimal expansion begins with an integer $\leqq p$ ($p = 1, 2, \ldots, 9$). Thus,

$$D_p = \bigcup_{n=-\infty}^{\infty} [10^n, (p + 1)10^n).$$

In some places $D_p$ will be spoken of as a subset of $N$; in such cases the context will make clear that

$D_p \cap N$ is meant. The mapping $\log_{10}: R^+\backslash\{0\} \to R$ will be denoted log, without subscript. $E_p$ will denote $\log D_p$, thus,

$$E_p = \bigcup_{n=-\infty}^{\infty} [n, n + \log(p+1)).$$

In the suggestive language of probability theory, the first digit phenomenon is usually expressed: $\text{prob}\{x \in D_p\} = \log(p+1)$. This assertion, whatever it may mean, will be called *Benford's Law* because it has been thought by many writers to have originated with the General Electric Company physicist Frank Benford [2]. Certainly Benford popularized the problem, and he may well have been unaware that the polymathic Simon Newcomb, primarily an astronomer but also sometime editor of *The American Journal of Mathematics*, had also formulated the same law 57 years earlier [1]. There is ample precedent for naming laws and theorems for persons other than their discoverers, else half of analysis would be named after Euler. Besides, even Newcomb implied that the observation giving rise to the Benford law was an old one in his day. One would hate to change the name of the law now only to find later that another change was called for.

**2. Empirical and other data.** Newcomb [1] opened his paper as follows. "That the ten digits do not occur with equal frequency must be evident to anyone making use of logarithm tables, and noticing how much faster the first pages wear out than the last ones. The first significant figure is oftener 1 than any other digit, and the frequency diminishes up to 9." However, Newcomb gave no actual numerical data.

Benford [2] gives a great deal; he summarizes the counts for each of twenty different tables with lengths ranging from 91 entries (atomic weights) to 5000 entries from a mathematical handbook ($n^{-1}$, $n^5$, etc.). These two tables obeyed Benford's law rather badly, in fact, while others of his listings, such as the street addresses of the first 342 persons named in *American Men of Science, 1934*, did better. What came closest of all, however, was the union of all his tables.

| $n =$ | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| Benford's Law | $L(n)$ | .301 | .176 | .125 | .097 | .079 | .067 | .058 | .051 | .046 |
| Stigler's Law | $S(n)$ | .241 | .183 | .145 | .117 | .095 | .076 | .060 | .047 | .034 |
| Benford's Data | $B(n)$ | .306 | .185 | .124 | .094 | .080 | .064 | .051 | .049 | .047 |
| Powers of Two | $P(n)$ | .30 | .17 | .13 | .10 | .07 | .07 | .06 | .06 | .05 |
| Electricity | $E(n)$ | .316 | .167 | .116 | .087 | .085 | .064 | .057 | .050 | .057 |
| Vancouver Tel. | $V(n)$ | .00 | .27 | .04 | .08 | .13 | .05 | .05 | .10 | .28 |
| Populations | $PP(n)$ | .190 | .200 | .185 | .168 | .098 | .065 | .043 | .037 | .013 |

TABLE 1

In Table 1 the first line $L(n) = \log(n+1) - \log n$ is Benford's law, and is compared with the data in the succeeding rows. $S(n)$ is also not empirical, but is the set of predicted frequencies given by George Stigler [5], based on hypotheses to be described later. $B(n)$ is the empirical frequency found by Benford [2] in his ensemble of 20,229 entries. $P(n)$ is the frequency of leading digit $n$ among the first hundred powers of 2, i.e. $2^0, 2, 2^2, \ldots, 2^{99}$; notice that $P(n) = L(n)$ about as nearly as 100 numbers can manage.

The row $E(n)$ was mailed to me by the head of the Electricity Board of Honiara in the (Br.) Solomon Islands. Upon reading of Benford's law in [22], he wrote, he went to the customer records of his 1243 electricity users, whose consumption in October, 1969, ranged from 1 KWH to over 40,000 KWH, and counted first digits for that month's consumption.

The row $V(n)$ I took from two columns of the 1974 Vancouver (Canada) telephone book, about 210 telephone numbers in all. A glance through the rest of the book confirms that no Vancouver telephone numbers begin with the digit 1.

The last row $PP(n)$ is taken from a table in *The World Almanac* (N. Y. Times, New York, 1971),

listing the populations of all 'Populated places with population at least 2500' in the United States, using census figures from 1960 and 1970. I used all the entries for five States.

Each listing in Table 1 illustrates some feature of the discussion to follow. Other empirical data are found in [2; 5; 6; 8; 24]. Papers containing numerical data based on mathematical considerations ($P(n)$ is such a set of data) include [3; 4; 5; 9; 15; 16; 17; 18; 24; 25; 27; 28]. Any reader can collect data of his own, or invent it, as like or as unlike any row of Table 1 as he wishes. In what sense, then, is $L(n)$ a 'Law'?

**3. A bit of philosophy.** Despite the obvious occurrence of natural tabulations which don't obey Benford's law, many authors (including the present author) have offered explanations of $L(n)$ which are purely mathematical in nature, as though the number system itself, along with the decimal numeration system, dictated these proportions.

Goudsmit and Furry [3] write, "It is merely the result of our way of writing numbers," though the sequel by Furry and Hurwitz [4] explicitly recognizes otherwise, and Warren Weaver [11] says it "is a built-in characteristic of our number system."

This point of view springs from the idea that there is some natural way to calculate a "density" for the set $D_p$ in $R^+$ or in $N$, and that this natural way yields $\log(p+1)$. Now it is true, as will be shown in Sections 4 and 5, that certain summability methods will indeed assign these 'correct' densities, but the quoted statements are nonetheless dead wrong. Certainly the compilations giving $V(n)$ and $PP(n)$ in Table 1 are written in "our way" but still violate what Goudsmit and Furry claim is the "mere ... result" of all this. Geometric sequences such as $\{2^n\}$, on the other hand, do have the property that randomly drawn finite samples tend to obey the Benford ratios ($P(n)$ in Fig. 1, for example).

This situation is exactly reversed if instead of looking for a "density" for $D_p$ we look for a density of the even positive integers. Everyone will agree usually that half of $N$ is made up of the evens; certainly all the most popular summability methods assign 1/2 as their density. But now if we look at the same numbers in the *World Almanac* that produced $PP(n)$, or the same columns of the Vancouver telephone directory that produced $V(n)$, we will find that in fact about half of each of them are even numbers; while samples drawn from the sequence $\{2^n\}$ most emphatically behave otherwise.

No purely mathematical argument can be expected, after all, to explain things actually found in the real world, like $B(n)$, without some correspondence between the hypotheses and structure of the mathematical argument on the one hand, and some observed facts and reliable laws of nature on the other. A strict and correct proof that in some precise sense half of $N$ is even, or that the density of $D_p$ is $\log(p+1)$, tells us nothing whatever about nature and $B(n)$ unless that 'precise sense' matches something relevant to the actual origin of $B(n)$.

The density arguments, reviewed in Sections 4 and 5 below, were generally given by their original authors without the least attempt at such justification, and the scale-invariance arguments of Section 6 are likewise philosophically barren. Not until Section 7, (Pinkham's second method) does a *scientific* theory appear, a formula that invokes observation in addition to calculation. The earlier arguments produce Benford's law uncritically and therefore predict its appearance in every possible context, even for $PP(n)$ and $V(n)$ where it fails. The statistical argument of Pinkham produces an approximation to Benford's law, with a criterion relating the closeness of the approximation to some other observable features of the phenomena.

**4. Density and summability arguments, discrete model.** A first attempt at density is the usual number-theoretic (or Cesaro) method. Call $\alpha_p(n)$ the characteristic function of $D_p$ in $N$, so that $\alpha_p(n) = 1$ if $n \in D_p$ and is otherwise 0. Then put

$$\alpha_p^1(n) = (1/n) \sum_{k=1}^{n} \alpha_p(k). \tag{4.1}$$

If $\lim_n \alpha_p^1(n)$ existed it would be the number-theoretic density of $D_p$, but the limit does not exist. In

Figure 1 of Section 5 below appears the graph of a function $\phi_1$, defined on $R^+$, which is for all practical purposes the graph of $\alpha_1^1$ in the sense that for $n \in N$, $|\phi_1(n) - \alpha_1^1(n)| \leqq 1/n$. Figure 1 shows the oscillatory nature of the averaging process for $D_1$ and much the same thing happens to $\alpha_p^1$ for the other values of $p$. Graphs like Figure 1 and Figure 2 appear in [**14**] and [**22**], and for other values of $p$ in [**2**] and [**11**].

When $\alpha_p^1$ is plotted on semilog paper, as in Figure 2 below, one is struck with the near-periodicity of the result and tempted to take an average height of the $n$th period and call the limit the density of $D_p$. This is most conveniently done by an integral approximation, so a discussion of the results will be deferred to Section 5.

B. J. Flehinger does something different. Since $\alpha_p^1$ doesn't converge, she reiterates the Cesaro process, putting

$$\alpha_p^t(n) = (1/n) \sum_{k=1}^{n} \alpha_p^{t-1}(k) \tag{4.2}$$

as $t = 2, 3, \ldots$. In [**14**] she proves that the successive functions oscillate ever more narrowly and that (see [**20**] for a clearer proof) the process converges to $\log(p+1)$ in the following sense:

$$\lim_t \liminf_n \alpha_p^t(n) = \lim_t \limsup_n \alpha_p^t(n) = \log(p+1).$$

The Flehinger limitation method is a regular method, that is, it yields ordinary limits when applied to convergent sequences. There is no shortage of regular methods, and an infinitude of them will *not* yield $\log(p+1)$ as the generalized limit of $\alpha_p(n)$, or even of $\alpha_p^t(n)$ for any given finite $t$. However, the Flehinger method has the property of being stronger than *all* iterations of the Cesaro methods, and it can be proved [**12B**; Section 6] that any *matrix* method having this property must agree with the Flehinger method whenever the latter applies. (Cf. also [**29**] and [**13B**] for relationships between these and yet other summability methods.)

An example of a method stronger than Flehinger's is the logarithmic matrix method $\mathcal{L}$, defined by

$$L_{nj} = (j \log_e n)^{-1} \text{ if } n \leqq j, \text{ and } L_{nj} = 0 \text{ if } n > j.$$

A sequence $\{s_n\}$ is called $\mathcal{L}$-summable if $\Sigma_j L_{nj} s_j$ converges in $n$. In the present case, then, where $s_n$ is $\alpha_p(n)$, the characteristic function of $D_p$,

$$\lim_n (\log_e n)^{-1} \sum_{j=1}^{n} \alpha_p(j)/j = \log(p+1).$$

This result was obtained directly by R. L. Duncan [**23**].

R. E. Whitney [**26**] has shown that the matrix method $\mathcal{L}$ can also be applied to the characteristic function of $D_p$ in the sequence of primes, with the same result. Thus, if $\pi_p(n) = 1$ when the $n$th prime begins with a digit $\leqq p$, and $\pi_p(n) = 0$ otherwise, then

$$\lim_n (\log_e n)^{-1} \sum_{j=1}^{n} \pi_p(j)/j = \log(p+1).$$

Unlike the case of $D_p$ in $N$, where one might suspect the Benford law by a careful examination of Figure 1 below, and adjust his choice of summation method accordingly, the same result for $D_p$ in the primes hardly seems intuitive. Of the 1125 primes less than 9999, about 14 percent begin with the digit 1.

J. Cigler, in a personal communication to me in 1969, called my attention to the relevance, in connection with such density arguments, of the notion of equidistributed sequences. Let $\{b_n\}$ be a sequence of real numbers in the interval $[0, 1)$; the sequence is called equidistributed if for each subinterval $[a, b) \subset [0, 1)$ we have

$$\lim_k \frac{1}{k} \sum_{n=1}^{k} \beta(n) = b - a, \tag{4.3}$$

where $\beta(n) = 1$ when $b_n \in [a, b)$ and $\beta(n) = 0$ if $b_n \notin [a, b)$.

Now let $\{a_n\}$ be a sequence in $R^+\backslash\{0\}$, and let $b_n = \log a_n \pmod 1$. Then $a_n \in D_p$ (i.e., $a_n$ has first digit $\leqq p$) if and only if $0 \leqq b_n < \log(p+1)$. If it turns out that $b_n$ is equidistributed on $[0, 1)$, then (4.3) holds with $[a, b) = [0, \log(p+1))$, and $\{a_n\}$ may be said to obey Benford's law in the strictest sense, the sense of ordinary density of $D_p$ in the sequence $\{a_n\}$. Cigler proposed calling such a sequence a strong Benford sequence.

The sequence of the positive integers themselves, $N$, is not a strong Benford sequence, but any geometric sequence $\{ar^n\}$ is, provided $r$ is not a rational power of 10. For, if $a_n = ar^n$, then $b_n = \log a + n \log r \pmod 1$, and it is well known (e.g., [4B; p. 390]) that arithmetic sequences with irrational spacing are equidistributed (mod 1). A heuristic 'proof' of this result may be found in [22].

The restriction "$r$ is not a rational power of 10" is not entirely needed for $\{ar^n\}$ to have a good approximation to the Benford law. If $r = 10^{p/q}$, then

$$b_n = \log a + \frac{np}{q} \pmod 1$$

is periodic, i.e., $b_{n+q} = b_n$ for all $n$. The finite range of $\{b_n\}$ is equally spaced and comes as close to equidistribution on $[0, 1]$ as the spacing $1/q$ allows. To be precise, we get instead of (4.3) the formula

$$\left| \lim_k \frac{1}{k} \sum_{n=1}^{k} \beta(n) - (b-a) \right| \leqq 1/q, \tag{4.4}$$

and since $q$ can seldom be a small integer, the Benford law is well approximated for all but a finite number of ratios $r$.

Empirical verification of Benford's law for geometric sequences can be seen strikingly in the first hundred powers of 2 (row $P(n)$ in Table 1), and E. Hafner [18] has added some interesting flourishes to compilations of this kind. Benford himself was well aware of this property of geometric sequences, so much so that he made it the philosophic rock on which to base his law. While mere Man counts arithmetically, $1, 2, 3, 4, \ldots$, says Benford, *Nature* counts $e^0, e^x, e^{2x}, e^{3x}, \ldots$, "and builds and functions accordingly." Therefore, Benford's argument goes on, 'naturally' derived data tend to come in mixtures of geometric sequences, which obey Benford's law. He cites numerous examples from science and technology to support this view, all of them variants on "Fechner's Law", a bit of 19th century scientism that says (roughly), "Response is proportional to the logarithm of the stimulus." If Fechner's law made sense and were true, it would still not be apparent why tabular data should favor lists of stimuli over lists of responses. However, for a hilarious demolition of Fechner's law in a typical application, see [3B].

The logarithmic summability method also corresponds to an equidistribution notion. If, as before, $b_n = \log a_n \pmod 1$, Cigler suggests calling $\{a_n\}$ a weak Benford Sequence if $\{b_n\}$ is $(1/n)$-equidistributed, which is to say, if

$$\lim_k (\log_e n)^{-1} \sum_{n=1}^{k} \beta(n)/n = b - a, \tag{4.5}$$

where $\beta(n)$, $a$ and $b$ are as in (4.3) above. What Whitney proved, then, is that $a_n =$ the $n$th prime defines a weak Benford sequence. Duncan's result in [23] now can be read as an alternate statement of the earlier known fact that $\{n\}$ is a weak Benford sequence. In [5B; 6B; 14B] are accounts of equidistribution (mod 1), and proofs that $\{\sqrt{n}\}$ and $\{Q(n)\}$ for any polynomial $Q$ are also weak Benford sequences.

Benford's observation that "the greatest variations from the logarithmic relation [i.e., Benford's Law] were found in the first digits of mathematical tables from engineering handbooks...", which led him to the conclusion that "the logarithmic law applies particularly to those outlaw numbers that are without known relationship..." was thus a bit hasty, the result of a preference for Cesaro frequency counts over a more sophisticated weighting. Even so, geometric sequences, which do obey his law, can hardly be called 'outlaw...without known relationship'!

Nor are geometric sequences the only kind of strong Benford sequences. D. Singmaster [**30**] calls $\{a_n\}$ asymptotically geometric if there is a geometric sequence $\{ar^n\}$ such that $\lim_n (a_n/ar^n) = 1$. In this case $\log a_n - n \log r$ converges, so that $\log a_n \pmod 1$ is just as equidistributed as $n \log r$. In other words, asymptotically geometric sequences are (unless $\log r$ is rational) strong Benford sequences.

Now Singmaster notices that the Fibonacci sequence is asymptotically geometric with the golden mean $(1+\sqrt{5})/2$ as limiting ratio. Furthermore, $\log[(1+\sqrt{5})/2]$ is irrational, hence the Fibonacci numbers form a strong Benford sequence.

Singmaster's observation follows no less than three earlier papers [**15**; **25**; **28**] giving empirical data supporting the same statement. Yet none of these four authors noticed any of the series of four articles by Brown, Duncan, Kuipers and Shiue [**8B**; **9B**; **10B**; **11B**] which proved that not only the Fibonacci numbers but (almost) all sequences $\{a_n\}$, defined by linear recursion, have the property that $\log a_n \pmod 1$ is equidistributed on $[0,1)$, i.e., $\{a_n\}$ obeys Benford's law. Benford's law, however, was not explicitly mentioned in these papers. The proof given by Brown and Duncan rests precisely on the fact that the sequences in question are asymptotically geometric.

Singmaster does notice that the interleaving of a finite number of asymptotically geometric sequences produces a strong Benford sequence, and proposes the converse as a conjecture.

**5. Density and summability arguments, continuous model.** Instead of looking at sequences, or subsets of $N$, one can look at $R^+$ and ask "What fraction of $R^+$ is occupied by $D_p$?" For convenience in this section we shall only consider the domain $[1,\infty)$. If $\alpha_p$ is the characteristic function of $D_p$ in $[1,\infty)$, let

$$\phi_p(x) = \frac{1}{x-1}\int_1^x \alpha_p(t)dt. \tag{5.1}$$

Explicit formulas are easily found. For $p = 1$, for example,

$$\begin{aligned}
\phi_1(x) &= 1 && \text{on } [1,2]\\
&= 1/(x-1) && \text{on } [2,10]\\
&= 1-8/(x-1) && \text{on } [10,20]\\
&= 11/(x-1) && \text{on } [20,100]\\
&= 1-88/(x-1) && \text{on } [100,200]; \text{ etc.}
\end{aligned}$$

The graph of $\phi_1$ is given in Figure 1, and in Figure 2 it is given again but with a logarithmically scaled $x$-axis to show its 'periodicity'. The maxima converge to 5/9 and the minima are 1/9. Similar graphs can be drawn for the other values of $p$.

George Stigler [**5**] obtains an 'average height' for $\phi_p$ by integrating over the $n$th cycle; he computes

$$\frac{1}{10^{n+1}-10^n}\int_{10^n}^{10^{n+1}} \phi_p(t)\,dt \tag{5.2}$$

and takes the limit as $n$ increases. His results are given in Table 1 as Stigler's Law $S(n)$.

Benford, using the same graph, gets prob $D_1 = \log 2$, which is .301 as against Stigler's .241; how? He does it by using Figure 2 instead of Figure 1, i.e., by getting the average height of the $n$th cycle as actually depicted geometrically in Figure 2 and taking the limit as $n$ increases. Thus Benford takes the limit of

$$\int_n^{n+1} \phi_p(10^x)dx, \tag{5.3}$$

which is not at all the same thing as Stigler's integral (5.2) under the relevant change of variable. (Both

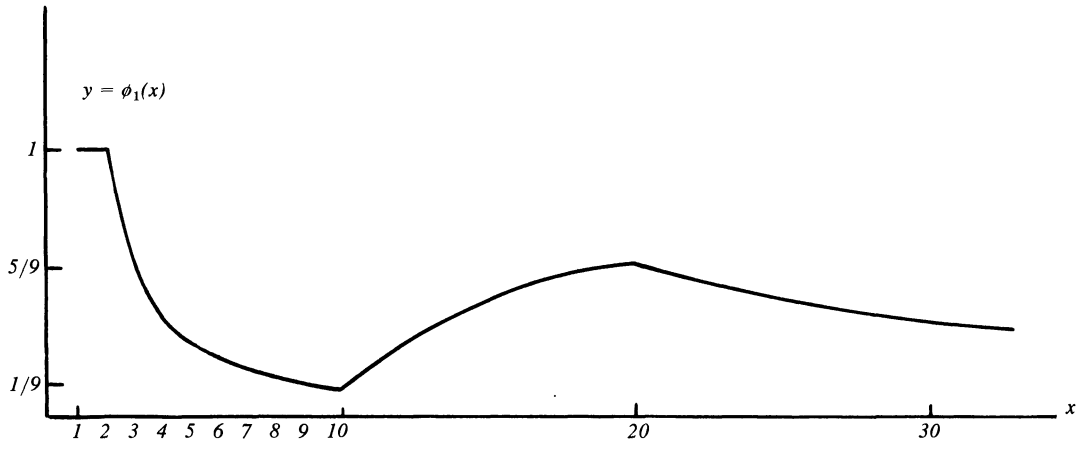



Fig. 1

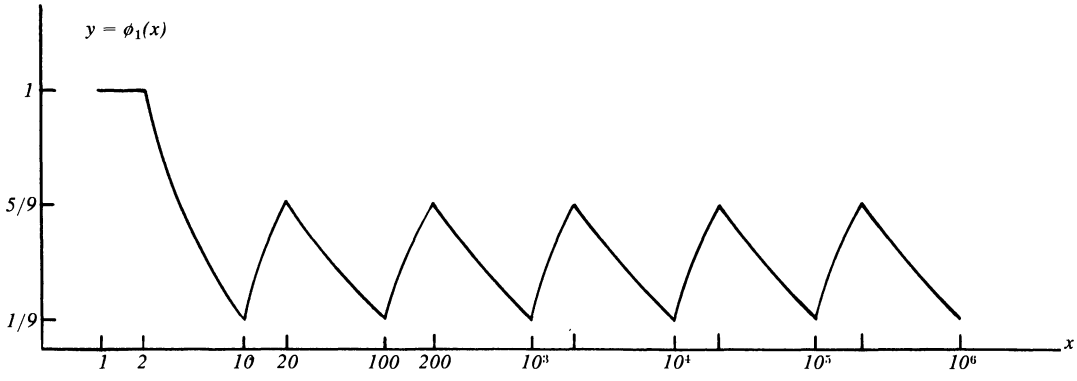



Fig. 2

Stigler's average and Benford's average will assign density 1/2 to the set of even integers, by the way. For this purpose (5.2) and (5.3) should be interpreted as the obviously related discrete sums rather than integrals.)

Warren Weaver, in his popularization "Lady Luck" [11], also draws a graph akin to Figure 2, but for $\phi_4$ rather than $\phi_1$, to exhibit the probability that a number will begin with 1, 2, 3 or 4. He uses the logarithmic horizontal scale, calling it inessential to his reasoning, as indeed it is since he uses Stigler's method of averaging. But he *thinks* his result is going to be Benford's, apart from errors of approximation, and refers to Benford's as the correct 'theoretical' law.

If Weaver didn't notice the difference between the two methods, certainly Benford didn't; he didn't even consider the 'somewhat distorted' (Weaver's words) horizontal scale of Figure 2 worth mentioning. It took Stigler, an economist, to observe that the two summability methods correspond to two different philosophical hypotheses concerning equiprobability. By implication, Stigler therefore showed something very many of the writers on the problem either ignored or explicitly denied: that mathematics alone cannot justify a first digit law.

**6. Probability interpretation of summability methods.** From the definition (5.1), $\phi_p(t)$ can be regarded as a conditional probability, the probability that a random variable $x$ is in $D_p$, given that $x$ is drawn from the rectangular distribution with support $[1, t)$. Let us denote this condition by the phrase "$\max x = t$"; also, let us denote the event "$x$ is drawn from a rectangular distribution with support $[1, t)$, where $t$ is somewhere in the interval $[a, b)$" by $\{\max x \in [a, b)\}$. Now Stigler's integral (5.2) can

be approximated by the Riemann sum

$$\sum_{i=1}^{M} \phi_p(t_i) \cdot (1/M), \tag{6.1}$$

where the points $t_0, t_1, \ldots, t_M$ form an equally-spaced partition of $[10^n, 10^{n+1}]$. In probability terms this sum can be rewritten

$$\sum_{i=1}^{M} \text{prob}\{x \in D_p \,|\, \max x = t_i\} \cdot \text{prob}\{\max x \in [t_{i-1}, t_i)\}, \tag{6.2}$$

provided we assume each of the $M$ disjoint events $\{\max x \in [t_{i-1}, t_i)\}$ has probability $1/M$.

Now for large $M$, the conditions "$\max x = t_i$" and "$\max x \in [t_{i-1}, t_i)$" are not very different in the first factor of each term of (6.2); therefore, by a sort of Duhamel's Principle, (6.2) may be replaced by

$$\sum_{i=1}^{M} \text{prob}\{x \in D_p \,|\, \max x \in [t_{i-1}, t_i)\} \cdot \text{prob}\{\max x \in [t_{i-1}, t_i)\} \tag{6.3}$$

as a Riemann approximation to (5.2). By the summation law for conditional probabilities, (6.3) reduces to

$$\text{prob}\{x \in D_p \,|\, \max x \in [10^n, 10^{n+1})\}, \tag{6.4}$$

which in turn is nothing but $\text{prob}\{x \in D_p\}$, since we are assuming, for the time being, that $\max x \in [10^n, 10^{n+1})$ is in fact the case. Stigler's ratios $S(p)$ of Table 1 come from these probabilities' limiting values as $n \to \infty$.

Stigler's method therefore amounts to imagining that we have a large number of tables of data with the largest entry $t_i$ in each table lying somewhere in the interval $[10^n, 10^{n+1})$ with equal probability, and that the $i$th table is a sample from the rectangular distribution supported by $[1, t_i)$; also that $n$ is large, and the same for all tables. (Actually, $n$ need not be very large for practical purposes; for $n = 2$ we are already within 1 percent of the values given as $S(p)$.)

Finally, we may drop the restriction that $n$ is the same for all tables by imagining that the tables before us are mixtures from the above situation for various values of $n$. The hypothesis that the largest entry is equidistributed on $[10^n, 10^{n+1})$ for a mixture of values of $n$ is translated by Stigler into the phrasing "The largest entry in each table is equally likely to begin with the integer 1, 2, ..., or 9." Stigler seems to think that this hypothesis, together with the hypothesis that the conditional distribution of the random entries in a table with a largest given entry is rectangular, produces his results, but I don't see that his phrasing is sufficient.

The corresponding interpretation of Benford's integral (5.3) proceeds from the approximating Riemann sum

$$\sum_{i=1}^{M} \phi_p(10^{t_i}) \cdot (1/M), \tag{6.5}$$

where the $t_i$ are equally spaced on $[n, n+1]$. In probability terms, as before, this becomes

$$\sum_{i=1}^{M} \text{prob}\{x \in D_p \,|\, \max x = 10^{t_i}\} \cdot \text{prob}\{\max x \in [10^{t_{i-1}}, 10^{t_i})\}, \tag{6.6}$$

analogously with (6.2), though now we must consider the events $\{\max x \in [10^{t_{i-1}}, 10^{t_i})\}$ all equal and of value $1/M$. As before, we may replace "$\max x = 10^{t_i}$" by "$\max x \in [10^{t_{i-1}}, 10^{t_i})$" in (6.6) and still have a Riemann–Duhamel approximation to (5.3). The summation law then reduces the new version of (6.6) to $\text{prob}\{x \in D_p \,|\, \max x \in [10^n, 10^{n+1})\}$, whose limit as $n \to \infty$ gives us Benford's law $L(p)$.

In Stigler's interpretation, then, Benford's universe of tables of data has each table drawn from a rectangular distribution on $[1, t_i)$ where the "largest entry" $t_i$ now has a skewed distribution on some

given $[10^n, 10^{n+1})$, skewed in the sense that $\log t_i$ is rectangularly distributed on $[n, n+1)$. (Of course, the same comments about large $n$, and mixtures of $n$, apply as before.)

Stigler takes this 'inconsistency' — the combination of a rectangular conditional distribution for random entry with skewed distribution of highest entry — as a flaw in Benford's analysis, but in fact both of these 'urn models' are equally artificial, although it is conceivable they — one or the other — might apply in some particular context like the assignment of street addresses or the populations of cities.

The discrete summability schemes of Section 4 above can also be tortured into probability interpretations, though none of the authors mentioned there (except Diaconis) does so. However, A. Herzel [9] *starts* with an urn model for the question "What is prob$\{x \in D_p\}$?" in the domain of positive integers $N$, and ends with summability questions related to Flehinger's and Stigler's.

Herzel imagines $M$ urns, the $n$th of which contains $n$ balls numbered 1 to $n$. The conditional probability of choosing a ball whose first digit is $\leqq p$, given that we choose from the $n$th urn, is of course Flehinger's first average $\alpha_p^1(n)$ ((4.1) above), which is asymptotically $\phi_p(n)$ of Figure 1. The total probability of choosing a ball in $D_p$ then depends on the *a priori* probabilities assigned for choosing among the urns. Herzel gives three schemes: equal probability, probability weighted linearly according to the size of the urn, and probability weighted as the square of the size of the urn.

None of these schemes produces a limit as $M \to \infty$, but Herzel obtains integral approximations and numerical results close to Benford's. Herzel's first scheme is the same as Flehinger's second approximation, and the others have justifications as arbitrary as the first. A vast amount of numerical data is given.

But the mere interpretation of an averaging device as a probability does nothing to answer the real-life question of why observed tables tend (or tend not) to obey Benford's (or Stigler's) law, unless we have some scientific information about the tabular entries enabling us to verify one or another of these 'urn model' hypotheses. It appears that in this form the scientific question is very hard to answer.

**7. Another probability interpretation: Scale-invariance.** Roger S. Pinkham [10], attributing the basic idea to R. Hamming, put forward an invariance principle attached to another sort of probability model, sufficient to imply Benford's law. If (say) a table of physical constants, or of the surface areas of a set of nations or lakes, is rewritten in another system of units of measurement, ergs for foot-pounds or acres for hectares, the result will be a rescaled table whose every entry is the same multiple of the corresponding entry of the original table. If the first digits of all the tables in the universe obey some fixed distribution law, Stigler's or Benford's or some other, that law must surely be independent of the system of units chosen, since God is not known to favor either the metric system or the English system. In other words, a universal first digit law, if it exists, must be scale-invariant. The simplest way to express scale-invariance, for $D_p$ in $R^+$ at any rate, is to ask that prob$\{x \in D_p\}$ = prob$\{x \in kD_p\}$ for all $k > 0$, whatever meaning we may attach to 'prob'.

Pinkham postulates, then, an 'underlying distribution of all physical constants' with a cumulative distribution function $F$, so that $F(x) =$ prob{physical constant is $\leqq x$}. Then since $D_p = \bigcup_{n=-\infty}^{\infty}[10^n, (p+1)10^n)$ and $kD_p = \bigcup_{n=-\infty}^{\infty}[k \cdot 10^n, k(p+1)10^n)$,

$$\text{prob}\{x \in D_p\} = \sum_{n=-\infty}^{\infty} [F((p+1)10^n) - F(10^n)], \text{ and} \tag{7.1}$$

$$\text{prob}\{x \in kD_p\} = \sum_{n=-\infty}^{\infty} [F(k(p+1)10^n) - F(k \cdot 10^n)]. \tag{7.2}$$

Under the reasonable assumption that $F$ is continuous, and that (7.1) = (7.2) for all $k > 0$ and all $p = 1, 2, \ldots, 9$, Pinkham proves that prob$\{x \in D_p\} = \log(p+1)$ for each $p$.

All depends on whether there *is* such a distribution function describing the universe of tabular data. As a scientific hypothesis it gives unease. For example, if $h$ is the real number such that

$F(h) = 1/2$, then half the numbers in the universe are less than $h$, which makes $h$ a most remarkable physical constant. Now what becomes of $h$ when we exercise our freedom to make scale changes?

D. Knuth [20] in fact points out that, science and philosophy aside, mathematics alone can prove no such $F$ exists. If instead of base 10 we used another base $b$ for our numeration system, the set of all numbers in $R^+$ whose initial digit is $\leqq p$ would be

$$\Delta_p = \bigcup_{n=-\infty}^{\infty} [b^n, (p+1)b^n), \quad \text{and}$$

$$\text{prob } \{x \in \Delta_p\} = \sum_{n=-\infty}^{\infty} [F((p+1)b^n) - F(b^n)],$$

and the same argument used by Pinkham to prove $\text{prob}\{x \in D_p\} = \log(p+1)$ will also produce the result $\text{prob}\{x \in \Delta_p\} = \log_b(p+1)$. Therefore

$$\sum_{n=-\infty}^{\infty} [F(rb^n) - F(b^n)] = \log_b r$$

for all integers $r$ and $b$ with $1 < r < b$. It is not hard to prove no such $F$ can exist.

All this is not to say, however, that the formula $\text{prob}\{x \in \Delta_p\} = \log_b(p+1)$ is wrong. It is in fact the correct generalization of Benford's law to tables written in the base $b$ notation, and it can be derived from any of the summability arguments used above for base 10. As will be seen below, it is as valid as Benford's law, *mutatis mutandis*, wherever Benford's law has validity. In the present context it is used only to show that the Pinkham–Hamming distribution function $F$ cannot exist, and not that deductions from the existence of such an $F$ are necessarily false.

In [19] the present author proposed another sort of probability model with scale-invariance. Instead of using some $F$ as above to give a countably-additive probability measure on $R^+$, a finitely-additive set function $\mu$ is used. Such functions, called Banach measures, exist with the properties: $\mu(R^+) = 1$; $\mu(A) \geqq 0$ for all $A \subset R^+$; $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A$ and $B$ are disjoint; and $\mu(kA) = \mu(A)$ for all $k > 0$ and $A \subset R^+$. The last property is the scale-invariance. All such measures (they exist in profusion) agree on the sets $D_p$, giving them measure $\log(p+1)$. One can even demand the additional property $\mu([0, h]) = 0$ for all $h \in R^+$, avoiding the philosophically awkward 'midpoint' of all physical constants since the resulting measure is concentrated in the neighborhood of infinity.

There are other ways to improve the same model. We can add the requirement $\mu(A + k) = \mu(A)$ for all $k \geqq 0$ and $A \subset R^+$. This means that an affine change of scale, as from Fahrenheit to Celsius temperatures, will preserve the Benford law. Again, if instead of $R^+$ for the underlying measure space we wish to use the space of positive rational numbers, or even the set of all rationals with terminating decimal expansions, Banach measures with the corresponding requisite properties can be shown to exist and give the Benford law. Terminating decimals seem particularly appropriate as the model for tabular data, but they cannot of course be used to underlie a countably additive non-atomic probability theory.

Bumby and Ellentuck [21] describe a third sort of model involving a form of scale-invariance, which has its points of mathematical interest but adds little to an understanding of the first-digit problem, except in that it reinforces the notion that every 'reasonable' notion of density should assign value $\log(p+1)$ to $D_p$ (this time as a subset of $N$).

Yet none of these arguments can finally be convincing, given the knowledge that any child can construct a list of numbers violating Benford's law. It would be perfectly consistent with every known theorem of mathematics to live in a universe whose *World Almanac* failed to contain a single entry beginning with an odd digit.

Besides, what can scale-invariance possibly have to do with the street addresses of the 342 American men of science canvassed by Benford?

**8. Statistical explanation: Pinkham's second method.** What is to my mind the true explanation is the statistical one given in the second part of Pinkham's paper [9]. It is a descriptive argument at bottom; anyone who thinks Isaac Newton didn't really explain the planetary motions, since he merely reduced them to an alternative formulation invoking an unexplained force of gravity, will also complain of what follows on the same philosophical grounds.

Suppose we are looking at a table of data drawn from some distribution $F$: $R^+ \to [0,1]$, so that $\text{prob}\{x \leqq a\} = F(a)$. We shall suppose $F$ not only continuous but thrice differentiable if need be, for convenience, just as $R^+$ is for convenience our underlying universe. $F$ is an approximation which can always be built to suit the finite situation at hand. By no means is $F$ to be construed as describing the whole numerical universe, or even the whole book of data before us; it is only to be consistent with our sample.

Then, as in Pinkham's scale-invariance argument but with a different philosophical interpretation,

$$\text{Prob}\{x \in D_p\} = \sum_{k=-\infty}^{\infty} [F((p+1)10^k) - F(10^k)]. \tag{8.1}$$

Let

$$G(x) = F(10^x); \tag{8.2}$$

then

$$\text{Prob}\{x \in D_p\} = \sum_{k=-\infty}^{\infty} [G(k + \log(p+1)) - G(k)]. \tag{8.3}$$

If we define

$$H(x) = \sum_{k=-\infty}^{\infty} [G(k+x) - G(k)] \text{ for all } x \in [0,1], \tag{8.4}$$

then (8.3) may be rewritten

$$\text{Prob}\{x \in D_p\} = H(\log(p+1)) \quad (p = 1,2,3,\ldots,9). \tag{8.5}$$

The statement of Benford's law is now simply

$$H(x) = x \text{ for } x = \log 2, \log 3, \ldots, \log 9; \tag{8.6}$$

and the scientific question becomes: how shall we recognize distributions $F$ whose corresponding $H$ have property (8.6)? And secondly, what reasons do we have to believe that our table of data does or does not derive from a distribution of this sort?

A sufficient condition for the exactness of Benford's law is obviously

$$H(x) = x \text{ for all } x \in [0,1), \tag{8.7}$$

and most of the analysis to be cited centers around (8.7) rather than (8.6). Let us denote the derivative $F'$ by $f$ and $G'$ by $g$ (these are the densities of their respective distribution functions); also $H'$ by $h$. Then from (8.4) we have, for all $x \in [0,1)$,

$$h(x) = \sum_{k=-\infty}^{\infty} g(x+k); \tag{8.8}$$

and the sufficient condition (8.7) becomes

$$h(x) = 1 \text{ for all } x \in [0,1). \tag{8.9}$$

In terms of random variables, if $F$ is the cumulative distribution function for the random variable $x$, then $G$ is the cumulative for the variable $\log x$ and $H$ for the variable $\log x \pmod 1$. We are looking for conditions on a variable $x$ which assure that $\log x \pmod 1$ is uniformly (i.e., rectangularly)

distributed on [0, 1]. We can, by the way, include distributions whose densities fail to exist here and there, as the next example will show, and only demand that (8.9) be true at all but a finite number of points.

One sort of distribution satisfying (8.9) precisely comes easily to mind: Suppose $F$ is such that for some fixed $\alpha \geqq 0$, $G$ turns out to be piecewise linear with slope $K_n$ on each interval $(\alpha + n, \alpha + n + 1)$, and such that $\Sigma K_n = 1$. Then $G' = g$ is a step-function such that (8.8) implies (8.9). Benford presciently exhibits this very phenomenon with $\alpha = \log 2$ (Fig. 3, taken from [2; p. 561]). $G$ is after all only $F$ plotted on semilog paper, so that Figure 3, though labelled as if it were $F$ for the street addresses of the scientists, is geometrically a graph for $G$, and would actually be $G$ if the abscissae were labelled with the logarithms of the numbers displayed.

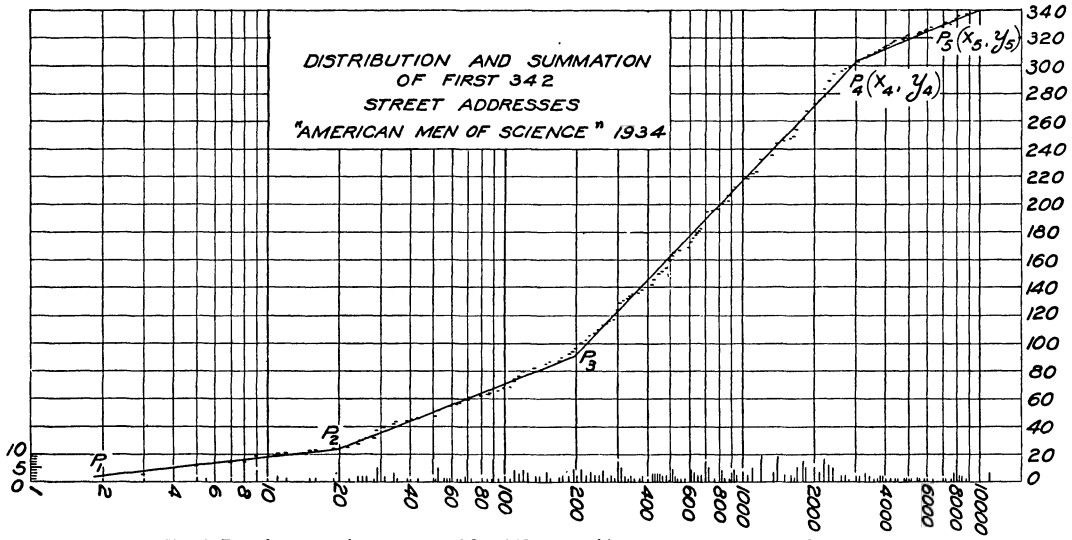



FIG. 3. Distribution and summation of first 342 street addresses, *American Men of Science*, 1934.

Had Benford been looking to verify (8.9) he would have been tempted to move the 'corner' $P_4$ of Figure 3 to a point above 2000, since the other corners are above 2, 20, 200 and 20,000, more or less (and slope zero after 20,000). This would have been not much of a distortion, and more convincing, than the use to which he did put Figure 3, which was to 'show' that the street addresses fell into a finite number of geometric sequences.

But a miracle like Figure 3 is too much to expect in general. The truly salient feature of Figure 3, and of any other distribution that obeys Benford's law more or less, inheres in its general shape, and not in any piecewise linearity.

Goudsmit and Furry [3] and especially Furry and Hurwitz [4] actually obtained formulas equivalent to (8.9) and understood perfectly well (though their language is by today's standards obscure) that for any given distribution some test of departure from (8.9) was needed. They actually calculated $\sup\{|h(x) - 1| : x \in [0, 1)\}$ for a number of popular distributions, including the Cauchy law which is given below as an example, but they gave no general criteria.

Pinkham's real contribution was to apply some Fourier transform theory to the matter and arrive at the following explicit formulas:

$$H(x) - x = \sum_{\substack{k \neq 0 \\ k=-\infty}}^{\infty} R(k)[1 - \exp(-i2\pi kx)], \text{ where} \tag{8.10}$$

$$R(k) = (4\pi^2 k^2)^{-1} \int_{-\infty}^{\infty} \exp(i2\pi kt)\, dg(t). \tag{8.11}$$

Sufficient conditions for the validity of these formulas are that $g$ be of bounded variation and that $\lim g(t) = 0$ as $|t| \to \infty$. For many popular distributions, including the Cauchy, these conditions are satisfied.

Since $|\exp(i2\pi kt)| = 1$, $|R(k)| \leqq (4\pi^2 k^2)^{-1} \operatorname{var}[g]$, and since

$$|1 - \exp(-i2\pi kx)| \leqq 2, \qquad |H(x) - x| \leqq \sum_{k=0} 2\operatorname{var}[g](4\pi^2 k^2)^{-1};$$

summing the series in (8.10) gives

$$|H(x) - x| \leqq (1/6)\operatorname{var}[g], \text{ for all } x \in [0, 1). \tag{8.12}$$

Pinkham stops with (8.12), but gives no examples to show his bound is practical. Feller [13] gets the rougher bound $|H(x) - x| < (x/2)\operatorname{var}[g]$ using the hypothesis that $g$ is unimodal and monotone on each side of its maximum (which is therefore $(1/2)\operatorname{var}[g]$). It is almost intuitively evident that when $g$ is 'wrapped up' mod 1 to give $h$ (by (8.8) above), then (8.9) fails by at most $(1/2)\operatorname{var}[g]$.

But these bounds are not in fact very good. For an example, the Cauchy distribution has density function $f(x) = 2a/\pi(x^2 + a^2)$ satisfying all the relevant hypotheses. A calculation shows that $\operatorname{var}[g] = 1.46$ approximately, so that for this case Pinkham's formula $|H(x) - x| < .25$ assures us only a miserable approximation to Benford's law. In fact, however, Furry and Hurwitz [4] have calculated $|H(x) - x| < .056x$ for the Cauchy distribution. In other words, a large sample from the Cauchy distribution will show $.284 <$ Frequency of $x \in D_1 < .318$, rather close to Benford's .301, while Pinkham's bound only gives $.05 <$ Frequency of $x \in D_1 < .55$.

At the suggestion of J. H. B. Kemperman I carried Pinkham's analysis one step further. Assuming $F$ (and therefore $g$) has enough derivatives, one can integrate by parts in (8.11), obtaining

$$R(k) = (8\pi^3 k^3)^{-1} \int_{-\infty}^{\infty} e^{i2\pi kt} dg'(t) \tag{8.13}$$

(provided $\lim_{|t| \to \infty} g'(t) = 0$, a reasonable assumption). Hence $|R(k)| \leqq (8\pi^3 k^3)^{-1} \operatorname{var}[g']$, and combining this with (8.10) yields the estimate

$$|H(x) - x| \leqq (4\pi^3)^{-1} \sum_{k \neq 0} 1/k^3 \operatorname{var}[g'], \text{ or} \tag{8.14}$$

$$|H(x) - x| \leqq .0194 \operatorname{var}[g'] \text{ approximately.} \tag{8.15}$$

In the case of the Cauchy distribution, to continue the example, it turns out that $\operatorname{var}[g'] = 2.53$, so that (8.14) gives $|H(x) - x| \leqq .0491$, about one-fifth the discrepancy allowed by Pinkham's formula.

Recently J. H. B. Kemperman, using a method quite different from that of Pinkham, has obtained the slightly better formulas (announced in [31]):

$$|H(x) - x| \leqq 1/8 \operatorname{var}[g], \text{ and} \tag{8.16}$$

$$|H(x) - x| \leqq 1/64 \operatorname{var}[g'], \tag{8.17}$$

with the additional information that 1/8 and 1/16 are the best possible coefficients. Indeed, he is able to describe the distributions which give rise to the worst possible cases. (They are quite jumpy.)

But in general even Kemperman's 'best possible' estimates in (8.16) and (8.17) are not very close to the true value of $|H(x) - x|$. The whole truth is contained in (8.10), and the shape of $F$ (and therefore of $g$ and $g'$) affects the coefficients $R(k)$ as given in (8.11) or (8.13). The integrals here are in general badly majorized by $\operatorname{var}[g]$ or $\operatorname{var}[g']$ for all $k$, because the integrands are periodic with period $1/k$ with values on the unit circle. For most $g$ the integrals should balance out to 0 or nearly so, rather than anything like $\operatorname{var}[g]$ or $\operatorname{var}[g']$. The literature still lacks a characterization in convenient terms predicting which distributions produce small discrepancies $|H(x) - x|$.

A direct study of (8.10) can throw light on an empirical observation. In Table 1, $PP(n)$ gave the first digit frequencies for populated places with populations $\geqq 2500$. $F$ has a very steep slope at 2500 so that $tf(t)$ has a peak there. Since (from (8.2)) $g(t) = F(10^t) \cdot 10^t \log_e 10$, we see that $dg$ and $dg'$ give a heavy contribution in the neighborhood of the single point log 2500. The integrands in (8.11) and (8.13) therefore have no reason to be small. The Vancouver telephone numbers, deliberately assigned so as to omit the first digit 1, plainly are drawn from an even more artificial distribution, also producing peaks in $g$ and $g'$.

**9. Mixtures of distributions; invariance properties again.** Some writers have pointed out that tables which occur naturally often represent distributions which are mixtures or composites of other distributions, and that the mixing process itself improves the approximation to Benford's law.

Furry and Hurwitz [**4**] prove the following: If $f$ is a probability density on $R^+$, and if we define $f^*f(x) = \int_{-\infty}^{\infty} (1/t)f(x/t)f(t)dt$, then $f^*f$ is also a density. In fact, $f^*f$ corresponds to the procedure of mixing all the rescaled distributions $F(x/t)$ in fractions proportional to $f(t)$, as $t$ runs through $R^+$. Reiterations of this procedure yield densities $f^{*n}$; and Furry and Hurwitz show that if $h_n$ is the function $h$ corresponding to $f^{*n}$, then $\lim_n |h_n(x) - 1| = 0$ uniformly in $x$. Thus any table in nature that can be construed as coming from high order mixtures of this sort can be expected to obey Benford's law (in the form (8.9)).

The Furry–Hurwitz convolution can also be interpreted as follows: If $X_i$ are identically distributed independent random variables each with probability density function $f$, then $f^{*n}$ is the probability density for the random variable $Y_n = \prod_{i=1}^n X_i$. Since $\log Y_n = \sum_{i=1}^n \log X_i$, the central limit theorem, which applies to sums of this sort under rather non-restrictive conditions, shows that the random variable $\log Y_n \pmod 1$ approaches uniform distribution as $n$ increases. This is an alternate statement of the Furry–Hurwitz result.

Adhikari and Sarkar [**16**; **17**] take $X_i$ as independent random variables uniformly distributed on $[0, 1]$ and prove directly, without reference to the above remarks, that the distribution of $\prod_{i=1}^n X_i$ tends to obey Benford's law as $n$ increases. They do the same for powers in place of products and for certain other multiplicative combinations of the $X_i$. They also present data from computer simulations to exhibit these phenomena and their rates of convergence. For example, in [**16**] they take 60,000 five-digit random numbers, $6700 \pm 100$ of which begin with each of the nine possible digits. Machine calculation showed that the 60,000 eighth powers of these numbers obey Benford's law to within 10 percent.

Actually one doesn't have to go this far. The reader can look for himself at the 81 products in the schoolboy notebook multiplication table and see that they are already noticeably closer to Benford's distribution than the rectangularly distributed margins.

Adhikari and Sarkar [**16**; Th. 3] also give a partial converse to the theorems on scale invariance: If $X$ is a random variable such that $H$, which is the distribution function of $\log X \pmod 1$, obeys (8.7) above, then the random variables $1/X$ and $cX$ for any $c \in R^+ \backslash \{0\}$ have the same property. For, if $X$ has the cumulative distribution function $F$ as above, and $\log X$ the distribution $G$, an easy calculation shows that $\log X$ has the distribution $G_1(x) = G(x - \log c)$ and $\log 1/X$ has the distribution $G_2(x) = 1 - G(-x)$. Thus $G_1$ and $G_2$ have exactly the same dispersion properties as $G$. In particular, if (8.7) holds for $X$, it holds for $cX$ and $1/X$.

Among other things, this observation contradicts the many writers who have said vaguely that Benford's law holds better when the distribution $F$ (or its density $f$) is 'widespread' or 'covers several orders of magnitude'. In fact, a change of scale makes no difference whatever; not only will a table obeying Benford's law continue to do so after a change of scale, but a table that fails to do so will neither improve nor grow worse by rescaling (or taking reciprocals).

One must be careful to interpret the Benford law for this purpose in the sense of (8.7), which is stronger than (8.6). Hilda Geiringer [**7**] was even led to doubt the validity of Benford's law because of this confusion. She observed (as Hamming and Pinkham did some years later) that if Benford's law

were true it should be scale-invariant, for the same reasons given by Pinkham. But she then imagined a table of 100 entries, 30 of which were the integer "1", 18 were "2", and so on in the proportions $L(n)$. Doubling every entry, however, would yield a table having very far from these first digit proportions.

The reason for this paradox is that Geiringer's table is a poor sample from any distribution which satisfies (8.7) although it may be a good sample from some crazy distribution satisfying only (8.6). A reasonable betting man confronted with a longish table he knows obeys Benford's law may be confident of its scale and inverse invariance.

This confidence was illustrated in [**22**; p. 118] when in preparing the article, I selected out of thin air a table of only 18 entries obeying Benford's law as well as so short a list can, and found the law so well preserved under the first four scale changes I chose ($c = 3$, 6, 9 and 12) that I published my first effort without the least change.

Finally, R. W. Hamming [**24**] adopts the finite point of view of a computing machine which for multiplicative purposes is indifferent to the position of the decimal point and considers the distribution whose density is given by

$$\begin{aligned} r(x) &= (\log e)/x \quad \text{if } 1/10 \leqq x \leqq 1 \\ &= 0 \qquad\quad \text{otherwise.} \end{aligned} \tag{9.1}$$

This "reciprocal distribution" satisfies Benford's law, and is in fact the only distribution supported on $[1/10, 1]$ which will satisfy (8.9) precisely.

In floating point arithmetic the product or quotient of two numbers in $[1/10, 1]$ is again placed in $[1/10, 1]$ by a shift of decimal point if necessary. With this definition, and defining the *distance* of an arbitrary probability distribution density $f$ (carried on $[1/10, 1]$) from $r$ as

$$D(f) = \sup\left\{\frac{|f(x) - r(x)|}{r(x)} : x \in [1/10, 1]\right\},$$

Hamming proves the following two theorems:

(a) *If a random variable $X$ has probability density $r$ and if $Y$ is any other random variable with density supported on* $[1/10, 1]$, *then $XY$, $X/Y$ and $Y/X$ all have the reciprocal distribution*;

(b) *If $X$ and $Y$ are any two random variables with densities supported on* $[1/10, 1]$, *and if $X$ has the probability density $f$ and $XY$ has density $g$, then $D(g) \leqq D(f)$.*

Equality in (b) is quite rare, and Hamming gives some further information on rates of convergence to $r$. He also has some theorems on sums, but they have rather artificial hypotheses and do not tend to explain the occurrence of Benford's law as (a) and (b) do for compilations which result from repeated multiplications.

**10. Final comments on the literature.** Certain items in the Bibliography have not yet been given sufficient attention. The book by Furlan [**6**] I have not myself seen, but Hilda Geiringer's review [**7**] makes it plain that the work is more mystical than mathematical, and offers insight to neither scientist nor mathematician. Apparently Furlan argues, with a superabundance of detail, that Benford's law reflects a profound 'harmonic' truth of nature, related to Benford's notion that nature counts in geometric sequences.

C. Gini [**8**] refers to Furlan's book in a footnote, rather guardedly ("... *un grosso volume, di vari punti di vista molto interessante.*"), but makes no use of it whatever. Gini's paper mainly gives empirical data, census figures supporting Benford's law and so on, but insists that the law cannot be a universal truth or a mere property of the number system; for comparison's sake he tries to fit another law to some of his data.

A. Herzel [**9**], writing at the same time as Gini, also does homage to Furlan — three pages' worth

— but ends up doubting that a universal exponential law of economics or sociology exists. Herzel's paper is mainly devoted to the urn models which in effect try to make $N$ into a probability measure space in which the sets $D_p$ have the right measure. Diaconis [29] does much more along these lines.

Varian's note [27] is a suggestion that Benford's law be used as a partial test of the honesty or validity of purportedly random scientific data. The suggestion is attractive. In fact, in Table 1 above, one can observe that $PP(9)$ is about 1/3 the size of $PP(8)$, which is very far from the ratio predicted by Benford's law. It is true that the $PP$ table doesn't satisfy the law overall, because of the 2500 threshold, but that really shouldn't affect $PP(9)/PP(8)$. One might therefore be led to suspect that census-takers have a tendency, when faced with a figure just a little below a power of 10, to round it up a little, putting it into the next order of magnitude. This may be wrong in the present case, but social scientists need all the tools of suspicion they can find.

The paper of J. Franel [2B] is a belated sequel to a famous argument of H. Poincaré [1B] in which the master attempted to show that the third digit of a table of logarithms should be as often odd as even. Franel shows this is not quite so, but that the probability of an odd $n$th digit converges to 1/2 as $n$ increases.

A good number of the writers cited above have also discovered, quoted or proved (in some sense), as Franel almost did, a second digit law related to Benford's. If $S_p$ is the set of all real numbers in $R^+$ whose second digit is $\leqq p$ $(p = 0, 1, 2, \ldots, 9)$, the law is

$$\text{Prob}\{x \in S_p\} = \sum_{k=1}^{9} [\log(k + (p+1)/10) - \log k], \tag{10.1}$$

and it follows immediately from any argument that assures (8.7). There are corresponding formulas for third and higher digits, all of which may be summarized in one diagram: The $C$ scale of a slide rule. The probabilities are just those fractions of the $C$ scale occupied by the sets in question.

Another popular generalization repeated very often is the substitution of some base $b$ other than 10 for this entire discussion. When $b = 2$ the first digit problem is of course trivial but otherwise there is nothing new in the generalization; every argument that applies to 10 applies to *b mutatis mutandis*.

Finally, it seems obligatory to describe Simon Newcomb's paper [1], in which, by the way, the second digit law is already given and the inessential character of the base 10 already noted. Newcomb notes that any positive number can be written $10^s$ for some real number $s$, and that since reducing $s$ modulo 1 does not affect the first digit behavior of $10^s$, we may as well assume $0 \leqq s < 1$. All that is needed now is some reasoning sufficient to give Newcomb's conclusion that "The law of probability of the occurrence of numbers is such that all mantissae of their logarithms are equally probable," i.e., that $s$ is uniformly distributed on $[0, 1)$.

Alas, the argument Newcomb does give begs the question, though it has at least the virtue of brevity. At the crucial point he says "... it is evident...". One is forced to conclude that the uniform distribution of $s$ (mod 1) was an inspired guess, akin to Buffon's guess that the angle of his needle, not its cosine or log sin, is what obeys a uniform distribution law (mod $\pi$).

One can easily invent a ('biassed'?) needle-tossing machine which violates Buffon's hypothesis, just as any phone company can print a directory violating Benford's law. What remains tantalizing is the notion that there is still some unexplained measure in the universe which says that the probability of such violations is small.

NOTE: References **32** through **37**, and **15B**, were added in proof. Of all these additions, only [33] invokes a method or point of view disjoint from those reviewed in this article. I am also in possession of recent preprints or otherwise unpublished memoirs on the first digit problem written by the following authors: Persi Diaconis (Stanford University, Stanford, Calif.), Robert J. Epp (University of California at Los Angeles, Calif.), and Dennis P. Allen, Jr. (Bell Telephone Laboratories, Holmdell, New Jersey 07733).

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## SOME INTERESTING PROPERTIES OF THE VARIATION FUNCTION

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For a function $f$ which is of bounded variation on $[a, b]$, $V_f$ denotes the function defined, for each $x$ in $[a, b]$, by $V_f(x) = V_a^x(f)$, where $V_a^x(f)$ denotes the total variation of $f$ on $[a, x]$. $V_f$, called the variation function of $f$ on $[a, b]$, possesses some rather interesting properties, particularly the property of inheriting certain properties from its parent function $f$. While some of the properties of $V_f$ are well known, others are not so well known, and it is the purpose of this paper to present a unified exposition of the properties of the variation function. In order to make this material more readily accessible to the advanced undergraduate or graduate student in mathematics, the author has given detailed proofs of all theorems in this paper. Although expository in nature, this paper contains some new results, namely Theorem 5, and Theorem 6 which is a generalization of a known result.

DEFINITION 1. The statement that the function $f$ is of **bounded variation** on $[a, b]$ means that $f$ is a function whose domain includes $[a, b]$ and there exists a nonnegative number $B$ such that if $\{x_i\}_{i=0}^n$ is any subdivision of $[a, b]$, then

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \leqq B. \tag{1}$$

The least such number $B$ is called the (total) variation of $f$ on $[a, b]$ and is denoted by $V_a^b(f)$. (Note: $V_a^a(f) = 0$.)

REMARK. Although some of the definitions and results given in this paper also apply to complex-valued functions, for the sake of simplicity this exposition will be restricted to real-valued functions.

It follows from a property of absolute values of numbers that the sum (1) is nondecreasing with respect to refinements.

As immediate consequences of Definition 1, we have the following lemmas, the proofs of which are left to the reader.

LEMMA 1. *If the function $f$ is of bounded variation on $[a, b]$ and $\varepsilon$ is a positive number, then there exists a subdivision $D$ of $[a, b]$ such that if $D'$ is a refinement of $D$ and $p$ and $q$ are two consecutive numbers of $D'$, then*

$$V_p^q(f) < |f(q) - f(p)| + \varepsilon.$$

LEMMA 2. *If $f$ is a function whose domain includes $[a, b]$ and $c$ is a number in $[a, b]$, then in order that $f$ be of bounded variation on $[a, b]$, it is necessary and sufficient that $f$ be of bounded variation on $[a, c]$ and on $[c, b]$. Moreover, in this case we have that*

$$V_a^b(f) = V_a^c(f) + V_c^b(f).$$

If $c$ is a number in the domain of the function $f$ and $f(x)$ approaches a (finite) limit as $x$ approaches $c$ from the right, then that right-hand limit will be denoted by $f(c+)$ or by $\lim_{x \to c^+} f(x)$. Similarly, if $f(x)$ approaches a (finite) limit as $x$ approaches $c$ from the left, then that left-hand limit will be denoted by $f(c-)$ or by $\lim_{x \to c^-} f(x)$.

THEOREM 1. *If the function $f$ is of bounded variation on $[a, b]$, then*

$$V_f(c+) - V_f(c) = |f(c+) - f(c)|$$

*for each number $c$ in $[a, b)$ and*

$$V_f(c) - V_f(c-) = |f(c) - f(c-)|$$

*for each number $c$ in $(a, b]$.*

*Proof.* Let $c$ be a number in $[a, b)$. Since $f$ is of bounded variation on $[a, b]$, the right-hand limit $f(c+)$ exists. Since $V_f$ is monotone nondecreasing on $[a, b]$, the right-hand limit $V_f(c+)$ also exists. Let $\varepsilon$ denote a positive number. There exists a positive number $\delta'$ such that $\delta' < b - c$ and if $c < x < c + \delta'$, then

$$\big|\, |f(x) - f(c)| - |f(c+) - f(c)| \,\big| < \varepsilon/2.$$

It follows from Lemma 1 that there exists a subdivision $D$ of $[a, b]$ such that $c$ belongs to $D$, and if $D'$ is any refinement of $D$ and $x$ is the immediate successor of $c$ in $D'$, then

$$V_f(x) - V_f(c) = V_c^x(f) < |f(x) - f(c)| + \varepsilon/2,$$

or

$$\big|\, [V_f(x) - V_f(c)] - |f(x) - f(c)| \,\big| < \varepsilon/2.$$

Let $\delta''$ be the distance between $c$ and the immediate successor of $c$ in $D$. Let $\delta$ be the smaller of the two positive numbers $\delta'$ and $\delta''$. Let $x$ be any particular number such that $c < x < c + \delta$. Since $c < x < c + \delta'$,

$$\big|\, |f(x) - f(c)| - |f(c+) - f(c)| \,\big| < \varepsilon/2.$$

Since there is a subdivision $D'$ of $[a, b]$ such that $D'$ is a refinement of $D$ and such that $x$ is the immediate successor of $c$ in $D'$,

$$\big|\, [V_f(x) - V_f(c)] - |f(x) - f(c)| \,\big| < \varepsilon/2.$$

Consequently,

$$\begin{aligned}|\,[V_f(x)-V_f(c)]-|f(c+)-f(c)|\,| &\leqq |\,[V_f(x)-V_f(c)]-|f(x)-f(c)|\,| \\ &\quad +|\,|f(x)-f(c)|-|f(c+)-f(c)|\,| \\ &< \varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Therefore $V_f(c+)-V_f(c)=|f(c+)-f(c)|$.

A similar argument will show that if $c$ is in $(a, b]$, then

$$V_f(c)-V_f(c-)=|f(c)-f(c-)|.$$

Being an immediate consequence of Theorem 1, the following corollary gives what is probably the best known property of the variation function.

COROLLARY 1.1. *If the function $f$ is of bounded variation on $[a, b]$ and $c$ is a number in $[a, b]$, then in order that $f$ be continuous on the right (left) at $(c, f(c))$, it is necessary and sufficient that the variation function $V_f$ be continuous on the right (left) at $(c, V_f(c))$.*

As a consequence of Theorem 1 and Corollary 1.1, we see that $f$ and $V_f$ are either both continuous at a number $x$ in $[a, b]$ or, if discontinuous at $x$, $f$ and $V_f$ have the same type of discontinuity there, and their jumps are equal except possibly for sign.

THEOREM 2. *If $f$ is of bounded variation on $[a, b]$, then $V_f$ is of bounded variation on $[a, b]$ and*

$$V_a^b(V_f)=V_a^b(f).$$

*Proof.* If $a \leqq x < x' \leqq b$, then $V_f(x')-V_f(x)=V_x^{x'}(f) \geqq |f(x')-f(x)| \geqq 0$. That is, $V_f$ is nondecreasing on $[a, b]$. Being nondecreasing on $[a, b]$, $V_f$ is of bounded variation on $[a, b]$ and

$$V_a^b(V_f)=V_f(b)-V_f(a)=V_a^b(f).$$

DEFINITION 2. The statement that **the graph of $f$ has length on** $[a, b]$ means that $f$ is a function whose domain includes $[a, b]$ and there exists a nonnegative number $B$ such that if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then

$$\sum_{i=1}^{n} \{(x_i - x_{i-1})^2 + [f(x_i)-f(x_{i-1})]^2\}^{1/2} \leqq B. \tag{2}$$

The least such number $B$ is called the length of the graph of $f$ on $[a, b]$ and is denoted by $L_a^b(f)$. (Note: $L_a^a(f)=0$.) The function $L_f$ defined, for each $x$ in $[a, b]$, by $L_f(x)=L_a^x(f)$ is called the "length function of $f$ on $[a, b]$."

It follows from the triangle inequality for the plane that the sum (2) is nondecreasing with respect to refinements.

As an immediate consequence of Definition 2, we have the following lemma, the proof of which is in many respects quite similar to that of Lemma 2.

LEMMA 3. *If $f$ is a function whose domain includes $[a, b]$ and $c$ is a number in $[a, b]$, then in order that the graph of $f$ have length on $[a, b]$, it is necessary and sufficient that the graph of $f$ have length on $[a, c]$ and on $[c, b]$. Moreover, in this case we have that*

$$L_a^b(f)=L_a^c(f)+L_c^b(f).$$

The following well-known result is a consequence of Definitions 1 and 2.

LEMMA 4. *If $f$ is a function whose domain includes $[a, b]$, then in order that the graph of $f$ have length on $[a, b]$, it is necessary and sufficient that $f$ be of bounded variation on $[a, b]$.*

Depending upon the inequality used in the proof of Lemma 4, one obtains either

$$V_a^b(f) \leqq L_a^b(f) \leqq V_a^b(f) + (b-a),$$

$$[V_a^b(f) + (b-a)] \cdot 2^{-1/2} \leqq L_a^b(f) \leqq V_a^b(f) + (b-a),$$

or

$$\{[V_a^b(f)]^2 + (b-a)^2\}^{1/2} \leqq L_a^b(f) \leqq V_a^b(f) + (b-a).$$

According to a footnote in [1], the following theorem was first discovered and proved by J. A. Clarkson. A proof of Theorem 3, which is different from the one given here, was also given in [1].

THEOREM 3. *If the function $f$ is of bounded variation on $[a, b]$, then*

$$L_a^b(V_f) = L_a^b(f).$$

*Proof.* Since $V_f$ is of bounded variation on $[a, b]$, its graph has length on $[a, b]$. Let $\{x_i\}_{i=0}^n$ be a subdivision of $[a, b]$. Then for each positive integer $i \leqq n$,

$$\{[\Delta_i V_f]^2 + [\Delta_i x]^2\}^{1/2} \leqq \Delta_i L_f,$$

where, for example, $\Delta_i V_f = V_f(x_i) - V_f(x_{i-1})$. Hence

$$\sum_{i=1}^n \{[\Delta_i V_f]^2 + [\Delta_i x]^2\}^{1/2} \leqq \sum_{i=1}^n \Delta_i L_f = L_a^b(f).$$

It follows that $L_a^b(V_f) \leqq L_a^b(f)$. Also, for each positive integer $i \leqq n$, $|\Delta_i f| \leqq \Delta_i V_f$. Consequently,

$$\sum_{i=1}^n \{[\Delta_i x]^2 + [\Delta_i f]^2\}^{1/2} \leqq \sum_{i=1}^n \{[\Delta_i x]^2 + [\Delta_i V_f]^2\}^{1/2} \leqq L_a^b(V_f).$$

It follows that $L_a^b(V_f) \geqq L_a^b(f)$. Therefore $L_a^b(V_f) = L_a^b(f)$.

Theorem 3 can be generalized to a curve $\Gamma$ in the complex plane where $\Gamma$ is defined parametrically.

REMARK. If $a \leqq x_1 < x_2 \leqq b$, then

$$0 < \{(x_2 - x_1)^2 + [f(x_2) - f(x_1)]^2\}^{1/2} \leqq L_f(x_2) - L_f(x_1).$$

Hence $L_f$ is increasing on $[a, b]$ and is of bounded variation on $[a, b]$ with

$$V_a^b(L_f) = L_f(b) - L_f(a) = L_a^b(f) = L_a^b(V_f).$$

That is, the variation of the length function is the length of the graph of the variation function on $[a, b]$.

DEFINITION 3. The statement that the **Hellinger integral** $\int_a^b (df)^2/dm$ exists means that $f$ is a function whose domain includes $[a, b]$, $m$ is an increasing function on $[a, b]$, and there exists a nonnegative number $B$ such that if $\{x_i\}_{i=0}^n$ is a subdivision of $[a, b]$, then

$$\sum_{i=1}^n \frac{[f(x_i) - f(x_{i-1})]^2}{m(x_i) - m(x_{i-1})} \leqq B. \tag{3}$$

The least such number $B$ is the Hellinger integral $\int_a^b (df)^2/dm$. (Note: $\int_a^a (df)^2/dm = 0$.)

The sums (3) are nondecreasing with respect to refinements. This can be shown by first adding a single point to a given subdivision $D$ of $[a, b]$ and using the inequality

$$\frac{(a_1 + a_2)^2}{b_1 + b_2} \leqq \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2},$$

where $a_1$ and $a_2$ are real numbers and $b_1$ and $b_2$ are positive real numbers.

Note that the Hellinger integral $\int_a^b (df)^2/dm$ is the limit under subdivision refinement of the sums

(3). That is, $\int_a^b (df)^2/dm$ is a number such that for each positive number $\varepsilon$ there exists a subdivision $D$ of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is any refinement of $D$, then

$$\left| \sum_{i=1}^{n} \frac{[f(x_i)-f(x_{i-1})]^2}{m(x_i)-m(x_{i-1})} - \int_a^b \frac{(df)^2}{dm} \right| < \varepsilon.$$

As an immediate consequence of Definition 3, we have the following lemma, the proof of which is, in many respects, also quite similar to that of Lemma 2.

LEMMA 5. *If $f$ is a function whose domain includes $[a, b]$, the function $m$ is increasing on $[a, b]$, and $c$ is a number in $[a, b]$, then in order that $\int_a^b (df)^2/dm$ exist, it is necessary and sufficient that both $\int_a^c (df)^2/dm$ and $\int_c^b (df)^2/dm$ exist. Moreover, in this case we have that*

$$\int_a^b \frac{(df)^2}{dm} = \int_a^c \frac{(df)^2}{dm} + \int_c^b \frac{(df)^2}{dm}.$$

Using Definition 3 and the Cauchy-Schwarz inequality, the following lemma can be proved.

LEMMA 6. *If $\int_a^b (df)^2/dm$ exists, then $f$ is of bounded variation on $[a, b]$ and*

$$V_a^b(f) \leqq \left\{ [m(b)-m(a)] \cdot \int_a^b (df)^2/dm \right\}^{1/2}.$$

The following theorem was discovered by Hellinger who gave a proof in his dissertation [3] in 1907. It was rediscovered by J. T. Darwin, Jr. who included a different proof in his dissertation [2] in 1963.

THEOREM 4. *If the function $f$ is of bounded variation on $[a, b]$ and the function $m$ is increasing on $[a, b]$, then in order that $\int_a^b (df)^2/dm$ exist, it is necessary and sufficient that $\int_a^b (dV_f)^2/dm$ exist. Moreover, in this case we have that*

$$\int_a^b \frac{(df)^2}{dm} = \int_a^b \frac{(dV_f)^2}{dm}.$$

*Proof.* Suppose $\int_a^b (df)^2/dm$ exists. Let $\{x_i\}_{i=0}^n$ be a subdivision of $[a, b]$. Then for each positive integer $i \leqq n$,

$$0 \leqq \Delta_i V_f = V_{x_{i-1}}^{x_i}(f) \leqq \left\{ [\Delta_i m] \cdot \int_{x_{i-1}}^{x_i} (df)^2/dm \right\}^{1/2}$$

by Lemma 6. It follows that

$$\sum_{i=1}^{n} \frac{[V_f(x_i)-V_f(x_{i-1})]^2}{m(x_i)-m(x_{i-1})} \leqq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \frac{(df)^2}{dm} = \int_a^b \frac{(df)^2}{dm},$$

where Lemma 5 was used to obtain equality on the right. Thus $\int_a^b (dV_f)^2/dm$ exists and $\int_a^b (dV_f)^2/dm \leqq \int_a^b (df)^2/dm$.

Suppose $\int_a^b (dV_f)^2/dm$ exists. Let $\{x_i\}_{i=0}^n$ be a subdivision of $[a, b]$. For each positive integer $i \leqq n$, $|\Delta_i f| \leqq \Delta_i V_f$. It follows that

$$\sum_{i=1}^{n} \frac{[\Delta_i f]^2}{\Delta_i m} \leqq \sum_{i=1}^{n} \frac{[\Delta_i V_f]^2}{\Delta_i m} \leqq \int_a^b \frac{(dV_f)^2}{dm}.$$

Therefore $\int_a^b (df)^2/dm$ exists and $\int_a^b (df)^2/dm \leqq \int_a^b (dV_f)^2/dm$. Consequently, $\int_a^b (df)^2/dm = \int_a^b (dV_f)^2/dm$.

DEFINITION 4. The statement that $f$ **satisfies a uniform Lipschitz condition of order $p$ with respect to $m$ on $[a, b]$** means that $f$ is a function whose domain includes $[a, b]$, $m$ is a real-valued increasing function on $[a, b]$, $p$ is a positive number, and there exists a nonnegative number $K$ such that if $x_1$ and $x_2$ are numbers in $[a, b]$, then

$$|f(x_1)-f(x_2)| \leqq K \cdot |m(x_1)-m(x_2)|^p.$$

The least such number $K$ is called the Lipschitz constant for $f$, $m$, and $p$ on $[a, b]$ and is denoted by $K_a^b(f, m, p)$.

THEOREM 5. *If the function $f$ is of bounded variation on $[a, b]$ and the function $m$ is increasing on $[a, b]$, then in order that $f$ satisfy a uniform Lipschitz condition of order 1 with respect to $m$ on $[a, b]$, it is necessary and sufficient that $V_f$ satisfy a uniform Lipschitz condition of order 1 with respect to $m$ on $[a, b]$. Moreover, in this case we have that $K_a^b(V_f, m, 1) = K_a^b(f, m, 1)$.*

*Proof.* Suppose $f$ satisfies a uniform Lipschitz condition of order 1 with respect to $m$ on $[a, b]$. Let $x_1$ and $x_2$ be numbers in $[a, b]$. Suppose $x_1 < x_2$. Let $\{t_i\}_{i=0}^n$ be a subdivision of $[x_1, x_2]$. Then for each positive integer $i \leqq n$,

$$|f(t_i)-f(t_{i-1})| \leqq K_a^b(f, m, 1) \cdot |m(t_i)-m(t_{i-1})|.$$

Thus

$$\sum_{i=1}^{n} |f(t_i)-f(t_{i-1})| \leqq K_a^b(f, m, 1) \cdot [m(x_2)-m(x_1)],$$

from which it follows that

$$|V_f(x_1)-V_f(x_2)| = V_{x_1}^{x_2}(f) \leqq K_a^b(f, m, 1) \cdot |m(x_1)-m(x_2)|.$$

If $x_2 < x_1$, a similar argument will show that

$$|V_f(x_1)-V_f(x_2)| \leqq K_a^b(f, m, 1) \cdot |m(x_1)-m(x_2)|.$$

Therefore $V_f$ satisfies a uniform Lipschitz condition of order 1 with respect to $m$ on $[a, b]$ and $K_a^b(V_f, m, 1) \leqq K_a^b(f, m, 1)$.

Suppose $V_f$ satisfies a uniform Lipschitz condition of order 1 with respect to $m$ on $[a, b]$. Then if $x_1$ and $x_2$ are numbers in $[a, b]$.

$$|f(x_1)-f(x_2)| \leqq |V_f(x_1)-V_f(x_2)| \leqq K_a^b(V_f, m, 1) \cdot |m(x_1)-m(x_2)|.$$

Therefore $f$ satisfies a uniform Lipschitz condition of order 1 with respect to $m$ on $[a, b]$ and $K_a^b(f, m, 1) \leqq K_a^b(V_f, m, 1)$. Hence $K_a^b(V_f, m, 1) = K_a^b(f, m, 1)$.

REMARK. Let $f$ be a real-valued function of bounded variation on a closed interval $[a, b]$ of the real axis and let $m$ be a monotone increasing real-valued function on $[a, b]$. It is easily seen that if $V_f$ satisfies a uniform Lipschitz condition of order $p$ with respect to $m$ on $[a, b]$, then $f$ does also. However, it has not been shown that the converse is true in case $p \neq 1$. In case $\int_a^b (df)^2/dm$ exists, then both $f$ and $V_f$ satisfy a uniform Lipschitz condition of order 1/2 with respect to $m$ on $[a, b]$ and

$$K_a^b(f, m, 1/2) \leqq K_a^b(V_f, m, 1/2) \leqq \left\{\int_a^b (df)^2/dm\right\}^{1/2}.$$

*Proof.* Let $x_1$ and $x_2$ be any two numbers in $[a, b]$ such that $x_1 < x_2$. From Lemma 6 we have that

$$\begin{aligned} V_f(x_2)-V_f(x_1) &\leqq \left\{\int_{x_1}^{x_2} (df)^2/dm\right\}^{1/2} \cdot [m(x_2)-m(x_1)]^{1/2} \\ &\leqq \left\{\int_a^b (df)^2/dm\right\}^{1/2} \cdot [m(x_2)-m(x_1)]^{1/2}, \end{aligned}$$

where Lemma 5 was used to obtain the last inequality. Therefore $V_f$ satisfies a uniform Lipschitz condition of order 1/2 with respect to $m$ on $[a, b]$ and

$$K_a^b(V_f, m, 1/2) \leqq \left\{\int_a^b (df)^2/dm\right\}^{1/2}.$$

Thus $f$ also satisfies a uniform Lipschitz condition of order 1/2 with respect to $m$ on $[a, b]$, and it is easily seen that

$$K_a^b(f, m, 1/2) \leqq K_a^b(V_f, m, 1/2).$$

Note that if $f$ is of bounded variation on $[a, b]$ and $f$ is not constant on any subinterval of $[a, b]$ so that $V_f$ is increasing on $[a, b]$, then $f$ satisfies a uniform Lipschitz condition of order 1 with respect to $V_f$ on $[a, b]$. This implies, by Theorem 3 of [4], that $\int_a^b (df)^2/dV_f$ exists and, by Theorem 4 of this paper, that

$$\int_a^b \frac{(df)^2}{dV_f} = \int_a^b \frac{(dV_f)^2}{dV_f} = V_a^b(f),$$

which expresses the variation of $f$ on $[a, b]$ as a Hellinger integral. Moreover, if $f$ is of bounded variation on $[a, b]$, the function $L_f$ is increasing on $[a, b]$ and, since

$$|f(x_1) - f(x_2)| \leqq |V_f(x_1) - V_f(x_2)| \leqq |L_f(x_1) - L_f(x_2)|$$

for $x_1$ and $x_2$ in $[a, b]$, both $f$ and $V_f$ satisfy a uniform Lipschitz condition of order 1 with respect to $L_f$ on $[a, b]$.

DEFINITION 5. The statement that $f$ **is absolutely continuous with respect to** $g$ **on** $[a, b]$ means that each of $f$ and $g$ is a function whose domain includes $[a, b]$ and for each positive number $\varepsilon$ there exists a positive number $\delta$ such that if $\{(a_i, b_i)\}_{i=1}^n$ is a finite collection of nonoverlapping subintervals of $[a, b]$ and

$$\sum_{i=1}^n |g(b_i) - g(a_i)| < \delta,$$

then

$$\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon.$$

If $f$ satisfies a uniform Lipschitz condition of order $p \geqq 1$ with respect to $m$ on $[a, b]$, then $f$ is absolutely continuous with respect to $m$ on $[a, b]$ but not conversely. The fact that if $f$ is of bounded variation on $[a, b]$, then

$$|f(x_1) - f(x_2)| \leqq |V_f(x_1) - V_f(x_2)| \leqq |L_f(x_1) - L_f(x_2)|$$

for $x_1$ and $x_2$ in $[a, b]$ implies that $f$ is absolutely continuous with respect to $V_f$ on $[a, b]$ and that both $f$ and $V_f$ are absolutely continuous with respect to $L_f$ on $[a, b]$.

THEOREM 6. *If the function $f$ is of bounded variation on $[a, b]$ and the function $m$ is increasing on $[a, b]$, then in order that $f$ be absolutely continuous with respect to $m$ on $[a, b]$, it is necessary and sufficient that $V_f$ be absolutely continuous with respect to $m$ on $[a, b]$.*

*Proof.* Suppose $f$ is absolutely continuous with respect to $m$ on $[a, b]$. Let $\varepsilon$ denote a positive number. There exists a positive number $\delta$ such that if $\{(a_i, b_i)\}_{i=1}^n$ is any finite collection of nonoverlapping subintervals of $[a, b]$ for which

$$\sum_{i=1}^n [m(b_i) - m(a_i)] < \delta,$$

then

$$\sum_{i=1}^n |f(b_i) - f(a_i)| < \varepsilon/2.$$

Let $\{(a_i, b_i)\}_{i=1}^n$ be a finite collection of nonoverlapping subintervals of $[a, b]$ such that

$$\sum_{i=1}^{n} [m(b_i) - m(a_i)] < \delta.$$

Then for each positive integer $i \leqq n$, $f$ is of bounded variation on $[a_i, b_i]$, so that there exists a subdivision $\{t_{j,i}\}_{j=0}^{k_i}$ of $[a_i, b_i]$ such that

$$V_f(b_i) - V_f(a_i) = V_{a_i}^{b_i}(f) < \sum_{j=1}^{k_i} |f(t_{j,i}) - f(t_{j-1,i})| + \varepsilon/2n.$$

For each positive integer $i \leqq n$, the intervals $(t_{j-1,i}, t_{j,i})$, $j = 1, \ldots, k_i$, are nonoverlapping, and

$$\sum_{i=1}^{n} \sum_{j=1}^{k_i} [m(t_{j,i}) - m(t_{j-1,i})] = \sum_{i=1}^{n} [m(b_i) - m(a_i)] < \delta.$$

Therefore

$$\sum_{i=1}^{n} |V_f(b_i) - V_f(a_i)| < \sum_{i=1}^{n} \left\{ \sum_{j=1}^{k_i} |f(t_{j,i}) - f(t_{j-1,i})| + \frac{\varepsilon}{2n} \right\}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $V_f$ is absolutely continuous with respect to $m$ on $[a, b]$.

Suppose $V_f$ is absolutely continuous with respect to $m$ on $[a, b]$. Let $\varepsilon$ denote a positive number, and let $\delta$ be the positive number for $V_f$ and $m$ corresponding to $\varepsilon$ as in Definition 5. Then if $\{(a_i, b_i)\}_{i=1}^n$ is any finite collection of nonoverlapping subintervals of $[a, b]$ such that $\Sigma_{i=1}^n [m(b_i) - m(a_i)] < \delta$, then

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| \leqq \sum_{i=1}^{n} |V_f(b_i) - V_f(a_i)| < \varepsilon.$$

Therefore $f$ is absolutely continuous with respect to $m$ on $[a, b]$.

If the function $f$ is of bounded variation on $[a, b]$, then both $f$ and $V_f$ are differentiable on $[a, b]$ with the possible exception of a set of Lebesgue measure zero. However, the set of numbers $x$ in $[a, b]$ at which $f'(x)$ exists is not necessarily the same as the set of numbers $x$ in $[a, b]$ at which $V_f'(x)$ exists. For example, if $f(x) = x$ for $x$ in $[0, 1/2]$ and $f(x) = 1 - x$ for $x$ in $(1/2, 1]$, then $V_f(x) = x$ for $x$ in $[0, 1]$ so that $V_f$ has a derivative at $1/2$ while $f$ does not. The following theorem gives a partial answer to the question concerning the relationship between $f'$ and $V_f'$.

THEOREM 7. *If $f$ is a differentiable real-valued function on a closed interval $[a, b]$ of the real axis such that $|f'|$ is bounded on $[a, b]$, then $V_f'(x) = |f'(x)|$ for all $x$ in $[a, b] - E$, where $E$ is a Lebesgue measurable subset of $[a, b]$ with Lebesgue measure zero.*

*Proof.* Let $M$ be a positive real number such that $|f'(x)| \leqq M$ for all $x$ in $[a, b]$. If $x_1$ and $x_2$ are distinct numbers in $[a, b]$, then by the Mean Value Theorem of differential calculus there exists a number $c$ between $x_1$ and $x_2$ such that

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$$

and hence we have that

$$|f(x_1) - f(x_2)| = |f'(c)| \cdot |x_1 - x_2| \leqq M \cdot |x_1 - x_2|.$$

Thus $f$ satisfies a uniform Lipschitz condition of order one on $[a, b]$. Hence $f$ is absolutely continuous on $[a, b]$. Therefore, for each $x$ in $[a, b]$,

$$V_f(x) = \int_{[a,x]} |f'(t)|\, dt,$$

where the integral here is the Lebesgue integral. Thus

$$V'_f(x) = |f'(x)|$$

for each $x$ in $[a, b] - E$, where $E$ is a Lebesgue measurable subset of $[a, b]$ with Lebesgue measure zero.

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## CORRECTIONS TO "TOPICS IN ORTHOGONAL FUNCTIONS"

(This MONTHLY, 82 (1975) 594–609)

J. J. PRICE

Two historical corrections should be made concerning a basic theorem discussed in the paper: for any complete orthonormal system $\Phi$ in $L^2(0,1)$, there exists a function in $L^2$ whose $\Phi$-Fourier series can be rearranged to diverge almost everywhere. First, this theorem was proved simultaneously by A. M. Olevskii and P. L. Uljanov [2, 3], but incorrectly attributed only to the latter author. Second, the special case of the theorem for the trigonometric system was asserted without proof by Kolmogoroff in 1927 [1]. This historical motivation for the general theorem was inadvertently omitted. We regret these oversights.

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## NOTICE

Beginning September 1, 1976, the address of Richard A. Brualdi, Editor of MATHEMATICAL NOTES and CLASSROOM NOTES, will be: Laboratoire Calcul des Probabilités, Université de Paris, T. 56, 4 Place Jussieu, 75–230 Paris, France.

# MATHEMATICAL NOTES

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## MULTIPLICATIVELY PERIODIC RINGS

TED CHINBURG AND MELVIN HENRIKSEN

**1. Introduction.** A ring $R$ is called *periodic* if for each element $a$ of $R$ there is a positive integer $n(a)$ such that $a^{n(a)+1} = a$. If there is a positive integer $n$ such that $a^{n+1} = a$ for all $a$ in $R$, then the smallest such $n$ is called the *period* of $R$, and $R$ is called a *J-ring* (see [7]). It is well known that every periodic ring is commutative [6, Chapter X].

A ring $R$ is called a *$p^k$-ring* in [8] if there is a prime $p$ and a positive integer $k$ such that $pa = 0$ and $a^{p^k} = a$ for all $a$ in $R$. In [7], J. Luh uses Dirichlet's Theorem on primes in an arithmetic progression to show that $R$ is a $J$-ring if and only if it is the direct sum of finitely many $p^k$-rings. In this note we prove the following generalization of Luh's result without using Dirichlet's theorem:

THEOREM 1. *A ring $R$ is periodic if and only if it is the union of a countable ascending chain $\{R(n)\}$ of $J$-rings such that every $J$-ring contained in $R$ is contained in some $R(n)$. Moreover, each $R(n)$ is the direct sum of finitely many $p^k$-rings.*

We use Theorem 1 to show that the $J$-subrings of a periodic ring form a lattice with respect to join and intersection (the *join* of two subrings is the smallest subring containing both of them).

After noting that every $J$-ring has nonzero characteristic, we determine for which positive integers $n$ and $m$ there exist $J$-rings of period $n$ and characteristic $m$. This generalizes a problem posed by G. Wene in [9].

**2. A basic lemma.** If $R$ is a ring and $n$ is a positive integer, let $\mathscr{A}(R, n) = \{a \in R : na = 0\}$, and for any $a \in R$, let $S(a)$ denote the subring of $R$ generated by $a$. Some parts of the following lemma are well known but appear in the literature only in the middle of proofs.

LEMMA 1. *Suppose $a$ is a non-zero element of a periodic ring $R$, $p$ is a prime, $n$, $r$ and $s$ are positive integers, $a^{n+1} = a$, and $(2a)^{s+1} = 2a$.*

(a) *$a^n$ is the identity element of $S(a)$.*

(b) *There is a non-zero square-free integer $m$, dependent only on $n$ and $s$, such that $a \in \mathscr{A}(R, m)$.*

(c) *If $pa = 0$, there is a positive integer $k$, dependent only on $n$ and $p$, such that $a^{p^k} = a$.*

(d) *If $pa = 0$, then $S(a)$ is isomorphic to the direct sum of finitely many finite fields of characteristic $p$.*

(e) *If $m = \prod_{i=1}^{r} p_i$ is the product of finitely many distinct primes $p_i$, then $\mathscr{A}(R, m)$ is the direct sum $\sum_{i=1}^{r} \oplus \mathscr{A}(R, p_i)$ of the rings $\mathscr{A}(R, p_i)$.*

(f) *If $R = \sum_{i=1}^{r} \oplus R_i$, where each $R_i$ is a $J$-ring of period $n_i$, then $R$ is a $J$-ring whose period is the least common multiple of $\{n_i : i = 1, \ldots, r\}$.*

*Proof.* The proof of (a) is left as an exercise.

If $a^{n+1} = a$ and $(2a)^{s+1} = 2a$, then by (a), $2a = (2a)^{s+1} = (2a)^{ns+1} = 2^{ns+1}a^{ns+1} = 2^{ns+1}a$. Hence $a$ has non-zero characteristic $m$. Since the only nilpotent element of $R$ is 0, $m$ is square free, so (b) holds.

In (c), suppose $n = p^e d$ for some integers $e \geqq 0$ and $d \geqq 1$ and that $(d, p) = 1$. By the Euler-Fermat Theorem [5, Chapter 6] there is a positive integer $k$ such that $p^k \equiv 1 \pmod{d}$. Then $(p^k - 1)p^e \equiv 0$

(mod $n$) so $a^{p^{k+e}} = a^{p^e}$ from part (a). Since $pa = 0$ we have

$$(a^{p^k} - a)^{p^e} = a^{p^{k+e}} - a^{p^e} = 0.$$

But $R$ has no nonzero nilpotents, so $a^{p^k} - a = 0$ and (c) holds.

If $pa = 0 \neq a$, then by (a), $S(a)$ is an algebra over the ring $Z_p$ of integers mod $p$. Since $a^{n+1} - a = 0$, there is a monic polynomial $\phi(x) \in Z_p[x]$ such that $S(a)$ and $Z_p[x]/\phi(x)Z_p[x]$ are isomorphic. Since $S(a)$ has no nonzero nilpotents, $\phi(x) = \Pi_{i=1}^{r} \phi_i(x)$ is a product of distinct irreducible elements $\phi_i(x) \in Z_p[x]$ and

$$Z_p[x]/\phi(x)Z_p[x] = \sum_{i=1}^{r} \oplus\, Z_p[x]/\phi_i(x)Z_p[x].$$

But each of these latter direct summands is a finite field, so (d) follows.

Part (e) follows from the well-known fact that every torsion abelian group $G$ may be represented as a direct sum of $p$-groups [**4**, p. 21]. Part (f) follows from (a), so the lemma is proved.

**3. The proof of Theorem 1 and some consequences.** Clearly the union of a chain of periodic rings is periodic, so it suffices to show that every periodic ring has the structure described in Theorem 1.

Let $\{p(i)\}$ denote the sequence of primes in numerical order, and for any positive integers $k$ and $r$, let $m(k) = \Pi_{i=1}^{k} p(i)$ and $P(r, k) = \{a \in \mathscr{A}(R, p(r)) \colon a^{p(r)^k} = a\}$. Since every periodic ring is commutative, each $P(r, k)$ is a $p(r)^k$-ring. Let $R(k)$ denote the subring of $R$ generated by $\bigcup_{i=1}^{k} P(i, k)$. Now, $R(k) \subset \mathscr{A}(R, m(k))$, and by Lemma 1(e), $\mathscr{A}(R, m(k)) = \Sigma_{i=1}^{k} \oplus\, \mathscr{A}(R, p(i))$. Therefore $R(k)$ is isomorphic to $\Sigma_{i=1}^{r} \oplus\, P(i, k)$, and hence is the direct sum of finitely many $p_i^k$-rings. Thus, $R(k)$ is a $J$-ring by Lemma 1(f), and $R(k) \subset R(k+1)$ since $P(i, k) \subset P(i, k+1)$ if $1 \leqq i \leqq k$.

If $n$ and $s$ are positive integers, let $T(n, s) = \{a \in R \colon a^{n+1} = a$ and $(2a)^{s+1} = 2a\}$. Clearly $\bigcup_{n,s=1}^{\infty} T(n, s) = R$, and if $T$ is a $J$-subring of $R$ with period $n$, then $T \subset T(n, n)$. Hence to complete the proof of Theorem 1, it suffices to show that given $n$ and $s$, there is a positive integer $k$ for which $T(n, s) \subset R(k)$.

By Lemma 1(b, e), there is a positive integer $r$ such that

$$T(n, s) \subset \mathscr{A}(R, m(r)) = \sum_{i=1}^{r} \oplus \mathscr{A}(R, p(i)).$$

If $1 \leqq i \leqq r$, then by Lemma 1(c), there is a positive integer $k^*(i)$ dependent only on $p(i)$ and $n$ such that if $a \in T(n, s) \cap \mathscr{A}(R, p(i))$, then $a^{p(i)^{k^*(i)}} = a$. Hence if $k(i) = \max(i, k^*(i))$, then $T(n, s) \cap \mathscr{A}(R, p(i)) \subset R(k(i))$. We conclude that if $k = \max(k(1), \ldots, k(r))$ then $T(n, s) \subset R(k)$, so by our previous remarks Theorem 1 follows.

Clearly the intersection of any two $J$-subrings of a periodic ring is a $J$-ring. By Theorem 1, the union of any two $J$-subrings of $R$ is contained in a $J$-subring of $R$, and so their join is a $J$-subring of $R$. Hence we have proved

COROLLARY 1. *The $J$-subrings of a periodic ring $R$ form a lattice with respect to the operations of intersection and join.*

By Theorem 1, every $J$-ring has finite characteristic. The next theorem describes the relation between the period and the characteristic of a $J$-ring.

THEOREM 2. *If $n$ and $m$ are positive integers, then there is a $J$-ring of period $n$ and characteristic $m$ if and only if $m = n = 1$ or $m = \Pi_{i=1}^{r} p(i)$ is a product of distinct primes and $n$ is the least common multiple of $\{p(i)^{k(i,j)} - 1 \colon i = 1, \ldots, r$ and $j = 1, \ldots, l(i)\}$ for some set of positive integers $\{k(i, j)\}$ and $\{l(i)\}$.*

*Proof.* Clearly $R$ has characteristic 1 if and only if $R = \{0\}$, so we suppose $m > 1$.

If $k$ is a positive integer and $p$ is a prime, let GF[$p^k$] denote the finite field with $p^k$ elements. It is well known (see [**1**, Chapter 5]) that GF[$p^k$] has characteristic $p$ and a cyclic multiplicative group.

Hence GF($p^k$] is a $J$-ring of period $(p^k - 1)$. Thus if $n, m, \{k(i,j)\}$ and $\{l(i)\}$ are as above and $m > 1$, then $R = \Sigma \oplus \{\mathrm{GF}[p(i)^{k(i,j)}]: i = 1, \ldots, r \text{ and } j = 1, \ldots, l(i)\}$ is a $J$-ring of period $n$ and characteristic $m$ by Lemma 1(f).

Conversely suppose $R \neq \{0\}$ is a $J$-ring of characteristic $m$ and period $n$. By Theorem 1, $R = \Sigma_{i=1}^{r} \oplus R(i)$ for some set of $p(i)^{k(i)}$-rings $R(i) \neq \{0\}$ having periods $n(i)$. Then $n = \text{L.C.M.}\{n(i): i = 1, \ldots, r\}$ by Lemma 1(f) and $m = \Pi_{i=1}^{r} p(i)$. If $0 \neq a \in R(i)$ let $n_a$ denote the period of $S(a)$. Clearly $n(i) = \text{L.C.M.}\{n_a: a \in R(i)\}$. By Lemma 1(d, f), $n_a = \text{L.C.M.}\{p(i)^{k(i,j)} - 1: j = 1, \ldots, l_a\}$ for some set of positive integers $\{k(i,j): j = 1, \ldots, l_a\}$, so Theorem 2 follows.

Suppose $n$ is the period of a $J$-ring $R$. In [9], G. Wene calls $n + 1$ the $\mu$-value of $R$, and asks for which positive integers $k$ there exist $J$-rings having $\mu$-value $k$. An answer to this question follows readily from Theorem 2. He also asks the reader to show that there are infinitely many $k$ that are not the $\mu$-value of any $J$-ring. The following corollary determines when an integer of the form $p^n + 1$ is the $\mu$-value of some $J$-ring.

COROLLARY 2. *Suppose $p$ is a prime and $n$ is a positive integer. Then $p^n$ is the period of some $J$-ring if and only if either*:
(a) *$p$ is odd, $n = 1$, and $p = 2^s - 1$ for some positive integer $s$, or*
(b) *$p = 2$, and $2^n + 1$ is a prime or $n = 3$.*

*Proof of* (a). It follows immediately from Theorem 2 that $p^n$ is a period of some $J$-ring if and only if $p^n = 2^s - 1$ for some positive integer $n$. In [3, Corollary 2], J. W. Cassells has shown that this equation has a solution if and only if $n = 1$, so (a) follows.

*Proof of* (b). By Theorem 2, $2^n$ is a period of some $J$-ring if and only if $2^n = p^s - 1$ for some odd prime $p$ and positive integer $s$. By [3, Theorem IV], this equation has a solution if and only if $s = 1$ or $n = 3$, so (b) holds.

Let $K$ denote the set of all positive integers $k$ for which there exist $J$-rings having $\mu$-value $k$. It follows from Corollary 2 that $p^n + 1 \in K$ if and only if $n = 1$ and $p = 2^s - 1$ is a Mersenne prime, $p^n + 1 = 9$, or $p^n + 1 = 2^n + 1$ is a Fermat prime. Consequently there are infinitely many integers of the form $p^n + 1$ that are not in $K$.

A more satisfactory solution of [9] would provide an efficient algorithm for deciding when a given positive integer is in $K$. It would also be interesting to determine the asymptotic density of $K$ if this density exists.

Theorem 1 reduces the problem of determining the structure of an arbitrary periodic ring to the study of $p^k$-rings. The structure of such rings is described by R. Arens and I. Kaplansky in [2, pp. 470–477].

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# ON THE LOCATION OF THE FRETS ON THE GUITAR

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In [1] J. M. Barbour presented an ingenious discussion of an approximate construction of the location of the frets on a guitar for the so-called equal temperament due to the Swedish musicologist D. P. Strähle (1743). The present author was the referee of Barbour's paper [1] and contributed two short footnotes on page 7 of [1]. The purpose of the present note is to present Barbour's analysis in the light of these footnotes.

**1. The problem and Strähle's solution.** It is known that if the 12 tones of an octave in the equal temperament are to be represented by the lengths of the strings on a monochord (or guitar), then their respective lengths are proportional to the terms of the geometric progression $r^n$ $(n = 0, 1, \ldots, 11)$, where $r = 2^{1/2}$. The open string on the guitar being represented by the interval $0 \leqq x \leqq 2$, the frets are to be placed at the twelve points $x = r^n$ $(n = 0, 1, \ldots, 11)$. The problem is to find a simple geometric construction giving acceptable approximations to these points.

Strähle's solution of this problem is described in Fig. 1. Let $VBC$ be an isosceles triangle having sides of 24, 24, and 12 units. Mark the point $A$ on $CV$ so that $CA = 7$ units, and produce $BA$ beyond $A$ so that $OA = AB$. If we divide the base $CB$ into 12 equal parts and project the 11 points of division from the center of projection $V$ onto the line $AB$, we obtain on $OB$ the 12 points $A, A_1, \ldots, A_{11}$. If $OB$ is the open string of the guitar, these 12 points represent the approximate location of frets.

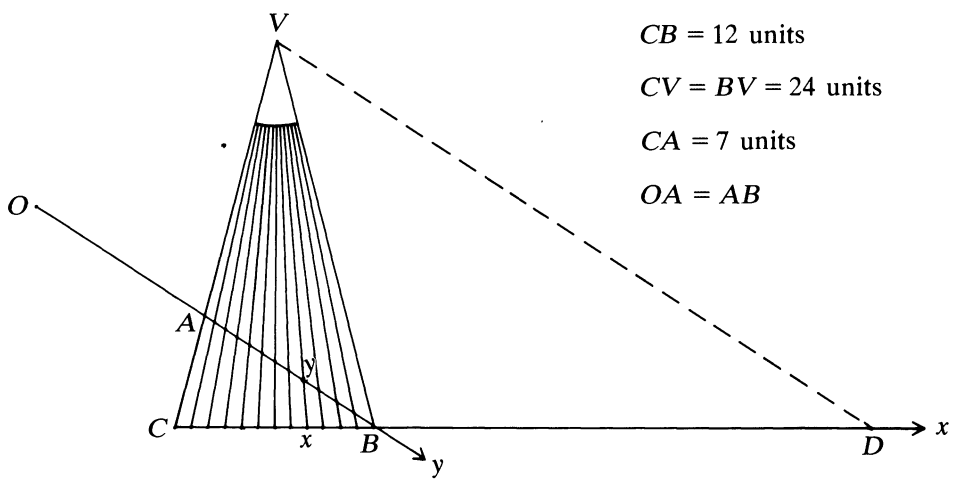



FIG. 1

**2. Approximating the exponential function by a linear fractional function.** Let $a$ and $b$ be given, $0 < a < b$, and let us consider the increasing exponential function

$$f(x) = a\,(b/a)^x, \qquad (0 \leqq x \leqq 1). \tag{2.1}$$

Notice that $f(0) = a$, $f(1) = b$. Let us approximate $f(x)$ in $[0, 1]$ by a linear fractional function

$$L(x) = \frac{\alpha x + \beta}{\gamma x + \delta}. \tag{2.2}$$

Since $L(x)$ depends on three arbitrary parameters, we expect to be able to choose these such that

$L(x)$ will interpolate $f(x)$ at the three points $x = 0$, $x = 1/2$, and $x = 1$. It is a matter of easy verification to see that

$$L(x) = a\frac{(b/a)x + \sqrt{b/a}(1-x)}{x + \sqrt{b/a}(1-x)} \tag{2.3}$$

is the required interpolating function because it satisfies the relations

$$L(0) = a = f(0), \quad L(1/2) = \sqrt{ab} = f(1/2), \quad L(1) = b = f(1). \tag{2.4}$$

If the ratio $b/a$ is not too large, it is reasonable to expect that $L(x)$, defined by (2.3), will be fairly close to $f(x)$ throughout the interval $0 \leqq x \leqq 1$.

Let us look at a numerical example: If $a = 1$, $b = 2$, hence

$$f(x) = 2^x, \tag{2.5}$$

and the approximation (2.3) becomes

$$B(x) = \frac{2x + \sqrt{2}(1-x)}{x + \sqrt{2}(1-x)}. \tag{2.6}$$

This is Barbour's approximation (see formula (4) on page 7 of [1]). For $x = 3/4$, when we expect about the largest error, we find

$$f(3/4) = 2^{3/4} = 1.68179, \quad B(3/4) = \frac{16 - 3\sqrt{2}}{7} = 1.67962.$$

We see that $f(3/4) - B(3/4) = .00217$. This is an error of .22% which is musically quite acceptable.

**3. Barbour's analysis of Strähle's construction.** Let us describe analytically Strähle's construction of Fig. 1: We consider the line $CB$ as the $x$-axis, the points $C$ and $B$ having the abscissae $x = 0$ and $x = 1$, respectively. Similarly, let $OB$ be the $y$-axis so that $O, A$, and $B$, correspond to $y = 0$, $y = 1$, and $y = 2$, respectively. Let the point $y$ be the projection of the point $x$ from the center of projection $V$. We use the fact that such a perspectivity is described by a linear fractional correspondence (or homographic transformation)

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}. \tag{3.1}$$

Let us find the values of $\alpha, \beta, \gamma, \delta$, in Fig. 1. We draw $VD$ parallel to $OB$ and intersecting $CB$ in $D$. By similar triangles we find that $x = CD = 24/7$. Thus in (3.1) $y = \infty$ if $x = 24/7$, and therefore (3.1) must be of the form

$$y = \frac{\alpha x + \beta}{(24/7) - x}.$$

Since $x = 0$ and $x = 1$ correspond to $y = 1$ and $y = 2$, respectively, we find that $\alpha = 10/7$ and $\beta = 24/7$. The final relation is $y = S(x)$, where

$$S(x) = \frac{24 + 10x}{24 - 7x}. \tag{3.2}$$

This, then, is Strähle's approximation. *How does it compare with Barbour's approximation* (2.6)?

For (2.6) we know, by (2.4), that

$$B(1/2) = \sqrt{2} = 2/\sqrt{2}. \tag{3.3}$$

On the other hand, (3.2) shows that

$$S(1/2) = 58/41 = 2/(41/29). \tag{3.4}$$

At this point we observe that the continued fraction expansion of $\sqrt{2}$ is

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\cdots,$$

whose successive convergents are (see [3, 166])

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \cdots. \tag{3.5}$$

On comparing the denominators of the last fractions in (3.3) and (3.4) we see that *we obtain Strähle's construction from Barbour's by replacing in* (3.3) *the denominator* $\sqrt{2}$ *by the fraction* 41/29. By a fundamental theorem of Lagrange we know that $41/29 = 1.41379$ *is the best approximation of* $\sqrt{2} = 1.41421$ *by rational fractions* $q/p$, *with* $q \leqq 41$ *or* $p \leqq 29$. It does seem remarkable that Strähle should have found his construction by a craftsman's sheer intuition without the benefit of mathematics. We refer the reader to [1] for interesting further background, historical information, references and tables of values.

Sponsored by the United States Army under Contract No. DA AG 29-75-C-0024.

The author thanks the referee for the delightful reference 2, which he wants to pass on to the readers of the MONTHLY.

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## AN ELEMENTARY PROOF THAT THE UNIT DISC IS A SWISS CHEESE

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After seeing an example due to Wolff [9], Borel [1] observed that if a set of disjoint discs is removed from the unit disc of $R^2$ in such a way that the area of the remaining region is zero, then the areas of the removed discs form a convergent series. Further, he noted that this series of areas converges slower than the series whose $k$th term is $\exp(-ek^2)$. Borel then asked what rate of convergence of the series of areas of removed discs one might expect (see Wesler [8]). A set $S$ in $R^2$ is called a Swiss Cheese if there exists a set of disjoint discs of radii $r_j$ belonging to $S$ so that when the discs are removed the remaining area is zero and $\Sigma_1^\infty r_j = \infty$. (It should be noted that the term "Swiss cheese" is used for a somewhat different type of set in the theory of uniform algebras of analytic functions [3].)

Mergelyan [7] has obtained a corollary to one of his theorems on uniform approximations that states: If $r_j < 1$ represent the radii of disjoint discs removed from the unit disc for which $\Sigma_1^\infty r_j^2 = 1$, then $\Sigma_1^\infty r_j = \infty$. The same result is also a special case of a theorem due to Wesler [8]. Using the Borel-Cantelli Lemma, Wesler proved that, in $R^n$, if one removes a set of disjoint discs of radii less than one from the unit disc in such a way that the sum of the volumes of the removed discs is one, then the sum of the surface areas of the removed discs is $\infty$.

This paper gives a simple proof of the two-dimensional result obtained by Mergelyan and Wesler. The proof given involves only elementary concepts and can be modified to yield the analogous result in $R^n$.

THEOREM. *If one removes from the unit disc a set of disjoint discs* $\{D_j\}_{j=1}^{\infty}$ *of respective radii* $\{r_j\}_{j=1}^{\infty}$ *where each* $r_j < 1$ *and* $\Sigma_1^{\infty} r_j^2 = 1$, *then* $\Sigma_1^{\infty} r_j = \infty$.

*Proof*: Assume the conclusion is false so that $\Sigma_1^{\infty} r_j = K < \infty$. Then there exists an $N > 0$ such that $\Sigma_1^{N} r_j > K - 1/10$. In order to reach a contradiction it is only necessary to show $\Sigma_{N+1}^{\infty} r_j > 1/10$.

Consider two cases: (1) at least one of the first $N$ discs is tangent to the unit circle, or (2) none of the first $N$ discs are tangent to the unit circle. In either case, it is possible to find an annulus $A$, whose outer rim is the unit circle, that does not intersect any of the first $N$ discs which are not tangent to the unit circle. Further, we choose the annulus $A$ so that its thickness (i.e.: one minus the radius of the circle which forms the interior rim of $A$) is less than $\min\{r_j/2: j \leqq N\}$. Denote the thickness of the annulus by $\Delta r$.

In case (1) we are interested in the region $B = A - \bigcup_{j \in T} D_j$ where $T = \{j: j \leqq N$ and $D_j$ is tangent to the unit circle$\}$. In case (2) the region of interest is just the annulus $A$.

Now, in case (1) we consider the portions of $B$ between two consecutive discs both tangent to the unit circle. (If only one of the first $N$ discs is tangent to the unit circle we consider it consecutive to itself.) Let $l_i$ represent the arc of the inner rim of the annulus $A$ between two consecutive discs tangent to the unit circle and let $L_i$ be the arc of the unit circle joining the points of tangency of the same two consecutive discs. Let $|l_i|$ denote the arc length of the arc $l_i$, etc. For all $i$ we may make:

$$|l_i| > \max\{|L_i| - 1/(10N),\ .99|L_i|\}. \tag{I}$$

This can be done by shrinking the annulus $A$, thus moving the $l_i$'s closer to the outer rim. Hence:

$$\Sigma_i |l_i| > \Sigma_i |L_i| - \Sigma_i\ 1/(10N) \geqq 2\pi - 1/10. \tag{II}$$

Next, consider those discs $D_j$ for which $j > N$. From this set of discs, consider those for which $D_j \cap B \neq \emptyset$. Each disc in this collection is either completely contained in $B$ or it is not completely contained in $B$. If $D_j$ is completely contained in $B$, translate $D_j$ along the radius of the unit circle determined by the center of $D_j$. This defines a projection, $\alpha_j$, of the disc $D_j$ onto the inner rim of the annulus $A$. If a disc $D_j$ is not completely contained in $B$, then $D_j \cap B$ may be projected in the same manner onto the inner rim of the annulus $A$. In both situations, the projection $\alpha_j$ will have the property $|\alpha_j| < \pi r_j$. This last relation follows from the fact that each $r_j$ must be less than the radius of the interior rim of the annulus. Since $D_j$ is a disc in the interior of the unit disc and $D_j \cap B \neq \emptyset$, the curvature of the circle determining $D_j$ must be greater than the curvature of the interior rim of the annulus. Now, $\pi r_j$ represents the length of the semi-circular arc associated with $D_j$ and $|\alpha_j|$ is the length of the projection of $D_j$ onto the inner rim of the annulus, so the relationship between curvatures forces $|\alpha_j| < \pi r_j$.

For each $i$, denote by $B_i$ that portion of the region $B$ subtended by the arc $l_i$. Since $\Sigma_j r_j^2 = 1$, there exists a finite set $F_i \subseteq S_i = \{j: j > N$ and $D_j \cap B_i \neq \emptyset\}$ such that the area of $\Sigma_{j \in F_i}(D_j \cap B_i)$ is greater than 99% of the area of $B_i$ for each $i$. From this we may conclude that $\Sigma_{j \in S_i} |\alpha_j| \geqq |l_i|/2$, for if not we would have $\Sigma_{j \in S_i} |\alpha_j| < |l_i|/2$ and thus $\Sigma_{j \in F_i} |\alpha_j| < |l_i|/2$. This yields

$$|l_i| - \Sigma_{j \in F_i} |\alpha_j| > |l_i| - |l_i|/2 = |l_i|/2,$$

so that the area of $B_i$ *not* covered by $\{D_j \cap B_i: j \in F_i\}$ must be greater than or equal to $(|l_i|/2)\Delta r$. It then follows from (I) that $(|l_i|/2)\Delta r \geqq (.99|L_i|)/2 > (.49)|L_i|\Delta r$, which is a contradiction. It thus follows that $\Sigma_{j \in S} |\alpha_j| \geqq \Sigma_i |l_i|/2$, where $S = \bigcup_i S_i = \{j: j > N$ and $D_j \cap B \neq \emptyset\}$.

The inequality $\Sigma_{j \in S} |\alpha_j| \geqq \Sigma_i |l_i|/2$ together with our previous result $\pi r_j > |\alpha_j|$ for all $j \in S$ yields:

$$\Sigma_{j \in S} \pi r_j > \Sigma_{j \in S} |\alpha_j| > \Sigma_i |l_i|/2.$$

This result together with (II) yields:

(III) $$\Sigma_{j\in S}\pi r_j > \Sigma_{j\in S}|\alpha_j| \geqq \Sigma_i |l_i|/2 \geqq (2\pi/2) - (1/20) = \pi - 1/20.$$

Since $\Sigma_{j=N+1}^{\infty}\pi r_j \geqq \Sigma_{j\in S}\pi r_j$, it follows from (III) that $\pi \Sigma_{j=N+1}^{\infty} r_j > \pi - 1/20$ and hence, dividing by $\pi$ we have

$$\Sigma_{N+1}^{\infty} r_j > 1 - 1/(20\pi) > 1/10.$$

This is the desired contradiction for case (1).

In case (2), a similar argument holds. In fact, the proof of case (2) is a special case of the proof of case (1) in which $\{L_i\} = \{L_1\}$, $L_1$ being the unit circle; and $\{l_i\} = \{l_1\}$, $l_1$ being the circle forming the inner rim of the annulus $A$. This completes the proof.

The similarity of this proof to the proof that the Hausdorff dimension [4] of the unit disc is two might lead one to wonder if the convergence of the sum of the areas of the covering discs is even slower, perhaps so slow that if $\Sigma_1^{\infty} r_j^2 = 1$, then $\Sigma_1^{\infty} r_j^{2-\varepsilon} = \infty$ for any $\varepsilon > 0$. Melzak [6] has shown that for arbitrarily small $\varepsilon > 0$ it is possible to find a packing for which $\Sigma_1^{\infty} r_j^{2-\varepsilon} = \infty$, while $\Sigma_1^{\infty} r_j^{1.999971} < \infty$ for the osculatory packing. Boyd [2] has shown that for the osculatory packing $\Sigma_1^{\infty} r_j^{1.300197} = \infty$ while $\Sigma_1^{\infty} r_j^{1.314534} < \infty$. For general packings, the best result known is due to Larman [5] who has shown $\Sigma_1^{\infty} r_j^{1.03} = \infty$.

We wish to thank the referee for his helpful suggestions.

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## RESEARCH PROBLEMS

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*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)*

### SUBSPACES OF NON-FIRST-COUNTABLE SPACES

FRANK SIWIEC

Our goal is to find a limited number of relatively simple topological spaces, $X_i$, each of which is non-first-countable, and such that every non-first-countable space contains a subspace homeomorphic with some $X_i$.

Let us restrict ourselves to regular $T_1$-spaces. Now consider three very well-known examples of countable, non-first-countable spaces due to Arens [1].

**Arens' $X$:** Consider the discrete union of countably infinitely many convergent sequences with limit (specifically, $\{0, 1, 1/2, 1/3, \ldots\}$ as a subspace of the real line). The space $X$ is obtained by identifying the limit points to one point and giving $X$ the quotient topology. This space $X$ is nowadays commonly called the "sequential fan."

**Arens' $Y$:** Let $Y = \{(0,0)\} \cup \{(1/m, 0) \mid m \in N\} \cup \{(1/m, 1/n) \mid m, n \in N\}$, where $N$ denotes the set of natural numbers. Each point $(1/m, 1/n)$ is itself an open set. A basic neighborhood of $(1/m, 0)$ is the column containing $(1/m, 0)$ with at most a finite number of points (different from $(1/m, 0)$) deleted. A neighborhood of $(0,0)$ must be a neighborhood of all but finitely many points $(1/m, 0)$.

**Arens' $Z$:** The space $Z$ is the subspace $Y - \{(1/m, 0) \mid m \in N\}$.

We will need some terminology. A space $X$ is **sequential** if every sequentially-open set in $X$ is open. Here, a set $U$ is said to be **sequentially-open** if whenever a sequence in $X$ converges to a point in $U$, the sequence is eventually in $U$. A space $X$ is **Fréchet** if whenever a point $x$ is an accumulation point of a set $A$, there is a sequence in $A$ which converges to $x$. A space $X$ is **countably bi-sequential** if whenever $x \in X$ and $\{A_n\}$ is a decreasing sequence of subsets of $X$ such that each $A_n$ has $x$ as a common accumulation point, then for each $n \in N$, there exists a point $x_n \in A_n$, such that $x_n \to x$. It is known [2, 9] and easy to see that the following are in order of increasing generality: first countable, countably bi-sequential, Fréchet, sequential.

It was shown in [9] that every space which is Fréchet but not countably bi-sequential contains (a subspace homeomorphic with) the sequential fan.

Commonly, a space which is sequential but not Fréchet contains Arens' $Y$. However, Franklin [3] gave an example of a space which is sequential and not Fréchet, but does not contain Arens' $Y$. Positive results may be found in [3, 5, 8]. Since Arens' $Z$ is a subspace of Arens' $Y$, the question arises under what conditions Arens' $Z$ is a subspace of a given non-Fréchet space.

We cannot always expect to find a countable subspace of interest, since $[0, \omega_1]$ is a *CM* space in the following sense. A space is said to be ***CM*** if every countable subspace is metrizable, or equivalently, every countable subspace is first countable.

The following may be easily proved:

A *CM* separable space is first countable.

A *CM* sequential space is countably bi-sequential (in fact, a *CM* space of countable tightness [9] is countably bi-sequential).

In a *CM* space, every countably compact subspace is sequentially compact.

*GO* spaces (= generalized ordered spaces, see for example [6]) are *CM*. Also $P$-spaces (see for example [7]) are *CM*.

The following conditions are equivalent for a space $X$: (a) $X$ is *CM* and every compact subspace is finite, (b) $X$ is *CM* and $X$ has no non-trivial convergent sequence, (c) every countable subset of $X$ is closed, (d) every countable subspace of $X$ is discrete.

The *CM* property is preserved by subspaces and by countable products.

PROBLEM. Let $X$ be *CM* and Fréchet, but not first-countable. If $X$ has a point which is not a $G_\delta$-set, does $X$ necessarily contain the one-point compactification of an uncountable discrete space? If each point of $X$ is a $G_\delta$-set, does $X$ necessarily contain a subspace "similar to" the following? Let $R'$ be the set of real numbers with the topology generated by having each point other than zero itself an open set. Neighborhoods of zero are of the form $U - \{x/n \mid x \in F, n \in N\}$, where $U$ is a usual neighborhood of zero and $F$ is a finite set not containing zero.

PROBLEM. If a space is *CM* and not Fréchet, does it necessarily contain one of the following two spaces?

(a) A *WON* space (well ordered net space [4]). We define a *WON* space $X$ by the condition that $X$ has exactly one non isolated point denoted by $x_0$, and a set $G$ containing $x_0$ is open if and only if $\operatorname{card}(X - G) < \operatorname{card} X$. Simple examples of *WON* spaces are a convergent sequence and the space

$[0, \omega_1]$ — {countable limit ordinals}, with the usual order topology on $[0, \omega_1]$. It is clear that a space $X$ contains a *WON* space as a subspace if $X$ has a point having a linearly ordered (by inclusion) base. It is also easily seen that every non-discrete *GO* space contains a *WON* space.

(b) Let $X' = \{0\} \cup \bigcup_{n=1}^{\infty} M_n$, where $M$ is a set of cardinality $\aleph_1$, and $M_n = \{1/n\} \times M$ for each $n \in N$. For the topology of $X'$, let each point other than 0 be itself an open set. A basic neighborhood of 0 is $\{0\} \cup (D_n - C)$, where $D_n = \bigcup_{i=n}^{\infty} M_i$ and $C$ is a countable set.

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## CLASSROOM NOTES

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### ANOTHER ELEMENTARY PROOF OF PEANO'S EXISTENCE THEOREM

CLIFFORD GARDNER

**1. Introduction.** In recent years there has been a continuing discussion in this journal of elementary proofs of Peano's existence theorem for ordinary differential equations. Here an *elementary* proof is one which does not use the Arzelà–Ascoli lemma (any bounded equicontinuous sequence of real functions of a real variable has a uniformly convergent subsequence). Kennedy [1] surveyed the literature and raised the question of the existence of elementary proofs of Peano's theorem. W. Walter [2] and J. Walter [3] gave proofs, and also valuable further references to the literature.

J. Walter's construction is especially simple. However, J. Walter's proof uses the theory of integration, and uniform convergence of sequences of functions. He suggests that a proof avoiding these notions may be of interest. Such a proof will be given here. By a variation of an argument given by El'sgol'ts [5] we show, without using the notions of integral, uniform convergence, or even uniform continuity, that the solution constructed by J. Walter is correct.

**2. The existence theorem.** This can be stated as follows:

*$x, y$ and bounded; that is, $K>0$ exists with*

$$|f(x, y)|<K$$

*for all $x, y$. Let $x_0, y_0$ be given numbers. Then there is a function $w(x)$ defined for all $x$ which is a solution of the initial-value problem*

$$w(x_0)=y_0$$

$$w'(x)=f(x, w(x)).$$

For the proof let $h_1, h_2, \ldots$ be a decreasing sequence of positive numbers tending to zero. Let $p_n(x)$ be the Euler polygon for the initial-value problem with step size $h_n$. That is, with

$$x_i = x_0 + ih_n \qquad (i \text{ an integer})$$

set $p_n(x_0)=y_0$ and for

$$x>x_0 \quad \text{and} \quad x_i<x\leqq x_{i+1} \quad \text{or}$$

$$x<x_0 \quad \text{and} \quad x_{i-1}\leqq x<x_i$$

set $p_n(x)=y_i+(x-x_i)f(x_i, y_i)$, where $y_i=p_n(x_i)$.

Following J. Walter, we define

$$w(x)=\limsup_{n\to\infty} p_n(x).$$

Clearly $|p_n(x)-p_n(x')|<K|x-x'|$ so

$$|p_n(x)-y_0|<K|x-x_0|$$

and therefore $w(x)$ is finite. Clearly $w$ satisfies the initial condition. We must show it satisfies the differential equation. Indeed we will show the following:

If a number $a$ and a number $\varepsilon>0$ are given, then if $\delta>0$ is determined so that

$$|f(a, w(a))-f(x, y)|<\varepsilon$$

is implied by $|x-a|<2\delta$ and $|y-w(a)|<(2K+1)\delta$, then if $0<|k|<\delta$ we have

$$|[w(a+k)-w(a)]/k-f(a, w(a))|\leqq\varepsilon.$$

To this end note that if $\sigma>0$, then for all large enough $n$ we have

$$p_n(a)<w(a)+\sigma, \qquad p_n(a+k)<w(a+k)+\sigma.$$

For infinitely many indices $n$, $w(a)-\sigma<p_n(a)$ and for infinitely many (possibly different) indices $n$,

$$w(a+k)-\sigma<p_n(a+k).$$

Therefore indices $j$, $m$ exist with $h_j<\delta$, $h_m<\delta$ and

$$w(a)-\sigma<p_j(a)<w(a)+\sigma,$$

$$w(a+k)-\sigma<p_m(a+k)<w(a+k)+\sigma,$$

and

$$p_j(a+k)<w(a+k)+\sigma,$$

$$p_m(a)<w(a)+\sigma.$$

We now define $p(x)$ for $x$ between $a$ and $a+k$ as follows:
If $w(a+k)-\sigma < p_j(a+k)$, then $p(x)=p_j(x)$.
If otherwise, but $w(a)-\sigma < p_m(a)$, then $p(x)=p_m(x)$.
Finally, if the polygons $y=p_j(x)$, $y=p_m(x)$ cross at an abscissa $c$ between $a$ and $a+k$ then let

$$p(x)=p_k(x) \qquad \text{for } x \text{ between } a \text{ and } c \text{ (inclusive)}$$

$$p(x)=p_m(x) \qquad \text{for } x \text{ between } c \text{ and } a+k \text{ (inclusive).}$$

Then the polygonal line $y=p(x)$ joining $(a, p(a))$ with $(a+k, p(a+k))$ has at each point $(x, p(x))$ on one of its sides the slope $p'(x)=f(x_i, y_i)$, where $(x_i, y_i)$ is a vertex adjacent to $(x, p(x))$ on one of the Euler polygons $y=p_k(x)$, $y=p_m(x)$. For given $x$ between $a$ and $a+k$ we clearly have

$$|a-x_i|<2\delta, \qquad |w(a)-y_i|<2K\delta+\sigma.$$

Taking $\sigma<\delta$, we then have

$$|f(a, w(a))-p'(x)|<\varepsilon.$$

Following El'sgol'ts [5] we note that the secant joining the points $(a, p(a))$ and $(a+k, p(a+k))$ on the polygon has a slope which lies between the maximum and the minimum of the slopes of the sides of the polygon. Hence

$$|[p(a+k)-p(a)]/k-f(a, w(a))|<\varepsilon.$$

It now follows that

$$|[w(a+k)-w(a)]/k-f(a, w(a))|<\varepsilon+2\sigma/k.$$

Now if we let $\sigma$ tend to zero we get the result:

$$|[w(a+k)-w(a)]/k-f(a, w(a))|\leqq\varepsilon.$$

As J. Walter points out [3] this construction does not work for systems of differential equations.

We note that inasmuch as our argument does not assume integration, it proves the existence of the Riemann integral of a continuous function (assume $f(x, y)$ does not depend on $y$). Thus the theories of integration and of solving a differential equation are unified. For a discussion along these lines see the paper of Osgood [4].

**3. Maximum and minimum solutions.** It is a well-known theorem that the initial-value problem has a maximum solution $w_1(x)$ and a minimum solution $w_2(x)$ between which all solutions lie. (I thought this theorem originated with Osgood [4] until I saw J. Walter's paper [3] in which it is ascribed to Peano [6].) The proofs given in textbooks all seem to rely on uniform convergence and integration. We shall now see that a proof can be given without these notions, along the same lines as our previous proof.

Let us define $w_1(x)$ to be the least upper bound of all solutions of the initial-value problem. It exists since at least one solution, namely $w(x)$, exists. (In passing let me remark that $w(x)$ and $w_1(x)$ are *not* necessarily the same.) We must show that $w_1(x)$ is a solution. Let numbers $a$ and $\varepsilon>0$ be given. Determine $\delta>0$ so that

$$|f(a, w_1(a))-f(x, y)|<\varepsilon$$

is implied by

$$|x-a|<\delta, \qquad |y-w_1(a)|<(K+1)\delta.$$

Then if $0<|k|<\delta$, we have

$$|[w_1(a+k)-w_1(a)]/k - f(a, w_1(a))| \leqq \varepsilon.$$

The proof is very similar to the proof in the previous section. We note that if $\sigma>0$, then solutions $u_1(x)$, $u_2(x)$ of the initial-value problem exist with $w_1(a)-\sigma<u_1(a)\leqq w_1(a)$ and

$$w_1(a+k)-\sigma<u_2(a+k)\leqq w_1(a+k).$$

Then we define

$$u(x)=u_1(x) \quad \text{or} \quad u_2(x)$$

(if these solutions do not cross between $a$ and $a+k$) or

$$u(x)=u_1(x) \qquad \text{for } x \text{ between } a \text{ and } c,$$

$$u(x)=u_2(x) \qquad \text{for } x \text{ between } c \text{ and } a+k,$$

(if the solutions $u_1(x)$, $u_2(x)$ cross at $c$ between $a$ and $a+k$). We note that $u(x)$ is a solution of the differential equation. Using the mean-value theorem, we get, taking $\sigma<\delta$,

$$|[u(a+k)-u(a)]/k - f(a, w(a))|<\varepsilon$$

whence

$$|[w_1(a+k)-w_1(a)]/k - f(a, w_1(a))|<\varepsilon+\sigma/k$$

whence, letting $\sigma$ tend to zero, the result.

**4. A generalization.** With a little bit of extra argument we can weaken the continuity requirement on $f(x, y)$. What is required is fairly obvious, so I won't go into detail, but merely state a result, as follows:

THEOREM. *If $f(x, y)$ is assumed bounded (but not necessarily continuous), then the function $w(x)$, constructed according to J. Walter's prescription, exists and is continuous (in fact Lipschitz continuous with constant $K$). Suppose $x=a$, $y=w(a)$ is a point of its graph where $f$ satisfies the following: a number $f(x+0, y)$ and a number $L$ with $L>K>0$ exist such that for any $\varepsilon>0$ there exists $\delta>0$ such that*

$$|f(x, y)-f(x', y')|<\varepsilon$$

*is implied by*

$$x<x'<x+\delta \quad \text{and} \quad |y-y'|<L|x-x'|.$$

*Then $w(x)$ has a right hand derivative at $a$ equal to $f(x+0, y)$:*

$$D_r w(a)=\lim_{\substack{k\to 0\\ k>0}} [w(a+k)-w(a)]/k = f(x+0, y).$$

This generalization includes many cases of interest where $f$ is not a continuous function of $x$.

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## A NOTE ON ZARISKI'S LEMMA

JOHN McCABE

Various elementary proofs of the Hilbert *Nullstellensatz* are to be found in the literature. It is well known that the *Nullstellensatz* may be derived from the statement known as the weak *Nullstellensatz*; this is done by the celebrated device called "Rabinowitsch's trick" [4]. Zariski [5] introduced a lemma from which the weak *Nullstellenatz* follows easily, and he proved this lemma using remarkably little algebraic machinery. Our purpose in this note is to give an alternative proof of Zariski's Lemma, a proof which will also use little in the way of algebraic preliminaries.

Since the Hilbert *Nullstellensatz* lies at the foundations of algebraic geometry, it is a matter of considerable interest to be able to prove this result as expeditiously as possible. In this regard, the reader might like to consult [1], [3], and [6], Volume II, pp. 164–167.

LEMMA (Zariski's lemma). *If $k$ is a field, if $R = k[x_1, \ldots, x_n]$ is a finite integral domain over $k$, which is a field, then $R$ is an algebraic extension of $k$.*

*Proof.* Assume that $R$ is not algebraic over $k$. We may assume the $x$'s have been so indexed that $x_1, \ldots, x_t$ are independent transcendentals over $k$, and that $x_{t+1}, \ldots, x_n$ are algebraic over $k(x_1, \ldots, x_t)$. Put

$$S = k[x_1, \ldots, x_t].$$

For an appropriate element $y \neq 0$ in $S$, the elements $yx_{t+1}, \ldots, yx_n$ will all be integral over $S$. It follows that $x_{t+1}, \ldots, x_n$ are all integral over $S[1/y]$. Since $k \subseteq S[1/y]$ and $x_1, \ldots, x_t \in S[1/y]$, we conclude that $R = k[x_1, \ldots, x_n]$ is integral over $S[1/y]$. Since $R$ is a field, $S[1/y]$ must be a field ([2], Proposition 5.7, p. 61).

Let $m$ be any maximal ideal of $S$. Since $S = k[x_1, \ldots, x_t]$ is not a field, $m \neq \{0\}$. Choose a non-zero element $f$ of $m$. In $S[1/y]$, $f$ is invertible so

$$1/f = g/y^N,$$

where $g \in S$ and where we may assume $N$ is a positive integer. We than have

$$y^N = fg.$$

Since $f \in m$, $y^N \in m$. This implies that $y \in m$.

Thus $y$ is in every maximal ideal of $S$. It follows that $1 + y$ is in no maximal ideal of $S$, and hence $1 + y$ is a unit of $S$ ([2], Corollary 1.5, p. 4). It's well known that the units of the polynomial ring $S = k[x_1, \ldots x_t]$ are exactly the non-zero elements of $k$, so

$$1 + y = c,$$

where $c \neq 0$ is in $k$. Thus $y = c - 1$, and $y$ is in $k$. Since $y \neq 0$, $y$ is a unit of $k$ and therefore a unit of $S$ also. It follows that $S[1/y] = S$, hence $S$ is a field.

But $S = k[x_1, \ldots, x_t]$ is certainly not a field. This contradiction establishes that $R$ must be algebraic over $k$.

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## MATHEMATICAL EDUCATION

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Your attention is drawn to the existence of the new quarterly "Journal of Personalized Instruction," 29 Loyola Hall, Georgetown University, Washington, D.C. 20057. Our continuing backlog prompts us to ask potential contributors of articles on personalized systems of instruction to keep the above journal in mind as a possible publishing alternative. — *The Editors.*

### IMPLICIT AXIOMS, $\omega$-RULE AND THE AXIOM OF INDUCTION IN HIGH SCHOOL MATHEMATICS

SHLOMO VINNER

**1. Implicit Axiom Systems.** The question of whether or not the axiomatic method should be taught to high school students is an extremely controversial one. One way of dealing with it is to describe the various aspects in detail and to argue the case for each of them. To a certain extent this has been done in [1] (especially in the papers of R. C. Buck, A. M. Gleason and L. Henkin). We shall confine ourselves here to only one aspect of the axiomatic method which we can call the naive aspect. In this, the axioms are taken as clear and well-known facts (or obvious and well-known statements)[1] of a known structure. These facts seem to be more fundamental than others and they have a special role in the mathematical activity of the high school student. To be more specific, certain facts[2] which often appear in the algebraic activity of the high school student obtain a special status after a certain period of time. The student is not only familiar with them but he also realizes that they are important and widely used in proofs and calculations. This process is implicit and the student is usually unaware of it. In most cases it is a long process. At the beginning of the process the facts are introduced to the student, and at the end they obtain their special status. For instance, at a certain age it takes some time to realize that $a + b = b + a$ for all numbers $a, b$. It takes perhaps more time to find out that this fact is extremely important in calculations and other arithmetical activities. However, although teachers tell their students that various facts are important, they do not call them axioms. So, the status of the

[1] We say "facts" instead of "statements," because the high school student usually does not make a distinction between semantics and syntactics. For example, to such a student it is a fact that $a + b = b + a$ for all numbers $a, b$.

[2] We shall confine ourselves here only to algebraic facts.

fundamental facts is, in a way, an informal one. We would, therefore, like to call these fundamental facts implicit axioms. More specifically, a mathematical fact becomes an implicit axiom for a student if:

(1) it is absolutely self-evident;

(2) it is widely or crucially used.

Of course, this is rather a relative notion and it is not well defined from the mathematical point of view. For example, for some students the formula $(a+b)(c+d)=ac+ad+bc+bd$ can be an implicit axiom. For others (hopefully very few and only at the elementary school level), even the formula $(a+b)c=ac+bc$ is not an implicit axiom.

What we mean by an implicit axiom system for a number system should now be clear. Usually it is quite different from an axiom system in the mathematical sense.

(1) An implicit axiom system can be dependent.

(2) The system does not necessarily contain all the axioms that are needed from the mathematical point of view.

(3) In the very early stages of learning, before the introduction of the number system has been accomplished, an implicit axiom system can even be inconsistent.

For example, a young child might think that there is a greatest number among natural numbers. An older child might think that all the numbers are rational; in fact some mathematicians thought this until Pythagoras. So, an implicit axiom system reflects the student's point of view about certain structures. If we adopt the naive approach to the axiomatic method, namely if we teach mathematics without presenting the formal axiomatic method and without declaring that certain statements are axioms, we can say that our aim in teaching is to make the implicit axiom system of the student equivalent, in the mathematical sense, to one of the possible axiom systems for the structure we teach.

It is interesting to determine the implicit axiom systems for students at various levels; certainly this question requires a development of suitable techniques, but we are not going to deal with it here. However, it does seem that the following might be taken as an implicit axiom system for the natural number system for high school students (grades 9–11).

For all natural numbers $a, b, c$:

(1) $a+b=b+a$

(2) $(a+b)+c=a+(b+c)$

(3) $ab=ba$

(4) $(ab)c=a(bc)$

(5) $(a+b)c=ac+bc$

(6) $a+0=a$

(7) $a\cdot 0=0$

(8) $a\cdot 1=a$.

We can see at once that this system is not equivalent to Peano's axiom system (or even to weaker axiom systems which can be found in the literature, see, for example [4]). However, using the above system, one can prove almost all statements about natural numbers that appear in high school algebra text books. Of course, one cannot prove the Factorization Theorem (every natural number is a prime or a product of prime numbers) nor the Division Theorem (for all natural numbers $a, b$ if $b\neq 0$ there exist natural numbers $q, r$ such that $a=qb+r$, $0\leqq r<a$), but these theorems do not usually appear in high school activities. They appear in elementary school in an implicit manner when children are asked to factorize a given natural number into prime numbers or to divide two given natural numbers and find the remainder. However, if these theorems do appear in a certain high school activity, one might consider them as two additional implicit axioms.[3]

[3] Note that the proposition: for every $a$, $a+1\neq 0$, has not been counted as an implicit axiom of the high school student despite the fact that a suitable version of it is necessary (from the mathematical point of view) for any axiom system of natural numbers. It has not been counted as an implicit axiom because it does not have a special status in high school activity that entitles it to be an implicit axiom. Although it is well known to the high school student, he does not pay any more attention to it than he does to propositions like $1\neq 0$ or $7\neq 9$ which do not seem to be important and very rarely (if ever) appear in mathematical arguments in high school.

Now, let us call the stage in which the high school student uses an implicit axiom system the naive stage, and let us call the stage in which the formal axiomatic method is introduced to the student (not necessarily in high school) the formal stage.

The question is asked whether we want students to pass from the naive to the formal stage in high school. We can see at least two reasons why this might be a sound approach:

(a) we want to teach the axiomatic method in an explicit and formal way in high school because we think it is an important chapter in mathematics and it can be taught at this stage in the student's development;

(b) we want to prove new theorems which cannot be proved on the basis of the implicit axiom system. We are therefore obliged to present a new axiom and such a presentation cannot usually be made by simply telling the students that a certain proposition is true. It requires a short discussion of the axiomatic method.

At the beginning of this paper we mentioned that there is no agreement between mathematicians and mathematics teachers about (a) above and at this time we do not want to discuss this controversy. However, suppose that one abandons (a), thinking that the axiomatic method is too difficult for high school students to grasp. What then can be done about (b)? Should a chapter based on a certain axiom, the presentation of which involves the axiomatic method, be abandoned?

In particular, the above situation occurs when we wish to teach mathematical induction. Up to this point, all algebraic activities can be developed in the naive stage. However, when presenting mathematical induction, few teachers can avoid the problems of the axiomatic method, especially if they refer to it as the axiom of induction. What should a teacher do when he thinks that teaching the axiomatic method may cause serious difficulty for the high school students[4], but he wants to teach mathematical induction?

We propose to suggest a way in which the above-mentioned dilemma can be avoided. However, in order to present our proposal we need to establish a certain background in mathematical logic. For those who are unfamiliar with notions from mathematical logic, a short survey follows.

**2. $\omega$-Rule.** $\omega$-rule is a rule of inference in an applied first-order predicate calculus which contains individual constants for all the natural numbers. Let us assume for the sake of simplicity that these individual constants are: $0, 1, 2, \ldots$. Let $\phi(x)$ be a formula with exactly one free variable $x$. The $\omega$-rule states: From the (infinitely many) premises, $\phi(0), \phi(1), \phi(2), \ldots$ conclude that $\forall x \phi(x)$ (namely, $\phi(x)$ for every $x$). This may require some explanation. First, we should explain what we mean by $\phi(0), \phi(1), \phi(2), \ldots$. This has the same meaning as $\phi(n)$, $n = 0, 1, 2, \ldots$. So it suffices to explain what we mean by $0, 1, 2, \ldots$. Usually mathematicians use three dots to denote infinite sequences.[5] So, $0, 1, 2, \ldots$ denotes:

(1) an infinite sequence whose first element is 0 and every other element is obtained by adding 1 to its predecessor.

By (1) we understand:

(2) all natural numbers.

However, in the English language (1) and (2) are different expressions, and it is interesting to ask how two different expressions can have the same meaning. Do (1) and (2) have the same meaning for the same reason that "The first President of the United States" and "George Washington" have the same meaning? We can say that (2) was chosen to denote the sequence defined in (1). We might also say that if we look at natural numbers as cardinals of the finite sets, then (1) and (2) have the same meaning because we believe that every natural number appears in the sequence defined in (1).

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[4] It certainly does, because the student is obliged to change his point of view about certain subjects and his whole way of thinking. This is not an easy thing to do.

[5] Here we speak about the naive sense of infinite sequence, namely a sequence which does not have a last element. By infinite sequence we do not mean a function whose domain is the set of all natural numbers.

*Whatever we say, however, we should state that* (1) *and* (2) *have the same meaning as a result of our arithmetical activities. We can say that* (1) *and* (2) *having the same meaning is a fundamental part of our arithmetical thinking, or, using our own terminology, it is an implicit axiom for our arithmetical activities.*

Now, when dealing with formal logic, we must remember that we are not concerned with models (or structures). We have only well-formed (but meaningless) formulae. So if, in such a formal logic, we have the premises $\phi(0), \phi(1), \phi(2), \ldots$ we cannot conclude that $\forall x \phi(x)$, unless there is a suitable rule of inference that enables us to do so. If a formal logic contains such a suitable rule (namely the $\omega$-rule) it is called an $\omega$-logic. There is, of course, a practical question to be raised about $\omega$-logics. The question is, how can we know that $\phi(0), \phi(1), \phi(2), \ldots$ without the previous knowledge $\forall x \phi(x)$? In other words, can we deduce $\phi(0), \phi(1), \phi(2), \ldots$ without first deducing $\forall x \phi(x)$? (If we cannot do this it means that we cannot use the $\omega$-rule to obtain new theorems.)

From more advanced studies in mathematical logic we know that there are cases in which we can deduce $\phi(0), \phi(1), \phi(2), \ldots$ and then use the $\omega$-rule to obtain $\forall x \phi(x)$ (which was unproved in the former stage). For more details on this, see, for instance, [**6**].

As often happens with many formal axioms and rules of inference, we call the informal rule, which is analogous to the $\omega$-rule, by the name of the formal rule. The informal $\omega$-rule is as follows: From the premises $\phi(0), \phi(1), \phi(2), \ldots$ conclude $\phi(n)$ for every natural number $n$.

Although some people may think that "$\phi(0), \phi(1), \phi(2), \ldots$" and "$\phi(n)$ for every natural $n$" are the same (and therefore the premises and conclusion of the $\omega$-rule are the same) we hope the reader can see that this is not the case. The identification of "$\phi(0), \phi(1), \phi(2), \ldots$" with "$\phi(n)$ for every natural number $n$" is made by the implicit informal $\omega$-rule. We therefore can conclude this section by saying that the $\omega$-rule is an implicit rule of inference for our arithmetical activities. It is so fundamental that many mathematicians do not usually notice that they are using it.

**3. Mathematical induction.** Let us now look at the way in which mathematical induction is presented to high school students. Let us choose its first-order formulation (relating to formulae).[6] It is introduced more or less in the following way. Let $\phi(x)$ be any formla in the algebraic language for natural numbers. Suppose we have found that $\phi(0)$ is true. Also suppose that for every natural number $n$, if $\phi(n)$ is true we can prove that $\phi(n+1)$ is also true. Then $\phi(n)$ is true for every natural number $n$.

Usually the teacher does not think the above is sufficient to allow the student to understand mathematical induction. Therefore, he tries to explain it more or less as follows: Assume we know that for every $n$, if $\phi(n)$ is true then $\phi(n+1)$ is also true. In particular, if $\phi(0)$ is true then $\phi(1)$ is true. But assume we find that $\phi(0)$ is true. Therefore $\phi(1)$ is also true. In the same way, we conclude that $\phi(2)$ is true and so on. We obtain that $\phi(0), \phi(1), \phi(2), \ldots$ are true. Therefore, $\phi(n)$ is true for every natural number $n$.

This explanation is usually sufficient to convince the student that mathematical induction leads to the right conclusions. If it is not sufficient, then there is little chance that something else will help him as the student is probably too immature to understand mathematical induction. This explanation, however, has the nature of a proof, and, in fact, most students regard it as such. Therefore, if the teacher does not introduce mathematical induction as an axiom, everything will probably go smoothly. However, if the teacher does declare that mathematical induction is an axiom, he is unable to avoid pointing out that axioms cannot be proved and we should not be surprised if the students are therefore confused.

Let us now look at the above explanation and determine whether or not it is a proof for mathematical induction. The crucial step in this proof is that from the assumption $\phi(0), \phi(1), \phi(2), \ldots$

[6] Similar arguments can be developed also for its second-order formulation (relating to sets).

we conclude that $\phi(n)$ for every $n$. Hence, we came to our conclusion by using the $\omega$-rule (although sometimes we are not aware of this fact or do not even know the $\omega$-rule). Since mathematicians often do not consider the above explanation as a proof for mathematical induction, it seems that they want to avoid the $\omega$-rule. There are probably several reasons for this. First, the $\omega$-rule is not formally introduced to mathematics students until they take a course in mathematical logic; sometimes they even have to wait for an advanced course in this subject. Therefore, there are many mathematicians who have not even heard of the $\omega$-rule. As noted above, however, this does not mean they don't use it implicitly. Second, the $\omega$-rule is an infinitary rule of inference and since in logic we are mainly concerned with ordinary first-order calculus, where formulae and rules are finitary, we are not inclined to adopt an infinitary rule and make it fundamental. This may explain why mathematicians avoid the $\omega$-rule, but it does not justify it. But, since mathematicians very often write $0, 1, 2, \ldots$, by which they mean all natural numbers, it shows that they are implicitly using the $\omega$-rule. Because of this, the $\omega$-rule becomes a legitimately implicit rule of inference for high school students and there is no reason why this should not be encouraged for mathematics freshmen. In fact, one should encourage the use of any implicit rule or axiom because later on it is always easy to make it explicit. From the mathematical point of view, nothing is wrong with using the $\omega$-rule (as implicit) to prove mathematical induction. We use many implicit logical axioms, mathematical axioms and rules of inference without thinking of making them explicit. There is no reason why we should avoid the $\omega$-rule any more than other implicit rules of inference. Now, it seems that the only reason it is avoided is a traditional one. The $\omega$-rule was formulated as a formal rule of inference only after the ordinary rules of the first-order calculus were formulated. Moreover, it has not become an elementary rule of mathematical logic because logics which include the $\omega$-rule do not have the same nice properties of ordinary first-order logic. Although some research concerning the $\omega$-rule has been done (see for instance [2], [3], [5], [6]) it seems that until now no attractive features had been found to make it more popular. This should not, however, prevent us from using the $\omega$-rule as an implicit rule of inference in high school. On the contrary, we should train ourselves (as mathematics teachers) to look at the $\omega$-rule as one of the most fundamental, implicit rules of our arithmetical thinking, to use it in high school, as we use many other implicit rules of inference and to prove mathematical induction by it. The benefits to be derived from this approach are obvious. We shall be able to teach mathematical induction in high school without getting into the problematic area of the formal explicit axiomatic method. As mentioned, we should avoid calling mathematical induction an axiom but instead should refer to it as a principle or method.

We claim that teaching with the implicit axiomatic approach does not contribute to the student's deductive ability less than the explicit formal approach and it saves him a lot of trouble and confusion. Note that one can have considerable ability in proving mathematical theorems even without knowing about axioms.

**4. Implicit Axioms in Other Domains of High School Mathematics.** We have confined our discussion to the system of natural numbers, but it is possible to generalize without any difficulty to other number systems that appear in high school mathematics. It is also possible to generalize it to geometry, but this would be much more complicated. Usually geometry is taught only in high school, using explicit axiomatic approaches. This is one of the many reasons that it is so difficult for high school students to learn geometry.

However, developing geometry with an implicit axiomatic approach should begin very early and special efforts must be made to develop new methods, new attitudes and new curricula.

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## PROBLEMS AND SOLUTIONS

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

### ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before December 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 2605*. *Proposed by Andreas P. Hadjipolakis, Anopolis Sfakion, Crete, Greece*

Consider a chessboard of odd order $n$ ($n \geqq 5$). Assign label $m$ to a cell of the chessboard if it can be reached by the knight in $m$ steps starting from the central cell, and this $m$ is minimal. Determine the number $K(n; m)$ of cells labelled $m$.

E 2606. *Proposed by R. S. Luthar, University of Wisconsin, Janesville*

Show that there are infinitely many integers $n$ such that $2n+1$ and $3n+1$ are both perfect squares, and that such $n$'s must be multiples of 40.

E 2607. *Proposed by the Eastern Montana College Problem Group*

Solve the functional equation

$$f(x, y)+f(y, z)+f(z, x)=3f(\tfrac{1}{3}(x+y+z), \tfrac{1}{3}(x+y+z))$$

in the class of all continuous functions $\mathbb{R}^2 \to \mathbb{R}$.

What can be said about the solutions in the class of all functions $\mathbb{R}^2 \to \mathbb{R}$?

E 2608*. *Proposed by Judith Q. Longyear, Wayne State University*

A child is riding in a train $n$ cars long and wishes to go exploring. An exploration may be described by listing in order the cars traversed; each exploration must end in the same car in which it began. How many explorations of length $k$ can it make?

Suppose we regard the exploration $(e_1, e_2, \ldots, e_k)$ to be equivalent with all explorations $(e_r, e_{r+1}, \ldots, e_k, e_1, \ldots, e_{r-1})$ $(r = 2, \ldots, k)$. How many non-equivalent explorations can it make?

E 2609*. *Proposed by Glen E. Bredon, Rutgers University*

Define integers $a_{ij}$ $(i, j \geq 1)$ by $a_{i1} = a_{1j} = 1$ and

$$a_{ij} = ia_{i,j-1} + ja_{i-1,j} \qquad (i, j \geq 2).$$

Show that

$$\sum_{i=1}^{2n-i} (-1)^{i-1} a_{i,2n-i} \equiv 1 \pmod 3.$$

E 2610. *Proposed by Hugh L. Montgomery, University of Michigan*

Let $f$ be a real-valued function defined on the unit square $[0,1] \times [0,1]$. Suppose that $f(x, y)$ is continuous in $x$ for each fixed $y$ and continuous in $y$ for each fixed $x$. Show that if $f^{-1}(0)$ is dense in the unit square then $f = 0$.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### Recall of "An Iterated Function"

E 984 [1951, 564]. *Proposed by Joseph Rosenbaum*

(a) Find $f(x)$ when $f[f(x)] = x^2 - 2$.

(b) More generally, find $f_1(x)$ when $f_n(x) = x^2 - 2$, where $f_n(x)$ is defined by the relation $f_{r+1}(x) = f_1[f_r(x)]$.

*Editor's Comment.* We are grateful to John S. Biggerstaff, Portland, Oregon, for drawing our attention to the fact that the solutions given in [1952, 252] do not really solve the problem: constraints on the unknown function such as continuity, etc., are not discussed, nor is the domain specified; thus a statement concerning the uniqueness of the solutions found cannot be made. The editors hereby solicit further solutions, in which such questions will be treated. The problem at a reasonable level of generality is likely to be difficult; see M. Kuczma, *Functional Equations in a Single Variable* (PWN, Warsaw, 1968), Chapter XV, and the references given there.

### Variations on a Well-known Limit

E 2125 [1968, 1007; 1969, 939]. *Proposed by F. A. Butter, Jr., California State College at Long Beach*

For every fixed real number $x$, the sequence $\{u_n(x)\}$, where

$$u_n(x) = (1 + n^{-1})^{n+x} \qquad (n = 1, 2, \ldots),$$

converges to $e$ as $n \to \infty$. Determine numbers $a$, $b$, $a < b$, such that $\{u_n(x)\}$ is monotone increasing for $x \in (-\infty, a)$ and monotone decreasing for $x \in [b, \infty)$. For $x \in (a, b)$ discuss the variation of $u_n(x)$ with $n$, and estimate bounds. (The cases $x = 0, 1$ have been widely treated.)

*Solution by the proposer (revised by the editor).* Let $F(z) = (1 + z^{-1})^{z+x}$ for $z \geq 1$. Then

$$(z^2 + z)F'(z) = F(z)(G(z) - x), \tag{1}$$

where $G(z) = (z^2 + z)\log(1 + z^{-1}) - z$. Since $G'(z) = (2z + 1)\log(1 + z^{-1}) > 0$ we conclude that when $z$ increases from 1 to $+\infty$ the function $G(z)$ strictly increases from $G(1) = \log 4 - 1$ to $G(+\infty) = 1/2$.

Using (1) we see that:

CASE 1. If $x \geqq 1/2$ then $F(z)$ is strictly decreasing in $[1, +\infty)$;

CASE 2. If $x \leqq G(1)$ then $F(z)$ is strictly increasing in $[1, +\infty)$;

CASE 3. If $G(1) < x < 1/2$ then $F(z)$ is strictly decreasing in an interval $[1, z_0(x)]$ and strictly increasing in $[z_0(x), +\infty)$.

In this last case $z_0(x)$ is the unique solution of $G(z) = x$ in $z > 1$.

Since $u_n(x) = F(n)$, $n = 1, 2, \ldots$, it follows from above that

(i) the sequence $u_n(x)$ is strictly decreasing if $x \geqq 1/2$;

(ii) the sequence $u_n(x)$ is strictly increasing if $x < a$, where $a$ is the solution in $x$ of the equation

$$2^{1+x} = F(1) = F(2) = \left(\tfrac{3}{2}\right)^{2+x},$$

i.e., $a = \log(9/8)/\log(4/3)$;

(iii) if $a \leqq x < 1/2$ then there is an index $k \geqq 2$ (depending on $x$) such that $u_n(x) > u_{n+1}(x)$ for $1 \leqq n \leqq k - 2$, $u_{k-1}(x) \geqq u_k(x)$ and $u_n(x) < u_{n+1}(x)$ for $n \geqq k$.

For instance, if $x = a$ then $k = 2$ and we have $u_1(a) = u_2(a) < u_3(a) < \cdots$.

*Editor's Comment.* The previously published solution by Michael Goldberg contained several gaps. We are grateful to M. T. Bird (San Jose, California) for pointing out these facts to us. The proposer also gives the reference to Polya and Szegö, *Aufgaben und Lehrsätze aus der Analysis* I, Springer 1925, pp. 30, 186 where it is shown that the sequence $\{u_n(x)\}$ is strictly decreasing if and only if $x \geqq 1/2$.

**Cutting Corners is not so Easy**

E 2452 [1974, 84; 1975, 169]. *Proposed by Joel Anderson, California Institute of Technology*

Starting with an arbitrary convex polygon $P_1$, a sequence of polygons is generated by successively "chopping off corners"; thus if $P_i$ is a $k$-gon, then $P_{i+1}$ is a $(k + 1)$-gon, etc. At the $j$th step, let $d_j$ be the altitude of the cut-off triangle, measured from the cut-off vertex. Prove or disprove: The series $\Sigma d_j$ converges.

*Solution by Herbert Taylor, Jet Propulsion Laboratory, Pasadena, California.* Let $P_\infty = \bigcap_{i \geqq 1} P_i$. We shall say that a point $A$ is a *vertex* if it is a vertex of some $P_i$. Similarly, we say that a line $L$ is a *side* if two consecutive vertices of some $P_i$ lie on $L$; more precisely, $L$ is then a side of $P_i$. If $L$ is a side of $P_i$ then it is also a side of $P_j$ for $j > i$. Hence $L \cap P_\infty$ is not empty and for each side $L$ we fix a point $X_L \in L \cap P_\infty$. Each vertex $A$ lies precisely on two sides $L'_A$ and $L''_A$. We choose the notation so that if $A$ is a vertex of $P_i$ and the boundary of $P_i$ is oriented in the counterclockwise direction, then the edge $[B, A] \subset L'_A$ of $P_i$ is oriented from $B$ to $A$.

Let $A$ be a vertex and $\alpha_A$ the angle between $L'_A = L'$ and $L''_A = L''$ containing $P_\infty$. Then we put $\theta_A = \pi - \alpha_A$ and

$$f(A) = \theta_A(\overline{AX}_{L'} + \overline{AX}_{L''}).$$

If moreover $A \notin P_\infty$ then there is a unique index $i$ such that $A \in P_i$ but $A \notin P_{i+1}$. Then instead of $A$ the polygon $P_{i+1}$ has two new vertices which we denote by $\sigma A$ and $\sigma^{-1}A$. The notation is chosen so that $\sigma A \in L'$ and $\sigma^{-1}A \in L''$. We denote by $d_A$ the altitude of the triangle $A$, $\sigma A$, $\sigma^{-1}A$ measured from $A$. If $A \in P_\infty$ we put $d_A = 0$. If a vertex $A \in P_\infty$ then $\sigma A$ and $\sigma^{-1}A$ are not defined.

If $A$ is a vertex and $A \notin P_\infty$ then we claim that

$$f(A) \geqq d_A + f(\sigma A) + f(\sigma^{-1}A). \tag{1}$$

Let $B = \sigma A$, $C = \sigma^{-1}A$, $L' = L'_A$, $L'' = L''_A$, $L = L''_B$, $X' = X_{L'}$, $X'' = X_{L''}$, $X = X_L$. Without loss of generality we may assume that $\theta_B \geqq \theta_C$. Then

$$\theta_B(\overline{AB} + \overline{AC}) \geqq \theta_B(\overline{BX} + \overline{XC}) \geqq \theta_B(\overline{BX}) + \theta_C(\overline{CX}),$$

$$\theta_C(\overline{AB} + \overline{AC}) \geqq \theta_C(\overline{AC}) \geqq \overline{AC}\sin(\theta_C) = d_A,$$

$$(\theta_B + \theta_C)(\overline{BX'} + \overline{CX''}) \geqq \theta_B(\overline{BX'}) + \theta_C(\overline{CX''}).$$

By adding these three inequalities we obtain (1) because $\theta_A = \theta_B + \theta_C$. It follows from (1) that

$$f(A) \geqq d_A + d_{\sigma A} + d_{\sigma^{-1}A} + f(\sigma\sigma A) + f(\sigma^{-1}\sigma A) + f(\sigma\sigma^{-1}A) + f(\sigma^{-1}\sigma^{-1}A),$$

and in general

$$f(A) \geqq \sum_{\substack{0 \leqq k \leqq m-1 \\ \varepsilon_i = \pm 1}} d(\sigma^{\varepsilon_1}\sigma^{\varepsilon_2}\cdots\sigma^{\varepsilon_k}A) + \sum_{\varepsilon_i = \pm 1} f(\sigma^{\varepsilon_1}\sigma^{\varepsilon_2}\cdots\sigma^{\varepsilon_m}A),$$

where we write $d_B = d(B)$, and if $\sigma^{\varepsilon_1}\sigma^{\varepsilon_2}\cdots\sigma^{\varepsilon_k}A$ is undefined, the corresponding terms should be omitted from these sums.

Hence, if $A$ is a vertex of $P_1$ then the series

$$\sum_{\substack{k \geqq 0 \\ \varepsilon_i = \pm 1}} d(\sigma^{\varepsilon_1}\sigma^{\varepsilon_2}\cdots\sigma^{\varepsilon_k}A)$$

is convergent and, consequently, $\Sigma d_A$ over all vertices $A \notin P_\infty$ is also convergent.

Also solved by Aage Bondesen (Denmark), C. S. Gardner, Andrew Harrington, L. E. Mattics, and Bryce Parry.

*Editor's Comment.* The hypothesis that $P_{i+1}$ has one vertex more than $P_i$ precludes chopping off through a vertex of $P_i$. Without this hypothesis the assertion of the problem becomes invalid. Counterexamples were supplied by Herbert Taylor and by Noël Cortey.

### Simple Diophantine Equations

E 2532 [1975, 400]. *Proposed by Erwin Just, Bronx Community College*

Solve the following Diophantine equations:

(1) $$x^m(x^2 + y) = y^{m+1},$$

(2) $$x^m(x^2 + y^2) = y^{m+1}.$$

*Solution by Robert Breusch, University of Waikato, New Zealand.*

EQUATION (1). If $m = 0$ the solutions are $x = 0$, $y$ arbitrary. If $m \geqq 1$ we claim that the solutions are

(3) $$x = b(b^m - 1), \qquad y = b^2(b^m - 1),$$

where $b \in \mathbf{Z}$ is arbitrary. It is easy to check that these are solutions. Let $(x, y)$ be a solution other than $(0, 0)$. Then $xy \neq 0$ and we can write $x = ac$, $y = bc$ where $a$ and $b$ are relatively prime and $a \geqq 1$. From (1) we obtain

$$a^m(a^2c + b) = b^{m+1}.$$

This implies $a = 1$, $c = b(b^m - 1)$ and consequently $x$ and $y$ are as given in (3).

EQUATION (2). If $m = 0$ we have only the solution $x = 0$, $y = 1$. If $m \geq 1$ we have only the trivial solution $x = y = 0$. Assume that there exists a non-trivial solution $(x, y)$. Then $xy \neq 0$ and we write again $x = ac$, $y = bc$ where $a$ and $b$ are relatively prime and $a \geq 1$. From (2) we get

$$a^m c(a^2 + b^2) = b^{m+1}.$$

This implies that $a = 1$ and then $1 + b^2$ divides $b^{m+1}$. Since $b \neq 0$ this is a contradiction.

Also solved by W. J. Blundon (Canada), Peter de Buda (Canada), Miltiades Demos, Francis Flanigan, Harvey Fletcher, M. G. Greening (Australia), Carl Hurd, Dennis Jesperson, Eleanor Jones, L. Kuipers (Switzerland), O. P. Lossers (Netherlands), Carolyn MacDonald, L. E. Mattics, Ram Murty & Kumar Murty (Canada), Gerson Robison, F. B. Strauss, and Temple University Problem Solving Group.

### Helping Professor Umbugio

E 2533 [1975, 401]. *Proposed by E. S. Pondiczery, Royal Academy of Poldavia (in exile)*

Our old friend Professor Euclide Paracelso Bombasto Umbugio, eminent numerologist from Guayazuela, is trying to calculate to an accuracy of 1% the sum of the reciprocals of the 8,877,690 positive integers whose decimal representations contain no repeated digits. Possessing only a desk calculator (Guayazuela is a poor country), he figures it will take him about one year to complete his calculations, performing them at the rate of one reciprocal per second, eight hours a day, six days a week. Help the Professor improve his efficiency.

*Editor's comment.* Although the Professor is too shy to ask, we know that he would like more accuracy in his sum. Perhaps some of the Professor's many friends who have access to high speed computers could help him out. There is a problem, however, in that the Professor would insist on paying for computer time out of his own pocket: since his salary is quite meager, direct calculation of the sum would be out of the question.

*Solution by Thomas Foregger, Bell Laboratories, Murray Hill, New Jersey.* We will show that the sum lies in the interval (8.857, 9.011) so that 8.934 will be within 1% of the sum. All displayed 10 digit numbers are rounded off, but are certainly accurate to 4 decimal places.

Define $S_n$ to be the sum of the reciprocals of all $n$-digit numbers having no repeated digits. Then

$$S_1 = \frac{7129}{2520} \doteq 2.828968254 \quad \text{and} \quad S_2 \doteq 2.091230332.$$

(The calculation of $S_2$ uses

$$S_2 = \sum_{n=1}^{100} \frac{1}{n} - \frac{12}{11} S_1 - .01$$

and

$$\sum_{n=1}^{100} \frac{1}{n} = \log(100) + \gamma + \frac{1}{2(100)} - \frac{1}{12(100)^2} + \varepsilon,$$

where $\gamma \doteq .5772156649$ is Euler's constant, and $0 < \varepsilon < 10^{-10}$). Let $n \geq 3$ and let $p$ be an $n$-digit number with no repeated digits. Suppose that the first digit of $p$ is $k$ and that the second digit is $j$. The number of such $p$'s is $8!/(10 - n)!$, and $p$ satisfies the inequalities

$$k \cdot 10^{n-1} + j \cdot 10^{n-2} \leq p < k \cdot 10^{n-1} + (j + 1) \cdot 10^{n-2}.$$

Hence

$$(1) \qquad 10^{2-n} \frac{8!}{(10-n)!} \sum_{\substack{j=0 \\ j \neq k}}^{9} \frac{1}{10k + j + 1} < S_n < \frac{8!}{(10-n)!} \cdot 10^{2-n} \sum_{k=1}^{9} \sum_{\substack{j=0 \\ j \neq k}}^{9} \frac{1}{10k + j}.$$

It is easy to check that

$$\sum_{k=1}^{9}\sum_{\substack{j=0\\ j\neq k}}^{9}\frac{1}{10k+j+1}=S_2-.09+\sum_{k=1}^{9}\frac{1}{11k(11k+1)}\doteq 2.013124699.$$

Summing the inequalities (1) for $n=3,4,\ldots,10$ we get

$$(2.013124699)\ 8!\cdot 10^{-8}\sum_{j=0}^{7}\frac{10^j}{j!}<\sum_{n=3}^{10}S_n<S_2\cdot 8!\cdot 10^{-8}\sum_{j=0}^{7}\frac{10^j}{j!}. \tag{2}$$

We find that

$$\sum_{j=0}^{7}\frac{10^j}{j!}\doteq 4850.682540$$

so that the right hand side of (2) is

$$(2.091230332)\ (4.032)\ (4.850682540)10^{-1}\doteq 4.090018245.$$

The left hand side of (2) is

$$(2.013124699)\ (4.032)\ (4.850682540)10^{-1}\doteq 3.937259623.$$

Hence $8.857458209<\sum_{n=1}^{10}S_n<9.010216831$.

The help was also offered by D. P. Laurie (South Africa), O. P. Lossers (Netherlands), and L. E. Mattics.

$$\binom{j+k}{k}+\binom{j+k-1}{k-1}$$

E 2534 [1975, 520]. *Proposed by Clark H. Kimberling, University of Evansville*

Consider the array of numbers $a(j,k)$ defined for $j,k=0,1,\ldots$ as follows: $a(j,0)=1$ for $j=0,1,\ldots$; $a(0,k)=2$ for $k=1,2,\ldots$; $a(j,k)=a(j-1,k)+a(j,k-1)$ for $j,k\geqq 1$. Prove the following:

(i) If $p$ is prime, then $p\mid a(j,p-j+1)$ for $j=2,3,\ldots,p-1$.

(ii) If $j+2k$ is prime, then it divides $a(j,k)$.

(iii) If $a(j,k)$ is prime, then it divides $a(mj,mk)$ for $m=1,2,\ldots$.

*Composite of solutions by M. G. Greening, University of New South Wales, Australia, and L. E. Mattics, University of South Alabama.* Let

$$F(s,t)=\sum_{j,k\geqq 0}a(j,k)s^jt^k.$$

Then from the definition of $a(j,k)$ we see that $(1-s-t)F(s,t)=1+t$. Hence

$$F(s,t)=\frac{1+t}{1-s-t}=(1+t)\sum_{n\geqq 0}(s+t)^n$$

and we have

$$a(j,k)=\binom{j+k}{k}+\binom{j+k-1}{k-1}.$$

(i) If $p$ is a prime and $2\leqq j\leqq p-1$ then $p$ divides both binomial coefficients

$$\binom{p}{p-j}\quad\text{and}\quad\binom{p+1}{p-j+1}.$$

Hence $p$ divides their sum $a(j,p-j+1)$.

(ii) This statement is false for $k = 0$. Let $k > 0$. Then the statement follows from the formula

$$(*) \qquad a(j,k) = \frac{2k+j}{k}\binom{k+j-1}{j}$$

and the inequality $k < 2k + j$.

(iii) Assume that $a(j,k)$ is prime. Then we must have $k > 0$. If $j = 0$ then $a(0,k) = 2$ and also $a(0,mk) = 2$. Thus the assertion is valid in that case. We may now assume that also $j > 0$. Then it is easy to check that

$$k < 2k + j \quad \text{and} \quad \binom{k+j-1}{j} \geqq k.$$

Since $a(j,k)$ is a prime, it follows from these inequalities and from (*) that $\binom{k+j-1}{j} = k$ and $a(j,k) = 2k + j$. Now it remains to note that

$$a(mj,mk) = \frac{2k+j}{k}\binom{mk+mj-1}{mj}.$$

Also solved by Günter Bach (Germany), Robert Breusch (New Zealand), R. W. Chamberlain, Peter de Buda (Canada), Angela DeLong, Michael Doob (Canada), Kay Dundas, Lorraine Foster, F. T. Howard, Carl Hurd, L. Kuipers (Switzerland), Jeff Lagarias, S. C. Locke (Canada), O. P. Lossers (Netherlands), Helen Marston, Richard Poppen, Daniel Rosenblum, James Singer, Richard Stevens, Gillian Valk, Charles Wexler, David Zeitlin, and the proposer.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before December 31, 1976.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

### Corrections

5687 [1969, 835; 1975, 766]. *Proposed by Z. Govindarajulu, University of Kentucky*
Change $H(\sigma) + ET_\sigma$ to $H(\sigma) = ET_\sigma$.
Change $H(\sqrt{2}/2) = \sqrt{\pi}I$ to $H(\sqrt{2}/2) = 2\sqrt{\pi}I$.

6071 [1976, 62]. *Proposed by J. G. Milcetich, Hunter College Campus School*
Change $|a_k| < \frac{1}{2}$ to $|a_k| \leqq \frac{1}{2}$.

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6102. *Proposed by Barbara Osofsky, Rutgers University*

Let $A$ and $B$ be nontrivial rotations of $\mathbb{R}^3$ about $l_1$ and $l_2$, respectively, which are axes through $(0,0,0)$ such that $A^2 = B^3 = \mathrm{Id}$. Hausdorff has shown that if $\cos 2\theta$ is transcendental, where $\theta$ is the angle between $l_1$ and $l_2$, then all relations between $A$ and $B$ are generated by $A^2 = \mathrm{Id}$ and $B^3 = \mathrm{Id}$. Show that the same is true for $\theta = \frac{1}{4}\pi$.

6103. *Proposed by Gérard Letac, Université Paul Sabatier, France*

Let $(X_n)_{n=1}^{\infty}$ be independent and identically distributed random variables, valued in a real vector space $E$ of finite dimension $d$. Let $Y$ be a random linear form on $E$ such that $\lim_{n\to\infty} Y(X_n)$ exists almost-surely (a.s.). If $d = 1$, it is easily proved that either $Y = 0$ a.s. or $X_n$ = constant a.s.. What happens if $d > 1$?

6104. *Proposed by Leonard W. Deaton, California State University at Los Angeles*

If $X$ and $Y$ are independent normal random variables with means $\mu_1$ and $\mu_2$ and variances $\sigma_1^2$ and $\sigma_2^2$ respectively, then the probability density function $f$ of the random variable $Z = X/Y$ is given by

$$f(x) = \frac{\sigma_1\sigma_2}{\pi(\sigma_1^2 + \sigma_2^2 z^2)}, -\infty < z < \infty.$$

Prove or disprove that $f$ is the probability density function of $Z$.

6105*. *Proposed by Harry D. Ruderman, Hunter College Campus School*

Prove that the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-1)^{[n\sqrt{2}]}}{n}.$$

Estimate its value.

6106*. *Proposed by D. S. Mitrinović and P. M. Vasić, University of Belgrade, Yugoslavia*

Find the general solution to the functional equation

$$\sum_{k=1}^{n} f(x^k) = \sum_{k=1}^{n} f(x^{-k}).$$

6107*. *Proposed by Roy E. DeMeo, Jr., Franklin Square, New York*

Let $\sigma(n)$ be the sum of all divisors of $n$. Let $A$ be the set of all rational numbers $\sigma(n+1)/\sigma(n)$. Determine the closure of $A$ in the set of real numbers.

## SOLUTIONS OF ADVANCED PROBLEMS

### The Equation $\partial f/\partial x = \partial f/\partial y$

5871 [1972, 780; 1973, 1150; 1975, 530]. *Proposed by P. R. Chernoff, University of California, Berkeley*

Let $f(x, y)$ be a real-valued function of two real variables which is separately differentiable. Assume that $\partial f/\partial x = \partial f/\partial y$ everywhere. Must there be a function $g$ of one variable such that $f(x, y) = g(x + y)$? What if we assume *a priori* that $f$ is jointly continuous?

III. *Comment by M. J. Pelling, University of Benin, Nigeria.* H. F. Royden [1975, 530] answered this question affirmatively assuming $f$ is jointly continuous. Using his result we may prove the following: With no continuity assumptions on $f$ there exist squares (whose union is dense in the plane) on which $f$ is expressible in the form $g(x + y)$.

*Proof.* It is known [1] that if a function of two variables is continuous in each variable separately then a partial derivative which is defined everywhere must be in Baire class 1. Thus $\partial f/\partial x$ and $\partial f/\partial y$ are in Baire class 1, so their points of continuity form dense $G_\delta$ sets. Since the intersection of two dense $G_\delta$ sets is also a dense $G_\delta$ set there exist points $P$, dense in the plane, at which both $\partial f/\partial x$, $\partial f/\partial y$ are continuous. Any such $P$ is therefore the center of a square $S$ on which $\partial f/\partial x$, $\partial f/\partial y$ are bounded and by an elementary argument it follows that $f$ is continuous on $S$. By Royden's result $f = g(x + y)$ in $S$, for some $g$.

#### Reference

1. G. P. Tolstov, On Partial Derivatives, Amer. Math. Soc. Transl., 69 (1952).

**Sum of Valencies for a Plane Graph**

5953. [1974, 176]. *Proposed by Paul Erdös, Budapest, Hungary, and G. B. Purdy, University of Illinois*

Let $G$ be a planar graph of $n$ vertices $x_1, x_2, \ldots, x_n$. Let $v(x_i)$ be the valency (or degree) of $x_i$ and let

$$v(x_1) \geqq v(x_2) \geqq \cdots \geqq v(x_n).$$

Determine (or estimate) $\Sigma_{i=1}^{k} v(x_i)$.

*Solution by Saul Stahl, Wright State University, Dayton, Ohio.* Let $q$ denote the number of edges of $G$; we show that

$$\sum_{i=1}^{k} v(x_i) \geqq 2q - \frac{3(n-k)(n+k-3)}{k+1} \qquad k = 1, 2, \ldots, n. \tag{1}$$

The reader may easily verify (1) when $n \leqq 4$, and hence we assume that $n \geqq 4$. For $m = 0, 1, \ldots, n-1$, let $f(m)$ denote the number of vertices of $G$ of degree at least $m$. We first show that

$$m \leqq 3 + \frac{3(n-4)}{f(m)} \qquad \text{for} \quad m = 0, 1, \cdots, n-1. \tag{2}$$

Since $n \geqq 4$ we may assume, without loss of generality, that $G$ is maximal planar and hence $q = 3n - 6$ and $v(x_i) \geqq 3$ for all $x_i$. Thus,

$$\begin{aligned} 6n - 12 = 2q = \sum_{i=1}^{n} v(x_i) &= \sum_{i=1}^{f(m)} v(x_i) + \sum_{i=f(m)+1}^{n} v(x_i) \\ &\geqq mf(m) + 3(n - f(m)) = (m-3)f(m) + 3n, \end{aligned}$$

and (2) now follows immediately. Noting that the inequalities $v(x_1) \geqq v(x_2) \geqq \cdots \geqq v(x_n)$ imply that $f(v(x_i)) \geqq i$ for each $i = 1, 2, \ldots, n$, we proceed as follows:

$$\begin{aligned} \sum_{i=1}^{k} v(x_i) &= 2q - \sum_{i=k+1}^{n} v(x_i) \geqq 2q - (n-k)v(x_{k+1}) \\ &= 2q - (n-k)\left[3 + \frac{3(n-4)}{f(v(x_{k+1}))}\right] \geqq 2q - 3(n-k)\left(1 + \frac{n-4}{k+1}\right) \\ &= 2q - \frac{3(n-k)(n+k-3)}{k+1}. \end{aligned}$$

As readers of the Chartrand, Geller and Hedetniemi paper, *Forbidden subgraphs*, may suspect, both (1) and (2) have analogs for acyclic and outerplanar graphs. Namely, if $G$ is outerplanar, then

$$(m-2)f(m) \leqq 2(n-3), \tag{2'}$$

$$\sum_{i=1}^{k} v(x_i) \geqq 2q - \frac{2(n-k)(n+k-2)}{k+1}. \tag{1'}$$

If $G$ is acyclic, then

$$(m-1)f(m) \leqq n-2, \tag{2''}$$

$$\sum_{i=1}^{k} v(x_i) \geqq 2q - \frac{(n-k)(n+k-1)}{k+1}. \tag{1''}$$

Also solved by E. F. Schmeichel.

### The Spectral Radius of $e^A e^{A^*}$

6003 [1974, 1121]. *Proposed by Emeric Deutsch, Polytechnic Institute of Brooklyn*

Let $A$ be a complex $n \times n$ matrix, show that

$$r(e^A e^{A^*}) \leqq r(e^{A+A^*}),$$

where $r$ denotes spectral radius.

*Solution by O. P. Lossers, Technological University, Eindhoven, the Netherlands.* Both $e^{A+A^*}$ and $e^A e^{A^*}$ are Hermitian positive definite. Therefore we have to show

$$\max_{|x_0| \leqq 1} x_0^* e^A e^{A^*} x_0 \leqq \max_{|x_0| \leqq 1} x_0^* e^{A+A^*} x_0. \tag{1}$$

If we denote the maximum eigenvalue of $A + A^*$ by $\lambda$, then the right-hand side of (1) equals $e^\lambda$. On the other hand, if $|x_0| \leqq 1$, the solution $x(t) = e^{A^* t} x_0$ of the differential equation $\dot{x} = A^* x$ satisfies

$$\frac{d}{dt} x^*(t)x(t) = x^*(t)(A + A^*)x(t) \leqq \lambda x^*(t)x(t),$$

hence

$$x_0^* e^{At} e^{A^* t} x_0 = x^*(t)x(t) \leqq e^{\lambda t} x_0^* x_0 \leqq e^{\lambda t}$$

from which the inequality (1) follows by the substitution of $t = 1$.

Also solved by Richard Datko, P. van der Driessche & D. D. Olesky (Canada), and the proposer.

### An Injective Map of $R \times R$

6004 [1974, 1121]. *Proposed by Oto Strauch, Bratislava, Czechoslovakia*

Find a concrete example of an injective function $f: R \times R \to R$ such that for every two real numbers $x, y$ it will be true that

$$x < y \Rightarrow x < f(x, y) < y.$$

*Solution by Wolfe Snow, Brooklyn College.* Write every real number $z$ as $[z] + .z_1 z_2 z_3 \ldots$, i.e., the integral part of $z$ plus an infinite decimal. Whenever there is a choice of an infinite string of consecutive zeros or nines, write nines if it is in the larger of $x, y$ and zeros if it is in the smaller of the two numbers (or if the two numbers are equal).

For $x < y$, construct $f(x, y)$ as follows: Begin with $[x]$ and keep on copying the decimal part of $x$ up to and including the first place, where $x_j < y_j$ (if $[x] < [y]$, simply copy $[x]$). Then keep on writing nines up to and including the first place where $x_j < 9$. No matter what our continuation is, we have achieved $x < f(x, y) < y$. The rest of our construction will be aimed at the injective property.

In the next place, put a 3. In the next place, put a 1 if $x$ and $y$ have the same sign, and a 2 if $x$ and $y$ have opposite signs. Then put a 0, then the first digit in $x$ after the decimal point, then a 0, then the first digit in $y$ after the decimal point, then a 0, then the first digit in $x$ before the decimal point, then a 0, then the first digit in $y$ before the decimal point, then a 0, then the second digit in $x$ after the decimal point, and so on. (Whenever there are no entries in places before the decimal point, place a 0.) This latter part of $f(x, y)$ makes $f$ injective; we know where this part of $f(x, y)$ begins by the fact that at least every alternate entry is a 0, whereas immediately preceding we have at least two consecutive nonzero digits.

If $y < x$, interchange the roles of $x$ and $y$ in the preceding construction. To preserve injectiveness,

simply put a 4 in the place where the preceding construction called for a 3 at the beginning of the last paragraph.

If $x = y$, let $f(x, y) = [x] + .0x_1 0x_2 0x_3 \ldots$. This is unequal to any of the above $f(x, y)$ since here no two consecutive nonzero digits appear after the decimal point.

Also solved by Anthony Barkauskas & David Bange, Ellen Hertz, Carl Hurd, Michael Josephy (Canada), Daniel Kleitman, Albert Leisinger, D. A. Overdijk (Netherlands), M. J. Pelling (Nigeria), David Pickard (Australia), Reinhard Wobst (Germany) and the proposer.

**Relatively Prime Matrices**

6006 [1975, 84]. *Proposed by Frank Uhlig, University of Würzburg, Germany*

Let $A_i$ be a finite family of complex square matrices that have no eigenvalues in common. Let $p_i$ be a family of real polynomials and define $B_i = p_i(A_i)$ for each $i$. If each $A_i$ is similar to a real matrix, prove that there is a real polynomial $p$ such that $p(A_i) = B_i$ for every $i$.

*Solution by Donald W. Robinson, Brigham Young University.* Let $m_i(x)$ be the minimum polynomial of $A_i$, $i = 1, \ldots, n$. Since each $A_i$ is similar to a real matrix, $m_i(x) \in R[x]$. Define $M_i(x) = \Pi_{j \neq i} m_j(x)$. Since $A_1, \ldots, A_n$ have no eigenvalues in common, then the g.c.d. of $(M_1(x), \ldots, M_n(x))$ is 1. Thus, there exist $N_1(x), \ldots, N_n(x) \in R[x]$ such that $\Sigma_{k=1}^n N_k(x) M_k(x) = 1$. In particular, $M_k(A_i) = 0$ if $k \neq i$ and $I_i = N_i(A_i) M_i(A_i)$, where $I_i$ is the identity matrix conformable with $A_i$. Define $p(x) = \Sigma_{k=1}^n N_k(x) M_k(x) p_k(x) \in R[x]$. Then for every $i$,

$$p(A_i) = \sum_{k=1}^{n} N_k(A_i) M_k(A_i) p_k(A_i) = I_i p_i(A_i) = p_i(A_i) = B_i.$$

Also solved by J. T. Arnold & R. L. Snider, Paul Chernoff, S. Ditor (Canada), D. Ž. Djoković (Canada), Kenneth Foote, Thomas Foregger, F. Gerrish (England), Charles Givens, Rolf Jeltsch, Otto Kuropatwa (Germany), Eric Lander, O. P. Lossers (Netherlands), Nicholas Passell, M. J. Pelling (Nigeria), Mary Powderly, Aaron Todd, William Watkins, J. H. Webb (South Africa), Helmuth Wielandt (Germany), and the proposer.

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## REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
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COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.*

*Elementary Probability Theory with Stochastic Processes.* By Kai Lai Chung. Springer-Verlag, New York, 1974. x + 325 pp. $12.00. (Telegraphic Review, January 1975.)

Springer-Verlag has launched itself into the field of undergraduate textbooks under the banner of a master, Professor Kai Lai Chung. The text, *Elementary Probability Theory with Stochastic Processes*, is an excellent book which promises to take its place as a favorite for first courses in probability theory

regardless of whether the students actually be undergraduates or first or second year graduate students seeking a first glimpse of probability theory. The book is set out in eight chapters with three short appendices. The first three chapters give a background in set theory, an introduction to the concept and structure of probability functions and a gem of a treatment of combinatorics. This is followed by a clear concise treatment of random variables as functions over random sample paths, their distributions and expectations, with an appendix giving more insight into the measure theoretic underpinnings. Then Professor Chung deals very clearly with conditioning concepts and their use by means of some classic probability problems, followed with a chapter on mean, variance and transforms. The Poisson and Normal distributions are treated together with the classical limiting theorems and the book closes with a chapter dealing with sums of independent random variables and makes a smooth transition from this to the topic of Markov chains including an insight into the strong Markov property, with an appendix on the Martingale concept. There is, in addition, at the end of the text a list of general references, which, for a man of Professor Chung's knowledge and experience, is surprisingly short and incomprehensive. Those familiar with Chung's graduate text, *A Course in Probability Theory*, will not find this surprising. The list of references in both texts appears to be inadequate. There are many, many problems at the end of each chapter. The number remains high from chapter to chapter throughout the entire book. It is clear that Professor Chung has a copious supply of probability problems and does not tire of presenting them. His creativity with problems and examples is probably unsurpassed. The problems cover a wide range of mathematical difficulty from straightforward application of rules to extremely difficult ones that will challenge the brightest students (and their professors!). Problems range from applied examples to theoretical developments, which latter provide an enrichment for the development in the text proper. Solution outlines and hints to some of the problems are provided at the end of the text. Once again, it is apparent that Professor Chung's interest in this aspect of the book did not flag as the work on it progressed. There are ample solution outlines for problems from beginning to end. Following these problem solutions is a table of values of the standardized normal distribution, the only table in the text, and an index that is not skimpy by any means.

The style of the text is not consistent. There is a discernible change in style between chapters 1–6 and chapters 7–8. This is a change for the better in my opinion. The result is a more readable, pleasant style which is more in keeping with that of a mathematics text (even for undergraduates).

While I find some fault with Professor Chung's style there can be no criticism of the organization and flow of ideas in the text. There is abundant reference in the development of topics to previous results and equations already derived which give to the work, for the most part, an organizational and logical tightness that is pleasing to mathematical minds. But "things are not quite what they seem," as will be noted in the latter part of this review. The overall development from beginning to end is superb. There is a beautiful coherent structure to the entire text and one senses in Chung a master at being able to steer the student along the straight path of essential probability exploring, as it were, an unknown woods without getting lost on some poorly marked trail. The discussion of probability functions is superb as is the magnificent treatment of transforms and the development of the central limit theorem. In connection with the latter, there is an extremely well done discussion of asymptotic theory in Chapter 7 that deserves to be presented in any undergraduate course. It is refreshing also to see Markov chain theory introduced in such a way that, without getting snagged along the way, the student is led to the concept of the strong Markov property — something most elementary courses in stochastic processes handle only cautiously. The chapter on combinatorics is outstanding. In short, through his text Chung introduces the student to a full, rich probability structure not often found in introductory courses. It is a sheer delight to observe Chung revealing to his readers all of the beauty of probability concepts. Chung has an unquestionable grasp of this subject and, as important, can convey this to the students in an inspiring way. For this reason this text will be an important contribution to the effort of attracting bright young mathematicians into the area of probability (and also into statistics).

Chung has added some historical spice to the development in the text by giving frequent references to people, dates, and in some cases, places. One finds these references interesting, indeed, yet only tantalizing, in most cases, as for example the reference to Norbert Wiener and those to Chebychev and Markov. I suppose such a brief historical reference might be adequate to send an historically minded student to the library. I would have preferred just two or three more lines as a part of each reference. Probabilists and statisticians do not afford adequate honor to their classic masters in textbooks. Chung's effort is in the right direction. Other authors would do well to follow this practice as a norm.

No textbook is without its faults and, unfortunately, Chung's book has two aspects which bother me. The first is substantive, the second more philosophical. Concerning the first, let me say that one aspect of Chung's text that catches one's attention almost immediately is the lavish use of equation reference numbers to refer to previous results in previous sections and chapters as the development progresses. It is the logical product in style of a logical mind. What Chung succeeds in doing is to force the student back constantly to material already covered. It is a built-in review mechanism. For topics in calculus Chung prods his readers: "Do you remember how to do that?" or "If you're not absolutely sure of this point, check your integral calculus text again." This is good. And on the surface of it the abundant equation and theorem references are good pedagogical tools. Unfortunately, when one tests the accuracy of these references one finds that a large number of them are incorrect. There are references to the wrong (in the sense of inappropriate) results, to sections containing theorems which do not exist. Several of these errors occur, for example, on pages 165 and 166. One can find at least twenty-five such incorrect references just from page 165 to the end of the text. To make matters worse, on page 166 Chung makes the following parenthetical remark, (with regard to a question of mathematical exigency), "These fussy details often detract from the main argument but are a necessary price to pay for mathematical rigor. The instructor as well as the reader is free to overlook some of these at his own discretion." Most of the trouble with these inaccuracies seems to stem from Chapter 5 Section 5. Perhaps Chung's editors suggested some extensive revisions, which were subsequently carried out, but the attendant equation reference changes were never made. This reflects, I think, Springer's haste to enter the market for undergraduate textbooks. Of course, such mistakes in a textbook can be labeled as trivial ones that will only reinforce the habit of having the student read very carefully the text, so that the basic arguments are understood.

But at the same time, Springer is somewhat misleading when it advertises this book. Springer quotes a review in *The American Mathematical Monthly* in which the book is described as a "... well-conceived and executed text ..." Well-conceived, yes. Six more months of work on the equation references and proof reading would have enabled Springer to claim legitimately that this is truly a well executed text also. Classes that elect to use Chung's book will have the task of ferreting out these troublesome mistakes.

The second point on which I find fault with Chung's text deals more with the philosophy of style and scholarly works. A textbook seems an inappropriate place for the comments and references to such a renowned probabilist as William Feller which appear, for example, on pages 107 and 233. That these comments appear, in the vein they do, only detracts from this text.

Despite these objections, though, *Elementary Probability Theory with Stochastic Processes* is a strong text for students, both undergraduate and graduate, who wish a full, clear insight into modern probability theory. Students who have the opportunity of working from this text will be exposed to probability at its finest. They, along with Daniel, will view the broad, rich structure of probability not often presented in a single course and be guided by an accomplished probabilist.

JEROME N. SENTURIA, University of Wisconsin-Madison

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L***. *Mathematics at a Glance, A Compendium.* Ed: W. Gellert, *et al.* VEB Biblio Inst, 1975, 760 pp. A translation of the popular (700,000 copies sold!) *Kleine Enzyklopädie der Mathematik.* A systematic, concise, lavishly illustrated synopsis of mathematics from arithmetic through, roughly, first year graduate topics. Theorems, definitions and examples are color coded to make scanning easy and effective. Each topic is enriched with historical detail and diverse applications. LAS

GENERAL, S**(15-18), P. *Jahrbuch Überblicke Mathematik 1976.* Benno Fuchssteiner, *et al.* Bibliographisches Inst, 1976, 204 pp, (P). Continues the tradition established with the 1975 yearbook (TR, December 1975). Expository articles, history, and notes at the level (at least) of advanced undergraduates in the U.S.. For example, this yearbook offers: Theory of Price Indices, Proof Machines, and The Source and History of the Fields Medal. JAS

GENERAL, P. *Lecture Notes in Mathematics-1-500: An Index and Other Useful Information.* Ed: A. Dold, B. Eckmann. Springer-Verlag, 1975, 78 pp, $2.90 (P). Indices by volume (including complete contents of conference proceedings), by author, and by subject, preceded by brief commentary on the history of the Lecture Notes series and followed by detailed instructions for preparation of manuscripts. Over 600,000 copies of LNM have been sold in the past 12 years. LAS

GENERAL, S*, P*, L***. *Annotated Bibliography of Expository Writing in the Mathematical Sciences.* Matthew P. Gaffney, Lynn Arthur Steen. MAA, 1976, xi + 282 pp, $7 (P). A well indexed survey of expository articles in English. Topics considered are as wide ranging as these telegraphic reviews and the levels by which the entries are grouped run from popular to research. This volume is designed to allow teachers to supplement their reading and teaching from a wide variety of sometimes unexpected sources. JAS

GENERAL, T(13-14: 1, 2). *The Nature of Modern Mathematics, Second Edition.* Karl J. Smith. Brooks/Cole, 1976, xiv + 619 pp, $13.95. Mathematics for non-science students. This revision of the first edition (TR, August-September 1975) contains new chapters on geometry and topology (including non-Euclidean geometry) and statistics. The chapter on the nature of computers has been revised to include a more thorough discussion of BASIC. JG

GENERAL, S(13-14), L?. *Introductory Mathematics for Engineers: Lectures in Higher Mathematics.* A.D. Myškis. Trans: V.M. Volosov. MIR, 1975, 814 pp. A non-rigorous introduction to calculus, analytic geometry, differential equations, linear algebra, probability theory and complex numbers. This well-written book, unfortunately, has very few examples and no problems. CEC

GENERAL, S. *SI Metric Units, An Introduction, Revised Edition.* H.F.R. Adams. McGraw, 1974, 114 pp, $3.95 (P). A comprehensive elementary handbook on the *Système International d'Unites,* referred, when conversions are necessary, to British units. LAS

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1973, V. 29.* AMS, 1976, iv + 267 pp, $36.80. Includes memorial tribute to and biography of A.G. Kurosh. LAS

BASIC, T(13: 2), S, L. *A Survey of General Mathematics.* James E. Shockley. HR&W, 1976, xv + 590 pp, $12.95. Blends cultural and technical topics: the former develops integers, rationals, and basic geometry while the latter develops linear and quadratic equations, and introductions to probability, statistics, and the computer. Many examples, illustrations, and exercises. Answers to selected exercises. Tables. References. Index. RJA

BASIC, T(13: 1), S. *Developing Skills in Algebra: A Lecture Worktext, Combined Second Edition.* J. Louise Nanney, John L. Cable. Allyn, 1976, viii + 528 pp, $12.95 (P). Concerns what is traditionally thought of as "high school algebra." Could be used as self-study material. Numerous examples and exercises. Clearly written and organized with special thought given to leaving ample space for working out problems in the text itself. Answers to exercises. Index. Diagnostic tests. RJA

BASIC, T(13: 1). *Modern University Math, V. I.* William W. Hokman, Neal C. Raber. Collegiate, 1974, vi + 346 pp, $10.95 (P). A workbook-type text which includes an introduction to elementary algebra in the first two-thirds and concludes with an introduction to combinatorics and probability. CEC

BASIC, T(13: 1). *Intermediate Algebra.* Richard B. Thompson. Prindle, 1976, 603 pp, $9.95 (P). A modularized workbook which divides the standard topics of intermediate algebra into fourteen units. The emphasis is on telling the student exactly what is expected of him and encouraging him to take responsibility for learning. CEC

BASIC, T(13: 1). *Basic Algebra for College Students.* Corrinne Pellillo and Charles Henry Brase. HM, 1976, x + 469 pp, $9.95 (P). Partially programmed combination textbook-workbook--including pretests, selftests, posttests--aimed at building arithmetical skills. Culminates with the quadratic formula. LCL

PRECALCULUS, T*(13). *Algebra for College Students.* Max A. Sobel, Norbert Lerner. P-H, 1976, xii + 544 pp, $11.95. A very nice precalculus text. Besides the basics (linear and quadratic equations, operations with algebraic expressions, logs, sequences and series) each chapter has a "Beyond the Horizon" section on topics ranging from logs in other bases to transfinite numbers. Plenty of exercises and chapter tests on each chapter. PJM

PRECALCULUS, T(13). *A Precalculus Course in Algebra and Trigonometry, Second Edition.* Earl W. Swokowski, *et al.* Prindle, 1976, ix + 850 pp, $13.95 (P). More solved examples, more exercises, more sections (including ones on graphs of quadratic functions, conic sections, linear programming), and a reordering of the material on trigonometry so that considerations of the unit circle are first. According to the preface: "Each sentence of the first edition was scrutinized to determine whether it could be improved." DFA

HISTORY, P. *Sophus Lie's 1884 Differential Invariant Paper.* Ed: Robert Hermann. Trans: M. Ackerman. Math Sci Pr, 1976, viii + 273 pp, $20 (P). A translation with extensive (over half the volume) commentary by Robert Hermann effecting not only a translation from German to English but from Lie to Cartan and a modern mathematical framework. JAS

HISTORY, T(15-16: 2), L. *An Introduction to the History of Mathematics, Fourth Edition.* Howard Eves. HR&W, 1976, xv + 588 pp, $12.95. Additions to earlier editions include material on early Chinese mathematics and on nineteenth and early twentieth century mathematics. The Problem Studies that conclude each chapter make this an attractive text, yet its coverage of contemporary mathematics is very thin. LAS

HISTORY, P, L*. *A History of Ancient Mathematical Astronomy, V. I-III.* O. Neugebauer. Springer-Verlag, 1975. *V. I:* xxi + 555 pp; *V. II:* 499 pp; *V. III:* 395 pp, $118.90 set. A monumental treatise beginning with the Almagest and ranging widely over Babylonian, Greek, Egyptian and Roman sources. V. III is devoted exclusively to appendices (on chronology, astronomy, mathematics), indices, and figures and plates referred to in V. I and II. LAS

HISTORY, L. *L.E.J. Brouwer, Collected Works 2: Geometry, Analysis, Topology and Mechanics.* Ed: Hans Freudenthal. North-Holland, 1976, xxvii + 706 pp, $99.95. This volume contains the greater part of Brouwer's published and extant unpublished work on classical mathematics, including letters to other mathematicians. Contains an English translation of an obituary by Freudenthal and Heyting, and a complete bibliography of Brouwer's works. V. I, edited by A. Heyting (TR, August 1975) contains Brouwer's philosophical papers. LAS

HISTORY, L. *Collected Papers of G.H. Hardy, V. VI.* Oxford U Pr, 1974, xii + 854 pp, $43.50. Sixth of seven planned volumes, this one contains papers (including joint papers with Littlewood and others) on series. Concludes, as does each volume in the series, with a complete Hardy bibliography. LAS

HISTORY, P. *Ricci and Levi-Civita's Tensor Analysis Paper.* Robert Hermann. Math Sci Pr, 1975, xi + 261 pp, $14 (P). A translation with extensive interspersed commentary with the aim of showing in a modern context what was achieved historically. JAS

FOUNDATIONS, T(18), P*. *Admissible Sets and Structures, An Approach to Definability Theory.* Jon Barwise. Springer-Verlag, 1975, xiii + 394 pp, $29.60. The theory is based on interactions between model theory, recursion theory and set theory: "...old boundaries disappear as notions merge, techniques complement one another, analogies become equivalences, and results in one field lead to results in another." Written for students and specialists--extensive instructional perspective, and exercises. LCL

FOUNDATIONS, S(15-17), P, L*. *The Ways of Paradox and Other Essays, Revised and Enlarged Edition.* W.V. Quine. Harvard U Pr, 1976, x + 335 pp, $15; $5.95 (P). Eight recent essays have been added to the 21 that comprised the original (1966) edition (TR, March 1967) of this well-known collection. Ranging in consistently forthright and nontechnical style from antinomies to algebraic logic, Quine here makes the flavor and substance of philosophical logic accessible to the educated layman. LAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-500: ISILC Proof Theory Symposion.* Ed: J. Diller, G.H. Müller. Springer-Verlag, 1975, viii + 383 pp, $13.20 (P). The proceedings of the symposion at Kiel in August 1974. The proceedings of the remaining parts of the summer institute and logic colloquium are covered in LNM-499. JAS

FOUNDATIONS, P. *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science, V. I-III.* Ed: W.L. Harper, C.A. Hooker. Reidel, 1976. *V. I: Foundations and Philosophy of Epistemic Applications of Probability Theory,* xi + 309 pp, $21.50 (P); *V. II: Foundations and Philosophy of Statistical Inference,* xi + 455 pp, $27 (P); *V. III: Foundations and Philosophy of Statistical Theories in the Physical Sciences,* xii + 241 pp, $17.50 (P). 35 papers (often plus ensuing discussion) from the May 1973 International Research Colloquium at the University of Western Ontario. Aimed, in part, at "removing the cloud hanging over many of the central methods of statistical testing now in constant use within the social and natural sciences." LAS

COMBINATORICS, T(16-17: 1), S, P, L. *Introduction to Combinatorics.* Ioan Tomescu. Trans: I. Tomescu, S. Rudeanu. Collet's, 1975, xiii + 249 pp, £8.50. Romanian original published in 1972. Tightly argued survey of standard topics including enumeration (inclusion-exclusion, Polya theory, Möbius functions), configurations (distinct representatives, Ramsey's theorem) with applications to graph theory (shortest paths, maximal independent sets, maximal and minimal colorings). Many results included among the exercises appear here in English for the first time. LCL

COMBINATORICS, S(13-16), P, L**. *Mathematics of Choice or How to Count Without Counting.* Ivan Niven. New Math. Libr., No. 15. MAA, 1975, xi + 202 pp, $4 (P). Reprint of the original edition first published in 1965 by Random House. Excellent exposition at an elementary level. LAS

NUMBER THEORY, P*. *Applications of Sieve Methods to the Theory of Numbers.* C. Hooley. Tracts in Math., No. 70. Cambridge U Pr, 1976, xiv + 122 pp, $18.95. An introduction to sieve methods directed at those with some background in the theory of numbers. The book contains an introductory chapter on general sieve methods and then gives several applications to well-known problems in number theory. Includes a bibliography. CEC

NUMBER THEORY, P. *Lecture Notes in Mathematics-504: Frobenius Distributions in $GL_2$-Extensions.* Serge Lang, Hale Trotter. Springer-Verlag, 1976, 274 pp, $10.30 (P). Concerned with distribution problems for primes which are related to elliptic curves. The authors conjecture asymptotic relations for the number of certain kinds of primes and present supporting arguments and numerical information for these conjectures. SG

NUMBER THEORY, T(14-16: 1). *Einführung in die Zahlentheorie I.* Ivan Niven, Herbert S. Zuckerman. Hochschultaschenbücher, B. 46. Bibliographisches Inst, 1976, xii + 201 pp, (P). A translation of the third edition. JD-B

NUMBER THEORY, S(13-16), P, L**. *Invitation to Number Theory.* Oystein Ore. New Math. Libr., No. 20. MAA, 1975, viii + 129 pp, $4 (P). Reprint of the original edition first published in 1967 by Random House. Excellent exposition at an elementary level. LAS

LINEAR ALGEBRA, S(13-14), L. *Vectors.* P.J. O'Neill. Reed, 1975, viii + 136 pp, $9.25 (P). A brief introduction to vector spaces, vector geometry, calculus of vector functions, and matrices. A good collection of examples and exercises. CEC

ALGEBRA, P, L. *Matrix Groups.* D.A. Suprunenko. Trans. Math. Mono., V. 45. AMS, 1976, viii + 252 pp, $31.20. An account of the classical results on the structure of the general linear group over a division ring. Also, the theory of periodic, solvable, nilpotent and locally nilpotent linear groups.LCL

ALGEBRA, P. *Rational Methods in Lie Algebras.* George B. Seligman. Lect. Notes in Pure and Appl. Math., V. 17. Dekker, 1976, viii + 346 pp, $29.50. An exposition of recent results on simple Lie algebras that are related to the theory of algebraic groups. However, the point is to give a "direct" derivation of these results based on techniques from linear algebra and the theory of assorted linear algebras. SG

ALGEBRA, P. *Proceedings of the Conference on Finite Groups.* Ed: William R. Scott, Fletcher Gross. Acad Pr, 1976, xiii + 565 pp, $27.50. From Park City, Utah, February 1975: local structure, known simple groups, representations, permutation groups, solvable groups. LAS

ALGEBRA, T(18: 1, 2), S*, L. *Introduction to Homological Methods in Commutative Rings.* Anthony V. Geramita, Charles Small. Queen's U, 1976, 352 pp, $9.25 (P). A treatment developed from a graduate course at Queen's University in 1974-75. A semester's material in commutative ring theory is assumed but all the rest is carefully developed, namely: a solid block of non-homological commutative algebra, basic homology theory with localization, and significant applications to regular local rings and the like. JAS

ALGEBRA, T(17-18: 1), S, P. *Ring Theory, Nonsingular Rings and Modules.* K.R. Goodearl. Pure and Appl. Math., V. 33. Dekker, 1976, viii + 206 pp, $24.50. A monograph, and a possible graduate text since there are many problems, on the theory of nonsingular rings and modules. Includes an extensive bibliography and historical notes giving the origins of most of the results. JD-B

FINITE MATHEMATICS, T*(15-16: 1, 2), L. *Discrete Mathematical Models with Applications to Social, Biological, and Environmental Problems.* Fred S. Roberts. P-H, 1976, xvi + 559 pp, $17.95. An impressive survey of modern mathematical models centered on graphs: tournaments, traffic flow, food webs, Markov chains and much more. Concludes with chapters on n-person games (and the Shapely value), Arrow's theorem on social welfare functions, and measurement theory. Presumes calculus, linear algebra and elementary probability. LAS

CALCULUS, T(13-14: 1, 2), S. *Applied Calculus.* Raymond F. Coughlin. Allyn, 1976, viii + 424 pp, $13.95. Attempts to provide "a straightforward, intuitive presentation interwoven with numerous mathematical models." "Focuses on applications derived from actual data" in physics, engineering, biology, sociology, ecology, business, and psychology. FLW

CALCULUS, S(13-14). *Mathematical Handbook: Higher Mathematics.* M. Vygodsky. Trans: George Yankovsky. MIR, 1975, 872 pp. Intended "as a source of definitions and factual material", this small (10.5 x 16.5 cm pages) thick book contains definitions and much factual material from analytic geometry, calculus, multivariable calculus, series and differential equations. Sort of an undergraduate almanac. TAV

CALCULUS, T(13-14: 1). *Calculus and Its Applications.* Stephen Willard. Prindle, 1976, vii + 310 pp, $12.95. A calculus text whose point of view is "practical rather than theoretical." So: no trig, no series, no Mean Value Theorem. Does include three chapters on multivariable extrema, and two chapters devoted to applications in business and economics. Applications in other areas are scattered throughout the book. PJM

CALCULUS, T(13: 1, 2), S. *The Brief Calculus with Applications in the Social Sciences, Second Edition.* James E. Shockley. HR&W, 1976, xiv + 414 pp, $13.95. Achieves a good blend of explaining the underlying principles as well as presenting examples and exercises. Good diagrams and illustrations. More substantial than most texts in this area. Appendix on review of algebra. Tables. Answers to selected exercises. Index. RJA

CALCULUS, S*(13), L. *The Behaviour of Functions.* G.H. Knight. Reed, 1975, viii + 136 pp, $9.25 (P). A brief but mathematically precise introduction to functions, limits, continuity, differentiability, sequences and series. Includes an adequate number of examples and exercises. CEC

CALCULUS, T(13-14: 1, 2), S. *A Brief Calculus with Applications to Business and Economics.* Marshall F. Ruchte, Roy W. Ryden, Howard E. Thompson. Cummings, 1976, viii + 293 pp, $10.95. Attempts to provide "a short and concise introduction that is careful in its development of the major concepts" with "realistic or at least plausible economic and business examples." Includes a short chapter on multivariate calculus. FLW

CALCULUS, T(13-14: 3). *Calculus, Second Edition.* Lipman Bers with Frank Karal. HR&W, 1976, x + 783 pp, $15. Like the first edition (TR, August-September 1969), this emphasizes applications, including new ones to economics and statistics. Intuitive throughout, but precise; a 48-page "Theoretical Supplement" is at the end. Contains historical notes and over 3500 problems. This edition has fewer, smaller pages than the earlier one. DFA

CALCULUS, T(14-16: 1), S, P, L**. *Foundations of Infinitesimal Calculus.* H. Jerome Keisler. Prindle, 1976, ix + 214 pp, $9.95 (P). An entirely self-contained advanced-calculus level supplement to the author's *Elementary Calculus* (TR, April 1976) containing detailed nonstandard (i.e., infinitesimal-based) proofs of all basic calculus theorems. Could serve nicely as an undergraduate seminar text (although it contains no problems) or as a source for mathematicians to learn infinitesimal methods. Contains an optional chapter designed to bridge the notational and conceptual gap between the elementary calculus treatment of infinitesimals and the model-theoretic treatment in the research literature. LAS

REAL ANLAYSIS, T*(15: 2), L*. *Methods of Real Analysis, Second Edition.* Richard R. Goldberg. Wiley, 1976, x + 402 pp, $14.95. A carefully conceived and executed text. While covering nearly all standard analysis topics up through Riemann integration, the author exploits the concept of a metric space to give the taste of mathematical generalization. The exercises are especially appealing and numerous. TAV

REAL ANALYSIS, L*. *Leçons sur les Séries Trigonométriques.* Henri Lebesgue. Blanchard, 1975, 128 pp, 28 F (P). A reprint of the original edition based on 1904-05 lectures at the Collège de France.LAS

DIFFERENTIAL EQUATIONS, P. *Dynamical Systems, An International Symposium.* Ed: Lamberto Cesari, Jack K. Hale, Joseph P. LaSalle. Acad Pr, 1976. *V. 1,* xxviii + 338 pp, $39; *V. 2,* xxii + 314 pp, $38. Proceedings of a symposium held in August 1974 at the dedication of the Lefschetz Center for Dynamical Systems at Brown University. LAS

DIFFERENTIAL EQUATIONS, T(14-15: 1, 2). *Elementary Differential Equations with Linear Algebra.* Ross L. Finney, Donald R. Ostberg, Robert G. Kuller. A-W, 1976, xii + 515 pp, $14.95. A revision of the 1968 text *Elementary Differential Equations* by D. Kreider, R. Kuller, D. Ostberg. Begins with first order equations, then turns to linear algebra, linear operators, and linear equations. Other topics: equations with constant coefficients, Laplace transform, power series, oscillation theory, systems of linear equations, numerical methods, existence and uniqueness, stability. Major change from first edition is inclusion of more applications. SG

DIFFERENTIAL EQUATIONS, T(14-15). *Ordinary Differential Equations with Applications.* Edward L. Reiss, Andrew J. Callegari, Daljit S. Ahluwalia. HR&W, 1976, xi + 387 pp, $12.95. Intended for basic course in differential equations. Covers homogeneous and inhomogeneous first to second order equations, boundary-value and eigenvalue problems, systems of linear differential equations with constant coefficients, second order systems with variables coefficients, power series solutions, nonlinear differential equations, numerical methods, Laplace transforms. Some applications (e.g., mechanics, population dynamics) are included. SG

DIFFERENTIAL EQUATIONS, P. *Singular Perturbations and Differential Inequalities.* Frederick A. Howes. Memoirs No. 168. AMS, 1976, iv + 75 pp, $7.20 (P). Uses several second-order differential inequality theorems to study singularly perturbed boundary value problems associated with the equation $\varepsilon y'' = f(t,y,y',\varepsilon)$, obtaining existence and asymptotic behavior of solutions as $\varepsilon \to 0^+$. DFA

OPTIMIZATION, T*(15-16: 1, 2), L. *Principles of Operations Research with Applications to Managerial Decisions, Second Edition.* Harvey M. Wagner. P-H, 1975, xii + 1039 pp, $19.95. A comprehensive, authoritative and realistic elementary text. Major changes from the highly regarded first (1969) edition: streamlining the presentation of network algorithms and of dynamic, integer and stochastic programming; updating the chapter on computer modelling; and addition of new exercises in most chapters. LAS

OPTIMIZATION, T(15:2), P, L. *Graph Theory, An Algorithmic Approach.* Nicos Christofides. Acad Pr, 1975, xv + 400 pp, $32.25. This book deals with the algorithmic and computational aspects of graph theory to the virtual exclusion of pure theory. Intended for those working in operations research, computer science and civil, transport and electrical engineering. Includes ample problem sets and thorough lists of references. CEC

OPTIMIZATION, T(14-15: 1), L. *Foundations of Mathematical Programming.* William W. Claycombe, William G. Sullivan. Reston, 1975, xiii + 288 pp, $13.95. Fundamental methods. Emphasis is on "how to" not "why" (despite the title). Brief introduction to linear algebra; classical optimization presuming calculus; linear programming, simplex algorithm, assignment and transportation problems; search methods. Appendix includes illustrative journal articles. Numerical examples and exercises. LP computer programs. RWN

OPTIMIZATION, T(14-17: 1, 2), S, P, L. *Methods for Linear and Quadratic Programming.* C. van de Panne. Stud. in Math. and Managerial Econ., V. 17. North-Holland, 1975, xii + 477 pp, $49.95. Provides "detailed expositions of the most important methods", relating these methods to each other. The treatment emphasizes methods and tableaux rather than theorems and proofs. Note the exorbitant price. FLW

ANALYSIS, S(15-17), L**. *Problems and Theorems in Analysis, V. II.* G. Pólya, G. Szegö. Trans: C.E. Billigheimer. Grund. math. Wissenschaften, B. 216. Springer-Verlag, 1976, xi + 391 pp, $45.10. Slightly revised and enlarged translation of the fourth (1971) edition. Topics: theory of functions, zeros, polynomials, determinants, number theory, geometry. (V. I translated as B. 193; TR, January 1973.) LAS

ANALYSIS, T(15-16). *Advanced Mathematics for Engineers, Special Courses.* A.D. Myškis. Trans: V.M. Volosov, I.G. Volosova. MIR, 1975, 795 pp. Suitable for a variety of applied courses. Treats functions of a complex variable, operational calculus, linear algebra, tensor analysis, calculus of variations, integral equations, ordinary differential equations. Has no problem sets, but lets reader fill in many details. Introductory, with many references for further study. DFA

ANALYSIS, P. *Multipliers of Pedersen's Ideal.* A.J. Lazar, D.C. Taylor. Memoirs No. 169. AMS, 1976, iii + 111 pp, $7.60 (P). Let A be a C*-algebra. Then A has a minimal-dense hereditary ideal K, known as Pedersen's ideal. A multiplier of K is a pair of functions S, T mapping K to K, for which xS(y) = T(x)y. This monograph studies the set of multipliers of K. PJM

ANALYSIS, S(13-16), P, L**. *An Introduction to Inequalities.* Edwin Beckenbach, Richard Bellman. New Math. Libr., No. 3. MAA, 1975, ix + 133 pp, $4 (P). Reprint of the original edition first published in 1961 by Random House. Excellent exposition at an elementary level. LAS

GEOMETRY, T**(14: 1), P. *Geometry: An Investigative Approach.* Phares G. O'Daffer, Stanley R. Clemens. A-W, 1976, xvi + 445 pp, $15; *Instructor's Guide to Accompany Geometry: An Investigative Approach,* 123 pp, (P); *Laboratory Investigations in Geometry,* viii + 128 pp, (P). Extensive use of pictures, diagrams and construction activities make this an attractive, exciting discovery experience and an invaluable resource for prospective elementary and junior high teachers. Topics explored include basic geometric and topological concepts, polygons and tessellations, transformations and measurement. The laboratory manual is independent of the text and includes materials for dissection puzzles. Geoboard activities and construction of tessellations and polyhedra. JNC

TOPOLOGY, T*(16-17: 1), S*, P, L*. *Graph Theory with Applications.* J.A. Bondy, U.S.R. Murty. Am Elsev, 1976, x + 264 pp, $19.50. Topics include connectivity, touring, matching, coloring, planarity, and network flows. Several carefully developed, meaningful applications of the theory. Stress on constructive proofs including several simple new proofs of major results. Pleasing format. LCL

PROBABILITY, T(18: 2), P*. *Stochastic Processes in Queueing Theory.* A.A. Borovkov. Trans: Kenneth Wickwire. Appl. of Math., No. 4. Springer-Verlag, 1976, xi + 280 pp, $29.80. A smooth translation of the 1972 Russian edition with additional material including exercises. A comprehensive work on the theory of queues with special emphasis on stability, asymptotic behavior and convergence to stationary processes. Of special value to one familiar with the field, the book is heavy going for the novice. TAV

PROBABILITY, T(16-17: 1, 2), S, P. *The Theory of Probability.* B.V. Gnedenko. Trans: George Yankovsky. MIR, 1976, 392 pp. Discusses various views of the nature of probability then builds from the Kolmogorov axioms to Markov chains, the law of large numbers, characteristic functions, the classical limit theorem, infinitely divisible distribution laws, stochastic processes, and queueing theory.FLW

PROBABILITY, P, L. *A Mathematical Theory of Evidence.* Glenn Shafer. Princeton U Pr, 1976, xiii + 297 pp, $17.50; $8.95 (P). A new mathematical theory of degrees of belief, greatly extending the earlier similar work of Arthur Dempster. Careful contrasts with the Bayesian theory reveal both similarities and differences that "should appeal to both sides of the controversy." Proofs are relegated to appendices. LAS

STATISTICS, T(15-17: 1, 2). *Introduction to Statistics and Probability.* Edward J. Dudewicz. HR&W, 1976, xi + 512 pp, $14.95. Presupposes calculus. Includes honors problems to challenge the better students. Considers both the classical and decision-theoretic approach to hypothesis testing.FLW

STATISTICS, T(15-16: 1, 2), S, P, L. *Probability and Statistics: Theory and Applications.* D.A.S. Fraser. Duxbury Pr, 1976, xi + 623 pp, $18.95. Presupposing multivariate calculus and some linear algebra, this text presents a new approach emphasizing the analysis of data, conditional probability, likelihood functions, and the unity of statistical inferences. Includes chapters on linear models and the design of experiments. FLW

STATISTICS, T*(13-14: 1, 2), S. *Introduction to Statistics: A Nonparametric Approach, Second Edition.* Gottfried E. Noether. HM, 1976, xv + 292 pp, $11.95. This approach presupposes only high school algebra, but gets quickly into hypothesis testing, and does eventually get to most of the standard topics. FLW

COMPUTER SCIENCE, P. *Structured Programming.* Ed: D. Bates. Infotech Inter, 1976, viii + 495 pp, $185. One of the "State of the Art" series featuring an editor's introduction linking key extracts from invited papers and conference presentations, followed by complete text of 20 invited papers, and several indices. Typewritten text at an unbelievable and indefensible price. LAS

COMPUTER SCIENCE, S(13-15). *Introducción al proceso de datos.* S. Dworatschek. Editorial Alhambra, 1974, viii + 471 pp, (P). A translation into Spanish of the German original, Walter de Gruyter (1971). Very elementary. JAS

COMPUTER SCIENCE, S(16-17), P, L. *Lectures on the Measurement and Evaluation of the Performance of Computing Systems.* Saul Rosen. SIAM, vii + 138 pp, $8 (P). Based on a series of lectures given at a regional conference sponsored by CBMS and NSF in 1974. Discussion of performance is based on the following topics: personal observation, accounting systems, software probes, hardware monitors, benchmark workload models, simulation, and mathematical models. References. RJA

COMPUTER SCIENCE, T(13-15: 1), S. *Problem Solving and the Computer: A Structured Concept with PL/1 (PL/C).* Joseph Shortt, Thomas C. Wilson. A-W, 1976, xii + 372 pp, $8.95 (P). Introductory programming text centered around PL/1. Emphasizes structured programming concepts from the very beginning; the GOTO statement is not introduced until the last chapter. Text is well organized and contains many examples. Provides a thorough treatment. Introduces various non-numeric algorithms at the appropriate points. Exercises. Several appendices. Index. RJA

APPLICATIONS, P. *Introduction a la Théorie des Sous-Ensembles Flous.* A. Kaufmann. Masson, 1975. *Tome II: Applications à la Linguistique, à la Logique et à la Sémantique,* xii + 235 pp, $33.80 (P); *Tome III: Applications à la Classification et à la Reconnaissance des Formes, aux Automates et aux Systèmes, aux Choix des Critères,* xii + 305 pp, $43.68 (P). "The theory of fuzzy sub-sets is a kind of mathematics which is well suited to the study of communication among humans and between humans and machines." Volume II is devoted to the development of fuzzy language, fuzzy logic, and fuzzy propositional calculus. Volume III applies all this to the study of classification and recognition of forms, of automata, and of choice of criteria and decision making. (Vol. I, TR, February 1975.) JAS

APPLICATIONS (BIOLOGY), S(14), L. *Mathematics for Ecologists.* I. Chaston. Butterworths, 1971, 132 pp, $5.95 (P). Presents facts from calculus, differential equations and linear programming necessary to the solution of the simplified biological problems which motivate the text. Very few exercises. JG

APPLICATIONS (CONTROL THEORY), P. *Théorie probabiliste du contrôle des diffusions.* Jean-Michel Bismut. Memoirs No. 167. AMS, 1976, xiii + 130 pp, $7.60 (P). Applications of stochastic processes, potential theory and convex analysis to the optimal control of diffusions. Very general results are obtained which include most previously known theorems on stochastic control. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-44: Gravitational Perturbation Theory and Synchrotron Radiation.* Reinhard A. Breuer. Springer-Verlag, 1975, vi + 196 pp, $8.20 (P). A theory of a gravitational synchrotron effect is developed as a possible means of explaining a certain puzzling observation in the study of gravitational radiation. JAS

APPLICATIONS (PHYSICS), P*. *Mathematical Cosmology and Extragalactic Astronomy.* Irving Ezra Segal. Pure and Appl. Math., V. 68. Acad Pr, 1976, ix + 204 pp, $22.50. An exposition of the author's cosmology theory which he has developed over the past 25 years. In addition to a complete mathematical development, the author discusses philosophical aspects of his work and the relationship of its predictions to observations. SG

APPLICATIONS (PHYSICS), T(18: 1). *Asymptotic Wave Theory.* Maurice Roseau. Appl. Math. and Mech., V. 20. North-Holland, 1976, x + 349 pp, $29.50. Objective: "A unified and rigorous treatment of the analytical methods for studying the asymptotic behavior of wave representations together with some typical results borrowed from hydrodynamics and elasticity theory." Applications are to the wave equation, scattering matrix theory, water wave problems, seismic waves, diffraction. DFA

APPLICATIONS (PHYSICS), P. *Problems in the Theory of Point Explosion in Gases.* V.P. Korobeĭnikov. Proc. of Steklov Inst. of Math., No. 119. AMS, 1976, iv + 311 pp, $37.20 (P).

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-45: Dynamical Concepts on Scaling Violation and the New Resonances in $e^+e^-$ Annihilation.* Ed: B. Humpert. Springer-Verlag, 1976, vii + 248 pp, $10.30 (P).

APPLICATIONS (PHYSICS), T(16-17: 1). *Continuum Mechanics, Concise Theory and Problems.* P. Chadwick. Wiley, 1976, 174 pp, $12.50. A concise yet mathematically complete treatment of the principles of continuum mechanics. For students with working knowledge of linear algebra, vector calculus and particle and rigid body mechanics. Purposely omits some physical considerations, motivating arguments and detailed interpretations to leave room for the individual teacher to exercise his own viewpoint. I-CH

APPLICATIONS (PHYSICS), T(14-15: 1), L. *Vector & Tensor Methods.* Frank Chorlton. Wiley, 1976, 335 pp, $22.50. This geometric presentation is more algebraic than some and emphasizes applications to the physical sciences and engineering. JAS

APPLICATIONS (PHYSICS), T(16-17: 1), S. *Elements of Analytical Dynamics.* Rudolph Kurth. Pure and Appl. Math., V. 105. Pergamon Pr, 1976, vii + 184 pp, $15. Grown from author's lecture notes, self-contained and well-referenced, this book covers the motion equation, Hamilton-Jacobi theory, Hamilton's principle and Jacobi's geometric interpretation of dynamics. Its content, serving well as mathematical background for theoretical physics and topological dynamics, makes it a respectable applied mathematics book. I-CH

APPLICATIONS (SOCIAL SCIENCE), P. *Mathematische Auswahlfunktionen und gesellschaftliche Entscheidungen.* Alexander van der Bellen. Birkhauser, 1976, 333 pp, Sfr. 44 (P). The concept of a choice function is used to develop a theory of corporate decision making. Most of the results are in the context of collective decisions. The necessary mathematical background is covered in the first chapter. JAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jennifer Galovich, St. Olaf; Steven Galovich, Carleton; Ih-Ching Hsu, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

*University of Miami*: Dr. Ronald Goldstein, Notre Dame University, has been appointed Associate Professor; Dr. Alexander Morgan, Yale University, has been appointed Instructor.

*Youngstown State University*: Assistant Professors John Buoni and J. D. Faires have been promoted to Associate Professors.

Professor Richard P. Bailey retired from the Naval Academy on July 1, 1975, after 30 years of service. On December 22, 1975, the Secretary of the Navy appointed him Professor Emeritus, U. S. Naval Academy.

Mrs. Louise H. Echols retired from Birmingham-Southern College with the title of Emeritus Associate Professor after 39 years of teaching, 17 at Birmingham-Southern College.

Assistant Professor F. N. Huggins, University of Texas at Arlington, has been promoted to Associate Professor.

Dr. John G. Kemeny, President of Dartmouth College, became the 25th recipient of Dickinson College's Priestley Memorial Award for his "contribution to the welfare of mankind through science".

Associate Professor J. E. Vaughan, University of North Carolina at Greensboro, has been promoted to Professor.

Professor Delbert R. Fulkerson, Cornell University, died on January 10, 1976, at the age of 51. Dr. Fulkerson was the Maxwell M. Upson Professor of Engineering, and Professor of Operations Research and of Applied Mathematics. He was a member of the Association for ten years.

Dr. Simon Kao, University of New Mexico, died on February 22, 1976. He was a member of the Association for twenty-four years.

Professor Leo Lapidus, Lewis and Clark College, died on December 21, 1975, at the age of 65. He was a member of the Association for thirty-nine years.

Dr. Samuel D. Lawn, Penn Yan, New York, died on December 3, 1975, at the age of 33. He was a member of the Association for twelve years.

Reverend John P. Murray, S. J., Fairfield University, died on January 7, 1976, at the age of 63. He was a member of the Association for twenty-eight years.

Dr. Izaak Opatowski, Seattle, Washington, died on February 29, 1976, at the age of 71. He was a member of the Association for thirty-six years.

Mr. Albert R. Wapple, Sanford, Florida, died on November 10, 1975. He was a Charter Member of the Association.

Mr. Glenn C. Wooters, Optical Instruments, Fullerton, California, died on December 28, 1975. He was a member of the Association for twelve years.

## THE 1976–77 SABBATICAL EXCHANGE INFORMATION SERVICE

For the past two years, the MAA Sabbatical Exchange Information Service (SEIS) has provided information to assist faculty members in universities and colleges (both two- and four-year) in arranging what might be called "no-cost sabbaticals." Reaction from participants has been enthusiastic, but SEIS remains under-used. Since the value of such an information exchange to its users increases with the number of listings, we are making an effort to broaden the visibility and hence the scope of SEIS. We ask readers not only to consider using SEIS, but also to remind colleagues of its existence. It costs nothing to be listed in SEIS, and the results can be a year of expanded horizons at little or no cost to faculty member or institution. Here is how SEIS operates:

Many MAA members are in institutions not offering faculty members a program of sabbatical leaves. Even in institutions with sabbatical leave programs individual faculty members often find themselves ineligible for such a leave at a time when the desire for one is strongest. We all recognize the rejuvenating effect of an occasional change of scene, even if it does not involve release from teaching duties. The Association therefore suggests that an occasional exchange between two faculty members of similar interests, training, and experience at different institutions could be of great benefit to the individuals and also to their institutions. The individuals and institutions all stand to gain from the refreshment of exchanged ideas and insights.

It is often possible for two such faculty members to trade identities, so to speak, for a year. Such an exchange might involve trading teaching responsibilities, living quarters, and some departmental responsibilities. The extent of the exchange would depend on the individual circumstances. It is suggested, however, that salaries should not be exchanged or even discussed. Each faculty member would remain on the payroll of his permanent institution and receive all of his normal fringe benefits. Financially, his institution would not recognize the exchange at all.

The MAA proposes to become involved only to the extent of assisting in bringing together like-minded mathematics faculty members who are interested in an exchange. The information exchange will be accomplished by the annual publication by the Association in December of a list containing the names, addresses, and other pertinent information about members of the Association interested in arranging a "Sabbatical Exchange" with a colleague in another institution. This list will be sent free of charge to all those on the list and to any other MAA member who requests it.

Members interested in being listed in December 1976 should write to "SEIS, The Mathematical Association of America, 1225 Connecticut Avenue, N.W., Washington, D.C. 20036," enclosing the following information about themselves:

1. Name
2. Institution
3. Department
4. Address
5. Rank
6. Major field of interest
7. Highest earned degree
8. Names of from one to five courses recently taught
9. Normal teaching load
10. Section of country preferred for visit: Northeast, Southeast, Northcentral, Southcentral, Northwest, Southwest.
11. Period for which exchange is desired, e.g., all of the academic year 1977–78, or the first two quarters of 1977–78, or the second semester of 1977–78, etc.

Communications must reach the Washington office by November 19, 1976, for inclusion in the December 1976 list.

## 1976 CONFERENCE OF THE ASSOCIATION FOR GENERAL LIBERAL STUDIES

The 1976 Conference of the Association for General Liberal Studies will be held October 28–30 in Boston, Massachusetts. The host institution is Boston University's College of Basic Studies, and the conference theme is "General Education Looks to the Future." For complete information regarding registration, reservations, or interest in serving as a conference participant, please write, as soon as possible, to: George R. Bond, Arrangements Chairman, 1976 AGLS Conference, College of Basic Studies, Boston University, 871 Commonwealth Avenue, Boston, MA 02215.

## ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION

Professor Walter E. Mientka of the University of Nebraska has been selected to succeed Professor Henry Miot Cox as Executive Director of the Annual High School Mathematics Examination.

A report on Honor Roll Students on the Examination has been prepared by Professor Cox and is available upon request.

# MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## ANNOUNCEMENT OF LESTER R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

The 1976 recipients of these Awards, selected by a committee consisting of E. F. Beckenbach, Chairman, Emil Grosswald, and I. J. Schoenberg, were announced by President H. O. Pollak at the business meeting on August 27, 1976, at the University of Toronto. The recipients of the Ford Awards for articles published in 1975 were the following:

M. L. Balinski and H. P. Young, *The Quota Method of Apportionment*, MONTHLY, 82 (1975), 701–730.

E. A. Bender and J. R. Goldman, *On the Applications of Möbius Inversion in Combinatorial Analysis*, MONTHLY, 82 (1975), 789–803.

Branko Grunbaum, *Venn Diagrams and Independent Families of Sets*, MATH. MAGAZINE, 48 (1975), 12–23.

J. E. Humphreys, *Representations of SL* $(2, p)$, MONTHLY, 82 (1975), 21–39.

J. B. Keller and D. W. McLaughlin, *The Feynman Integral*, MONTHLY, 82 (1975), 451–465.

J. J. Price, *Topics in Orthogonal Functions*, MONTHLY, 82 (1975), 594–609.

DAVID P. ROSELLE, *Secretary*

## THE 1976 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The 37th annual William Lowell Putnam Mathematical Competition will be held at participating institutions on Saturday, December 4, 1976. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship and is administered by The Mathematical Association of America. All colleges and universities in Canada and the United States may register eligible undergraduates. Registration forms will be mailed to institutions that participated in the 36th competition by September 21, 1976. Other institutions that wish to enter undergraduates should request registration forms from Dr. A.P. Hillman, Director; The William Lowell Putnam Mathematical Competition; 709 Solano Dr., S. E.; Albuquerque, New Mexico, 87108. Completed registrations must be received by the director no later than October 26, 1976.

Further details are given in the Announcement Brochure that is mailed with the registration material. Reports of previous competitions, including examination questions and outlines of solutions, are in past issues of this MONTHLY; the most recent of these reports were in the issues of Nov. 1975, Dec. 1974, Nov. 1973, and Feb. 1973.

## APRIL MEETING OF THE OKLAHOMA-ARKANSAS SECTION

The thirty-seventh annual meeting of the Oklahoma-Arkansas Section of the MAA was held at Central State University, Edmond, Oklahoma, on April 4–5, 1975. There were 111 members, 100 nonmembers attending, making a total of 211. Professor J.R. Hodges of the University of Arkansas at Little Rock presided over the meeting.

At the dinner held on Friday evening in the Central Cafeteria, Mr. D.R. Leftwich of Oklahoma Christian College received an award for scoring the highest on the William Lowell Putnam Competition among the participants from Oklahoma and Arkansas. The invited address entitled "On the Nature of Mathematics Work in Industry" was given on Friday afternoon by Dr. H.O. Pollak of Bell Laboratories. Friday evening, Professor J.C. Brixey of the University of Oklahoma gave the third N. A. Court Lecture. The title was "Memories of Professor Court and the Early Days of the Oklahoma Section."

The following officers were elected: Chairman, Professor J.H. Yates, Central State University; Past-Chairman, Professor J.R. Hodges, University of Arkansas at Little Rock; First Vice-Chairman, Professor T.H. Fay, Hendrix College; Second Vice-Chairman, Professor Gus Pekara, South Oklahoma City Community College; and Secretary-Treasurer, E.K. McLachlan, Oklahoma State University. The following were appointed as co-chairmen of the High School Mathematics Contest: For Arkansas, Professor Claude Duplissey of the University of Arkansas at Little Rock, and for Oklahoma, Professor Thomas Cairns of the University of Tulsa.

In Oklahoma 40 schools involving 2185 students and in Arkansas 42 schools involving 2188 students participated in the high school contests.

The following papers were presented:

1. *A characterisation of H-closed Urysohn spaces*, by L.L. Herrington, University of Arkansas at Pine Bluff, and P.E. Long, University of Arkansas at Fayetteville.

2. *The fixed-point question à la Kuratowski, Hamilton,...*, by Slster Jo Ann Mark, Oklahoma State University.

3. *An order characterization of the compact open topology of $C(X)$ and the Mackey topology on its dual*, by Tak Juim Kong, University of Arkansas at Fayetteville.

4. *On Biesterfeldt's completion axiom spaces*, by R.J. Gazik, Arkansas State University.

5. *Vocational opportunities in mathematics for graduates with bachelors*, by Terral McKellips and Loyal Farmer, Cameron College.

6. *Oscillation criteria for linear second order complex differential equations*, by J.R. Choike, Oklahoma State University.

7. *Stability of a structural type for solutions of Volterra integral equation*, by J.B. Bennett, Arkansas State University.

8. *Error analysis of linear systems and a determinant identity*, by Franklin Kemp, Amoco Production Research Company.

9. *Sequential and nonsequential confidence intervals of constant width for a simple linear regression model*, by Jin T. Chen, Central State University.

10. *On the existence of certain free groups*, by Leslie Lynne Berry, Hendrix College.

11. *Computer generation of graphs of a function on a line printer*, by Dixie Johnson and Terry Gett, Oklahoma Baptist University.

12. *On the unique factorization of continuous functions*, by M.G. Tiefenback, Hendrix College.

13. *A graphical solution to the cannibals and missionaries problem*, by Carolyn Rhodes, Oklahoma State University.

14. *On some fixed point theorems*, by Patricia Guinn, Hendrix College.

15. *Quadratic residue patterns, based on Gauss' lemma*, by Karen E. Hicks, Oklahoma State University.

16. *Strict convexifiability of spaces of continuous functions*, by John Wolfe, Oklahoma State University.

17. *General decomposition theorems for M-convex sets in the plane*, by Marilyn Breen and David Kay, University of Oklahoma.

18. *Extreme Lipschitz functions*, by Jerry Johnson, Oklahoma State University.

19. *Injective and projective complete semilattices*, by J.L. Williams and Naoki Kimura, University of Arkansas.

20. *On epic implies dense for Hausdorff groups: An open problem*, by T.H. Fay, Hendrix College.

21. *The conjugacy problem in metabelian groups*, by J.S. Boler, Oklahoma State University.

22. *Extension-closed classes of residually Finite Groups*, by T.K. Teague, Hendrix College.

23. *Tensor product of commutative semigroups*, by James Anderson, Northern Arizona University and Naoki Kimura, University of Arkansas.

24. *On constructing reflective and coreflective hulls with sources and sinks*, by J.A. Tiller, Hendrix College.

25. *N-Goofspiel*, by Ernest Brickell, Oklahoma State University.

26. *Programming a mate-in-one*, by W.R. Orton III, Hendrix College.

27. *Sequences that generate primitive Pythagorean triplets*, by Andrew Simoson, Oral Roberts University.

28. *Group operations on the power set*, by Carolyn Bean, Hendrix College.

29. *A new way to develop the natural numbers*, by D.C. Kay, University of Oklahoma.

30. *Use of computer in course in theory of numbers*, by Dale Woods and Joe Flowers, Northeast Missouri State University.

31. *Current results on the distribution of prime numbers*, by Robert Curry, Claremore Junior College.

32. *Four years of IPI at Oklahoma State*, by D.D. Fisher and Bill Davis, Oklahoma State University.

33. *An introduction to mathematical thought: Mathematics for the humanities*, by Leon Semkoff and Larry Vickers, Claremore Junior College.

34. *Pocket calculators: A comparison*, by Annette Cooper, Oklahoma State University Technical Institute.

35. *Characterizing certain sets with functions having closed graph*, by Travis Thompson, University of Arkansas at Fayetteville.

36. *Characterizations of completely Hausdorff-closed spaces*, by L.L. Herrington, University of Arkansas at Pine Bluff.

37. *When local properties become global properties in topological spaces*, by W.D. Hammers, University of Arkansas at Fayetteville.

38. *Covering axioms in bitopological spaces*, by J.A. Wiley, Southeastern State University.

39. *Functions which preserve compactness and connectedness*, by T.R. Hamlett, University of Arkansas at Fayetteville.

E.K.McLACHLAN, *Secretary-Treasurer*

## NOVEMBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Valparaiso University at Valparaiso, on Saturday, November 15, 1975, with approximately 50 persons in attendance. The Chairman of the Section, Rodney Hood of Franklin College, presided.

Chairman Hood initiated the custom of an inaugural address with the topic, "Rest Points and Tangent Spheres."

The following papers were presented:

1. *A probabilistic estimate of normality*, by G.J. Sherman, Rose-Hulman Institute of Technology.
2. *Convergence theorems for sequences of nonlinear semigroups on Banach Spaces*, by Yen Tzu Fu, Indiana State University, Evansville.
3. *Individualization in the calculus course*, by D.L. Neuhouser, Taylor University.
4. *Indiana House Bill* 246, 1897, *revisited*, by A.E. Hallerberg, Valparaiso University.
5. *Some applications of n-person Cooperative Games*, by John Sorenson, Valparaiso University.

At the business meeting Maynard Mansfield, Purdue University, Fort Wayne, gave the Governor's report, and Duane Deal, Ball State University, urged members to write Governor Otis Bowen concerning Rules 46 and 47 for teacher certification.

D.E. WILSON, *Secretary*

## NOVEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA - VIRGINIA SECTION

The annual Fall meeting of the Maryland–District of Columbia–Virginia Section of the MAA was held November 22, 1975, at Georgetown University, Washington, D.C. One hundred seven persons attended of whom ninety-nine were members of the Association. Professor Ronald Davis, chairman of the Section, presided.

During the morning session there were contributed papers and a short business meeting. Following lunch, there were additional contributed papers and an invited speaker. The invited speaker, Dr. Leon Greenberg of the University of Maryland, presented a talk titled "Groups and Riemann Surfaces."

The contributed papers presented were:

1. *Mathematics of Flatland Mirrors*, by W.A. Barwick, Jr., College Park, Maryland.
2. *Placement and Prescription: Another approach to individualized instruction*, by William Steger, Essex Community College.
3. *Here's the Last Flaw*, by C.J. Maloney, Bethesda, Maryland.
4. *Mathematics in Colonial America*, by Sister Marie Augustine Dowling, College of Notre Dame of Maryland.
5. *The relationship between multinomials and nth dimensional Pascal's Spaces*, by A.P. Crotts, Jr., Georgetown University.
6. *A mathematics preparatory program for minority students: a post-mortem examination*, by S.E. Goodman, University of Virginia.
7. *A note on Nim*, by B.L. Schwartz, Department of the Navy.
8. *A course in the application of mathematics: A model building approach*, by D.C. Cathcart, Salisbury State College.
9. *Variously restricted occupancy theory, or, how (unrestricted) formal "communes" may facilitate mathematical education*, by John Hays, Reston, Virginia.
10. *Optimal allocation of inspection resources*, by Martin Pearl, National Bureau of Standards.
11. *A linear algebra approach to non-stationary Markov Chains*, by J.C. Hennessey, Loyola College.
12. *Some aspects of gambling*, by Anthony Mucci, University of Maryland.
13. *Algorithms for transit information*, by Judith F. Gilsinn, National Bureau of Standards.
14. *How to trivialize many results in topology*, by R.A. Herrmann, U.S. Naval Academy.
15. *Two notes on numerical integration*, by T.S. Schreiber, Fairfax, Virginia.

J.M. SMITH, *Secretary*

## NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The fiftieth annual meeting of the Philadelphia Section of the MAA was held at Franklin and Marshall College, Lancaster, Pennsylvania, on November 22, 1975. The Section Chairman, Professor Eugene Klotz, presided at the meeting. A total of 135 persons attended, including 108 members of the Association.

The following papers were presented:

*Relations between the application of mathematics and the teaching of mathematics*, by H.O. Pollak, President of the MAA.

*How do we tell them they need us*?, A panel discussion by Louis Hoelzle, James Rubillo, and Benjamin Volker, all of Bucks County Community College.

*How to choose k out of n*, by Herbert Wilf, University of Pennsylvania.

*Uniformly distributed sequences, stationary processes, and the ergodic theorem*, by Bennett Eisenberg, Lehigh University.

A special session for students heard the following twenty minute papers:

*Dedekind's foundation for a Lobachevskian idea*, by Valerie Treichel, Muhlenberg College.

*Independence ratios of graphs on $S_n$*, by Paul Edelman, Swarthmore College.

*Mineralistic compositions of the earth's mantle-geophysical linear programming models*, by Ellen Raber, Lafayette College.

The top performer from the Section in the 1974 Putnam Competition was Mark Anderson, Lehigh University. He was also the top performer in 1973 and was awarded an additional one year membership in the MAA.

Section Officers elected for 1976 are: E.A. Klotz, Swarthmore College, Chairperson; D.W. Schattschneider, Moravian College, Vice-Chairperson; P.E. Bedient, Secretary-Treasurer, Franklin and Marshall College. Newly elected members of the Executive Committee are: C.G. Denlinger, Millersville State College; G.J. Porter, University of Pennsylvania.

P.E. BEDIENT, *Secretary-Treasurer*

## FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifty-third annual meeting of the Louisiana-Mississippi Section was held at the Buena Vista Hotel, Biloxi, Mississippi, on February 13–14, 1976. It was hosted by the University of Southern Mississippi, Hattiesburg, with Dr. James Caveny in charge. Presiding was Dr. Carol Ottinger, Mississippi University for Women.

The Friday afternoon sessions consisted of two invited addresses and a panel discussion of remedial needs and ways to fill these needs. Professor R.F. Arenstorf, Vanderbilt University, gave a lecture on "Periodic Earth-Moon-Bus Orbits" followed by a motion picture film on these orbits. Professor Betty Hinman, Second Vice-President MAA, and University of Houston, spoke on "Some Problems in Mathematics Curricula and Student Placement". The panel discussion was arranged by Professor Ottinger and by Professor B.R. Sneed, Itawamba Junior College, who presided.

Contributed papers were presented on Saturday morning in three sections: Mathematical Notes, Classroom Notes, and Undergraduate Projects.

1. *Semigroups associated with automata*, by Margaret M. LaSalle, University of Southwestern Louisiana.
2. *Applications of restrictedness in elementary topology*, by J.E. Kimber, Rust College.
3. *On the computation of generalized inverses of matrices*, by J.D. Gilbert, Louisiana Tech University.
4. *A note on attractors for compact sets*, by J.L. Soloman, Mississippi State University.
5. *Boundary value functions for nonlinear differential equations*, by James Spencer, Louisiana Tech University.
6. *An application of Jordan algebra in general relativity*, by J.F. Reed, McNeese State University.
7. *Some subseries of the harmonic series and their convergence*, by J.R. Clark, University of Mississippi.
8. *A non-utopian approach to remedial mathematics*, by A.J. Hulin, Helen Hays, and Joe Albree, University of New Orleans.
9. *Finite and infinite sets and two results in ring theory*, by Carroll F. Blakemore, University of New Orleans.
10. *Applications of the weak law of large numbers*, by Jerome Goldstein, Tulane University.
11. *Stokes' theorem and the fundamental theorem of algebra*, by R.J. Knill, Tulane University.

12–15. *The two-phase simplex algorithm I and II, Post optimality analysis in linear programming*, and *A linear programming system for the Honeywell* 6000, by Chris Waguespack, senior student at Nicholls State University.

On Saturday, Professor Arenstorf delivered a lecture entitled "On the Distribution of Lattice Points on Spheres." During the Business Meeting which followed, the following officers were elected: Chairman, J.E. Keisler, Louisiana State University; Mississippi Vice Chairman T.A. Atchison, Mississippi State University; Louisiana Vice Chairman, C.R. Smith, Northeast Louisiana University. A certificate for outstanding performance in the Section on the Putnam competition was awarded to W.C. Davenport, Jr., of the University of New Orleans. Congratulations and applause were given to those present who had been MAA members for forty years or more.

J.R. FOOTE, *Secretary-Treasurer*

## MARCH MEETING OF THE FLORIDA SECTION

The ninth Annual Spring Meeting of the Florida Section of the MAA was held on March 5, and 6, 1976, at Florida A & M University in Tallahassee, Florida.

Six invited addresses were presented as follows: "The Relationship Between the Application of Mathematics and the Teaching of Mathematics," Dr. H.O. Pollak, Bell Laboratories, and President of MAA; "Projective Modules," Professor Hyman Bass, Columbia University; "The Initial Value Problem for First Order Systems of Differential Equations," Professor J.A. Donaldson, Howard University; "Math Topics for Liberal Arts Students," Professor Paul McDougle, University of Miami; "Relations, Discrete and Topological," Professor A.R. Bednarek, University of Florida; "A Mathematical Subtlety — and Where it has Led," Professor G.W. Medlin, Stetson University.

In conjunction with the meeting there was a State Articulation Conference which began with a panel discussion on a proposed new project to have local fall meetings. The following talks were presented to the Conference: "A Joint Teaching Venture Between HCC and USF," Frederic Zerla, University of South Florida, and Donald C. Rose II, Hillsborough Community College; "The Elementary Education Math Sequence," James Gard, University of South Florida, Homer Gerber, Florida Technological University, Marilyn L. Repsher, Jacksonville University; "An Interdisciplinary Approach to Computers and Society," Don Hill, Gwen Humphries, Andy Foster, and Wendell Meeks, all of Florida A & M University.

A Saturday morning session was sponsored by Pi Mu Epsilon the Mathematics Honorary Fraternity. The following talks were presented: "Zorn's Lemma and Ring Theory," Robert Tubbs, Florida Epsilon Chapter, University of South Florida; "Rational Cuts and the Development of the Real Numbers," David Kerr, Florida Epsilon Chapter, University of South Florida; and "The Five Regular Polyhedra," B.M. Castellano, Florida Epsilon Chapter, University of South Florida.

The following papers were presented to the section:

1. *A surprising result in Number Theory*, by E.D. Nichols, Florida State University.
2. *On generalized inverses of matrices*, by T.L. Wade, Jr., Florida State University.
3. *On a problem of rushing*, by Orville Harrold, Florida State University.
4. *The mathematician in government*, by W.H. Green, John F. Kennedy Space Center.
5. *Calculators in college algebra*, by Alan Wayne, Holiday, Florida.
6. *On the confinement of monkeys in preparation for the assault on Shakespeare's works*, by R.E. Dalton, Naval Underwater Systems Center.
7. *Subgroups, cosets, and an application to codes*, by J.R. Weaver, University of West Florida.
8. *Using simulation to introduce probability distributions in elementary statistics*, by D.L. Sherry, University of West Florida.
9. *Orders of ternary cubic forms*, by R.M. Caron, Florida Technological University.
10. *The existence of a maximal separable subalgebra in a Galois extension*, by John Cruthirds, Florida State University.
11. *A stochastic formulation of sampling procedures*, by Robert Jernigan, University of South Florida.
12. *A curriculum in computer and information sciences*, by John Grant, University of Florida.
13. *An identity element for a Galois algebra*, by Theresa E. Early, Florida State University.

The luncheon-business meeting was held Saturday, March 6, 1976. Chairman Bill Rice presided at the meeting. Committee reports were presented and Professor Beverly Brechner of the University of Florida was elected chairman-elect; Professors Anthony Shershin of Florida International University and Ignacio Bello of Hillsborough Community College, were elected as Vice-Chairmen. Professor F.L. Cleaver of the University of South Florida was elected Secretary for a three year term.

F.L. CLEAVER, *Secretary*

## MARCH MEETING OF THE SOUTHEASTERN SECTION

The fifty-fifth annual meeting of the Southeastern Section was held on March 26–27, 1976, at Central Piedmont Community College in Charlotte, North Carolina. A total of 265 persons attended the meeting, including 195 members of the Association. The local arrangements were shared by Central Piedmont Community College and the University of North Carolina at Charlotte and coordinated by R.A. Morrow.

Three invited addresses were given: J.V. Brawley (Section Lecturer) of Clemson University on "Mathematical Cryptosystems," Ivan Niven (Past First Vice-President of the MAA) of the University of Oregon on "Maxima and Minima without Calculus," and M.S. Klamkin of the University of Waterloo on "Applied Mathematics in the

University." There was a Symposium on Mathematics in the Two-Year Colleges chaired by J.E. Cicero of Clayton Junior College, with presentations by H.E. Hall, J.R. Long, H.H. Marshall, B.A. Rice, and T.R. Thomson.

There were eight sessions for contributed papers. The presiders were: J.R. Wesson (Chairman of the Section), R.A. Morrow, R.A. Dobyns, and E.V. Haynsworth for the general sessions and F.L. Cook, G.D. Faulkner, J.R. Garrett, P.A. Hartman, H.V. Park, J.F. Schell, J.R. Smith, and A. Sobczyk for the special sessions. Two MAA films were shown on Friday evening.

Officers elected for 1976–77 were: Chairman, Emilie V. Haynsworth, Auburn University; Chairman-elect, Ivey C. Gentry, Wake Forest University; Vice Chairman, R.A. Dobyns, Carson-Newman College, and Section Lecturer, Lida K. Barrett, University of Tennessee at Knoxville.

At the business meeting, the winner of the $25 prize from the Section for the best performance on the Putnam Examination in the Section was announced to be M.E. Hoffman of the Georgia Institute of Technology. The Section voted to hold its 1978 meeting at Clemson University.

The following contributed papers were presented:

1. *Bayes' rule of p-th power loss function*, by Ritu Gupta, University of Tennessee, Knoxville.
2. *Some identities for a recursive sequence of order k*, by W.R. Jernigan, East Carolina University.
3. *A program to compute the number of groups of a given square-free order*, by Thorna Humphries, Bennett College.
4. *Continuous convergence in $C(X)$*, by Mackey Lewis, East Carolina University.
5. *Monthly problem E 2552*, by Bertha Scarborough, Bennett College.
6. *Some novel uses of the second derivative test in elementary differential equations*, by J.V. Baxley, Wake Forest University.
7. *Interdisciplinary activities: three case studies*, by J.J. Buckley, University of South Carolina.
8. *Attitudes toward mathematics in siblings*, by W.R. Spickerman, East Carolina University.
9. *An undergraduate mathematical modeling course*, by G.M. Eargle, Appalachian State University.
10. *Non-traditional introduction to the derivative concept in calculus for business and economics*, by L.M. Perry, Appalachian State University.
11. *How to embed a Steiner triple system into a Kirkman triple system*, by C.C. Lindner, Auburn University.
12. *Conics in $PG_{10_{nd}}$*, by C.R. Wylie, Furman University.
13. *On the N canonical Fibonacci representations of order N*, by Robert Silber, North Carolina State University.
14. *The j-order Radon index in $(R^d, Q^d)$*, by Gerald Thompson, Augusta College.
15. *The Ramsey number $R(4,4;3)$ revisited*, by Andrew Sobczyk, Clemson University.
16. *A batch processing approach to supplemental instruction for elementary statistics courses*, by Herman Senter and Carl Ahlers, Clemson University.
17. *The $3 \times 3$ non-returning random walk: an example of student-faculty teamwork in research*, by Thad Dankel, Jr., and Barbara Ann Greim, University of North Carolina at Wilmington.
18. *FORTRAN as an introduction to computers for non-science majors*, by R.L. Anderson, Deering Milliken Service Corporation.
19. *Computer and calculator inaccuracy — some elementary examples*, by J.C. Kropa, Clayton Junior College.
20. *CAI at Bennett College — a case study in progress*, by Nellouise D. Watkins, Bennett College.
21. *A computerized visual display of the central limit theorem*, by L.L. Long and J.T. Browning, Tennessee Technological University.
22. *Two new visual aids for teaching probability*, by J.P. Thomas, Western Carolina University.
23. *Mathematics for the consumer: an alternative to liberal arts mathematics*, by R.P. Webber, Longwood College.
24. *A mastery approach to developing skills in college algebra*, by H.T. Matthews, University of Tennessee, Knoxville.
25. *Operational and intuitive algebra*, by Donald Cook, Appalachian State University, and Betsey Whitman, Florida A&M University.
26. *Individualized mathematics courses at Shelby State Community College*, by Karen A. Bowyer, Shelby State Community College.
27. *A report on "One-on-One" mathematics*, by D.M. Jordan, University of South Carolina.
28. *Groups of square-free order, an algorithm*, by James Alonso, Bennett College.
29. *Quasi-injective S-systems and their s-endomorphism semigroup*, by A.M. Lopez, Jr. and J.K. Luedeman, Clemson University.
30. *A ring of infinite diagonal matrices*, by D.R. Stone, Georgia Southern College.
31. *The quotient ring of a semigroup ring*, by J.A. Bate and J.K. Luedeman, Clemson University.
32. *A commutative nonassociative algebra*, by Byung-Gu Han, East Tennessee State University.
33. *A condition equivalent to Frechet differentiability*, by G.W. Painter, Barber–Scotia College.
34. *Operator methods in finite difference equations*, by E.D. Huthnance, Newberry College.

35. *Strong bounded variation in topological vector spaces*, by T.W. Goodman, Catawba College.
36. *Two point estimates of a zero of* $f(z)$, by C.H. Frick, White Rock, South Carolina.
37. *Total sequences in a topological vector space*, by John Nichols, University of Tennessee, Knoxville.
38. *Countability in Urysohn's metrization theorem*, by Arthur Charlesworth, Queens College.
39. *Some remarks on circulant and block circulant matrices*, by R.L. Smith, Auburn University.
40. *Extensions of Blattner's theorem*, by Edward Blair, University of Tennessee, Knoxville.
41. *A linear algebra problem from algebraic coding theory*, by W.F. Lyle, III, Clemson University.
42. *A characterization of simple random variables by moments*, by R.M. Norton, The College of Charleston.
43. *From Pascal's triangle to a pyramid*, by S.C. Saxena, University of South Carolina, Coastal Carolina.
44. *Integral algorithms for computing zeros and fixed-points of* $f(x)$, by Shan Manickam, Western Carolina University.

J.D. NEFF, *Secretary-Treasurer*

## MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The fifty-sixth regular meeting of the Southern California Section of the MAA was held on 13 March 1976 at San Diego State University. The Chairman of the Section, Professor J.M. Hood of Occidental College, presided. The registered attendance was 86.

At the business meeting, the results of the election of section officers for 1976–77 were announced as follows: Chairman, Paul Yale, Pomona College; 1st Vice-Chairman, John Todd, California Institute of Technology; 2nd Vice-Chairman, Alice Huffman, California State Polytechnic University, Pomona; Secretary-Treasurer, Edmund Deaton, San Diego State University; Program Chairman, R.F. Brown, University of California, Los Angeles. The luncheon speaker was Professor Einar Hille, University of California, San Diego; his talk was entitled "Rambling Thoughts."

The following program was presented:

*The calculus of set-valued derivatives*, by Hubert Halkin, UCSD.

*Panel Discussion*: *The mathematical training of students interested in the social and biological sciences.* Robert Pendleton, Whittier College (Chairman); Harry Cromer, SD Evening College; Frank Flanigan, SDSU; Larry Gardner, Southwestern College; Rodolfo DeSapio, UCLA.

*Teacher Certification and the Ryan Act. The Elementary Program*, by John McGhee, CSU Northridge and UCLA.

*Teacher Certification and the Ryan Act. The Secondary Program*, by Paul White, USC.

*Struggling with the Bieberbach conjecture*, by Carl FitzGerald, UCSD.

*Homogeneous quadratic systems of differential equations in the plane*, by Frank Farris, Jr., Pomona College.

*The mathematics of gas exchange of asthma*, by John Evans, UCSD.

*Differential geometry and mathematical geodesy*, by Nathaniel Grossman, UCLA.

JOHN GREEVER, *Secretary-Treasurer*

## NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association representing the Sections indicated:

| | |
|---|---|
| KANSAS | Richard E. Shermoen, Washburn University |
| MISSOURI | Charles J. Stuth, Stephens College |
| NEW JERSEY | Eileen Poiani, St. Peters College |
| NORTHEASTERN | Donald L. Kreider, Dartmouth College |
| OHIO | Robert L. Wilson, Ohio Wesleyan University |
| PACIFIC NORTHWEST | John Reay, Western Washington State College |
| SEAWAY | Erik Hemmingsen, Syracuse University |
| SOUTHEASTERN | Trevor Evans, Emory University |
| SOUTHWESTERN | James E. Nymann, University of Texas at El Paso |

The highest percentage of voters was 36%, occurring in the Missouri Section. The Kansas Section was the runner-up with 30%.

A.B. WILLCOX, *Executive Director*

## CALENDAR OF FUTURE MEETINGS

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

Fifty-seventh Summer Meeting, University of Washington, August 14–16, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, St. Francis College, Loretto, Pennsylvania, April 22–23, 1977.

FLORIDA, University of South Florida, Tampa, March 4–5, 1977.

ILLINOIS, Chicago Loop College, Chicago, May 6–7, 1977.

INDIANA, Manchester College, North Manchester, November 6, 1976.

INTERMOUNTAIN

IOWA, Drake University, Des Moines, April 22–23, 1977.

KANSAS, Tabor College, Hillsboro, March or April 1977.

KENTUCKY, early April. Deadline for papers 6 wks. bef. mtg.

LOUISIANA-MISSISSIPPI, University of New Orleans, Louisiana, February 25–26, 1977.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Virginia Commonwealth University, Richmond, Virginia, November 20, 1976.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Eastern Michigan University, Ypsilanti, May 6–7, 1977.

MISSOURI, University of Missouri, St. Louis, April 29–30, 1977.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 15–16, 1977.

NEW JERSEY, early November and early May.

NORTH CENTRAL, South Dakota State University, Brookings, October 22–23, 1976.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Marshall University, Huntington, West Virginia, October 22–23, 1976.

OKLAHOMA-ARKANSAS, Oral Roberts University, Tulsa, Oklahoma, April 1–2, 1977.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 wks. bef. mtg.

PHILADELPHIA, Montgomery County Community College, Blue Bell, Pennsylvania, November 20, 1976.

ROCKY MOUNTAIN, Metropolitan State College, Denver, Colorado, April 29–30, 1977.

SEAWAY, Broome County Community College, Binghamton, N.Y., November 5–6, 1976.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, Loyola Marymount University, Los Angeles, March 12, 1977.

SOUTHWESTERN, Phoenix College, Phoenix, Arizona, Spring 1977.

TEXAS, Baylor University, Waco, April 1–2, 1977.

WISCONSIN, University of Wisconsin, Oshkosh, Spring 1977.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 20–26, 1977.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, San Francisco, October 29–30, 1976.

AMERICAN MATHEMATICAL SOCIETY, St. Louis, Missouri, January 27–30, 1977.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of North Dakota, Grand Forks, June 13–16, 1977.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, Chase Park Plaza Hotel, St. Louis, Missouri, January 27–28, 1977.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, Hamilton, Ontario, June 2, 1977 (tentative date).

FIBONACCI ASSOCIATION, University of Santa Clara, October 16, 1976 (Fall Research Conference).

INSTITUTE OF MATHEMATICAL STATISTICS, Seattle, Washington, August 14–18, 1977.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cincinnati, Ohio, April 20–23, 1977.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami Beach, November 3–5, 1976.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Commodore Perry Hotel, Toledo, Ohio, November 11–13, 1976.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Georgia Institute of Technology, Atlanta, October 18–20, 1976.

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AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 8

CODEN: AMMYAE

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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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# MATHEMATICS AND SEX

JOHN ERNEST*

Our title might well evoke a smile. To a rationally minded person it would be hard to imagine two subjects which are less related. Yet it is one of those irrational features of our cultural and social structure that a person's sex is, and has been, very relevant to that individual's opportunities in the world of mathematics.

Some fifty years ago, one of the great mathematicians of all time, Emmy Noether (1882–1935), encountered opposition to her admission to the faculty at the University of Göttingen solely because of her sex. The distinguished mathematician David Hilbert, a professor at Göttingen, was incredulous: "Meine Herren, I do not see that the sex of the candidate is an argument against her admission as a Privatdozent. After all, the Senate is not a bathhouse." Emmy Noether was not admitted (at that time) to the all male faculty at Göttingen, although she delivered many series of advanced lectures on her work, which had to be announced under the name of Hilbert (cf. pp. 142–143 of **[62]** or **[39]**). (Numbers in brackets refer to the bibliography at the end of the paper.)

At the time this study was initiated (Fall 1973) the mathematics faculty at the University of California at Santa Barbara consisted of 32 members (all male) while the secretarial staff consisted of 6 members (all female). These curious figures served as our initial stimulus, causing us to wonder about the forces which could bring about such gross sex differences. While few could claim to be totally dispassionate in such matters, we began by trying to put aside our preconceptions and prejudices and to impartially examine these sex differences in mathematics as the curious phenomenon that it truly is. We hope the reader will attempt to do the same in reading this report, which is a compilation of the major findings of a freshmen (oops—freshpeople) seminar on women in mathematics, which met in the Fall quarter of 1973, and ran over into the Winter quarter of 1974, under the auspices of the Mathematics Department of the University of California at Santa Barbara.

Our first impulse was to concentrate on the question of discrimination in employment. Indeed this is the main emphasis of the Department of Health, Education and Welfare and hence also the main emphasis of most affirmative action programs on college campuses. We soon realized, however, that sexism at the professional level was just a continuation of pervasive patterns extending all the way down into the elementary grades. Thus there are really (at least) two basic issues: equal opportunity in employment and equal opportunity in mathematical education (training and counseling). Our curiosity became centered around the second one. Therefore, this report will contribute little to the question of employment discrimination, even though a study of women in mathematics is admittedly incomplete without investigation of this critical dimension. One invaluable (albeit irksome) aspect of the new affirmative action procedures mandated by HEW is the keeping of detailed records of every step in the employment process. (We hope other investigators will be able to make extensive use of this new source of data.) This is a prime example of the measuring process itself affecting the phenomenon being examined. The rules are in fact intended to make both administrators and faculty more sensitive and attentive to the basic requirements of fairness and objectivity in personnel decisions. Thus we may never fully know the extent to which past prejudicial attitudes and discriminatory recruitment procedures (overt or covert, conscious or unconscious) have produced the extreme sex differences prevalent in the mathematical community. While our study offers little new information on this vital question, we do give a detailed report (in our fifth section on the mathematics profession) of some of the recent ('75–'76) employment figures, which indicate that the proportion of

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* This paper is a report of an undergraduate research seminar whose members were: John Harris, Donna Hoffman, Cindy Hunt, Cindy Lasher, Denise Parent, Marilynn Payne, Judy Riggle, Jennifer Silver, Cindy Snow, David Wink.

faculty who are women, remains considerably below the proportion of mathematics Ph.D.s who are women.

In the last four decades about seven percent of the Ph.D.s in mathematics were earned by women [**45**]. This ratio of women earning Ph.D.s in mathematics is beginning to show an upturn, the current figure being about 10 percent [**5**]. Nevertheless, the proportion remains extraordinarily low. We thus became convinced that there is an urgent need for affirmative action at the academic level as well, in order to unblock the beginning of the pipeline. We have therefore elected to concentrate our attention on sex differences in mathematical education, beginning at a very early age (second grade) and following through to the research mathematician. We have also been concerned (see section 3 below) with the effect of these sex differences in mathematical training on the participation of women in many other scientific and technological fields requiring some degree of mathematical sophistication. Our studies confirm the hypothesis of the sociologist Lucy Sells [**64**], [**65**], that mathematics is a "critical filter" tending to eliminate women from many fields, from chemistry, physics and engineering, to architecture and medicine. This conclusion lends greater import and urgency to this study, and to the need for effective changes in the mathematical training and counseling of women.

The first section of this report deals primarily with student attitudes from the second grade through high school. Our second section examines the attitudes of teachers. Our third section examines sex differences in the extent of mathematical training in the high schools. Here we identify the inadequate mathematics training for women (as compared to that for men) as a major culprit in closing off scientific and technological opportunities for women. In section 4 we go on to examine the situation at the university level, both undergraduate and graduate. Here our results are somewhat special, describing the situation only for the UCSB campus, and then for a rather limited time period. Nevertheless we feel these results are likely to be indicative of sex differences to be found in the mathematics programs of most universities. In section 5 we return for a closer look at the mathematics profession. We conclude, in section 6, with some observations and recommendations.

**1. Student attitudes in elementary and secondary school.** To obtain some data on this subject, we constructed a simple questionnaire consisting of three questions. The first question was designed to determine which subjects students liked best (and least) and to discover whether there were statistically significant sex differences in these preference patterns. (Here we obtained one of our most unexpected results.) The second question attempted to uncover possible sexist attitudes in the family by finding out whether the person (mother or father) who helped with the homework varied from subject to subject. Here we obtained a rather striking, albeit not unexpected, result. Our third question examined peer group attitudes (who does better — boys or girls? — in various subjects). In the case of all three questions we obtained statistically significant results (significance level $< .05$).

We had a total sample of 1324 questionnaires, distributed near the end of 1973, among grades 2 through 12. Most of the sample came from schools in Southern California, although one portion of the sample came from the East Coast. (The geographical differences were not significant.) We had more than 100 questionnaires for each grade except the fifth, where we had 81.

In the first question we asked the students to rank the four subjects, mathematics, English, science, and social studies (mathematics was replaced by arithmetic and English by reading, in the lower grades) on the basis of which they liked the most to the one which they liked the least. There were statistically significant sex differences in the three subjects, English, science and social studies (*very* striking in the case of English and science, but in all three cases the significance level was less than .01). As one might expect, the boys tended to prefer science and the girls English. Social science got a somewhat better rating from the boys than the girls. What we found totally unexpected — contradicting our preconceptions when designing this question — is that in terms of *liking* the subject, mathematics was the only subject which exhibited no sex differences. In fact the preference patterns in mathematics were almost identical for boys and girls, in marked contrast to the three other subjects. (For the statistics buff we mention that we computed the $\chi^2$-statistic for the data contingency table

associated with the following table. Assuming no sex differences, the probability of obtaining a $\chi^2$-statistic that small (or smaller) is .09, indicating that the preference patterns in mathematics of men and women are remarkably similar.)

PREFERENCE PATTERN FOR MATHEMATICS

| Rating for Mathematics | Boys | Girls |
|---|---|---|
| Liked best | 30% | 29% |
| Liked 2nd best | 23% | 24% |
| Liked 3rd best | 19% | 18% |
| Liked least | 27% | 29% |

This pattern of no sex difference in mathematics preference pretty much held up through the grades, although for both boys and girls its popularity went down in the high school years. While this result really surprised us, we later discovered that the same outcome (for the eighth grade) was obtained by L. R. Aiken [2]. The finding indicates that there is nothing intrinsic in arithmetic or mathematics that makes it more appealing or enjoyable to one sex than the other. This is a hopeful sign as other findings, to be discussed later, show that, nevertheless, when mathematics becomes optional (in high school and college) far fewer women take it. We would conjecture that men take more mathematics not for the superficial reason that they like mathematics more than women, but because, whether they like it or not, they are aware that such courses are necessary prerequisites to the kinds of future occupations, in medicine, technology or science, they envision for themselves. If this conjecture is true, (and it does receive some confirmation in the studies of Lynn Fox [24] and Elizabeth Haven [30]) then we expect that an upgrading of the counseling program in high schools, designed to make women aware of the full range of career opportunities and of the courses they must take in high school in order to keep all their options open, could have a very significant effect (cf. [65]).

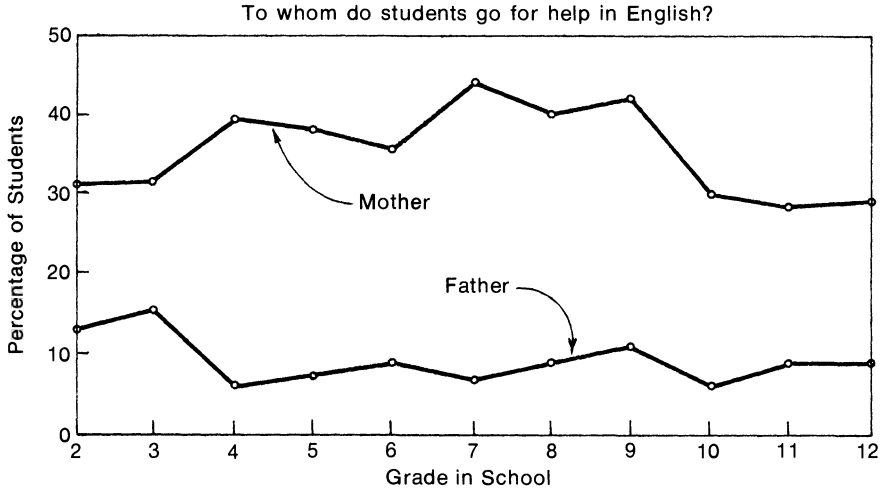
Our second question to the school children concerned whom they got help from in the various subjects. Here there were no statistically significant sex differences (i.e., between boys' and girls' responses). For both girls and boys it is true that the mother helps more than the father, except in the higher grades, where the father helps more in mathematics (and, to a lesser extent, in science). The following two graphs, indicating this pattern through the grades, for English and mathematics, are quite striking. Beginning in the sixth grade the father becomes the "authority" on mathematics and continues this role through high school. This fact alone must have its subtle influence on a young girl's (or boy's) attitude. (See graphs on page 598.)

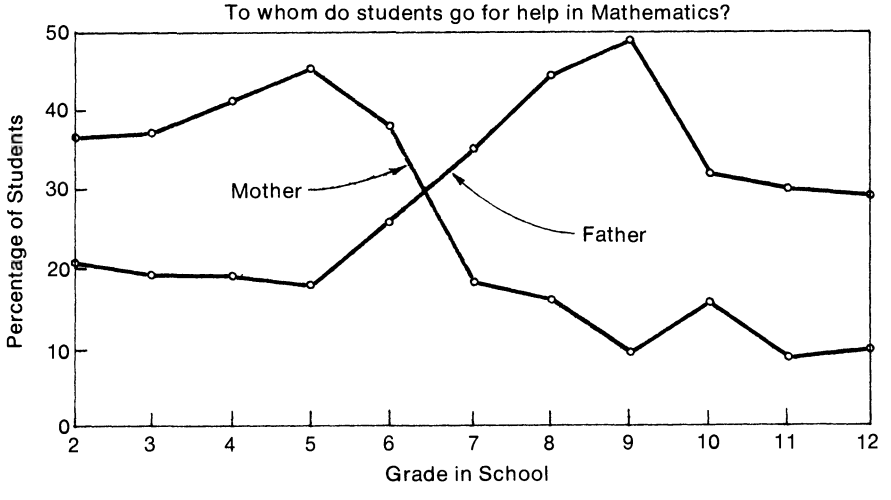
Martha Smith, a mathematician at the University of Texas has written us:

> I once heard a sociological theory to the effect that the crucial factor in whether a young woman became an "academic achiever" was her father's attitude towards her — the attitude most conducive to producing an academic achiever being one which showed approval both of his daughter's "feminine" pursuits and of her intellectual ones. Reflection on my own situation and that of friends who are women and mathematicians doesn't cause me to doubt this theory — e.g., my father seemed equally pleased when I baked an apple pie, accomplished some academic achievement, or helped him saw wood.

Other women mathematicians have also referred, in correspondence with us, to the positive influence of their fathers on their intellectual development. Further, Ravenna Helson [31], in a study of highly creative women mathematicians, found that most (2/3) had professional men as fathers and that interviewers judged that they had identified primarily with their fathers.

Of course it is hardly a new psychological insight that our parents have a tremendous influence on our personality development. (L. R. Aiken [2] also found that a child's attitude towards mathematics as strongly influenced by his or her parents.) Yet we feel still more must be done to make parents aware of the great danger of imposing, consciously or unconsciously, sexist roles on their children and the importance of offering their children encouragement and support to aspire to whatever they truly want to be.




The third question in our school questionnaire attempted to examine peer group attitudes by asking the students who they thought did better (boys or girls) in various subjects. In the elementary grades there were very substantial differences between the responses of the boys and the girls. Due undoubtedly to a kind of sexual competition fostered in the lower grades, the boys felt that boys do better — in all subjects and the girls felt that girls do better — in all subjects. However, by high school these sharp differences, while still statistically significant, become muted and there emerges an overall peer group attitude concerning competence in various subjects. If we take the top four grades, 9 through 12, together (sample size 506) we get the following table. The 32% in the table means that 32% of the sample (boys and girls together) feel that boys do better than girls in mathematics.

| | Math | English | Science | Social Studies |
|---|---|---|---|---|
| Boys do better | 32% | 4% | 47% | 21% |
| Girls do better | 16% | 52% | 5% | 15% |
| no difference | 52% | 44% | 48% | 64% |

We have recently found out that the Stanford Center for Research and Development in Teaching, about the same time we did this study in Southern California, distributed a questionnaire to 1886 high school students in the San Francisco area (cf. [14]). Many of the questions applied to each of the four specific fields of study, mathematics, English, social studies and business/vocational courses. While the primary aim was to discover the roots of student failure, some interesting sex differences relating to student attitudes towards mathematics were uncovered. We quote from an expository article on this study by the sociologist Sanford Dornbusch [14].

> One of our questions had asked the student: when you get a poor grade, which reason do you think usually causes the poor grade? There were four alternative answers: I had bad luck, I didn't work hard enough, the teacher didn't like me, and I'm not good at this subject. Most students gave lack of effort as the reason for receiving a poor grade in every subject. However when it came to math, 26 percent of the females gave lack of ability as the basis for a poor grade as compared to 15 percent of the males. Female students in every ethnic group in San Francisco were more than three times as likely to give, "I'm not good in math" as the basis for a poor grade as "I'm good in math" as the basis for a good grade. In the suburbs, the pattern was similar but the ratio was two-to-one. This pattern was found in no other subject for females and in no subject for males.

Elizabeth Fennema and Julia Sherman [21] have studied how sexual stereotyping affects attitudes and proficiency in mathematics. They conclude: "there is, then, an accumulation of evidence which points to the conclusion that sexual stereotyping of mathematics as a male domain operates through a myriad of subtle influences from peer to parent and within the girl herself to eventuate in the fulfillment of the stereotyped expectation of a 'female head that's not much for figures'."

This stereotype, as well as the attitudinal patterns of students (both self-image and peer group) discussed earlier, are unfortunately reinforced by the expectations of teachers. Hence the topic of our next section.

**2. Teacher attitudes.** We next asked two questions of a small sample of elementary and high school teachers (24 women and 3 men). For these teachers at least, mathematics was liked better than the other subjects. Forty-four percent of the teachers listed it as their favorite subject to teach (from a list of four, mathematics, science, English and social studies) while another third of them listed it as their second most liked subject. Nevertheless, their responses to a second question indicated attitudes (concerning sex differences in aptitude and performance with respect to particular subjects) very similar to that obtained for the peer group. Forty-one percent of the respondents felt boys did better in science while only one felt girls did better in science. Almost two–thirds (63%) of the respondents felt girls did better in English, while no one felt boys did better. Finally 41% felt boys did better in mathematics while no one felt girls did better.

Given this prevalent attitude (that boys do better in mathematics) among a substantial portion of both peers and teachers, perhaps we should digress here to examine the evidence concerning this question. We tested for sex differences in performance (i.e., class grades) in a number of large elementary mathematics courses at UCSB and in no case did we find any statistically significant sex differences.

As a glance at the bibliography will indicate the literature on sex differences in mathematical ability is enormous. The topic has fascinated many people, including teachers, educational researchers and psychologists. Yet the results are very mixed and often confusing. There is no consensus as to what "mathematical ability" is. Some researchers measure computational proficiency, others geometric or algebraic aptitude while the psychologists will usually measure some more precise and specific intellectual function. J. E. Garai and A. Scheinfeld [28] give a survey of pre–1960 studies which tend to confirm that boys are more proficient in mathematics. For example, in 1942 a study was conducted involving a sample of 50,000 students from some 300 schools, where the authors concluded: "Girls have maintained a consistent and, on the whole, significant superiority over boys in the subject tests, save in arithmetic, where small, insignificant gains favor the boys" [71]. However, the more recent work (since 1960) has been surveyed by Elizabeth Fennema and no such clear picture emerges. For example, one study of third and sixth graders indicates that girls are in fact better at arithmetic [80]

while another [3] finds no significant sex differences in problem solving ability among seventh graders. Perhaps the most controversial paper is that of Richard Stafford [70] who claims "it appears that in general there is an underlying hereditary component for a proficiency in quantitative reasoning which fits the sex-linked recessive model fairly well." This hypothesis has received little acceptance (e.g., [20] and [36]). Most studies emphasize cultural and environmental influences rather than intrinsic biological differences (for example [21], [61] and [74]).

The findings on sex differences in intellectual functioning which were produced by psychologists are the ones most carefully defined and consistently replicable (cf. [36], [49], [64], [68], [69], [77]). Of particular interest are the results which show that males perform slightly better in certain spatial visualization tasks. For example, studies indicate that men are more likely to be "field independent" than women (see pp. 21–26 of [68]). Field independence means the perceiving of items independently from their backgrounds. It is an ability to overcome the influence of the embedding context. For example, in a rod and frame test the subject is asked to place a rod vertically in a frame which gives misleading background cues. In this and similar tasks, males have a measurable superiority. Since this is obviously a very specialized intellectual function, it would seem quite misleading to refer to this field independence ability as an "analytic cognitive approach" or "analytical ability," as it is often referred to in the literature. We feel that the attempts at interpreting these specialized results of psychology in terms of general mathematical ability are highly speculative and go far beyond that which is justified by the data. Furthermore, as Fennema and Sherman point out [21], the development of spatial visualization skills may be closely related to sexual stereotypes and hence may be largely the result of cultural conditioning. These considerations have implications for school curricula. Indeed Sheila Tobias [73] has suggested that exercises involving spatial relations and field independence be made part of the elementary school curriculum, particularly for girls.

Almost all the pertinent educational research investigated proficiency and aptitude for school mathematics, usually at a very elementary level. The studies dealt with arithmetic or various kinds of problem solving, and did not attempt to measure creativity and imagination in modern, highly abstract mathematics. (However, see [31] and [32].) I do not think that one can extrapolate these educational and psychological findings to the level of mathematics done by the research mathematician. Creative mathematical ability is a rich mixture of interrelated components, involving motivation, intelligence, imagination and many skills, including computational, spatial *and verbal* skills. I personally feel that modern mathematical development suffers from the defective formulation and articulation of the conceptual structure underlying mathematical research and that this state of affairs is not unrelated to the limited female participation in the creation of new mathematics (cf. [1]).

Returning to the case of school mathematics, where male performance is found to be superior, we may be observing the so-called "Pygmalion effect" in education, according to which the student performs, to some (measurable) extent, in response to the expectations of the teacher. In our small survey for example, almost half the teachers expect their male students to do better in mathematics, while *none* of them expect the female students to do better.

Of course the various sex differences in intellectual functioning, such as space perception discussed above, could be related to aptitude in mathematics. These differences, however, are subtle, being far smaller than differences among individuals of either sex. These minor differences (which may themselves be culturally induced and thus modifiable) might explain some small difference in the percentage of women mathematicians. However, we have found nothing in these psychological studies which helps us to explain the enormous sex differences actually found in the mathematics profession.

There is a mathematics course at UCSB designed specifically for future school teachers. We distributed special questionnaires to such a class consisting of 11 men and 64 women. We were discouraged to find that 26% of these future teachers indicated they are indifferent towards mathematics, while another 14% stated they actually dislike or hate it. Thus 40% of these prospective teachers are likely to transmit something less than a positive attitude towards mathematics to their students.

We distributed a similar questionnaire to a number of the large lower division undergraduate courses asking the students to indicate their attitude towards mathematics by marking one of the five possibilities (love it, like it, indifferent to it, dislike it, hate it). No statistically significant sex differences were found in the responses. We also asked the students to indicate what they felt were the major influences determining their attitude. Among those indicating an extreme attitude (either loving it or hating it) one of the most mentioned factors was a particular teacher they had had in their prior schooling. It is largely for this reason that we were so distressed to find that 40% of the mathematics class for future teachers indicated either a negative or indifferent attitude towards mathematics.

These findings indicate to us that certain adjustments should be considered, both in the training of school teachers and in their employment in the classroom. We need instructors who are competent in mathematics, who love the subject, who enjoy teaching it, and who will not project sexist expectations on the students. Our small sample of 75 prospective teachers indicates that about one out of seven either dislike or hate mathematics. This makes it quite likely that an elementary school child will at some point be influenced in a detrimental way by one of his or her teachers. We feel the child should be protected from this possibility. Perhaps those teachers who feel uncomfortable with certain mathematics subject matter and do not enjoy teaching it, should be able to enlist the aid of eager and competent specialists, much as music teachers are currently used. Such specialists would be carefully trained and highly sensitive to the prevailing sexist attitudes concerning mathematics. In particular they would encourage girls to enjoy and to excel in mathematics. If most of these mathematical specialists were women, we would have the added advantage that the female students would be presented with a positive role model.

**3. The inadequate mathematical training of women in high school.** In a perceptive article [**63**] entitled *Women in Science: why so few?* Alice Rossi has described the many social and psychological influences which restrict women's choice of careers in science. (Cf. also [**79**].) In this section we shall describe another factor which restricts women's choices which, with some effort, may be more quickly corrected than the deep rooted causes of which Alice Rossi speaks. Whether through inadequate or even misleading counseling, or by personal choice, women simply are not taking enough of the optional mathematics courses offered in high school to prepare them to enter college programs in science, engineering or other "hard" disciplines. Without unusual and early efforts to make up this deficiency in their first year of college, all of these career options become effectively closed.

The sociologist Lucy Sells (cf. [**64**] and [**65**]) shared with us a fact sheet (dated December 15, 1973) which she developed on women in higher education at Berkeley, and we quote here two of her points.

1. In a systematic random sample of freshmen admitted at Berkeley in Fall, 1972, 57% of the boys had taken four full years of mathematics, including the trigonometry–solid geometry sequence, compared with 8% of the girls. The four year mathematics sequence is required for admission to Mathematics 1A, which in turn is required for majoring in every field at the University *except* the traditionally female, and hence lower paying, fields of humanities, social sciences, librarianship, social welfare, and education.

2. Among students earning the Bachelor's degree in the 21 largest letters and science departments, there is a strong and statistically significant relationship between having a one-year college mathematics requirement in the curriculum, and having less than one third of the degrees in the department earned by women.

This second result clearly underlines the tragic consequences of the incredible sex difference uncovered in the first finding. It is for this reason that Lucy Sells refers to mathematics as a "critical filter" in cutting down the percentage of women in many fields other than mathematics. Thus substantial changes in student and teacher attitudes towards girls taking mathematics, and particularly changes in the academic counseling procedures at the high school level, are likely to have a significant effect on the percentage of women majoring in disciplines which use mathematics as well as in mathematics itself (cf. [**24**]).

There is additional evidence that mathematics plays the role of a critical filter. For the period 1972–75, the percentage of Ph.D.s earned by women is below ten percent for the following fields: geography, astronomy, economics, mathematics, religion, computer science, applied mathematics,

geology, agriculture, atmospheric science, business administration, physics, engineering and operations research [54]. The only subject that does not fit the hypothesis, religion, is a very special case, since most ministers are men. In contrast, women received more than 25% of the doctorates (in the same period) in each of the following fields: home economics, art history, Romance languages, Germanic languages, comparative literature, social work; health sciences, English, speech, psychology, anthropology, library science, linguistics, education, classics, microbiology and sociology.

We found the result (1. above) of Lucy Sells' concerning the inadequate mathematical training of entering women students at Berkeley so remarkable that we decided to repeat the experiment at UCSB. We took a random sample of the files of freshmen (50 men and 50 women) who had been admitted to UCSB in the fall of 1973. Perhaps because this was done a year later (the situation is in great flux) and because UCSB is a rather different campus than Berkeley, we obtained somewhat different results. The percentage of entering men with four years of high school mathematics was considerably smaller (36%) while the percentage of entering women with four years of high school mathematics was somewhat larger (16%). Nevertheless, the sex difference is still enormous (and statistically significant) and thus confirms Lucy Sells' findings at Berkeley. Other voluntary (and hence non-random) samples taken at UCSB also showed a significant sex difference in high school mathematical training. (We conjecture that these kinds of sex differences prevail at most schools throughout this country — and even the world [35], [58].)

We have recently learned that IBM has funded a Mathematics Association of America proposal for a secondary school lectureship program, designed to interest more women in mathematics. This program will be administered by the Association's Committee on Secondary School Lecturers. Under this grant, women mathematicians will visit secondary schools in the New York–New Jersey, Chicago, and San Francisco Bay areas. To the usual aims of a secondary school lectureship program, this program will add the very important dimensions:

1. To interest girls in studying mathematics by providing a role model.

2. To impress advising and guidance personnel with the importance of mathematics for girls and the career opportunities for women scientists and mathematicians. Included among such personnel are the professional counselors, State curriculum consultants, Career Day organizers, and parents. In order to have the widest possible impact on guidance personnel, women mathematicians participating in this program shall also be made available to address mathematics teachers at workshops and district meetings, spend some time talking with appropriate personnel at the State level in the various Departments of Education, take part in school Career Days, and possibly even in PTA meetings.

We believe that efforts like this will be most helpful in changing some of the patterns described in this section.

The relevance of mathematics to many career options outside of mathematics has also been observed by the Carnegie Commission on Higher Education in its recent report [11] on opportunities for women in higher education. They state (in referring to women on page 64 of the report): "Not only have they preferred fields that lead to traditionally female professions, but they also tended to avoid fields requiring extensive application of mathematical reasoning." We concur completely with the following Carnegie Commission recommendation:

> Recommendation 7 (page 79 of [11]): Because of the evidence that many women enter college with inadequate mathematical training, special provision should be made to ensure that women desiring to major in fields calling for extensive use of mathematics are encouraged to make up this deficiency in order to enter the fields of their choice.

Specifically we recommend that every college and university offer, *for credit*, appropriate pre-calculus courses. This brings us to the next stage of our investigation, sex differences in mathematical education after high school.

**4. Sex differences in mathematical training at the university level.** We began by looking for sex differences in the elementary calculus sequence, beginning in the fall of 1971 and extending to the

winter quarter of 1973, at UCSB. Math 34AB is the short (two quarter) course while Math 3ABC, 4AB, is the long (five quarter) course. (This long course branches in the second year. The figures for 4A include two other parallel sections, 5A and 4AH. The figures for 4B include 5B.)

We first note that the enrollment of women in these basic courses is disproportionately low. Women comprised only about a third of the class, even though women were in the majority in the freshmen class of 1971. This undoubtedly is one of the inevitable corollaries of the deficient mathematical training women receive in high school, which we described in the previous section. Nevertheless, we feel that improved counseling and other remedial aids at the college level can significantly improve these enrollment figures.

ENROLLMENT, FALL 1971

| | Freshman Class | Math 34A | Math 3A |
|---|---|---|---|
| Male | 1001 (45%) | 133 (63%) | 328 (64%) |
| Female | 1228 (55%) | 78 (37%) | 184 (36%) |

We found no statistically significant sex differences in the grades achieved in the various courses. The Carnegie Commission ([11], pp. 50–51) has found that women tend to get better grades in college — in all fields. Nevertheless, in *every one* of the five places (one in the short sequence, four in the long sequence) where a student has an opportunity to drop, the attrition rate was greater for women than men. The attrition rates for 34A → 34B, 3C → 4A and 4A → 4B were particularly large, with the women's attrition rate almost double that of men. This is particularly disconcerting as the original enrollment of women was already disproportionately low. These rather large attrition rates are difficult to interpret as various majors require only certain portions of the complete sequence.

ATTRITION RATES FOR CALCULUS SEQUENCES, BEGINNING FALL 1971

| Course | Male | Female |
|---|---|---|
| 34A → 34B | 33% | 58% |
| 3A → 3B | 33% | 37% |
| 3B → 3C | 26% | 35% |
| 3C → 4A | 30% | 51% |
| 4A → 4B | 22% | 41% |
| 3A → 3C | 51% | 59% |
| 3A → 4A | 66% | 80% |
| 3A → 4B | 73% | 88% |

The above table is read as follows: 33% of the men who completed 34A did not complete 34B. We did not actually follow each individual student through the sequence and hence the figures are subject to considerable error due to students transferring to other schools, and other students joining the sequence at a later stage.

The enrollment of women in the honors calculus section (which begins with the second quarter) is considerably lower than in the regular sections. While women comprise about 35% of the second quarter calculus enrollment, they comprise (as an average over the last four years) only about 20% of the honors section enrollment. The author taught this honors section in the winter of 1971 and there were nine men and no women enrolled. The following year the author was again scheduled to teach the honors calculus. This time he visited all the regular sections, described the nature of the honors section, and specifically encouraged women to enroll. In the winter of 1972 the enrollment was again nine students, but this time there were six men and three women. We strongly recommend that every mathematics department look into their honors sequence and determine the percentage of female

participation. We recently received a letter from Professor Joan Birman telling us of her similar experiences in the honors sequence at Barnard–Columbia (in alphabetical order).

> I learned last year, to my astonishment, that for about four years running the honors calculus course had been all male, in spite of the fact that admission was based on an open competitive examination. This fall, one of the senior mathematics majors and myself made an intensive effort to encourage women to *try the exam*! The typical answer was, "I know I won't pass it," — to which we replied over and over, "Well, if you try it, at worst you will confirm what you already know, and only an hour of time will have been lost." After three days of such advising, the big day came, the exam was given, and this year the class has five men and five women!

These experiences indicate that we can anticipate considerable change in the enrollment of women in mathematics courses, if only a greater effort is made to encourage them to do so. A tiny survey was conducted, at Stanford, of women majoring in natural sciences, mathematics and engineering (cf. [14]). These women reported having received less encouragement to study mathematics than did any group of Stanford males, even those males who were majoring in history or the humanities. Numerous other studies confirm Joan Birman's experience, namely that a little encouragement can make a tremendous difference in female participation and performance levels in mathematics. (Cf. [8], [9], [10], [23], [30], [43], [50], [61], [73].)

We next examine the attrition of mathematics majors. The Carnegie Commission [11] has made the following observation:

> When we consider the fields in which students major in college, we shall find that almost as large a percentage of women as of men receive bachelor's degrees in mathematical science, although women are considerably less well represented than men among recipients of degrees in fields calling for the use of mathematics as a tool, such as the natural sciences, engineering and economics.

The statistics on mathematics majors at UCSB do not seem to follow this national pattern. We have followed the class of '72 and the class of '73 and found that women got less than half as many degrees in mathematics as men. Here again we find rather heavy attrition. For example, when the class of '72 began in the fall of '68, about the same percentage of men and women indicated their desire to major in mathematics. (70 women and 63 men indicated their intention to major in mathematics. This represents 4.4% of the women in the class and 4.5% of the men.) These percentages continually decline in the following years, the attrition being higher for women than for men. Finally at graduation time there were only 20 women mathematics majors (1.8% of the women graduates) and 48 men mathematics majors (3.1% of the men graduates). For the class of '73 the pattern is the same. Only 16 women (1.5% of the women graduates) and 45 men (2.8% of the men graduates) received bachelor's degrees in mathematics. Of course the attrition figures may be partially caused by students transferring to other campuses of the UC system.

We conclude that it is very much in the university's own interest to invest in subsidized child care centers, better academic and career counseling, and more flexible degree programs (among other things being recommended by various women's groups and the Carnegie Foundation [11]) to try to decrease this sizable attrition among women.

In going from the bachelor's degree into a graduate program we once again find that attrition takes a higher toll for women than for men. (At this point we may marvel that there are as many women mathematicians as there are!) For example Lucy Sells [64] has found that "there is a statistically significant drop in the proportion of women earning the bachelors degree in the physical sciences at Berkeley (26%) and the proportion of women applying to graduate school in the physical sciences (11%)." In the fall of 1972, 15% of the applicants to the graduate program in mathematics at UCSB were women, while in 1973 it was 16%. Averaged over these two admission periods, the acceptance rate was 78% for the men and 88% for the women.

We feel an effort can and should be made to increase the proportion of women in both the undergraduate and graduate mathematics program. For example, women students at MIT compiled an excellent booklet [53] about the school's opportunities for women and mailed it to 10,000 female high school juniors. This resulted in 1400 inquiries from women about admission to MIT, nearly four times as many as the year before. The engineering department at Berkeley has also used a special

recruiting program, with some success. At Mills College, an all women's school in Oakland, California, Lenore Blum found that the mathematics enrollment there nearly doubled after she made a concerted effort to encourage women to enter science, including the development of special programs, counseling and publicity [8].

Considering the importance of encouragement, the dearth of women faculty in mathematics, physics, chemistry and engineering can only have an inhibiting affect on female enrollments in these subjects. The mathematician Martha K. Smith has written us concerning this matter:

> Role models play an often neglected but, to my mind, important role in education. If a young girl never encounters a woman in mathematics, it is quite reasonable that she should conclude that mathematics isn't something women do (whether for lack of ability or lack of opportunity). Perhaps more likely, she may not even think of the possibility of a woman mathematician.
>
> I know I felt a certain amount of relief when, the summer after my first year in graduate school, I met a woman mathematician for the first time. There was living evidence that what I wanted to do was not impossible.
>
> To reverse these stereotypes won't be easy, but I think it's possible. An effort should be made to get all the trained women mathematicians possible into positions where they will have an impact on both male and female students and colleagues. Women need to see examples of practicing mathematicians; men need to become accustomed to accepting a woman as an honest-to-goodness colleague rather than a curiosity.

In regard to role models, we recommend that books like [13], [40], [42], [43], [59], and articles like [12], [39], [41], [57], [75], [78] be available in every mathematics reading room, both undergraduate and graduate.

**5. Women in the mathematics profession.** We have been most fortunate that a large number of women mathematicians have corresponded and shared with us their experiences, attitudes and insights. They have enabled us to go beyond statistical averages to see the complexity and variability of human personality and creativity. Women, just like men, have various reasons for being attracted to mathematics, few of them having anything to do with their sex. They also hold diverse opinions regarding the various matters considered in this report. Their suggestions have done much to shape our study. Their ideas permeate this report. We therefore wish to express our deep appreciation to the following women mathematicians who have consulted with us: Joan Birman, Judy Bruckner, Mary Gray, Susan Gerstein, Mary Elizabeth Hamstrom, Eleanor G. Jones, Nancy Kopell, Tilla Milnor, Cathleen Morawetz, Alice Schafer, Marianne Smith, Martha K. Smith, Ann Stehney, Diane Stuebing, Olga Taussky Todd, Karen Uhlenbeck.

Just over half of our respondents indicated they had experienced some form of discrimination in their professional lives. Essentially all of them indicated they had experienced sexist attitudes of some form.

Professor Martha Smith (whom we have already quoted) has put forth the idea that many women are dissuaded from a mathematical career by societal stereotypes:

> Many people on hearing the words "female mathematician" conjure up an image of a six-foot, gray-haired, tweed suited oxford clad woman... This image, of course, doesn't attract the young woman who is continually being bombarded with messages, direct and indirect, to be beautiful, "feminine" and to catch a man.

We have found the notion that women who excel in mathematics are less feminine to be utter nonsense. For example, in Ravenna Helson's study [31] of creative women mathematicians, her findings did not "show the creative woman to be more masculine, if one means by this that they might have been expected to score higher on measures of masculinity–femininity, or dominance, assertiveness, or analytical ability." In another (older) study [44] involving undergraduate students at UCLA Philip Lambert found that among women students "mathematics majors were not only equal to non-mathematics majors in feminity, but significantly more feminine." This finding was sufficiently surprising that Lambert took a second sample, which confirmed the result.

Joan Birman, a professor at Columbia University was influenced earlier in her life by the "unfeminine" stereotype. "Why didn't I study mathematics at age 21? I felt it was not a 'feminine'

thing to do. I'm afraid that it seems to me that this is a continuing problem for many young women." As we have just noted, studies indicate that, if anything, women mathematicians tend to be more feminine than the average. We thus feel it is time for society to completely do away with the ridiculous prejudice that a female mathematician is somehow less of a woman because of her intellectual pursuits.

When discussing the stereotype of a female mathematician, colleagues sometimes refer to Emmy Noether. (Cf. [13], [39], [75], and [78].) Since she died the same year I was born I have no first hand knowledge of her traits. In making her way to the top of a male dominated profession, she may not have met everyone's expectations of the way a female should look and behave. Hermann Weyl described Emmy Noether quite candidly in the eulogy he wrote for her: [59]

> It was only too easy for those who met her for the first time, or had no feeling for her creative power, to consider her queer and to make fun at her expense. She was heavy of build and loud of voice, and it was often not easy for one to get the floor in competition with her. She preached mightily, and not as the scribes. She was a rough and simple soul, but her heart was in the right place. Her frankness was never offensive in the least degree. In every day life she was most unassuming and utterly unselfish; she had a kind and friendly nature. Nevertheless, she enjoyed the recognition paid her; she could answer with a bashful smile like a young girl to whom one had whispered a compliment. No one could contend that the Graces had stood by her cradle; but if we in Göttingen often chaffingly referred to her as "der Noether" (with the masculine article), it was also done with a respectful recognition of her power as a creative thinker who seemed to have broken through the barrier of sex.

It was the Russian mathematician P. S. Alexandroff who had dubbed her "*der* Noether." Nevertheless, Alexandroff, who knew her well, spoke eloquently of her feminine characteristics, in an address to the Moscow Mathematical Society in 1935:

> A spokesman of the most abstract areas of mathematical science, she distinguished herself at the same time by a surprising sensitivity in understanding the great historical movements of our epoch; always vitally interested in politics, hating war with her whole being, and hating chauvinism in all its manifestations, she never in the area knew any vacillation.
>
> Such was Emmy Noether, the greatest of women mathematicians, a great scientist, an amazing teacher, and an unforgettable person. True, Weyl has said that "the Graces did not stand at her cradle," and he is right, if one has in mind the generally known heaviness of her appearance. But here Weyl is speaking of her not only as a great scholar, but also as a great woman. And she was that — her femininity appeared in that gentle and subtle lyricism which lay at the heart of the far-flung but never superficial concerns which she maintained for people, for her profession, and for the interests of all mankind. She loved people, science, life, with all the warmth, all the cheerfulness, all the unselfishness, and all the tenderness of which a deeply sensitive — and feminine — soul is capable [39].

This problem of the "unfeminine" stereotype of the female mathematician is only one of many faced by the woman considering mathematics as a career. Professor Joan Birman has discussed other misunderstandings with us:

> The female students I know also seem to often misunderstand the nature of mathematics. It appeals to them (I think) because they are systematic, neat, logical and orderly. They do not seem to appreciate the creative, imaginative, esthetic aspect of mathematics, and indeed are often bewildered by it when they first encounter it in advanced courses. I hope this is something which is changing, as more young women become aware of themselves as questioning, growing, thinking people. Conformity is really deadly for mathematics, because it is not possible to learn the subject unless you are constantly questioning whether you understand it. To become a mathematician it is also necessary to a) have a good deal of tolerance for frustration, b) to not be distressed by the distrust or fear other people have of those who are "smart" and c) to not be afraid to be wrong or make mistakes — all of these seem to be problems for women.

There are additional accounts of the experiences of women in the mathematical profession. Sónja Kovalévsky [40], a nineteenth century Russian mathematician, Else Hoyrup [34], a Danish mathematician, and Marian Boykan Pour-El [43], a professor at the University of Minnesota, have all written about their professional development. Clearly, given the circumstances described in this report, women have had to have considerable courage and motivation, to become creative mathematicians. This was recognized by Karl Friedrich Gauss, who wrote the following words to the great French mathematician, Sophie Germain (1776–1831):

> But when a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without a doubt she must have the noblest courage, quite extraordinary talents and a superior genius [**59**].

In the past, women have had to suffer considerable employment discrimination. Even when Emmy Noether finally obtained the position of *nichtbeamteter ausserordentlicher Professor* at Göttingen in 1922, she still was not paid a salary. She received only a modest remuneration for some teaching in algebra [**59**]. Fortunately we are beginning to see some very healthy changes in the employment situation for women mathematicians. The nepotism rules, invariably discriminatory against the wife, are rapidly falling away. Under the watchful eye of the Department of Health, Education and Welfare, academic and industrial institutions are adopting widespread procedures to ensure that hiring patterns are free of sex bias. We strongly support such affirmative action to ensure that all potential female candidates are considered. Such increased recruitment efforts can only enlarge the list of qualified candidates and thus result in the *raising* of standards. To immediately dispose of a red herring, let us state emphatically that none of us believe a *less* qualified mathematician should be hired, just because she is female.

An examination of the latest employment figures for mathematics departments clearly establishes the need for affirmative action procedures. We have already noticed that at every rung of the ladder leading to the Ph.D. in mathematics, the attrition rate has been higher for women than for men. We will now see that that pattern continues for the professional success ladder beyond the Ph.D. During the past four decades about 7% of the mathematics Ph.D.s were earned by women [**45**]. The Carnegie Commission on Higher Education (p. 89 of [**11**]) has indicated that the percentage of Ph.D.s in mathematics has steadily declined over the years. The most recent data suggests we are beginning to see an upswing. In the period '69–'72, 7.3% of the Ph.D.s in mathematics were earned by women, while the corresponding figure for the period '72–'75 is 9.1% [**54**]. In the most recent survey [**5**], it was found that, for the academic year '74–'75, women earned 103 of the 1022 Ph.D.s in mathematics, which is just over ten percent.

While my own university (UCSB) now ('75–'76) has two women mathematicians on the faculty, they are both employed as temporary lecturers. Thus we still have no women on the regular ladder faculty. Our campus is not unique. The same situation (no women on the regular ladder faculty) prevails at other schools — University of Chicago, Princeton, Yale, Harvard, MIT, to mention a few of which the author is personally aware. (This statement refers to the 1975–1976 academic year. By regular ladder faculty we mean tenured associate professors and full professors, and assistant professors who will be considered for tenure. It does not include instructors, lecturers, temporary and terminal positions or visiting professors.)

For all doctorate granting mathematics departments in the United States, women comprised 4.8% of the ladder faculty (174 out of 3614) in the 1975–76 academic year. This is up slightly from the previous year, when they represented 4.7% of the ladder faculty. The percentages become smaller when we restrict ourselves to the tenured faculty. Women comprised 4.5% (121 out of 2713) of the tenured faculty, which represents a slight *decrease* from the previous year's figure of 4.6% (120 out of 2620). Indeed, for all doctorate granting mathematics departments in the country, there was an increase of 93 tenured faculty members (over the previous year), but an increase of only one woman tenured faculty member. All of these statistics, and the ones to follow, are based on the results of the nineteenth annual American Mathematical Society survey [**5**], conducted by the Committee on Employment and Educational Policy. Of course the results are based only on the returns received. In the case of the above figures the return rate was two-thirds (104 out of 156).

If we examine the next rung of the success ladder and look at the figures only for the 27 most prestigious research departments, the percentages are almost halved. (Here there was a return rate of 20 out of 27 or 74%.) In the academic year '75–'76, 3 percent (27 out of 882) of the ladder faculty are women. This figure is up from the previous year (2.2%) due to an increase in the number of female

assistant professors. The percentage of women on the tenured faculty is still smaller, 14 out of 673, or 2%. This is in fact a slight *decrease* from the previous year when the percentage was 2.3% (15 out of 658). The attrition rate for women, which we have observed to be higher than for men at every rung of the success ladder, takes its final toll at the very top. At the most prestigious schools, women represent only 1.6% (8 out of 490) of the full professors.

The above figures are convincing arguments for the urgency of affirmative action at the employment level. If current patterns prevail, not only are women's expectations for success severely limited, but the mathematical, scientific and academic growth of our culture is also being restricted through the inadequate utilization of our human and intellectual resources. Nevertheless, we do not believe that affirmative action should concentrate exclusively at the employment level. We would be distressed if universities, in fear of losing federal funding, develop programs only in response to HEW threats and emphasize those paper–work procedures designed to protect them in case of HEW audits. We believe our study has established the great need for affirmative action at the academic level — from elementary school through college. It doesn't help much to carefully distribute the few drops from the end of a pipeline when the pipe is clogged at a much earlier point. If university administrators are sincere in their desire to open up opportunities for women (and we believe they are) then they must put far more resources into academic affirmative action, even if this is not where the federal government is putting on the pressure. There is no dearth of suggestions in this regard, from information and recruiting efforts at the high school level, subsidized child care centers, improved career counseling, more flexible requirements including external degree programs and part time study, to mention just a few. There are many such recommendations throughout this report. We shall discuss later some recent efforts and experiments to improve mathematical education for women.

We are pleased to observe that the "women's movement" is alive and well in the mathematics profession and is continuing to have a very beneficial effect (cf. **[6]**, **[29]**, **[34]**, **[51]** and **[52]** for example). More women are appearing as invited speakers at professional meetings. They are being nominated and elected to important positions in the professional societies. Scientific meetings often have special sessions concerned with women in mathematics (cf. **[22]**, **[25]**, **[29]**, **[32]**, **[47]**, **[50]**). The American Mathematical Society has recently compiled a directory of women mathematicians **[4]**. There is an active organization concerned with women's rights: The Association for Women in Mathematics (address: Department of Mathematics, Wellesley College, Wellesley, MA 02181). The pioneering work of Mary Gray (American University) in founding the Association for Women in Mathematics some five years ago, has recently been recognized by her election as the (first woman) Vice–President of the American Mathematical Society, the professional organization of research mathematicians. Mary Gray was the first president of AWM. The current president is Lenore Blum, of Mills College. The Association issues a regular and lively newsletter, jointly edited by Judith Roitman and Alice Schafer, both of Wellesley College. It also runs an employment information service (under the direction of Judy Green of Rutgers–Camden) which helps women become aware of job opportunities as they develop. We recommend this organization to any mathematician (male or female) who is concerned with the issues raised in this report.

**6. Conclusion.** We have tried to discuss our results and make some tentative conclusions and recommendations in each of the appropriate sections. The findings of all sections tend to support one basic recommendation of the Carnegie Commission on Higher Education (pp. 56–57 of **[11]**).

> RECOMMENDATION 1: The first priority in the nation's commitment to equal educational opportunity for women should be placed on changing policies in pre–elementary, elementary and secondary school programs that tend to deter women from aspiring to equality with men in their career goals. This will require vigorous pursuit of appropriate policies by state and local boards of education and implementation by school administrators, teachers and counselors.
>
> For example, high school counselors and teachers should encourage women who aspire to professional careers to choose appropriate educational programs. They should also encourage them to pursue mathematical studies throughout high school, because of the increasing importance of mathematics as a background, not

only in engineering and the natural sciences, but also in other fields, such as the social sciences and business administration.

The topic we have taken up is vast and has many ramifications. Our investigation should be looked upon as a pilot study. We hope that this preliminary data will suggest many interesting projects to educators, sociologists and psychologists. We fully recognize that we are trying to measure something that is in great flux. We therefore recommend that every mathematics department begin to keep annual records of sex differences in enrollment rates, attrition rates, degrees granted and graduate admissions and support. This will help alert the department to areas which need attention as well as to indicate if new policies are having the desired effect. It will be most interesting to observe how these patterns change as our schools and society move away from sexual stereotypes and bias.

There are a number of major investigations currently underway, relating to the subject matter of this essay, which will help to further clarify the situation. For example, the mathematician Edith Luchins [47] of Rensselaer Polytechnic Institute, is probing into the whole question as to why there are so few women mathematicians and what can be done about it, under a grant from the National Science Foundation. She is gathering information, using questionnaires as well as interviewing many women mathematicians. Similarly the mathematician Marta Levine of the University of Cincinnati is doing a study, with NSF support, to try to identify why qualified women do not pursue mathematical careers. She has interviewed many female and male mathematicians to get their response to such questions as to whether or not discrimination actually exists. Among students she is concentrating on four grade levels, fourth, ninth, twelfth and college sophomores. Here she is primarily concerned with the role of the educator in developing mathematical self-confidence and interest. A particularly valuable aspect of her study is that she intends to interview a random sample of mathematics teachers and guidance counselors or academic advisors at each grade level. The educator Elizabeth Fennema and the psychologist Julia Sherman are continuing their collaboration (cf. [21] and [22]) at the University of Wisconsin at Madison, under an NSF grant. They are investigating sex differences in mathematics achievement and related cognitive and affective factors, involving students in grades six through twelve. Finally, at my home institution, University of California at Santa Barbara, the social psychologists Marilynn Brewer and Myrtle Blum are making a longitudinal study to assess the relationship, across time, of identification with sex roles and feminine stereotypes on the one hand and attitudes and competency feelings towards, and participation in, mathematics and science, on the other. Clearly our understanding of the mystifying extreme sex differences in mathematics will be greatly improved when the results of these various interrelated studies come in.

As educators, however, we need to go beyond the accumulation of new knowledge. These insights will be but the first step. We agree with the basic philosophy of Renata Tesch [72] that a major function of educational research is to bring about positive change. We also agree with the eloquent and impassioned plea [19] of Elizabeth Fennema that the educational establishment must accept these extreme sex differences in mathematics as a major failure of our pedagogical system, and begin to make those changes which will ensure that women are no longer disadvantaged by inadequate mathematical training. We therefore wish to briefly survey some of the recent efforts and experiments aimed at rectifying, enriching and invigorating mathematics education for women.

At my own institution, UCSB, we have instituted remedial pre-calculus courses, for credit, with a module approach which enables each person to proceed at his or her own pace, in making up deficiencies in high school mathematics. There is also a special course, under the auspices of the Council of Experimental Courses at UCSB, given by Julian Weissglass and Gail Gliner, concerned with the problem of "mathophobia." Through general reading, graphing exercises and weekly one-to-one tutoring, the course aims at both making students with poor backgrounds functionally literate in mathematics, as well as overcoming their fears and blocks.

In a similar vein, Sheila Tobias [73] of Wesleyan University has opened a "Math Anxiety Clinic" on her campus. Here people who are willing to acknowledge that they have emotional blocks when it

comes to math, can be interviewed, tested, diagnosed and helped in a variety of ways, including therapy and counselling. In addition to this psychological approach, more standard remedial sessions were established, where students can back track and clarify some of the most elementary beginnings of mathematics.

Lenore Blum is chairperson of the Department of Mathematics and Computer Science at Mills College, a small college for women with about 850 undergraduates. She has developed a program [8] for providing academically as well as psychologically easy access to the mathematics program. Her carefully designed pre-calculus course in combination with her remedial workshops have managed to nearly double the enrollment in beginning calculus, within one year.

A rather interesting and slightly controversial phenomenon is the development of all female mathematics classes. For example, Carolyn MacDonald of the University of Missouri, Kansas City, created a project, supported by the National Science Foundation, to increase women's participation in science, primarily by designing an appropriate mathematics program (cf. [50]). She taught a special section of a pre-calculus mathematics course, for women only. The purpose was to ensure an encouraging and supportive environment where women students, no matter what their backgrounds, could succeed. This course was augmented by counselling and tutoring sessions. She reports a high degree of success, even among the students with very weak backgrounds. A few years ago, Esther Silberstein, then a teaching assistant at Berkeley, experimented with all women tutorial sections in upper division courses. She reports particular success and enjoyment in one such seminar on algebraic topology, involving six students. There has also been some experimentation with all female math classes at the pre-college level. Nancy Krienberg is involved in such a program at the Lawrence Hall of Science at U.C. Berkeley. The Hall found that its after-school and weekend classes, designed for ages 6 through adult, were about three fourths male. Therefore, a special eight week course was developed called "Math for Girls." Some two hundred girls, ages 6 to 14, have participated so far. A basic discovery approach is augmented by puzzles, games and computer activities designed to show that mathematics can be as much fun as it is challenging.

We have, for reason of time, necessarily ignored many fascinating and important questions in this study. We have purposely tried not to discuss aspects of sexism and women's rights that do not relate specifically to mathematics. Nevertheless, many of these (such as the toys children play with, the games they play, sex roles in movies and television, etc.) clearly do have their effect on female attitudes towards mathematics. For an excellent essay covering these more general concerns we refer the reader to the book [27] by Nancy Frazier and Myra Sadker entitled *Sexism in School and Society*.

A very serious problem, which has not been adequately researched in this study, is whether mathematics is being presented, both in lectures and textbooks, in a manner which tends to discourage or deter women. Certainly the higher attrition rates for women in the calculus sequence are a source of great concern. Since the vast majority of mathematicians are men, most of the textbooks and lectures are also developed by men. There are studies which show the approach and problem context do have their effects on the interest and response of women. (Cf. [9], [10], [55] and [56].) As more women become involved in mathematics perhaps we can expect some subtle changes in its formulation, presentation and exposition.

Winifred Jay [37] and Clarence Schminke [38] have found considerable sex bias in elementary school mathematics texts. Additional studies at the high school and college level would be beneficial. The crucial task of eliminating sexual bias and stereotypes from the mathematics curriculum will require the sincere and persistent efforts of educators and mathematicians for some time to come. We seem to be taking that first decisive step by becoming aware and sensitive to the problem.

We would like to conclude with a short discussion of certain moral issues which have been lying beneath the surface of this study. The question is this: Have we made an implicit assumption that dropping out of a degree program in general, and a mathematics program in particular, is necessarily bad? Given the extremely depressed (and depressing) employment prospects for young mathematicians (both men and women) perhaps those who drop out, or don't even begin the training process, are

in fact making the correct choice, not only in terms of economic consequences, but also in terms of fulfilling their human potential.

We do not claim that the goal is the elimination of all measurable sex differences in all human pursuits. We all have different opinions on the extent to which that would be desirable. But we are all agreed that the many sex differences in mathematical training and attitudes described in this report are *not* the result of free and informed choice. If they were then the low enrollments and high attrition rates for women in mathematics would be a matter of less concern. The immorality of these sex differences lies precisely in the fact that they are the result of many subtle (and not so subtle) forces, restrictions, stereotypes, sex roles, parental–teacher–peer group attitudes, and other cultural and psychological constraints which we haven't begun to fully understand. Before we can hope that each individual child and young adult will make these choices freely and wisely we must work towards a society generally (and an academic program specifically) which ensures that the freedom of opportunity to become whatever that individual is truly capable of becoming, is not compromised by such a chance event as the child's sex at birth. We believe that each of us, male or female, who have children or anticipate having children, would desire such a society for them. And those of us who are mathematicians and whose lives have been so greatly enriched as a result would wish that this opportunity be available, regardless of sex, to anyone who finds they have the aptitude, interest and creative and intellectual ability to do it.

Albert Einstein, on the occasion of the death of the mathematician Emmy Noether in 1935, wrote the following lines in a letter to the New York Times (see [39] for the full text of this beautiful letter).

> Beneath the effort directed toward the accumulation of worldly goods lie all too frequently the illusion that this is the most substantial and desirable end to be achieved; but there is, fortunately, a minority composed of those who recognize early in their lives that the most beautiful and satisfying experiences open to human kind are not derived from the outside but are bound up with the individual's own feeling, thinking and acting ... However inconspicuously the lives of these individuals run their course, nonetheless, the fruits of their endeavors are the most valuable contributions which one generation can make to its successors.

We believe that we must continue the effort to bring about a sexually unbiased social and intellectual environment where many other Emmy Noether's will be able to flourish and grow. We shall all be the beneficiaries of their creative endeavors.

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## COVERING SPACES OF ALGEBRAIC GROUPS

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Let $G$ be a simply connected topological group and $D$ a discrete normal subgroup of $G$. Then $D$ can be identified with the fundamental group of the topological group $G/D$: for $p: G \to G/D$ can be shown to be a covering space projection, so that $G$ is the universal covering space of $G/D$, and the action of $D$ on $G$ by left translation exhibits $D$ as the group of deck transformations of the covering space projection $p$. (As an example, think of the case $G = \mathbb{R}$ and $D = \mathbb{Z}$. Then $G/D$ can be regarded as the unit circle in the complex plane and $G \to G/D$ becomes the map $\exp(2\pi i(\ ))$.)

The converse of this result is also true: if $G$ is a connected topological group with a simply connected universal covering space $\bar{G}$ and if $p: \bar{G} \to G$ is the covering space projection, then $\bar{G}$ can be endowed with the structure of a topological group so that $p$ is a group homomorphism. Then the kernel of $p$, which is a discrete normal subgroup of $\bar{G}$, will be isomorphic to the group of deck transformations of $p$, and hence isomorphic to the fundamental group of $G$.

We can think of this result as expressing a connection between the group law and the topology on $G$: the topological complications of $G$, which are reflected in the existence of covering spaces of $G$, are tied to the group properties by the group structures on the covering spaces. In this paper we examine the analogous connection when $G$ is an *algebraic* group.

Let $k$ denote an algebraically closed base field (we fix this notation throughout). Let $GL_n(k)$ denote the group of invertible $n \times n$ matrices over $k$. An algebraic group $G$ is a subgroup of $GL_n(k)$ which is determined by polynomial conditions on the matrix coordinates of $GL_n(k)$. (Technically, such a $G$ is an *affine* algebraic group; there are non-affine algebraic groups but these will not be discussed here.) For example, $G_m(k)$, the multiplicative group of the field $k$, is the same as $GL_1(k)$ and hence can be regarded as an algebraic group. Also $G_a(k)$, the additive group of $k$, can be regarded as an algebraic group by identifying it with a subgroup of $GL_2(k)$ as follows:

$$G_a(k) = \left\{ \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \mid x_{21} = 0, x_{11} = x_{22} = 1 \right\};$$

and $SL_2(k)$ is the subgroup of $GL_2(k)$ determined by

$$SL_2(k) = \left\{ \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \mid x_{11}x_{22} - x_{12}x_{21} = 1 \right\}.$$

Algebraic groups are a special kind of *algebraic variety*. (An algebraic variety is a subset of $k^{(n)}$ determined by polynomial conditions in the coordinates of $k^{(n)}$.) To see this, it suffices to exhibit $GL_n(k)$ as an algebraic variety. If we consider $k^{(n^2+1)}$ with last coordinate $y$ and first $n^2$ coordinates $x_{ij}$, $1 \leqq i, j \leqq n$, then $GL_n(k)$ corresponds to $(n^2+1)$-tuples which satisfy the polynomial condition

$y \cdot \det((x_{ij})) = 1$ under the correspondence which sends the invertible matrix $A$ in $GL_n(k)$ to the tuple whose last entry is $1/\det A$ and whose entry in the $x_{ij}$ coordinate is the $(i, j)$-entry of $A$.

There is a notion, which is examined below, of a covering space for algebraic varieties, and the existence of covering spaces of an algebraic variety reflects some of the geometrical complications of the variety. In attempting to lift the group law from an algebraic group $G$ to a covering space of $G$ as an algebraic variety, we are, as in the case of topological groups, examining the connections between the geometry of $G$ regarded as an algebraic variety, and the group law on $G$.

We begin with a brief summary and review of the topological notion of covering space and the techniques used to put a group structure on the universal covering space of a topological group. Next, we look at the notion of covering space of an algebraic variety by analogy with the topological case. It will turn out that the topological techniques for lifting group structures to the universal cover, however, do not have a complete analogue for algebraic varieties. Instead, as we are successively less demanding about the group structure, we find an increasing hierarchy of "universal" covers.

**1. Topological covering spaces.** We recall the basic notions of topological covering spaces ([7, Chapter 2] is a good reference for this material) beginning with the definition:

DEFINITION. Let $p: X \to Y$ be a continuous surjection of topological spaces. We call $p$ a *covering space projection*, and refer to $X$ as a *covering space* of $Y$, if there is a discrete space $D$ so that every point of $Y$ has a neighborhood $U$ for which $p^{-1}(U)$ is homeomorphic to $U \times D$ in such a way that the following diagram commutes:

$$\begin{array}{ccc} U \times D & \to & X \\ \downarrow & & \downarrow \\ U & \to & Y \end{array}$$

where the left map is projection on the first factor, the right map is $p$, the top map is the given homeomorphism, and the bottom map is the inclusion.

(Another way of phrasing the condition on $U$ is to say that $p^{-1}(U)$ is the disjoint union of open sets homeomorphic to $U$ by the restriction of $p$.)

We give the definition in detail because we want to construct an analogous definition for algebraic varieties. With that end in mind, we remark that $p^{-1}(U)$ above has a categorical description: $p^{-1}(U)$ is the fibre product $U \times_Y X$ of $U$ with $X$ over $Y$. (We recall that if $A$, $B$, $C$ are sets and $f: A \to C$ and $g: B \to C$ are functions then $A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}$.)

If $X$ is any space and $D$ a discrete space then $X \times D \to X$ is, in a trivial way, a covering space projection. To focus on non-trivial covering space projections, it is customary to restrict to the case of connected covering spaces of connected spaces.

We recall further that a *universal covering space* of a connected space $X$ is a covering space projection $\bar{X} \to X$ which dominates every other connected covering space $X' \to X$ in the sense that there is a continuous function $\bar{X} \to X'$ such that the diagram

$$\begin{array}{ccc} \bar{X} & \to & X' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes.

Finally, we recall how to put a group structure on the universal covering space of a topological group (see [1, Section VIII, p. 50]): let $G$ be a connected topological group with identity $e$ and simply connected universal covering space $\bar{G}$, and let $p: \bar{G} \to G$ be the covering space projection. Choose $\bar{e} \in p^{-1}(e)$. Consider the diagram

$$\begin{array}{ccc} \bar{G}\times\bar{G} & & \bar{G} \\ {\scriptstyle p\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G\times G & \xrightarrow{m} & G \end{array}$$

where $m$ is the group law on $G$. We want to cover $m$ by a map $\bar{m}\colon \bar{G}\times\bar{G}\to\bar{G}$ so that the diagram commutes. This can be done since $\bar{G}\times\bar{G}$ is simply connected and maps to $G$ by $m(p\times p)$, while $p$ is a covering space projection, so there is such a map $\bar{m}$ which can be uniquely specified by requiring $\bar{m}(\bar{e},\bar{e})=\bar{e}$. This uniqueness then implies that $\bar{m}$ is a group law on $\bar{G}$ with $\bar{e}$ as identity, and the commutativity of the diagram shows that $p$ is a homomorphism.

For later reference we highlight the two main facts which make the above argument work:

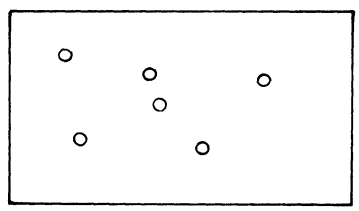
*Fact* 1: If $p\colon X'\to X$ is a covering space projection with $X'$ connected and locally connected, and if $q\colon \bar{X}\to X$ is any continuous function from the simply connected space $\bar{X}$ to $X$, then there is a continuous function $f\colon \bar{X}\to X'$ such that $pf=q$. If $\bar{x}\in\bar{X}$ and $x'\in X'$ are such that $p(x')=q(\bar{x})$, then $f$ can be taken so that $f(\bar{x})=x'$, and this uniquely determines $f$ [**1**, Prop. 1, p. 50].

*Fact* 2: If $X_1$ and $X_2$ are simply connected topological spaces, so is $X_1\times X_2$ [**1**, Prop. 4, p. 52].

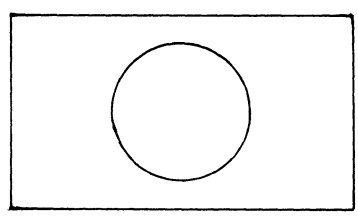
**2. Étale covering spaces of algebraic varieties.** We now turn to the problem of finding a suitable notion of covering space for algebraic varieties. The algebraic varieties we are dealing with are the affine algebraic varieties; that is, the subsets of $k^{(n)}$ determined by polynomial conditions on the coordinates of $k^{(n)}$. A morphism $f\colon X\to Y$ of algebraic varieties $X\subseteq k^{(n)}$ and $Y\subseteq k^{(m)}$ is by definition the restriction of a function $k^{(n)}\to k^{(m)}$ given by polynomials in the coordinates of $k^{(n)}$ and $k^{(m)}$. (A good basic reference for affine varieties is [**6**, Chapter I, section 2].)

We are looking for conditions under which a morphism of algebraic varieties should be called a covering space projection. There is a natural topology on an algebraic variety, the Zariski topology, whose basic closed sets are the zeros of polynomial functions. As a first attempt at a definition of covering space projection of algebraic varieties we can try merely adapting the definition in Section 1, using the Zariski topology and requiring all our maps to be morphisms of algebraic varieties. This will be an extremely restrictive formulation, however. The following example will illustrate this:

*Example* A. Suppose $k=\mathbb{C}$ and $X=Y=k-\{0\}=G_m(k)$, and let $p\colon X\to Y$ be $p(z)=z^2$. Clearly $p$ is an everywhere two-to-one morphism of algebraic varieties. Polynomial functions on $\mathbb{C}$ have only finitely many zeros, so a typical Zariski open set $U\subseteq Y$ looks like

$U=$ 

Suppose $p^{-1}(U)$ is homeomorphic to $U\times D$ as in the definition of covering space in Section 1. Pick a disk containing all the missing points of $U$ and let $V$ be its complement

$V=$ 

Then $V$ is a subset of $U$ so $p^{-1}(V)$ should be homeomorphic to $V \times D$, but in fact $p^{-1}(V)$ looks just like $V$ itself. This example shows that there are no Zariski open subsets of $Y$ satisfying the definition of covering space projection. On the other hand, $p: X \to Y$ is a homomorphism of algebraic groups with discrete kernel $\{\pm 1\}$, and any good definition of algebraic covering spaces should certainly include situations like this.

The difficulty in Example A can be linked to the failure of the implicit function theorem to hold for algebraic varieties. The derivative of $p$ in Example A is everywhere non-zero and so $p$ locally, in the usual topology on $\mathbb{C}$, has inverses by the implicit function theorem. Then, since $p$ is finite to one and a local homeomorphism, it is a covering space projection. But $p$ is not a local homeomorphism in the Zariski topology.

Grothendieck introduced a device to circumvent the lack of an implicit function theorem for algebraic varieties, namely the concept of an étale neighborhood. Roughly speaking, this is something which would be a neighborhood if the implicit function theorem were true.

In order to give a more precise definition, we will need the notion of tangent spaces of algebraic varieties. Readers unfamiliar with this notion should see [**6**, Chapter II, section 1] for an elementary discussion of tangent spaces.

DEFINITION. A morphism of algebraic varieties $p: U \to Y$ is an *étale neighborhood* if

(1) $p$ is an open map in the Zariski topology;

(2) for every $x$ in $U$ the induced map $(Dp)_x: T_x(U) \to T_{p(x)}(Y)$ on tangent spaces is an isomorphism.

This definition is not quite accurate. We are tacitly assuming in it that $U$ and $Y$ are non-singular, so they have tangent spaces. When no confusion arises, we denote the étale neighborhood $p: U \to Y$ by $U$ alone.

Now we define covering space of algebraic varieties by translating the topological definition but with "étale neighborhood" replacing "open set."

DEFINITION. A morphism $p: X \to Y$ of algebraic varieties is an *étale covering space projection* if there is a finite variety $D$ such that given $y \in Y$ there is an étale neighborhood $q: U \to Y$ with $y \in q(U)$ such that $U \times_Y X$ is isomorphic to $U \times D$ under an isomorphism which makes the following diagram commute:

$$\begin{array}{ccc} U \times D \cong U \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\quad q \quad} & Y, \end{array}$$

where the left map is projection on the first factor.

If we take $X = Y \times D$ with $D$ finite, then the first projection $X \to Y$ is an étale covering space (for we can take $1: Y \to Y$ as the required étale neighborhood of any point of $Y$). Such trivial étale covering spaces are always present, and not of much interest. We can disregard these spaces by focusing on connected covering spaces, which we shall call non-trivial.

We see next that the Example A above is a non-trivial étale covering space projection. In that example, we have $p: X \to Y$ where $X = Y = k - \{0\}$ and $k = \mathbb{C}$, and $p(z) = z^2$. Then for all $z \in X$, the induced map $(Dp)_z$ is $[2z]$ which is never 0 since $0 \notin X$. Thus $(Dp)_z$ is always an isomorphism. We claim that $p$ is actually an étale neighborhood of $Y$: we need to see that $p$ is an open mapping for the Zariski topologies on $X$ and $Y$. But an open subset $U$ of $X$ is just a subset of $X$ with finite complement, and since $p$ is surjective this means that $p(U)$ also has finite complement, so $p(U)$ is open in $Y$. Every point $y$ in $Y$ belongs to $p(X)$; so to show that $p: X \to Y$ is an étale covering space

projection, it is sufficient to show that $X \times_Y X$ is isomorphic to $X \times D$ for some finite variety $D$. Now

$$X \times_Y X = \{(z_1, z_2) \in X \times Y \mid z_1^2 = z_2^2\}.$$

If $z_1^2 = z_2^2$, then $z_1 = \pm z_2$, and we have an isomorphism

$$X \times \{-1, 1\} \to X \times_Y X$$

given by sending $(z, 1)$ to $(z, z)$ and $(z, -1)$ to $(z, -z)$, so that the required diagram in the definition of étale covering space commutes.

This example points out that even if an étale covering space projection is non-trivial, it still may be the case that there is a single étale neighborhood such that restricting to that neighborhood displays the coverings as a product of the base variety with a finite set.

The covering space in Example A is essentially just an algebraic version of a topological covering space. In fact, if $k = \mathbb{C}$, the implicit function theorem implies that an étale covering space of complex algebraic varieties is a topological covering space, when the varieties are regarded as topological subspaces of $\mathbb{C}^{(n)}$.

To find examples which do not arise for topological reasons, for instance, in case $k$ has characteristic $p > 0$, we need a remark about tangent spaces of hypersurfaces. Let $f$ be in $k[x_1, \cdots, x_n]$ and let $Y \subseteq k^{(n)}$ be $f^{-1}(0)$, the variety defined by $f$. If the gradient of $f$ at $y$ is non-zero, we can compute

$$T_y(Y) = \left\{ (a_1, \ldots, a_n) \,\middle|\, \sum a_i \frac{\partial f}{\partial x_i}(y) = 0 \right\}.$$

If $p: Y \to k^{(n-1)}$ is the restriction to $Y$ of the projection on the first $n-1$ coordinates of $k^{(n)}$, then $(Dp)_y(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1})$, and $(Dp)_y$ is an isomorphism of tangent spaces if $a_n = 0$ whenever $a_1 = \cdots = a_{n-1} = 0$, i.e., if $(\partial f/\partial x_n)(y) \neq 0$.

*Example* B. Let $k$ have characteristic $p > 0$. Let $f \in k[x_1, x_2]$ be $f = x_2^p + x_2 + x_1$, let $Y = f^{-1}(0) \subseteq k^{(2)}$ and let $q: Y \to k^{(1)}$ be projection on the first coordinate. Since $k$ is algebraically closed, for every $a \in k$ we can find $b \in k$ such that $b$ is a root of $x^p + x + a$, and hence $(a, b) \in Y$ with $q(a, b) = a$. Thus $q$ is surjective. Now $Y$ is one-dimensional, as is $k^{(1)}$, so Zariski open subsets of them are sets with finite complement, and thus $q$ is an open map for the Zariski topology. Finally, $(\partial f/\partial x_2)(y) = 1$ for all $y$ in $Y$, so $(Dq)_y$ is an isomorphism on tangent spaces. Thus $q$ is an étale covering space projection.

It is possible to show that the covering $Y \to k^{(1)}$ of Example B is not trivial, i.e., that $Y$ is not of the form $k^{(1)} \times D$ for $D$ finite. An example such as this can arise only in characteristic $p > 0$, for by [4, Lemma 1, p. 353] every étale covering space $Y \to k^{(1)}$, where $k$ has characteristic zero, must be trivial.

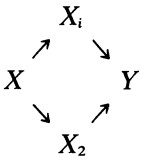
We next want to define the analogue of universal covering space and fundamental group for algebraic varieties. We shall only give the basic outlines of these constructions; for precise descriptions and proofs [5] is a good reference. We shall get the universal covering by "backing up" along all coverings, and get the fundamental group by knowing that it is sufficient to back up along coverings to which a group is associated.

Let $p: X \to Y$ be an étale covering space projection. A morphism $\sigma: X \to X$ is called a *deck transformation* if $p\sigma = \sigma p$. It turns out that deck transformations are automorphisms when $X$ is connected, and that the set of all deck transformations of $p$ forms a finite group, which we denote $\mathrm{Aut}_Y(X)$. We call $p$ a *normal* étale covering space if given $x_1, x_2 \in X$ for which $p(x_1) = p(x_2)$ there is $\sigma \in \mathrm{Aut}_Y(X)$ with $\sigma(x_1) = x_2$.

Now we suppose that $Y$ is a connected algebraic variety. Then the following properties are the basis for the "backing up" program described above:

(P1) If $X$ is connected and $p: X \to Y$ is an étale covering space, then there is a connected étale covering space $X' \to X$ such that the composite étale covering space is normal [5, 4.4.1.8, p. 77].

(P2) If $X_i \to Y$ for $i = 1, 2$ are étale covering spaces with $X_i$ connected, there is a connected variety $X$ and étale covering spaces $X \to X_i$ such that the diagram




commutes.

Let $Y$ be a connected algebraic variety. Then $\tilde{Y}$, the *universal étale covering space* of $Y$, is the projective limit of the set of connected étale covering spaces of $Y$. Property (P2) shows that this projective limit will be connected. By (P1), it is enough to just take the inverse limit over the set of normal étale covering spaces. This inverse system of normal covering spaces $X$ over $Y$ gives rise to an inverse system of their finite groups of deck transformations $\mathrm{Aut}_Y(X)$. The inverse limit of this system of finite groups will be a profinite group—a totally disconnected compact Hausdorff topological group—which we call the *étale fundamental group* of $Y$ and denote by $\hat{\Pi}_1(Y)$.

This universal covering space and fundamental group will behave analogously to the similarly named topological objects.

We next consider some of their properties.

We can regard $\hat{\Pi}_1(Y)$ as a group of automorphisms of $\tilde{Y}$. Then if $X \to Y$ is any connected étale covering space, there is a closed subgroup $H$ of $\hat{\Pi}_1(Y)$ such that $X$ can be identified with the space $\tilde{Y}/H$ of $H$-orbits in $\tilde{Y}$. (If $y \in \tilde{Y}$, its $H$-orbit is the set of all elements of $\tilde{Y}$ of the form $h(y)$ for $h \in H$.)

We call $Y$ *simply connected* if $\hat{\Pi}_1(Y) = 1$. (By the preceding paragraph, this is equivalent to saying that if $X \to Y$ is a connected étale covering space then $X = Y$.) If $Y$ is simply connected, $q: Y \to Z$ a morphism, and $p: X \to Z$ an étale covering space, then there exists a morphism $f: Y \to X$ such that $pf = q$. If $y \in Y$ and $x \in X$ with $q(y) = p(x)$, we can choose $f$ so that $f(y) = x$, and this choice specifies $f$ uniquely.

Finally, we remark that if $\mathscr{C}$ is any class of étale covering spaces of the connected variety $Y$ such that (P1) and (P2) are satisfied by $\mathscr{C}$, then we can construct a "universal cover" and "fundamental group" with respect to $\mathscr{C}$.

**3. Étale covering spaces of algebraic groups.** At the end of Section 1 above we reviewed how to put a group structure on the universal covering space of a topological group, and we highlighted the two main points of that argument, which are labeled *Fact* 1 and *Fact* 2. As we have just remarked, *Fact* 1 also holds for algebraic varieties which are simply connected in the étale sense. If we knew *Fact* 2 for algebraic varieties, then we could carry over the proof from the topological case to the algebraic case and make the universal étale covering space of an algebraic group into a group by lifting the group law from the original algebraic group to it.

Unfortunately, *Fact* 2 is false in the étale setting: a product of simply connected algebraic varieties need not be simply connected. One way to see this is to find an example of an algebraic group whose universal étale cover is not an algebraic group, such as the example in [**4**, Remark 2, p. 364].

Nonetheless, even though the product formula of *Fact* 2 is in general false, we can look at more restricted situations and hope to get some version of the product formula there. The idea is to restrict to a class $\mathscr{C}$ of étale coverings of the given algebraic group which is large enough so that (P1) and (P2) hold so we can talk about universal covers relative to $\mathscr{C}$, but small enough so that the product formula holds for $\mathscr{C}$.

A suitable class $\mathscr{C}$ turns out to be the following:

DEFINITION: Let $p$ be the characteristic of $k$ and let $Y$ be an algebraic variety. A connected étale covering space $X \to Y$ is *prime-to-p* if there is a connected étale covering space $X' \to X$ such that the

composite $X' \to Y$ is a normal étale covering space and the number of preimages in $X'$ of a point in $Y$ is always prime-to-$p$.

The properties (P1) and (P2) are satisfied by prime-to-$p$ étale coverings and thus for a connected algebraic variety $Y$ there are the notions of a *universal prime-to-p étale covering space* $\tilde{Y}^{(p)}$ of $Y$ and a *prime-to-p fundamental group* $\hat{\Pi}_1^{(p)}(Y)$ of $Y$.

We call $Y$ *prime-to-p simply connected* if $Y = \tilde{Y}^{(p)}$, or, equivalently, if $\hat{\Pi}_1^{(p)}(Y) = 1$. Then the product of prime-to-$p$ simply connected varieties turns out to be simply connected by the following deep theorem:

SMOOTH BASE CHANGE THEOREM [2, Prop. 2.1.5.11, p. 135]: $\hat{\Pi}_1^{(p)}(X_1 \times X_2)$ *is isomorphic to* $\hat{\Pi}_1^{(p)}(X_1) \times \hat{\Pi}_1^{(p)}(X_2)$.

So we can completely mimic the topological arguments for algebraic groups as long as we stay prime-to-$p$, and conclude: if $G$ is a connected algebraic group, then $\tilde{G}^{(p)}$ carries an algebraic group structure (actually a projective limit of such) so that $\tilde{G}^{(p)} \to G$ is a homomorphism.

This conclusion is a complete answer to our original problem if $p = 0$, for then $\tilde{G}^{(p)}$ is the same as $\tilde{G}$. In the case $p > 0$, however, it may happen that $\tilde{G}^{(p)}$ is distinct from $\tilde{G}$. For example, it is known that if $G = G_a(k)$, then $\tilde{G}^{(p)} = G$ [4, Lemma 2, p. 354], while we saw in Example B that there are coverings of $G_a(k)$ which are not $G_a(k)$ itself.

It is possible, of course, that even if $\tilde{G}^{(p)}$ is not $\tilde{G}$ there are no larger coverings of $G$ than $\tilde{G}^{(p)}$ which carry algebraic group structures. We shall examine Example B again to see that this is not the case. We recall that $k$ has characteristic $p > 0$, that $f \in k[x_1, x_2]$ is $f = x_2^p + x_2 + x_1$, that $Y = f^{-1}(0)$ and that $q: Y \to k^{(1)}$ by projection on the first coordinate is an étale covering space projection. Now $k^{(1)} = G_a(k)$, and we will show that $Y$ is an algebraic group such that $q$ is a homomorphism. Let $(x, y)$ and $(x', y')$ be in $Y$ and define $(x, y) + (x', y') = (x + x', y + y')$. Then

$$(y + y')^p = y^p + y'^p = -y - x - y' - x' = -(y + y') - (x + x'),$$

so $(x + x', y + y') \in Y$. It is immediate that $Y$ is an algebraic group and that $q$ is a homomorphism. Thus $Y$ is a group which is an étale covering space of $G_a(k)$, and $Y$ is not a prime-to-$p$ covering of $G_a(k)$.

It is natural, in light of the preceding example, to look at all group coverings, as well as just those prime-to-$p$. We do this by introducing a new class of coverings:

DEFINITION: Let $G$ be a connected algebraic group. The map $p: G' \to G$ is a *group covering* if $G'$ is a connected algebraic group and $p$ is a group homomorphism which is also an étale covering space projection.

The properties (P1) and (P2) hold for group coverings, and thus there is a *universal group cover* $\tilde{G}^{(grp)}$ and a *group fundamental group* $\hat{\Pi}_1^{(grp)}(G)$.

Now starting with $G$, we have the universal covers $\tilde{G}^{(grp)}$ and $\tilde{G}^{(p)}$, and since prime-to-$p$ covers are group covers we have a map $\tilde{G}^{(grp)} \to \tilde{G}^{(p)}$. Example B showed that the map is not a bijection. As before, $\tilde{G}^{(grp)}$ carries a projective limit of algebraic group structures and $\tilde{G}^{(grp)} \to G$ is a homomorphism with kernel $\hat{\Pi}_1^{(grp)}(G)$. It turns out that $\hat{\Pi}_1^{(grp)}(G)$ is abelian: for if $f: G' \to G$ is a group covering, then the kernel $K$ of $f$ is a finite normal subgroup of the connected group $G$. If $x$ is in $K$, then $y \to x^{-1}y^{-1}xy$ is a morphism from $G$ to $K$. Since $K$ is discrete and $G$ connected, the morphism must be constant, and evaluating at the identity shows this constant value to be the identity. But then $x$ and $y$ commute for all $y$ in $G$, so $K$ is central, hence abelian. Now $\hat{\Pi}_1^{(grp)}(G)$ is a projective limit of kernels of group coverings, and so it is abelian.

Group coverings of $G$ are normal étale coverings with abelian group of deck transformations. It is possible to show that for suitable $G$ there are coverings of this nature which are not group coverings [4, Remark 2, p. 364]. To investigate all such coverings, we shall introduce another type of universal cover:

DEFINITION: A connected étale covering space $X \to Y$ of algebraic varieties is an *abelian covering* if it is normal and $\mathrm{Aut}_Y(X)$ is abelian.

Abelian coverings satisfy (P1) and (P2) and there is a *universal abelian covering* $\tilde{Y}^{(ab)}$ and an abelian *fundamental group* $\hat{\Pi}_1^{(ab)}(Y)$.

Thus for a connected algebraic group $G$ we have $\tilde{G}^{(ab)} \to \tilde{G}^{(grp)} \to \tilde{G}^{(p)} \to G$, and all spaces may be distinct.

Our next example will show that $\tilde{G}^{(ab)}$ and $\tilde{G}$ may be distinct:

*Example* C. Let $k$ have characteristic 3, and let $f \in k[x_1, x_2]$ be $f = x_2^3 + x_1 x_2 + 1$. Let $E = f^{-1}(0) \cap \{G_m(k) \times k\} = \{(x, y) \in k^{(2)} \mid x \neq 0 \text{ and } f(x, y) = 0\}$ and let $q: E \to G_m(k)$ be projection on the first factor. Since $\partial f/\partial x_2 = x_1$, then $(\partial f/\partial x_2)(z) \neq 0$ for all $z \in E$, and so $q$ is an étale covering space projection. If we let $Y \to E$ be the smallest étale covering space such that $Y \to G_m(k)$ is normal, then it can be shown that the group of deck transformations of $Y$ over $G_m(k)$ is $S_3$ [3, Example 3.3]. (The point is that the discriminant of $f$, $-4x_1^3 - 27 = -x_1^3$, is not a square in the coordinate ring of $G_m(k)$.) Thus $E \to G_m(k)$ is not an abelian covering.

For a connected algebraic group $G$ we have constructed a tower of covers $\tilde{G} \to \tilde{G}^{(ab)} \to \tilde{G}^{(grp)} \to \tilde{G}^{(p)} \to G$, and none of the maps need be bijections. In a sense, the fact that the second map is not a bijection can be regarded as an answer to our original problem of lifting the algebraic group structure to the universal cover, since $\tilde{G}^{(grp)}$ is the largest cover for which a lift of algebraic group structures is possible. However, we can say a little more: if we relax our demand for algebraic group structures, we have the following positive result.

THEOREM [3, Thm 2.2]: *Let $G$ be a connected algebraic group. The $\tilde{G}$ carries the structure of an abstract group such that*

(1) *for all $x$ in $\tilde{G}$, the map $\tilde{G} \to \tilde{G}$ by $y \to xy$ is a morphism of (projective limits of) algebraic varieties.*

(2) *$\tilde{G} \to G$ is a homomorphism.*

We call such an object a *left algebraic group.* (The leftness is by choice; the construction could be carried out to produce a right algebraic group structure on $\tilde{G}$, but in general this is a different group.) Now the kernel of $p: \tilde{G} \to G$ can be identified with $\hat{\Pi}_1(G)$, and each of $\hat{\Pi}_1^{(p)}(G)$, $\hat{\Pi}_1^{(grp)}(G)$, and $\hat{\Pi}_1^{(ab)}(G)$ are characteristic subgroups of $\hat{\Pi}_1(G)$ (so they are normal in $\tilde{G}$), and we can identify $\tilde{G}^{(p)}$, $\tilde{G}^{(grp)}$ and $\tilde{G}^{(ab)}$ as the group quotients of $\tilde{G}$ by $\hat{\Pi}_1^{(p)}(G)$, $\hat{\Pi}_1^{(grp)}(G)$ and $\hat{\Pi}_1^{(ab)}(G)$ respectively.

We recall that our original goal was to show that algebraic group structures can be lifted to universal covering spaces. What we have found is a number of answers: as we were less demanding about the type of lifting we got larger coverings. If we wanted to follow the topological analogue, we needed to stay prime-to-$p$. If we wanted to lift algebraic group structures, we could enlarge to group coverings. If we only looked for normal coverings with deck transformation groups like those of group coverings, we could enlarge to abelian coverings. And if, finally, we would tolerate group structures which are algebraic on one side only, we could get all the way up to the universal cover.

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# MATHEMATICAL NOTES

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## A NOTE ON THE TRIGAMMA FUNCTION

HAROLD RUBEN

The purpose of this note is to bring attention to the fact that the formula

$$\frac{d^2}{dz^2}\log\Gamma(z+1)=\frac{1}{z+1}+\frac{1!}{2}\frac{1}{(z+1)_2}+\frac{2!}{3}\frac{1}{(z+1)_3}+\cdots,\quad \mathrm{Re}(z)>-1, \tag{1}$$

where $(z+1)_j=(z+1)(z+2)\cdots(z+j)$, is a direct consequence of the well-known formula of Gauss for the value of the hypergeometric function with unit argument. The series in (1) converges absolutely and uniformly on the region $\mathrm{Re}(z)>-1+\delta$ $(\delta>0)$, and serves also as an asymptotic development for the trigamma function when $-\pi/2<\arg(z+1)<\pi/2$. In fact, we have

$$F(a,b;c;1)=\sum_0^\infty\frac{(a)_j(b)_j}{j!(c)_j}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},\quad \mathrm{Re}(c-a-b)>0,\ c\neq 0,-1,-2,\ldots,$$

and therefore in particular

$$\frac{\Gamma(c)\Gamma(c-2b)}{\Gamma^2(c-b)}=\sum_0^\infty\frac{\{(b)_j\}^2}{j!(c)_j},\quad \mathrm{Re}(c-2b)>0,\ c\neq 0,-1,-2,\ldots. \tag{2}$$

(1) follows from (2) on letting $b\to 0$. It is actually slightly more convenient to reparametrize (2) by setting $z=c-b$, $h=b$ and taking logarithms before allowing $h$ to tend to 0. Thus,

$$\frac{\log\Gamma(z+h)-2\log\Gamma(z)+\log\Gamma(z-h)}{h^2}=\frac{1}{h^2}\log\left(1+\frac{\{(h)_1\}^2}{1!(z+h)_1}+\frac{\{(h)_2\}^2}{2!(z+h)_2}+\cdots\right),$$

$\mathrm{Re}(z-h)>0$; $z+h\neq 0,-1,-2,\ldots$; $h\neq 0$, which gives

$$\frac{d^2}{dz^2}\log\Gamma(z)=\sum_1^\infty\frac{\{(j-1)!\}^2}{j!(z)_j},\quad \mathrm{Re}(z)>0,$$

as in (1), when $h\to 0$. [Here $\lim_{h\to 0}\{(h)_j\}^2/\{(z+h)_jh^2\}=1^2\cdot 2^2\cdots(j-1)^2/(z)_j$ has been used. Justification of the term by term limiting process is elementary and is omitted.] Absolute and uniform convergence of the series in (1) on the region $\mathrm{Re}(z)>-1+\delta$ $(\delta>0)$ follows from Raabe's test and

$$\left|\sum_1^\infty\frac{(j-1)!}{j\cdot(z+1)_j}\right|<\sum_1^\infty\frac{(j-1)!}{j\cdot\delta(\delta+1)\cdots(\delta+j-1)}<\infty$$

(the ratio of the $j$th to the $(j+1)$th term in the dominating series being $(1+(1/j))(1+(\delta/j))$, while the asymptotic nature of the series follows from

$$\begin{aligned}\left|\sum_{j=m+1}^\infty\frac{(j-1)!}{j\cdot(z+1)_j}\right|&=\left|\frac{m!}{(m+1)\cdot(z+1)_{m+1}}\cdot\sum_{q=0}^\infty\frac{m+1}{m+1+q}\frac{(m+1)_q}{(z+m+2)_q}\right|\\&<\left|\frac{m!}{(m+1)\cdot(z+1)_{m+1}}\right|\cdot\sum_{q=0}^\infty\frac{m^q}{(m+1)_q}\\&<e^m\cdot\frac{m!}{m+1}\left|\frac{1}{(z+1)_{m+1}}\right|=o(|m\text{th term}|),\qquad |z|\to\infty.\end{aligned}$$

(1) should be contrasted to the familiar result

$$\frac{d^2}{dz^2}\log\Gamma(z+1)=\frac{1}{(z+1)^2}+\frac{1}{(z+2)^2}+\cdots.$$

(1) can, of course, be obtained independently. For example, it follows from Nörlund's series [1, p. 261]

$$\psi(a-z)=\psi(a)-\left\{\frac{(z)_1}{(a)_1}+\frac{1}{2}\frac{(z)_2}{(a)_2}+\frac{1}{3}\frac{(z)_3}{(a)_3}+\cdots\right\},$$

valid for $\mathrm{Re}(a-z)>0$ and $a\neq 0,-1,-2,\ldots$, in which $\psi(z)=(d/dz)\log\Gamma(z)$, by differentiation at $z=0$, and can also be obtained by a slight modification of standard theory for the development of functions in series of inverse factorials (Whittaker and Watson [2, pp. 142–144]). (We remark in passing that the last method is here rather clumsy, and the coefficients of $1/(z+1)_j$ are obtained as linear functions of the Bernoulli numbers.) Nevertheless, the proof of this note is simple and suggestive.

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## AN ANCIENT UNFAIR GAME

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The game of dreydel has been played by Jews for many years on the festival of Chanukah. (Some historians date it back about 2000 years and some date it from the early middle ages.) What we have discovered is that, surprisingly, such an ancient game is also an unfair game.

The dreydel is a four-sided top whose sides we'll denote by the letters *N*, *G*, *H*, and *S* (corresponding to the Hebrew letters Nun, Gimel, Hay, and Shin). The game is played with any number of players, each of whom contributes one unit to the pot to start the game. The players take turns spinning the dreydel until each player has had his turn, at which time the players spin again in the same order. The game continues until some mutually agreed stopping point.

The payoffs (to the spinning player) corresponding to each of the four equally likely outcomes are:

| | |
|---|---|
| *N*: no payoff | *H*: half the pot |
| G: entire pot | *S*: puts one unit into the pot. |

When one player spins a *G*, he collects the entire pot, and all the players then contribute one unit to form the new pot.

We will prove the following:

MAIN THEOREM. *Let $X_n$ be the payoff on the n-th spin and let p be the number of players. Then, the expected value of $X_n$, $E(X_n)$, is*

$$\frac{p}{4}+\left(\frac{5}{8}\right)^{n-1}\left(\frac{p-2}{8}\right).$$

We thus see that if $p > 2$ (i.e., if there are more than two players), the expected payoff on the $n$th spin is a strictly monotonic decreasing function of $n$. Therefore, the first player (whose spins correspond to $n = 1, p+1, 2p+1, 3p+1, \ldots$) has a term-for-term greater expected payoff than the second player (whose spins correspond to $n = 2, p+2, 2p+2, 3p+2, \ldots$) and who in turn has a greater expected payoff than the third player, etc. Thus, the first player has an unfair advantage over the second player, who in turn has an unfair advantage over the third player, etc. Furthermore, this unfairness is accentuated if a stopping rule is used which does not guarantee an equal number of turns to each player (e.g., until the 20th $G$).

A moment's thought convinces us that on any spin one of the most important factors is the number of spins since the last $G$. We shall therefore need the following:

LEMMA. *Let $Y_k$ be the amount in the pot at the start of the $k$-th spin, given that there were no $G$'s on the first $k-1$ spins. Then, $E(Y_k)$, the expected value of $Y_k$, is*

$$2 + (\tfrac{5}{6})^{k-1}(p-2).$$

*Proof*: The lemma is obviously true for $k = 1$ as $E(Y_1) = p$.

Now assume the lemma is true for $k$ and that there were no $G$'s on the first $k$ spins. Then, on the $k$th spin there were 3 equally likely outcomes ($N$, $H$, and $S$) which would produce pots which are, respectively: the same as, half of, and one more than the pot at the start of the $k$th spin. Thus,

$$\begin{aligned} E(Y_{k+1}) &= \tfrac{1}{3}\left(E(Y_k) + \tfrac{1}{2}E(Y_k) + E(Y_k) + 1\right) \\ &= \tfrac{5}{6}E(Y_k) + \tfrac{1}{3} \\ &= \tfrac{5}{6}\left(2 + (\tfrac{5}{6})^{k-1}(p-2)\right) + \tfrac{1}{3} \\ &= 2 + (\tfrac{5}{6})^{k}(p-2) \end{aligned}$$

and the lemma is proven by induction (we could also have proven the lemma by solving the recursion formula $E(Y_{k+1}) = \frac{5}{6}E(Y_k) + \frac{1}{3}$ with the initial condition $E(Y_1) = p$).

We are now ready for the

*Proof of Main Theorem*: The expected payoff, $E(X_n)$, depends on the last occurrence of a $G$ (if there were any previous $G$'s).

If the last $G$ was on the $(n-1)$st spin, then there were $E(Y_1)$ units in the pot at the start of the $n$th spin. Thus with probability $\frac{1}{4}$ the payoff would be each of 0, $E(Y_1)$, $\frac{1}{2}E(Y_1)$, and $-1$. Since there is a probability of $\frac{1}{4}$ that there was a $G$ on the $(n-1)$st spin, we get a contribution (to $E(X_n)$) of

$$\tfrac{1}{4}\left[\tfrac{1}{4}\left(0 + E(Y_1) + \tfrac{1}{2}E(Y_1) - 1\right)\right] = \tfrac{1}{16}\left(\tfrac{3}{2}E(Y_1) - 1\right).$$

If the last $G$ was on the $(n-2)$nd spin, there was no $G$ on the $(n-1)$st spin and therefore $E(Y_2)$ in the pot at the start of the $n$th spin. Therefore, with probability $\frac{1}{4}$ the payoff would be each of 0, $E(Y_2)$, $\frac{1}{2}E(Y_2)$ and $-1$. Since there is a probability of $\frac{1}{4}\cdot\frac{3}{4}$ of the last $G$ being on the $(n-2)$nd spin, we get a contribution (to $E(X_n)$) of

$$\tfrac{1}{4}\cdot\tfrac{3}{4}\left[\tfrac{1}{4}\left(0 + E(Y_2) + \tfrac{1}{2}E(Y_2) - 1\right)\right] = \tfrac{1}{16}\tfrac{3}{4}\left(\tfrac{3}{2}E(Y_2) - 1\right).$$

In general, if the last $G$ was on the $(n-k)$th spin ($k = 1, 2, \ldots, n-1$) which occurs with probability $\frac{1}{4}(\frac{3}{4})^{k-1}$ we get a contribution (to $E(X_n)$) of

$$\tfrac{1}{4}(\tfrac{3}{4})^{k-1}\left[\tfrac{1}{4}\left(0 + E(Y_k) + \tfrac{1}{2}E(Y_k) - 1\right)\right] = \tfrac{1}{16}(\tfrac{3}{4})^{k-1}\left(\tfrac{3}{2}E(Y_k) - 1\right).$$

Finally, if there were no previous $G$'s, which occurs with probability $(\frac{3}{4})^{n-1}$, we get a contribution (to $E(X_n)$) of

$$(\tfrac{3}{4})^{n-1}\left[\tfrac{1}{4}\left(0 + E(Y_n) + \tfrac{1}{2}E(Y_n) - 1\right)\right] = \tfrac{1}{4}(\tfrac{3}{4})^{n-1}\left(\tfrac{3}{2}E(Y_n) - 1\right).$$

The sum of all these is, of course, $E(X_n)$. Therefore,

$$E(X_n) = \tfrac{1}{16}\sum_{k=0}^{n-2} (\tfrac{3}{4})^{k-1}(2+\tfrac{3}{2}(\tfrac{5}{6})^{k-1}(p-2)) + \tfrac{1}{4}(\tfrac{3}{4})^{n-1}(2+\tfrac{3}{2}(\tfrac{5}{6})^{n-1}(p-2))$$

$$= \frac{p}{4} + \left(\frac{p-2}{8}\right)(\tfrac{5}{8})^{n-1}.$$

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## CROWDED POSETS AND RAMSEY'S THEOREM

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In their recent paper on topological groups, Mukhin and Starukhina give a basic result on lattices ([1], Lemma 1; restated in our Proposition 3). Our aim here is to generalize their result and to supply an improved proof based on the combinatorial theorem of Ramsey.

**Posets.** Our discussion uses little more than the basic definitions and standard notations concerning *posets* (or *partially ordered sets*). Two elements $x, y$ of a poset $A$ (with partial order $\leqq$) are *comparable* if either $x \leqq y$ or $y \leqq x$. A *chain* is a set of pairwise comparable elements and an *antichain* is a set of pairwise incomparable elements. By $\omega$ and $\omega^*$ we mean the usual posets formed by the nonnegative integers and by the nonpositive integers. The poset $A$ *has* (respectively *omits*) the poset $C$ if some (respectively no) isomorphic copy of $C$ occurs as a sub-poset of $A$. (Posets which omit $\omega^*$ are variously called "well-founded" or "partially well-ordered" or "with minimum condition" or "with descending chain condition".) Given two posets $A$ and $B$, their *direct product* $A \times B$ is the poset whose elements are the ordered pairs $(a, b)$ with $a$ in $A$ and $b$ in $B$, and such that the relation $(a, b) \leqq (c, d)$ holds between two such pairs in $A \times B$ if and only if both $a \leqq c$ (in $A$) and $b \leqq d$ (in $B$).

**Ramsey's theorem.** A version of the well-known "pigeonhole principle" asserts that when a "large" set $S$ is covered by a "few" sets $Q1, Q2, \cdots, Qk$, then at least one $Qi$ covers a "rather large" subset $T$ of $S$. Ramsey's theorem ([2], or cf. chapter 1 of [3] or chapter 4 of [4]) generalizes this principle from coverings of $S$ itself to coverings of the $r$-subsets of $S$ (i.e., the subsets of $S$ of some fixed finite size $r$). When $S$ is infinite we have:

1. THEOREM (Ramsey). *Suppose given an infinite set $S$ and positive integers $k, r$ and sets $Q1, Q2, \cdots, Qk$ such that every $r$-subset of $S$ belongs to at least one of the sets $Qi$. Then there is an infinite subset $T$ of $S$ such that some one $Qi$ contains every $r$-subset of $T$.*

This theorem has an important direct application to posets. Given an infinite poset $A$, let $S$ be a denumerable subset and let $f$ be a one-one function on $S$ onto the nonnegative integers. For $i = 1, 2$ or 3, define $Qi$ to be the set of all pair-subsets $\{a, b\}$ of $S$ for which the following condition (i) holds:

(1) $a, b$ are incomparable in $A$;

(2) $a < b$ (in $A$) and $f(a) < f(b)$, *or* $b < a$ and $f(b) < f(a)$;

(3) $a < b$ and $f(a) > f(b)$, *or* $b < a$ and $f(b) > f(a)$.

Choose $T$ and $Qi$ by Ramsey's theorem. Then (according as $i$ is 1, 2 or 3) the sub-poset of $A$ induced by $T$ is either an infinite antichain or is isomorphic with $\omega$ or is isomorphic with $\omega^*$. That is, we have:

2. COROLLARY. *Every infinite poset has an infinite antichain or has $\omega$ or $\omega^*$.*

**Crowding.** In [1] a poset is called *crowded* if every antichain is finite. The authors of [1] are interested in classifying topological groups having a crowded lattice of closed subgroups, but they first

discuss the more general question: *when is the direct product $A \times B$ of posets $A, B$ a crowded poset?* Our Theorem 4 below answers this question. The "necessity" part of the theorem is more or less contained in the discussion of [1] but the "sufficiency" part strongly improves the best sufficiency result in [1], namely the following (Lemma 1(c) of [1]):

3. PROPOSITION. *If $A$ is a crowded lattice which omits $\omega^*$, then $A \times \omega$ is crowded.*

4. THEOREM. *Suppose $A, B$ are posets. In order that $A \times B$ be crowded it is necessary and sufficient that $A$ and $B$ both be crowded and that at least one of the following conditions holds: $A$ is finite, $B$ is finite, $A$ and $B$ both omit $\omega$, $A$ and $B$ both omit $\omega^*$.*

*Proof of necessity.* Suppose $A \times B$ crowded. Fix $b$ in $B$. $A$ is isomorphic with the sub-poset $A \times \{b\}$ of $A \times B$; since sub-posets of crowded posets are crowded, $A$ is crowded. Similarly, $B$ is crowded. Suppose $A$ and $B$ are infinite and that they do not both omit $\omega$ and that they do not both omit $\omega^*$. We complete the proof of necessity by obtaining a contradiction to the crowding of $A \times B$. By Corollary 2, *either* $A$ has $\omega$ and $B$ has $\omega^*$ *or* $A$ has $\omega^*$ and $B$ has $\omega$. In either case, $A \times B$ has $\omega \times \omega^*$; but in $\omega \times \omega^*$ an infinite antichain is formed by the "diagonal" set $D$ of all pairs $(n, -n)$ for nonnegative integers $n$.

*Proof of sufficiency.* Suppose $A$ and $B$ are both crowded. If one of $A, B$—$B$, say—is finite, then $A \times B$ is a finite union of crowded posets (namely, of posets of the form $A \times \{b\}$ with $b$ in $B$), and therefore is crowded. Suppose then that $A$ and $B$ are both infinite. Then $A$ and $B$ both omit $\omega$ or both omit $\omega^*$, say both omit $\omega$. Suppose $A \times B$ has an infinite antichain $S$. Applying Ramsey's theorem with $k = 5$ and $r = 2$, we shall obtain a contradiction.

Given any pair $s, t$ of distinct elements of $S$ we can write $s = (a, b)$ and $t = (c, d)$, where $a$ and $c$ belong to $A$ and $b$ and $d$ belong to $B$. Using this notation we define $Qi$ (for $i = 1,2,3,4$ or 5) to be the set of all pairsubsets $\{s, t\}$ of $S$ for which the following property (i) holds: (1) $a = c$ ("$A$-equal"); (2) $b = d$ ("$B$-equal"); (3) $a$ and $c$ are incomparable in $A$ ("$A$-incomparable"); (4) $b$ and $d$ are incomparable in $B$ ("$B$-incomparable"); (5) $a < c$ but $b > d$, or $a > c$ but $b < d$ ("crossed").

The sets $Qi$ contain all pairs from the antichain $S$, so by Ramsey's theorem $S$ has an infinite subset $T$ with all pairs of distinct elements of $T$ belonging to some common $Qi$. We claim that $i = 5$. Indeed, let $C$ be the domain of $T$ (i.e., the set of first members of pairs in $T$) and let $D$ be the range of $T$ (i.e., the set of second members of pairs in $T$). If $i = 1$, the fact that $T$ is an infinite antichain makes $D$ an infinite antichain in $B$, and if $i = 4$, $D$ is again an infinite antichain in $B$. The crowding of $B$ therefore ensures that $i$ is not 1 and not 4. Similarly, the crowding of $A$ ensures that $i$ is not 2 and not 3. Hence $i$ is 5. $C$ is then an infinite chain. Since $A$ omits $\omega$, Corollary 2 ensures that $C$ has $\omega^*$. By the "crossed" property of pairs of $T$, $D$ has $\omega$, contradicting $B$ omitting $\omega$.

REMARKS. (1) The above sufficiency proof circumvents an apparent error in the proof of Proposition 3 given in [1]. (2) Theorem 4 has the following consequence. Suppose the poset $C$ is the direct product of finitely many factors, one of which is the infinite poset $A$. Then $C$ is crowded if and only if $A \times B$ is crowded for each factor $B$ different from $A$.

Following [1] we define a poset to be *finitely crowded* if there is a finite bound for the size of antichains. Note that if $A, B$ are both infinite, then $A \times B$ is not finitely crowded. Indeed, by Corollary 2, $A \times B$ is either not crowded or else has one of $\omega \times \omega$, $\omega^* \times \omega^*$, $\omega \times \omega^*$. The last of these three posets is not crowded. The other two posets are not finitely crowded; for, given any positive integer $m$, each of these posets has an antichain of size $m+1$ formed by the pairs $(a, b)$ such that $|a + b| = m$.

If we replace "crowded" by "finitely crowded" throughout Theorem 4 and its proof, the necessity part remains valid, and so does the sufficiency part for $A$ or $B$ finite. Hence we obtain the following result (quoted in [1] as Lemma 1(d)):

5. THEOREM. *$A \times B$ is finitely crowded if and only if $A$ and $B$ are both finitely crowded and at most one of $A, B$ is infinite.*

**Acknowledgment.** The author wishes to thank the referees and also his colleagues at St. Mary's College, in particular Professor T. Erickson, for helpful suggestions concerning the presentation.

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## ON THE CONVERGENCE OF A SEQUENCE OF $m$ TIMES DIFFERENTIABLE FUNCTIONS

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In this paper we extend a well-known theorem from classical analysis, which is obtained by putting $m = 1$ in our theorem. Assuming this classical result, we prove the following generalization:

THEOREM. *Assume that every term of the sequence* $\{f_n\}_{n=1}^{\infty}$ *is a real-valued function which is m times differentiable on an open finite interval* $(a, b)$, *further, that there exist m points* $x_0, \ldots, x_{m-1}$ *in* $(a, b)$ *such that the sequence* $\{f_n(x_i)\}_{i=1}^{\infty}$ *converges for* $i = 0, 1, \ldots, m-1$. *If there is a function* $g\colon (a, b) \to \mathbb{R}$ *such that* $f_n^{(m)} \to_{n\to\infty} g$ *uniformly on* $(a, b)$, *then there exists a function* $f\colon (a, b) \to \mathbb{R}$, *satisfying the following*:

(a). $f$ *is m times differentiable on* $(a, b)$ *and* $f^{(m)}(x) = g(x)$ *for every* $x \in (a, b)$.

(b) $f_n^{(i)} \to_{n\to\infty} f^{(i)}$ *uniformly on* $(a, b)$ *for* $i = 0, 1, \ldots, m$.

*Proof*: As mentioned above, we will assume the theorem valid in the situation $m = 1$. Now we assume for simplicity $x_0 = 0$, and consider the sequence

$$\{f_n^{(m-1)}(x) - f_n^{(m-1)}(0)\}_{n=1}^{\infty}.$$

Evaluated at $x = 0$, this sequence obviously converges, and since its derived sequence equals $\{f_n^{(m)}(x)\}_{n=1}^{\infty}$, we conclude, by our assumption, that there exists a function $g_1$ satisfying the conclusions in the theorem (for $m = 1$). Especially we have that

$$f_n^{(m-1)}(x) - f_n^{(m-1)}(0) \xrightarrow[n\to\infty]{} g_1(x)$$

uniformly on $(a, b)$.

As the next step, consider the sequence

$$\{f_n^{(m-2)}(x) - f_n^{(m-2)}(0) - x f_n^{(m-1)}(0)\}_{n=1}^{\infty}.$$

Again, this converges at $x = 0$, and its derived sequence equals $\{f_n^{(m-1)}(x) - f_n^{(m-1)}(0)\}_{n=1}^{\infty}$, so by the same argument as above, we obtain a function $g_2$ such that

$$f_n^{(m-2)}(x) - f_n^{(m-2)}(0) - x f_n^{(m-1)}(0) \xrightarrow[n\to\infty]{} g_2(x)$$

uniformly on $(a, b)$.

After $m$ steps with this procedure, we can conclude that the sequence

$$\left\{f_n(x) - f_n(0) - \cdots - \frac{1}{i!} x^i f_n^{(i)}(0) - \cdots - \frac{1}{(m-1)!} x^{m-1} f_n^{(m-1)}(0)\right\}_{n=1}^{\infty}$$

converges (uniformly) to a function $g_m(x)$ on $(a, b)$.

Since the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges for $x = x_i$, $i = 0, 1, \ldots, m-1$, we get that the sequence

$$\{P_n(x)\}_{n=1}^{\infty} = \left\{ f_n(0)x + \cdots + \frac{1}{(m-1)!} f_n^{(m-1)}(0) x^{m-1} \right\}_{n=1}^{\infty}$$

converges for $x = x_i$, $i = 1, 2, \ldots, m-1$.

Define $\alpha_i = \lim_{n\to\infty} P_n(x_i)$, $i = 1, 2, \ldots, m-1$ and $\beta_i^{(n)} = P_n(x_i) - \alpha_i$. Then

(1) $$P_n(x_i) = \alpha_i + \beta_i^{(n)}, \quad \text{where} \quad \beta_i^{(n)} \xrightarrow[n\to\infty]{} 0, \qquad i = 1, \ldots, m-1.$$

Now the determinant

(2) $$\begin{vmatrix} x_1 & \cdots & x_1^{m-1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_{m-1} & \cdots & x_{m-1}^{m-1} \end{vmatrix}$$

is essentially a Vandermonde determinant, and since $x_i \neq 0$ and $x_i \neq x_j$, $i \neq j$ for $i, j = 1, 2, \ldots, m-1$, this is different from zero. This implies that the set of equations (1) can be solved uniquely for every $n = 1, 2, \ldots$ with respect to $y_n^i = (1/i!) f_n^{(i)}(0)$, $i = 1, 2, \ldots, m-1$. By Cramer's rule and the continuity of determinants it follows that $y_n^i \to_{n\to\infty} \gamma^i$, $i = 1, 2, \ldots, m-1$, where $\gamma^i$ is the quotient of two determinants with (2) as denominator and (2) with the $i$'th column replaced by the column $[\alpha_1, \ldots, \alpha_{m-1}]^T$ as numerator.

From this we conclude that the sequence $\{f_n^{(i)}(0)\}_{n=1}^{\infty}$ converges for $i = 1, 2, \cdots, m-1$.

Now we are in a position to apply the theorem (for the situation $m = 1$) successively, and we easily establish the general version.

REMARK: By considering one-sided derivatives at the end-points, the theorem remains equally valid for a closed interval.

We finish by showing that the condition in the theorem on the number of points where $\{f_n(x)\}_{n=1}^{\infty}$ converges cannot be weakened. We show this by the following example:

Define the function $f_n$ on an open interval $(a, b)$ by

$$f_n(x) = n \prod_{i=1}^{m-1} (x - x_i), \qquad x_i \in (a, b)$$

for $i = 1, 2, \ldots, m-1$, $n = 1, 2, \ldots$.

The sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges only for $x = x_i$, $i = 1, 2, \ldots, m-1$, that is, for at most $m-1$ points. Each $f_n$ is, of course, $m$ times differentiable, and we find

$$f_n^{(m-1)}(x) = (m-1)!\, n \quad \text{and} \quad f_n^{(m)}(x) = 0.$$

Here $\{f_n^{(m)}\}_{n=1}^{\infty}$ converges uniformly on $(a, b)$, but since $\{f_n^{(m-1)}(x)\}_{n=1}^{\infty}$ diverges for every $x \in (a, b)$, it is impossible to find a function which satisfies the conclusion of the theorem with respect to this sequence.

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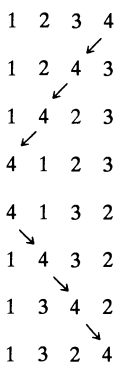
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## GENERATING PERMUTATIONS WITH NONDISTINCT ITEMS

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One problem of common interest is to generate all permutations (see [5]) when all items are distinct, the algorithm in [3] and [6] generates all permutations by adjacent-transpositions. In the present paper, we generate all permutations when not all items are distinct. Our algorithm is a natural extension of Johnson's method and uses Lehmer's method [4] as a subprocedure. For other approaches to the same problem, see [1, 2].

For four distinct items, the first 8 permutations generated by Johnson's algorithm would be:




Note that the largest item 4 first sweeps from right to left and then from left to right where the relative positions of all other digits remain unchanged during each sweep. In our algorithm for repeated items, if there are two or more largest items, these largest items also sweep from right to left and left to right just as in Johnson's algorithm. Our algorithm reduces to Johnson's when all items are distinct and to Lehmer's algorithm when there are only two kinds of items.

In Lehmer's algorithm for generating combinations, a binary sequence represents a possible selection. The sequence 01011 means that the second, fourth and fifth items are selected. All binary sequences of five digits with three 1's correspond to all possible combinations of three items out of five items. But we can also view the same algorithm as an algorithm for generating all permutations with two kinds of items, say $\alpha$ and $\beta$. Since Lehmer's algorithm generates all combinations lexicographically, the algorithm essentially changes the sequence 00111 step-by-step to 11100. This can be viewed as sweeping the block of 111 from right to left.

Let the multiset under consideration consist of the first $n$ integers $(1, 2, \ldots, n)$ and each integer $i$ has $c_i$ copies in the multiset. In the following context, we use *items* to denote distinct integers when possible confusion arises. The central idea of the algorithm is to interweave the $c_n$ copies of $n$ into each permutation of $n-1$ integers to get all possible "lexicographical" permutations for that $(n-1)$-integer permutation. In other words, for each permutation consisting of $n-1$ integers, a block of $n$'s is interwoven from one end to the other in such a way that all possible $n$-integer permutations are generated with the relative positions of the elements of the original $(n-1)$-integer permutation unchanged. The permutations are lexicographically increasing or decreasing, depending upon whether the $n$-block moves from right to left or from left to right.

For each item an arrow is associated with it to tell which direction it should be moved to. We define *downward neighbor* of an integer $p$ to be the integer adjacent to $p$ in the direction pointed by the arrow of $p$ and the *upward neighbor* to be the other adjacent neighbor. If there are two or more copies of item $p$ in a permutation, the rightmost (leftmost) $p$ is said to be the *upwardmost* one if the arrow of $p$ is pointing to the left (right). And an integer $p$ in a given permutation is *mobile* if the downward neighbor of $p$ is smaller than $p$. For example, if the sequence of integers and their arrows are as shown,

$$\vec{1}\ \overleftarrow{2}\ \vec{4}\ \overleftarrow{3}\ \vec{4}$$

then the mobile integers are $\overleftarrow{2}$, $\overleftarrow{4}$ and $\overleftarrow{4}$. The rightmost $\overleftarrow{4}$ is the upwardmost one. For the sequence

$$\overrightarrow{4}\ \overrightarrow{4}\ \overrightarrow{3}\ \overrightarrow{4}\ \overrightarrow{3}\ \overrightarrow{2}\ \overleftarrow{1},$$

the leftmost $\overrightarrow{4}$ is the upwardmost one but is not mobile.

Now we are able to state the algorithm as follows:

**Algorithm**

STEP 0. Put all integers in nondecreasing order as a sequence with all arrows pointing to left.

STEP 1. Output the present sequence. If there is no mobile integer, stop.

STEP 2. Among all largest mobile integers $p$ (if more than one), let $p^*$ be the upwardmost one. Interchange $p^*$ with its downward neighbor.

STEP 3. Assume that the arrow of $p^*$ is pointing to the left. Among all integers larger than $p^*$ and to the right of $p^*$, let the leftmost integer occupy the position $i$ in the sequence. Move all other $p$'s (if any) which are $p^*$'s upward neighbors to the position $i-1$, $i-2$, etc. (A new permutation has just been generated.) Do the symmetric thing if the arrow of $p^*$ is pointing to the right.

STEP 4. Change all directions of arrows of items larger than $p$. Return to step 1.

Note that step 3 is the same idea as in Lehmer's algorithm to maintain "lexicographical" order and to generate all possible permutations.

Let us list down the first several permutations generated by the algorithm for the multiset $\{1,1,2,2,2,3\}$; (i.e. $n=3$, $c_1=2$, $c_2=3$, $c_3=1$). An integer without an arrow is assumed to have an arrow pointing to the left.

$$\begin{array}{l}
1\ 1\ 2\ 2\ 2\ 3\\
1\ 1\ 2\ 2\ 3\ 2\\
1\ 1\ 2\ 3\ 2\ 2\\
1\ 1\ 3\ 2\ 2\ 2\\
1\ 3\ 1\ 2\ 2\ 2\\
3\ 1\ 1\ 2\ 2\ 2\\
\overrightarrow{3}\ 1\ 2\ 1\ 2\ 2\\
1\ \overrightarrow{3}\ 2\ 1\ 2\ 2\\
1\ 2\ \overrightarrow{3}\ 1\ 2\ 2\\
1\ 2\ 1\ \overrightarrow{3}\ 2\ 2\\
1\ 2\ 1\ 2\ \overrightarrow{3}\ 2\\
1\ 2\ 1\ 2\ 2\ \overrightarrow{3}\\
1\ 2\ 2\ 1\ 2\ \overleftarrow{3}\\
\overleftarrow{3}\ 1\ 2\ 2\ 1\ 2\\
\overrightarrow{3}\ 1\ 2\ 2\ 2\ 1\\
1\ 2\ 2\ 2\ 1\ \overrightarrow{3}\\
2\ 1\ 1\ 2\ 2\ \overleftarrow{3}\quad \text{Note that the rightmost } \overleftarrow{2}\text{'s are not pushed "too far."}\\
\cdots\cdots\cdots
\end{array}$$

**Proof of the algorithm.** The proof is by induction on the number of items.

(1) If there are only two kinds of items, we can consider the larger items as $\beta$'s and the smaller items as $\alpha$'s. Then the algorithm generates all permutations just like Lehmer's algorithm.

(2) When there are $k$ items:

(a) By the algorithm, the $k$-block always "sweeps" from one end to the other for each $(k-1)$-item permutation. Consider $k$'s to be $\beta$'s and all other integers to be $\alpha$'s. By case 1), all possible $k$-item permutations with the relative positions of $k-1$ items in the original permutation fixed are generated by the sweep.

(b) As the $k$-block gets to each end, it stays there without any effect on the next step of the

algorithm, just as though the algorithm were applied to only $k-1$ items. By induction hypothesis all possible permutations of $k-1$ items are generated while $k$-block is at each end.

By (a) and (b), all possible permutations of $k$ items are generated. Q.E.D.

This research was supported by National Science Foundation DCR75-06270 and U.S. Army Research Office DAAG29-76-G-0031.

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## GENERALIZATION OF RICHERT'S THEOREM

J. L. BROWN, JR.

**1. Introduction.** The subject theorem asserts that each natural number $N > 6$ can be expressed as a sum of distinct primes; Richert's proof ([1]–[2]) relies on the result (Bertrand's postulate) that for the sequence $P_1 = 2$, $P_2 = 3$, $P_3 = 5, \ldots$ of positive primes, we have $P_{n+1} \leqq 2P_n$ for $n \geqq 1$. Moreover, Sierpiński ([3], pp. 143–144), in his development of Richert's result, shows that similar theorems may be obtained for more general sequences $\{S_i\}_1^\infty$ of positive integers satisfying $S_{n+1} \leqq 2S_n$ for sufficiently large $n$, say $n \geqq r$. This latter constraint on the growth of the sequence is coupled with a direct verification of the expansion for a consecutive run of integers to yield an inductive proof that all sufficiently large integers are representable as a sum of distinct terms from the sequence $\{S_i\}$.

Our purpose in this note is to give an alternative condition on the growth of the sequence $\{S_i\}$ which is weaker than $S_{n+1} \leqq 2S_n$ for $n \geqq r$, yet sufficient (in conjunction with an appropriate verification) to yield the expansion of all sufficiently large integers.

As applications, two results due to Dressler ([4]–[5]) are proved:

(i) Every positive integer $> 9$ can be written as a sum of distinct odd primes.

(ii) Every positive integer $> 45$ is the sum of distinct (odd) primes each of which is at least 11. (This result is misquoted in the MR abstract.)

A new result along these same lines is similarly easy to verify:

(iii) Every positive integer $> 57$ is the sum of distinct (odd) primes each of which is at least 13.

We also prove Sprague's result [6] stating that every integer $N > 128$ can be written as a sum of distinct terms from the sequence $\{1, 4, 9, 16, 25, 36, \ldots\}$ of positive squares. In addition, Sprague [7] provided a (nonconstructive) proof that for each integer $n \geqq 2$, there exists a largest positive integer $r_n$ that is *not* expressible as a sum of distinct $n$th powers of the positive integers. Thus $r_2 = 128$ and it has recently been found [8] with the aid of a computer that $r_3 = 12{,}758$.

**2. Generalization.** In the following, we shall use $\alpha_i$ to denote a generic binary expansion coefficient having either the value zero or one; a sum in the form $\Sigma_i^j$ with $j < i$ will be interpreted as zero.

LEMMA 1. *Let $\{S_i\}_1^\infty$ be a sequence of positive integers, and assume there exist fixed integers $r \geqq 0$, $K_0 \geqq 0$ such that*

(i) $S_{n+1} \leqq S_{r+1} + \Sigma_{r+1}^n S_i$ *for all* $n \geqq r$.

(ii) *Each integer $N$ in the range $K_0 \leqq N < K_0 + S_{r+1}$ has an expansion in the form*

$$N = \sum_1^r \alpha_i S_i.$$

*Then any positive integer $N$ satisfying $K_0 \leqq N < K_0 + S_{r+1} + \Sigma_{r+1}^n S_i$ for some $n \geqq r$ is expressible in the form*

$$N = \sum_1^n \alpha_i S_i.$$

*Proof*: The statement is true for $n = r$ by assumption (ii). For a proof by induction, assume that the Lemma is true for some $n_0 \geqq r$. We must show that $K_0 \leqq N < K_0 + S_{r+1} + \Sigma_{r+1}^{n_0+1} S_i$ implies an expansion at the form $N = \sum_1^{n_0+1} \alpha_i S_i$. Clearly it suffices to consider only those values of $N$ satisfying

$$K_0 + S_{r+1} + \sum_{r+1}^{n_0} S_i \leqq N < K_0 + S_{r+1} + \sum_{r+1}^{n_0+1} S_i.$$

Then, using (i),

$$K_0 \leqq K_0 + S_{r+1} + \sum_{r+1}^{n_0} S_i - S_{n_0+1} \leqq N - S_{n_0+1} < K_0 + S_{r+1} + \sum_{r+1}^{n_0} S_i,$$

so that by the induction hypothesis, there exist binary coefficients $\{\alpha_i\}$ such that

$$N - S_{n_0+1} = \sum_1^{n_0} \alpha_i S_i,$$

and the lemma follows.

Note that the conditions of Lemma 1 are sufficient for every integer $N \geqq K_0$ to have an expansion as a sum of distinct terms from $\{S_i\}$. The following lemma shows that when the condition used by Sierpiński holds, the verification of condition (i) in Lemma 1 is greatly simplified.

LEMMA 2. *Let $\{S_i\}_1^\infty$ be a sequence of positive integers and let $r \geqq 0$ be such that*

$$S_{n+1} \leqq 2S_n \quad \text{for} \quad n \geqq r+1.$$

*Then* (i) *of Lemma* 1 *holds for all $n \geqq r$.*

*Proof*: Note that (i) of Lemma 1 is always true for $n = r$. If (i) is true for some arbitrary $n_0 \geqq r$, then

$$S_{n_0+2} \leqq 2S_{n_0+1} = S_{n_0+1} + S_{n_0+1} \leqq S_{n_0+1} + S_{r+1} + \sum_{r+1}^{n_0} S_i = S_{r+1} + \sum_{r+1}^{n_0+1} S_i,$$

which is the statement of (i) for $n = n_0 + 1$. The lemma is immediate by induction. The theorems of Dressler are now simple consequences of Lemma 1.

THEOREM 1. (Dressler [4]): *Every positive integer $> 9$ can be written as a sum of distinct odd primes.*

*Proof*: Let $S_1 = 3$, $S_2 = 5$, $S_3 = 7, \ldots$ be the sequence of odd primes. Then, in Lemma 1, we take $K_0 = 10$ and $r = 6$. We see from Lemma 2 and Bertrand's postulate that (i) of Lemma 1 holds for all $n \geqq 6$. Direct verification that each of the integers in the range $[10, 29)$ is expressible as a sum of distinct terms from the set $S_1, \ldots, S_6$ concludes the proof.

THEOREM 2. (Dressler [5]) *Every positive integer* $>45$ *is expressible as a sum of distinct primes each of which is at least* 11.

*Proof*: Here $S_1 = 11$, $S_2 = 13$, $S_3 = 17, \ldots$; $K_0 = 46$, $r = 10$, and the verification requires that each integer in the range $[46, 93)$ be expressible as a sum of distinct terms from $S_1 = 11, \ldots, S_{10} = 43$.

THEOREM 3. *Every positive integer* $>57$ *is expressible as a sum of distinct primes each of which is at least* 13.

*Proof*: Take $S_1 = 13$, $S_2 = 17$, $S_3 = 19, \ldots$ as the sequence of primes $\geq 13$. Then with $K_0 = 58$, $r = 10$, verify directly that each integer in $[58,111)$ has an expansion as a sum of distinct terms from $S_1 = 13, \ldots, S_{10} = 47$. The theorems of Richert and Sprague are also amenable to proof using Lemmas 1 and 2.

THEOREM 4. (Richert, [1], [2]) *Let* $P_1 = 2$, $P_2 = 3$, $P_3 = 5$, $P_4 = 7$, $P_5 = 11$, $P_6 = 13, \ldots$ *be the sequence of positive primes. Then for every integer* $N \geq 7$, *we have* $N = \Sigma_1^\infty \alpha_i P_i$ *where* $\{\alpha_i\}$ *are binary coefficients.*

*Proof*: In Lemma 1, set $S_i = P_i$ for $i \geq 1$ and take $K_0 = 7$, $r = 5$. By Lemma 2, in conjunction with Bertrand's postulate, (i) holds for all $n \geq r = 5$. For (ii), we verify directly that each integer in the range $[K_0, K_0 + P_6) = [7,20)$ can be expressed as a sum of distinct terms chosen from $P_1, \ldots, P_5$. Then Lemma 1 yields the desired expansion for all $N \geq K_0 = 7$.

THEOREM 5. (Sprague, [6]) *Let* $\{S_i\}_1^\infty$ *be the sequence of squares*: $S_1 = 1$, $S_2 = 4, \ldots, S_n = n^2, \ldots$. *Then every integer* $N > 128$ *can be expressed as a sum of distinct squares.*

*Proof*: Note here that $S_{n+1} \leq 2S_n$ for $n \geq 3$. In Lemma 1, we take $K_0 = 129$, $r = 10$, so that (i) holds (Lemma 2) for all $n > 10$. One can then verify (ii) that each integer $N$ in the range $[K_0, K_0 + 121) = [129,250)$ has an expansion using only the terms, $1, 4, \ldots, 100$, and the result follows from Lemma 1.

**Note**: If $S_1 = 1$, $r = 0$ and $K_0 = 0$, then (ii) is vacuously satisfied and (i) yields (iii) $S_{n+1} \leq 1 + \Sigma_1^n S_i$ for $n > 0$ as a sufficient condition for every integer $N \geq 0$ to be expressible as a sum of distinct terms from $\{S_i\}$. A sequence possessing this latter property is termed *complete* and it is known ([9], [10]) that (iii) is also a necessary condition for completeness if $S_1 = 1$ and $\{S_i\}$ is a nondecreasing sequence of positive integers.

Lastly, we give a simple example using completeness to see that condition (i) of Lemma 1 is weaker than its alternative $S_{n+1} \leq 2S_n$ for $n \geq r$. Consider the sequence defined by

$$S_1 = 1, S_2 = 2, \begin{cases} S_{2n+1} = S_{2n} + 1 & \text{for} \quad n \geq 1 \\ S_{2n} = 2S_{2n-1} + 1 & \text{for} \quad n \geq 2; \end{cases}$$

i.e., the sequence 1, 2, 3, 7, 8, 17, 18, 37, 38, ....

Clearly (i) is satisfied with $r = 1$, $K_0 = 0$, while $S_{n+1} \geq 2S_n$ does not hold for all $n \geq r$ no matter how large $r$ is taken.

Two other references of related interest are given in [11] and [12].

*Acknowledgement*: The author would like to thank B. J. Entsminger for computational assistance and the referee for supplying some helpful comments and additional references.

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## RESEARCH PROBLEMS

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*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)*

### TWENTY QUESTIONS CONCERNING CONWAY'S SYLVER COINAGE

RICHARD K. GUY

"After the revolution, the Emperor abolished all pre-existing currency, and introduced a new monetary unit, the Oonit. The masters of the Imperial Mint, Hi and Lo, were alternately to decide each new denomination, and after each decision, sufficiently many coins of this value were to be struck. Each denomination was to be an integral number of Oonits, and no value was to be chosen which could be made up from coins already minted. All went well until Hi ordered a one Oonit coin, and the Workers of the Mint were thereafter thrown into unemployment. They rose in a body, and threw the unfortunate Hi from the tower at the quiet end of the capital, which has been known as the Hi Tower ever since."

JHWH [**9**] Conway

Consider the game, *Sylver Coinage*, defined as follows. Two players alternately name positive integers, subject only to the **sum condition**, that the integer named at any stage be not expressible as a sum of previously named integers, with repetitions allowed in forming this sum. The player who names 1, so ending the game, is the loser.

Sylvester [**14**] proved that any sequence satisfying the sum condition terminates. At any stage we have a set $M$ of non-negative integers. If $M$ is not empty, define $g = g(M)$ as the g.c.d. of the members of $M$. After a finite number of moves we arrive at a position with a smaller value of $g$; finally we reach a position with $g = 1$, and after finitely many further moves the game terminates. However, the game does not terminate after a *bounded* number of moves, nor is there a number $n$ such that after at most $n$ moves it necessarily becomes bounded. Indeed, consider the sequence of properties:

$P_0$: *there is some n such that after at most n moves the game ends.*

$P_k$ $(k > 0)$: *there is some n such that after at most n moves the game has some property h with* $h < k$.

For no finite $k$ does Sylver Coinage have property $k$, but it does have property $\omega$.

Define a *position* as an arbitrary set of non-negative integers. For any position $M$, let $M^+$ be the set of all finite sums of members of $M$, with repetitions allowed, and 0 included as the empty sum. Play from $M$ depends only on $M^+$, so we call $M$ and $N$ **equivalent** if $M^+ = N^+$. A classical theorem of von

Neumann and Morgenstern [**10**] asserts that a game of bounded play has a winning strategy for one of the players, and this has been generalized [**3**, **4**, **8**] to all terminating games. All published proofs seem to use transfinite induction, but there is also a simple non-constructive proof.

Follow Guy and Smith [**7**, see also **13**] and assign a position to the **outcome class** $\mathcal{P}$ if the Previous player has a winning strategy, and to the outcome class $\mathcal{N}$ if the Next player has such a strategy. A **follower** of a position $M$ means a position obtainable from $M$ by a single legal move, and a **later** position one obtainable from $M$ by any finite non-empty sequence of such moves. A **reply** to $M$ is a move which is legal from $M$, and a **good reply** is one which leads to a $\mathcal{P}$**-position** (position belonging to outcome class $\mathcal{P}$). If some follower of a given position has status $\mathcal{P}$, then that position has status $\mathcal{N}$, while if all followers have status $\mathcal{N}$ the original position had status $\mathcal{P}$. If a position had no status, the same must be true of one of its followers, and so on. So (using the axiom of choice in the general case; it is not needed for Sylver Coinage) we obtain an infinite sequence of legal moves, showing that the play need not terminate. The non-constructive nature of this proof suggests that even when a game is defined in a completely effective and computable way it need have no effectively computable strategy. Indeed this has been shown by Rabin [**11**], who has produced a terminating game satisfying the strongest of computability conditions which can be proved to have no computable winning strategy for either player. It is not obvious *a priori* (indeed not obvious at all) that Sylver Coinage has computable winning strategies from arbitrary initial positions; it is the contrary possibility from which the game derives much of its interest.

Another pleasing feature of the game is that although the first player has a winning strategy, this does not affect the playability of the game, since there does not seem to be any simple rule determining this strategy. Unlike the more familiar number-theoretic game of Nim [**2**], Sylver Coinage can still be interesting when played between experts. The first player to name 1, 2 or 3 loses, since a good reply to either of 2 or 3 is the other. We leave the reader to confirm that for each of the pairs $(4, 6)$, $(6, 9)$ and $(8, 12)$, either member is a good reply to the other; and also to find the non-constructive proof of R. L. Hutchings' beautiful theorem that the first player can win by naming a prime greater than 3. Here are twenty unanswered questions concerning Sylver Coinage.

1. Is there an effective technique for computing the status of a general position $M$?

2. Is there an effective technique for producing good replies when such exist?

If we could find a bound for the good replies to $M$ in terms of the largest member of $M$, the answers to questions 1 and 2 would be "yes". These questions are of interest even in restricted circumstances, say for positions with $g = 3$.

3. What is the status of the position $\{n\}$ for $n$ of the form $2^a 3^b$ (only a finite number of them can be $\mathcal{P}$-positions)? In particular:

4. What is the status of $\{16\}$?

5. What is the status of $\{18\}$?

It is conjectured that 24 is a good reply to 16, and that 12 and 27 are good replies to 18. Conway offers \$50 for the first establishment of the status of a number $2^a 3^b \geqq 16$.

6. Is $M$ a $\mathcal{P}$-position whenever $2M$ is a $\mathcal{P}$-position?

7. Is $M$ a $\mathcal{P}$-position whenever $3M$ is a $\mathcal{P}$-position?

Questions 6 and 7 derive from experimental evidence. The answer to 6 is "yes" if $M = \{2, 3\}, \{4, 6\}, \{4, 6, 4n + 1, 4n + 3\}$ or $\{4, 5, 11\}$. However, there is also evidence for the converses, whose falsity is evinced by the positions $\{5, 6, 19\}$ and $\{10, 12, 38, 7\}$ for question 6 and by $\{4, 5, 11\}$ and $\{12, 15, 33, 5\}$ for question 7, all these being $\mathcal{P}$-positions.

8. If the game is played "between intelligent players" is it always the case that the first person to make the game bounded is the loser?

This question is not completely well posed and is one of a number of places where Game Theory leaves the realm of Pure Mathematics and takes off into Psychology and Philosophy. An intelligent but cautious first player will always play a prime greater than 3 (if he is very cautious he will play 5, but

he may still lose on time if his opponent replies 4287503, say) and since the second player has to make the game bounded, the answer appears to be "yes". There may be winning strategies however, beginning with numbers of the form $2^a3^b$ (but not with any other composite numbers), and Lorne Houten and Wayne Erickson (written communication) offer the $\mathcal{P}$-position $\{16, 24, 28, 5\}$ as evidence that the answer is "no". It remains to establish that this position can arise in play between intelligent players!

9. Is there a winning strategy of bounded length?

10. Is there an $\mathcal{N}$-position with $g > 1$ for which all good replies lead to positions with $g = 1$?

The good reply 7 to $\{10, 12, 38\}$ shows that we cannot replace "all" by "some" in question 10. The author [**6**] has calculated $G(M)$, the Sprague-Grundy function [**1**, **5**, **12**] for a number of positions $M$ (*Sylver Coinage* can be considered as a last-player-winning game if the move 1 is prohibited). For example $G(\{2\}) = \omega$, $G(\{3\}) = 1$, $G(\{4, 5\}) = 3$ and $G(\{4, 10\}) = 1$.

11. Is $G(\{4\}) = \omega + 1$?

12. What is $G(\{6\})$?

13. What is the least ordinal which is *not* the $G$-value of any Sylver Coinage position? The answer is at most $\omega^2$.

Consider positions $M$ of the form $\{m, m_1, \ldots, m_{m-1}\}$, where $m_i \equiv i, \bmod m$. If for each $i$, $m_i$ is the least $n \in M^+$ which is congruent to $i, \bmod m$, then $M$ is called **$m$-reduced**. A pair $(a, b)$ for which $\{4, a, b\}$ is a $\mathcal{P}$-position with the 4-reduced form $\{4, a, 2a, b\}$ is called a **4-pair**. If $\{4, a, b\}$ is a $\mathcal{P}$-position with $ab$ odd, then just one of $(a, b)$, $(b, a)$ is a 4-pair. Conway and M. J. T. Guy have calculated all 4-pairs with $a < 100$ and R. Gerritse has extended this to $a \leqq 707$. The first few are $(5, 11)$, $(7, 13)$, $(9, 19)$, $(15, 33)$, $(17, 43)$, $(21, 51)$, $(23, 57), \ldots$ .

14. Does $b/a$ tend to a limit as $a \to \infty$ while $(a, b)$ remains a 4-pair?

Conway can prove that $2 < b/a < 3$ and can get better bounds on the assumption that $b$ is a monotonic function (separately for $a = \pm 1, \bmod 4$) of $a$. A pair $(a, b)$ for which $\{5, a, b\}$ is a $\mathcal{P}$-position is called an **$\alpha$-pair**, a **$\beta$-pair** or a **$\gamma$-pair** according as its 5-reduced form is $\{5, a, 2a, b, 2b\}$, $\{5, a, 2a, b, a + b\}$ or $\{5, a, 2a, 3a, b\}$. If $\{5, a, b\}$ is a $\mathcal{P}$-position and $ab$ is not a multiple of 5, then $(a, b)$ or $(b, a)$ is an $\alpha$-pair, $\beta$-pair or $\gamma$-pair. Here are some examples:

$\alpha$-pairs: $(2, 3)$, $(7, 8)$, $(17, 18)$, $(22, 23)$, $(39, 41)$, $(42, 43), \ldots$ ,

$\beta$-pairs: $(16, 18)$, $(18, 14)$, $(26, 23)$, $(28, 29)$, $(29, 27)$, $(38, 39)$, $(42, 41), \ldots$ ,

$\gamma$-pairs: $(4, 11)$, $(6, 19)$, $(9, 31)$, $(12, 33)$, $(13, 37)$, $(21, 69)$, $(24, 71)$, $(32, 103)$, $(34, 101)$, $(36, 109)$, $(67, 218)$, $(82, 268)$.

15. Does $b/a \to 1$ as $a \to \infty$ while $(a, b)$ remains an $\alpha$-pair?

16. Does $b/a \to 1$ as $a \to \infty$ while $(a, b)$ remains a $\beta$-pair?

17. Does $b/a \to 3$ as $a \to \infty$ while $(a, b)$ remains a $\gamma$-pair?

A stronger conjecture than that implied by question 15. is the *$\alpha$-hypothesis*:

18. If $(a, b)$ is an $\alpha$-pair is $|a - b| = 1$ or 2?

A noticeable feature is that the existence of a $\gamma$-pair $(a, b)$ seems to "prompt" others $(c, d)$ with $b \approx c$. For example $(4, 11)$ prompts $(9, 31)$, $(12, 33)$ and $(13, 37)$ which in turn prompt $(32, 103)$, $(34, 101)$ and $(36, 109)$, while $(6, 19)$ prompts $(21, 69)$ and $(24, 71)$ which prompt $(67, 218)$, or is this another example of the law of small numbers? An example of an unprompted $\gamma$-pair is $(82, 268)$.

19. Are there infinitely many $\gamma$-pairs?

The table gives the status of, and usually the good replies to, each position $M$ which is a subset of $\{6, 7, 8, 9, 10, 11, 12\}$. The set of even members of $M$ is given on the left, and the set of odd members at the head of the table. An entry with closed braces is the complete set of good replies to the relevant $M$, so "{}" denotes a $\mathcal{P}$-position. If the right brace is omitted, there *are* good replies (so that $M$ is an $\mathcal{N}$-position) but not all of these are necessarily given. The last entry, for example, should contain (at least) all primes greater than 3. The five entries "{" indicate known $\mathcal{N}$-positions with no known good replies.

| | {7, 9, 11} | {7, 9} | {7, 11} | {7} | {9, 11} | {9} | {11} | {} |
|---|---|---|---|---|---|---|---|---|
| {6, 8, 10} | {} | {11} | {9} | {} | {4, 5, 7} | {5} | {} | {4, 7, 11} |
| {6, 8} | {10} | {} | {} | {9, 10, 11} | {4, 5} | {5, 7} | {7, 10} | {4} |
| {6, 10} | {8} | {} | {15} | {8, 9} | {4} | {7} | {8, 13} | {4} |
| {6} | {} | {8, 10, 11} | {8, 9} | {16} | {4, 7} | {} | {26} | {4, 9} |
| {8, 10, 12} | {4, 5, 6} | {4} | {13} | {5, 6} | {13, 14, 15} | {23} | {6} | {14} |
| {8, 12} | {5} | {6} | {6} | {5} | {} | {11, 15} | {9, 13} | {} |
| {10, 12} | {4} | {4, 6} | {16} | {} | {} | {11, 13, 14} | {9 | {7 |
| {12} | {6} | {15} | {27} | {10 | {8, 10} | {6} | { | {8 |
| {8, 10} | {4, 5, 6} | {4} | {} | {5, 6, 11} | {23} | {13, 15} | {6, 7} | {22} |
| {8} | {5} | {6} | {6, 10} | {5} | {12} | {21 | { | {12 |
| {10} | {4} | {4, 6} | {8} | {12} | {12} | { | { | {5 |
| {} | {6} | {19, 24} | {24, 34} | {} | { | {6} | {} | {5, 7, 11, 13, |

20. Can the reader complete the table?

"The reader who, armed with this table, ventures to play the game in polite society, should be aware of an important point in the etiquette of the game. It is customary for a player who feels that his victory is assured, to resign by naming one of the numbers 1, 2 or 3. This custom is said to have arisen from the tradition that Hi, who had foreseen the fate which was about to befall his beloved twin brother, nobly took this fate upon himself."

J. H. Conway

I am indebted to J. H. Conway for the material between the two quotations. I also thank R. Gerritse and C. A. B. Smith for constructive suggestions. Gerritse has done extensive computer calculations: he also suggests a misère form of the game in which the first person to mention a prime (or 1) is the loser.

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## CLASSROOM NOTES

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### A NEW APPLICABLE PROOF OF THE EULER CIRCUIT THEOREM

ALAN TUCKER

One of the mainstays of many liberal-arts courses in mathematical concepts is the Euler Circuit Theorem. The theorem is also the first major result in most graph theory courses. In this note, we give an application of this theorem to street-sweeping and, in the process, find a new proof of the theorem. A *graph* consists of a finite set of *vertices* and a set of *edges* each linking some pair of distinct vertices (several different edges may link the same two vertices). A graph is *connected* if there exists a path of edges between any two vertices. The number of edges incident to a vertex is called the *degree* of the vertex. A circuit passing through every edge just once (and every vertex at least once) is called an *Euler circuit.*

THEOREM. *A graph possesses an Euler Circuit if and only if the graph is connected and each vertex has even degree.*

Euler "proved" this theorem in generalizing the answer to the question of whether there existed a route in old Koenigsberg that crossed each bridge among some islands in the River Pregel exactly once. (Euler's argument has been translated [2] and makes very instructive reading.) This theorem is in a way very surprising, since simply stated properties are usually extremely difficult to characterize in graph theory. For example, no characterization is known for graphs with a circuit that passes through each *vertex* exactly once (this "paradox" should be stressed to students). Equally surprising is the elementary proof available for this theorem. The standard proof may be sketched as follows (necessity is trivial):

(i) Start at a chosen vertex $x$ and arbitrarily trace out a route which never repeats an edge; since the even-degree condition assures that each vertex arrived at can be left, the route must inevitably terminate at $x$.

(ii) If the route does not contain all edges, choose a vertex $x'$ on this route and incident to an unused edge; repeat step (i) starting from $x'$ and using the graph consisting only of the unused edges; and incorporate the resulting circuit into the original route (the graph of unused edges has even-degree vertices).

(iii) Repeat step (ii) until all edges are included in the route. In a rigorous proof, a bit of care is required to show that a vertex $x'$ always exists.

Consider the following application of the Euler circuit concept. We want to route street-sweeping vehicles in a district of a city. The problem breaks naturally into temporal pieces, so that during a given one- or two-hour period, which repeats every two days, one works with just a particular subset of curbsides in the district (the curbsides freed for sweeping by no-parking regulations). We make an associated graph $G$ with a vertex for each street corner in the district and an edge for each block-long curbside to be swept. In practice, the edges must be oriented in the direction of traffic on the side of the street where the curb is, and one uses an oriented version of the Euler circuit theorem, but we shall ignore directions in this discussion. Each edge has a number representing the time needed to sweep it.

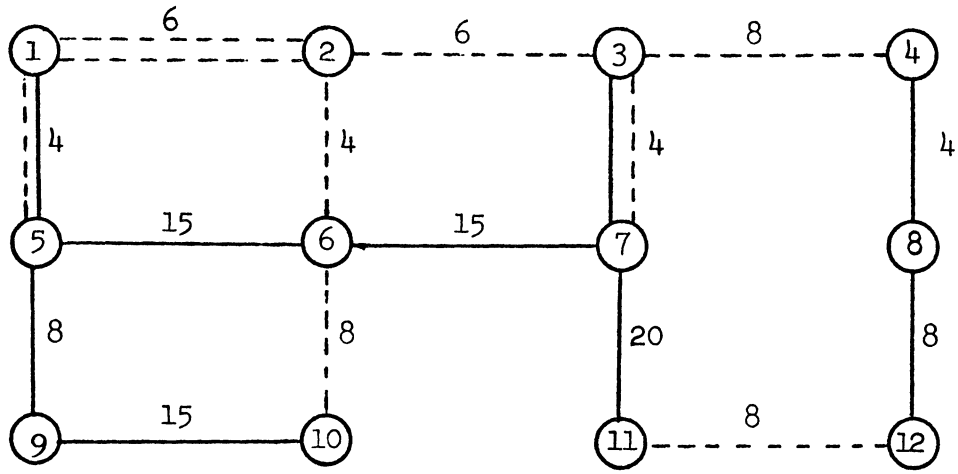



FIG. 1:
Graph for sample sweeping problem.
Edges to be swept are solid, additional deadheading edges are dashed. Edge numbers are sweeping times (deadheading times are half as long). If no U-turn at vertex 1 is permitted, then this set of dashed edges minimizes dead heading.

Our goal is a minimal-time tour that traverses all edges in the graph. Ideally we want a tour covering each edge of $G$ just once; i.e., an Euler circuit. If the time required to sweep all edges exceeds the time period, we still seek an Euler circuit and later break it into appropriate pieces (see [3] for details). Normally $G$ will not satisfy the conditions in the theorem, and so *deadheading* edges, edges where the sweeper raises its broom and moves toward some other corner where sweeping continues, are added to $G$ to enlarge $G$ into a graph $G'$ satisfying the conditions of the theorem; see Figure 1. (It takes half as much time to deadhead along an edge as to sweep it.) The deadheading edges may be copies of edges in $G$ or they may represent other (unswept) streets in the district. Once we know that $G'$ satisfies the conditions required for an Euler circuit, the question arises of which Euler circuit is "best" — there was a lot of choice in the construction outlined above. The best type of tour from a practical standpoint is a circuit with the fewest turns, especially U-turns or left turns which take extra time and tie up traffic. For technical reasons, it is also most important to switch sides of the street as infrequently as possible. Suppose that at every corner we assign a penalty to each different possible pairing of two incident edges at the corner (e.g., a pairing that represents going straight through an intersection has penalty 0, a U-turn has 10; note that in the undirected case we cannot distinguish between a right and a left turn). Then the ideal Euler circuit in $G$ would be one such that *at each individual corner* the sum of the penalties caused by the circuit's pairings is minimized. Clearly, the construction of an Euler circuit outlined above is a "global" sort of procedure that cannot be made sensitive to these "local" questions. On the other hand, the key condition of even degrees in our theorem is a local condition. Indeed, this condition says merely that the incident edges at any particular vertex can be paired off. Suppose then that at each vertex we make a set of pairings whose sum of penalties is minimal. Suppose we "tie together" the ends of the paired edges at each vertex; e.g., if edges $e_1, e_2$ are paired at vertex $v$, any path that enters (leaves) $v$ on edge $e_1$ must leave (enter) on edge $e_2$; see Table 1. Then looking at the whole graph, we see that the edges form a collection of chains which, having no ends, must be circuits; see Table 2. At this point, we can break up a couple of the pairings to effect a combination of the circuits into one circuit. Thus we have a near-ideal Euler circuit. In practice, when edges are paired to minimize penalties in a graph with about 200 edges, no more than 5 or 6 circuits are usually formed, and so the joining of circuits affects only a few of the minimized pairings.

| *Vertex* | *Pairings* |
|---|---|
| 1 | 2–1–5, 2–1–5 |
| 2 | 6–2–1, 3–2–1 |
| 3 | 7–3–2, 7–3–4 |
| 5 | 1–5–9, 1–5–5 |
| 6 | 5–6–7, 10–6–2 |
| 7 | 6–7–3, 11–7–3 |

TABLE 1: A sample set of pairings of edges incident to each vertex of graph in Figure 1. Underlining indicates deadheading edges. The table is interpreted as follows: the sequence 2–1–5 means a deadheading edge between vertex 2 and vertex 1 is followed (or preceded) by a sweeping edge between vertex 1 and vertex 5. Vertices not in table have degree 2 and hence have trivial pairings.

*Circuits*: 1–5–9–10–6–2–1–5–6–7–3–2–1, 7–3–4–8–12–11–7

*Sweeper Tour*: 1–5–9–10–6–2–1–5–6–7–3–4–8–12–11–7–3–2–1

TABLE 2: By tracing along the edges of the graph in Figure 1 in accordance with the pairings given in Table 1, we obtain the two circuits listed in this table. Then we combine the circuits at a common vertex, say vertex 3, that is, the pairings at vertex 3 are switched to: 7–3–4, 7–3–2. Now we have the desired sweeper tour.

The preceding reasoning also yields a new proof of our theorem. Formally, the proof goes: at each vertex arbitrarily pair off, and link together, the incident edges. The result is a set of chains which, having no ends, must be circuits. Repeatedly combine any two circuits with a common vertex until we have vertex-disjoint circuits. Since the graph is connected, there is just one circuit — an Euler circuit. This proof has the advantage that it uses the even-degree constraint directly and it immediately accomplishes the key step of getting each edge on a circuit. The standard proof uses the constraint indirectly and requires many iterations to pick up all the edges. Also, a rigorous statement of the standard proof is longer. On the other hand, this new construction builds circuits indirectly while the other construction is direct. Finally, the new method works very well for the many variations of the Euler circuit result that exist in graph theory. For example, it makes it easy to prove the following lemma, a nice homework exercise, which is a simplified version of the result needed in the sweeping problem to show which deadheading edges should be added to build $G'$.

*Exercise*: Let $G$ be a connected graph with $2k$ vertices of odd degree and suppose we want to augment $G$ with a set of duplicates of certain edges of $G$ so that the augmented graph has all even-degree vertices. Then a minimal-sized augmenting set $S$ of such duplicates must consist of $k$ (shortest) paths between certain pairs of odd-degree vertices (an edge on $m$ of the paths is duplicated $m$ times in $S$).

A detailed discussion of the street sweeper problem is contained in a CUPM applied mathematics classroom modules project [3]. Also see [1].

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## UNIFORM CONVERGENCE AND GRAPH CONVERGENCE

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The purpose of this note is to pin down the intuitive feeling that a sequence of functions $f_n$ should converge uniformly to a function $f$ if and only if the graphs $\Gamma(f_n)$ are approaching $\Gamma(f)$. The non-uniform convergence of $f_n(x) = x^n$ on $[0, 1]$, for instance, should be related to the fact that the $\Gamma(f_n)$ have a vertical line segment in their limit. This intuition turns out to be justified subject to a compactness condition. The results here are not new—they can essentially be found in §§ 17, 21, 29, and 42 of [1]—but they are little known, and there seems to be no easily accessible presentation. Students in calculus courses usually learn to study functions via their graphs, and mentioning this interpretation of uniform convergence might help them acquire a feeling for it. At a higher level, the propositions can provide interesting exercises for topology students practicing compactness arguments.

**I.** Our first task is obviously to figure out what exactly is meant by convergence of graphs. More generally, given a sequence $S_n$ of sets in some metric space $X$, when should we say $S_n$ has a limit? Probably the most elementary definition is the point-by-point one: the $S_n$ should ultimately stay close to every point that they repeatedly approach. Formally, we define **Lim sup** $(S_n)$ to be the set of points that $S_n$ repeatedly approaches, i.e., the $q$ in $X$ such that every neighborhood of $q$ meets infinitely many $S_n$. Similarly we let **Lim inf** $(S_n)$ be the set of points that $S_n$ ultimately stays close to, i.e. the $q$ in $X$ such that every neighborhood of $q$ meets all but finitely many $S_n$. If these two sets coincide, we define **Lim** $(S_n)$ to be their common value.

Before going on to the theorem, it might help if we make some preliminary observations about these concepts. First, obviously Lim inf $(S_n) \subseteq$ Lim sup $(S_n)$. Second, the definitions can be paraphrased as follows: a point $q$ is in Lim inf $(S_n)$ iff there is a sequence of points $p_n$ in $S_n$ converging to $q$, and $q$ is in Lim sup $(S_n)$ iff there is a subsequence $S_{n_i}$ with points $p_i$ in $S_{n_i}$ converging to $q$. Finally, Lim inf $(S_n)$ and Lim sup $(S_n)$ are always closed, so that Lim $(S_n)$ is closed when it exists. Indeed, take (say) $x$ in the closure of Lim inf $(S_n)$. Given $\varepsilon > 0$, we can find $q$ in Lim inf $(S_n)$ with $d(x, q) < \varepsilon/2$. For all sufficiently large $m$ there is a point $p_m$ in $S_m$ with $d(q, p_m) < \varepsilon/2$, and then $d(x, p_m) < \varepsilon$; thus $x$ is in Lim inf $(S_n)$.

THEOREM. *Let $X$ and $Y$ be compact metric spaces. Let $f_1, f_2, \ldots$ be functions from $X$ to $Y$, with graphs $\Gamma(f_1), \Gamma(f_2), \ldots$ in $X \times Y$. Then* Lim $\Gamma(f_n)$ *exists and is the graph of a function $f$ if and only if $f_n$ converges uniformly to $f$ and $f$ is continuous.*

*Proof.* Continuity of $f$ may be a bit unexpected, but it is an automatic (and familiar) consequence of the fact that $\Gamma(f) =$ Lim $\Gamma(f_n)$ is closed. Indeed, suppose $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$. By compactness of $Y$ we can pass to a subsequence and assume that $f(x_n)$ converges to some other value $y \neq f(x)$. But $\langle x_n, f(x_n)\rangle$ is in $\Gamma(f)$ and converges to $\langle x, y\rangle$, so by closure $\langle x, y\rangle$ is in $\Gamma(f)$, which is impossible since $y \neq f(x)$.

We still must show, of course, that $f_n$ converges uniformly to $f$. If not, then for some $\varepsilon > 0$ we have a subsequence $f_{n_i}$ and points $x_i$ such that $d(f_{n_i}(x_i), f(x_i)) \geqq \varepsilon$. Passing to subsequences we may assume the $x_i$ converge to a point $x$ and the $f_{n_i}(x_i)$ converge to a point $y$. Then $\langle x, y\rangle = \lim \langle x_i, f_{n_i}(x_i)\rangle$ is in Lim sup $\Gamma(f_n) = \Gamma(f)$, and so $y = f(x)$. But since $f$ is continuous, $f(x) = \lim f(x_i)$. Thus $f(x_i)$ and $f_{n_i}(x_i)$ are approaching the same point, which is absurd because they are supposedly at distance at least $\varepsilon$ from each other.

Assume now conversely that $f_n \to f$ uniformly with $f$ continuous. Obviously each $\langle x, f(x)\rangle = \lim \langle x, f_n(x)\rangle$ is in Lim inf $\Gamma(f_n)$, and we must show only Lim sup $\Gamma(f_n) \subseteq \Gamma(f)$. If $\langle x, y\rangle$ is in the Lim sup, there is a sequence $\langle x_i, f_{n_i}(x_i)\rangle$ converging to it. For large $n_i$ we know $f_{n_i}(x_i)$ is uniformly close to $f(x_i)$,

which by continuity is close to $f(x)$. Thus $f_{n_i}(x_i)$ is converging to $f(x)$, and $\langle x, y\rangle = \langle x, f(x)\rangle$ is in $\Gamma(f)$. ■

**II.** It may be interesting to see what the proof of a standard uniform convergence theorem looks like from this point of view.

THEOREM. *Let $f_n$ be a sequence of real-valued functions on $[0,1]$ converging pointwise to a continuous function $f$, and assume that each $f_n$ is an increasing function. Then the convergence is uniform.*

*Proof.* As $f_n(0)$ is a convergent sequence, it is bounded below; similarly $f_n(1)$ is bounded above. Since each function is monotone, all values of all $f_n$ thus lie in a fixed compact interval, and the previous theorem applies. Since $\langle x, f_n(x)\rangle \to \langle x, f(x)\rangle$ we have $\Gamma(f) \subseteq \operatorname{Liminf} \Gamma(f_n)$; for $y \neq f(x)$ we must show that $\langle x, y\rangle$ is not in $\operatorname{Limsup} \Gamma(f_n)$.

Suppose $y > f(x)$ (the proof for $y < f(x)$ is similar). By continuity of $f$ there is an interval $[a, b]$ around $x$ on which $f$ is less than some $y - \varepsilon$. For all large $n$ we have $f_n(b) < f(b) + \varepsilon/2$, and so $f_n(b) < y - \varepsilon/2$; since $f_n$ is monotone, $f_n(x') < y - \varepsilon/2$ for all $x' \leqq b$. Thus $[a, b] \times (y - \varepsilon/2, \infty)$ is a neighborhood of $\langle x, y\rangle$ which for large $n$ does not meet $\Gamma(f_n)$. ■

The reader might enjoy using the same method to prove Dini's theorem: Let $X$ be a compact metric space. Let $f_n$ be a sequence of continuous real-valued functions on $X$ converging pointwise to a continuous function $f$, and assume $f_1 \leqq f_2 \leqq \cdots$. Then the convergence is uniform.

**III.** One who knows about nets of functions and uniformity on compact Hausdorff spaces can easily generalize the main theorem to that setting. More interesting is the question whether we can generalize to non-compact spaces. As the definitions stand, the answer is no. Consider for instance the functions on $[0,1]$ given by

$$f_n(x) = \begin{cases} n, & 1/2n \leqq x \leqq 1/n \\ 0, & \text{otherwise;} \end{cases}$$

the $f_n$ converge pointwise non-uniformly to $f(x) \equiv 0$, and yet $\operatorname{Lim} \Gamma(f_n) = \Gamma(f)$. But clearly here the problem is in the definition of graph convergence. The $\Gamma(f_n)$ are running off to infinity, and there is no point which could appear in the Limsup to indicate this behavior.

We therefore need to try a more global approach to set convergence. We say that two sets $S$ and $T$ in a metric space are **no more than $\varepsilon$ apart** from each other if each point of $S$ is within $\varepsilon$ of some point of $T$, and vice versa. Then we can say that a sequence $S_n$ **Hausdorff-converges** to $S$ if for every $\varepsilon > 0$ the sets $S$ and $S_n$ are ultimately no more than $\varepsilon$ apart.

This definition clearly eliminates any problem with sets running off to infinity. It has one minor quirk left, coming from the fact that $S$ and its closure are never more than $\varepsilon$ apart. If, for instance, we let

$$f_n(x) = \begin{cases} 1, & 0 \leqq x \leqq 1/n \\ 0, & \text{otherwise} \end{cases}$$

and

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise,} \end{cases}$$

then $f_n$ converges non-uniformly to $f$ and $\Gamma(f_n)$ Hausdorff-converges to $\Gamma(f)$. But this can be eliminated by restricting to closed sets; we then get a definition which agrees with the previous one for compact spaces and yields a more general theorem. Specifically, we have:

THEOREM. *Let $X$ be a compact metric space, $S_n$ a sequence of subsets of $X$, and $S$ a closed subset. Then $S_n$ Hausdorff-converges to $S$ if and only if* $\operatorname{Lim}(S_n) = S$.

THEOREM. *Let $X$ and $Y$ be metric spaces, $X$ compact. Let $f$ and $f_1, f_2, \ldots$ be functions from $X$ to $Y$ with $f$ continuous. Then $f_n$ converges uniformly to $f$ if and only if $\Gamma(f_n)$ Hausdorff-converges to $\Gamma(f)$.*

Both of these are proved by compactness arguments very much like the previous proofs.

**IV.** Compactness of $Y$ has been eliminated as a hypothesis, but compactness of $X$ remains; to see why that is necessary, consider the following example. Let $f$ be a function on the positive reals which is zero, except that around each integer $m$ it has a narrow peak of height 1 and base-width $1/2m$. For $n \geq 2$, let $f_n$ have the same graph, except that the $n$th peak is shifted over to be centered at $n + (1/n)$. Then $\Gamma(f_n)$ Hausdorff-converges to $\Gamma(f)$, but $f_n$ does not approach $f$ uniformly. What happens here is that $\Gamma(f_n)$ is near $\Gamma(f)$ because it is *horizontally* close, while for uniform convergence we would need it lying in a *vertically*-thin neighborhood of $\Gamma(f)$. This is no longer a problem of definition; the original intuitive feeling is wrong for $X$ not compact.

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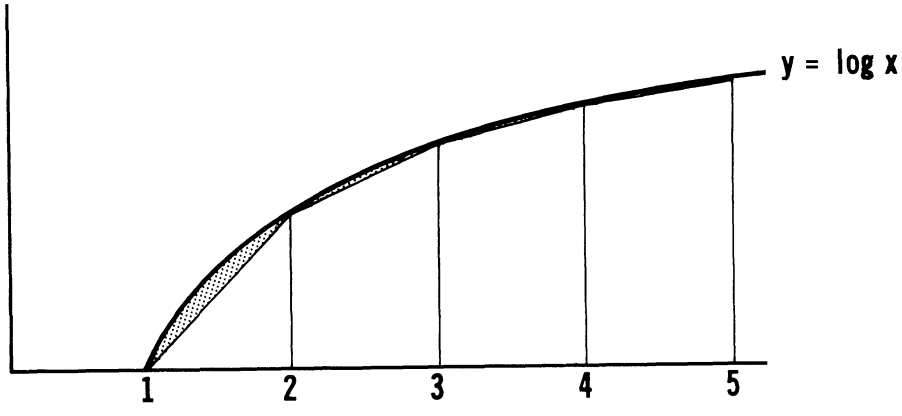
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## THE ERROR OF THE TRAPEZOIDAL METHOD FOR A CONCAVE CURVE

S. K. STEIN

Consider the area of the region between the curve $y = \log x$ and the polygonal approximation formed by using as vertices the points $(n, \log n)$, $n = 1, 2, 3, \ldots$ . This (unbounded) region is indicated by the shading in the accompanying diagram.

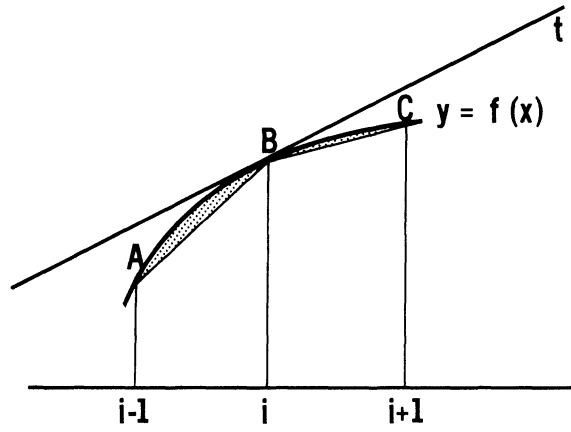



That this region has finite area can provide a basis for establishing Stirling's estimate of $n!$, as in ([2]: 361–363) and shown below in Theorem 3. This note presents a short geometric proof that the area is finite; this proof uses only the concavity of the function $\log x$. This result, stated in Theorem 1, is equivalent to ([1]: 170). Theorem 2 is a generalization.

THEOREM 1. *Let $f$ be a function defined for $x \geq 1$ such that $f(x) \geq 0$, $f^{(1)}(x) \geq 0$, and $f^{(2)}(x) \leq 0$ for all $x \geq 1$. Let $n$ be a positive integer and let $T_n$ be the trapezoidal estimate of $\int_1^n f(x)dx$ based on the numbers* $1, 2, \ldots, n$. *Then*

$$\int_1^n f(x)dx - T_n \leqq \frac{f(2)-f(1)}{2}.$$

*Proof.* For any integer $i = 2, 3, \ldots, n-1$, consider the following diagram, which is the key to the proof.




The line $t$ is tangent to the graph of $f$ at $B = (i, f(i))$. The areas of the shaded regions represent the errors of two adjacent summands in the trapezoidal approximation.

When the shaded region from $B$ to $C$ is translated by the vector $B - C$, no point of the resulting region $R$ lies below $t$. Thus $R$ and the shaded region from $A$ to $B$, which have the common right-hand point $B$, share no interior points. Now for each $i = 2, 3, \ldots, n-1$ translate the corresponding region $BC$ so that the right-hand end point coincides with the point $(2, f(2))$. Observe that no two of the resulting translates share an interior point with each other or with the shaded region $AB$ associated with the case $i = 2$. Thus the total error of the trapezoidal approximation is at most the area of the right triangle with vertices, $(1, f(1))$, $(1, f(2))$, and $(2, f(2))$, which has area $(f(2)-f(1))/2$. This proves the theorem.

Theorem 1 has two immediate consequences, which deserve mention. First of all, it shows that the infinite series which describes the error of the trapezoidal method over the interval $[1, \infty)$ using the numbers $1, 2, 3, 4, \ldots$, is convergent:

$$\sum_{n=1}^{\infty} \left( \int_n^{n+1} f(x)dx - \frac{f(n+1)+f(n)}{2} \right) \leqq \frac{f(2)-f(1)}{2}.$$

Also, with the aid of Theorem 1 and the fact that $\int_1^n f(x)dx - T_n$ is nonnegative, it is easy to establish the inequalities

$$-\frac{f(1)}{2} - \frac{f(n)}{2} \leqq \int_1^n f(x)dx - \sum_{i=1}^{n} f(i) \leqq \frac{f(2)}{2} - f(1) - \frac{f(n)}{2},$$

which, as in ([1]: 172), can be used to estimate $\Sigma_{i=1}^{n} f(i)$.

The same type of argument that proved Theorem 1 establishes the following more general theorem.

THEOREM 2. *Let $f$ be a function defined for $x \geqq 1$ such that $f(x) \geqq 0, f^{(1)}(x) \geqq 0$, and $f^{(2)}(x) \leqq 0$ for all $x \geqq 1$. Let $1 = a_1 < a_2 < a_3 < \cdots < a_n < \cdots$ be an infinite sequence such that the sequence of*

*differences $a_2 - a_1, a_3 - a_2, \ldots$ is bounded, say by b. Then the series*

$$\sum_{n=1}^{\infty} \left[ \int_{a_n}^{a_{n+1}} f(x)dx - \frac{f(a_{n+1}) + f(a_n)}{2} (a_{n+1} - a_n) \right]$$

*is convergent, and its sum is at most $b^2 f'(1)/2$.*

If the sequence of differences, $a_{n+1} - a_n$, is unbounded, the series of Theorem 2 need not be convergent. For example, let $f(x) = \sqrt{x}$ and $a_n = n^2$. In this case

$$\int_{a_n}^{a_{n+1}} f(x)dx - \frac{f(a_{n+1}) + f(a_n)}{2} (a_{n+1} - a_n) = 1/6$$

for all $n$.

THEOREM 3. *There is a constant c such that $n! \sim c\sqrt{n}(n/e)^n$.*

*Proof.* Let $a_n$ denote the difference between $\int_1^n \log x dx$ and the trapezoidal approximation, $T_n$, based on the values $1, 2, \cdots, n$. By Theorem 1, $\lim_{n\to\infty} a_n$ exists. Let $a$ denote this limit.

Now, $\int_1^n \log x dx - T_n = a_n$ or

$$x \log x - x\Big|_1^n - \sum_{k=1}^{n-1} \frac{\log(k) + \log(k+1)}{2} = a_n$$

or

$$n \log n - n + 1 - (\log 2 + \log 3 + \cdots + \log n) + \frac{\log n}{2} = a_n.$$

Application of the exponential function to both sides yields

$$\frac{e\sqrt{n}(n/e)^n}{n!} = e^{a_n}.$$

Thus, $n! \sim e^{1-a}\sqrt{n}(n/e)^n$, and the theorem is established.

The fact that the constant $c$ in Theorem 3 equals $\sqrt{2\pi}$ is usually established by utilizing Wallis's theorem:

$$\lim_{n\to\infty} \frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} = \sqrt{\pi}.$$

This limit is determined by evaluating $I(n) = \int_0^{\pi/2} \sin^n \theta d\theta$ and exploiting the inequalities $I(2n+1) < I(2n) < I(2n-1)$.

Incidentally, Stirling's formula implies that the shaded region described in the opening paragraph has area $1 - \log\sqrt{2\pi}$.

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middle-two-thirds Cantor set) $C'$ which consists of all real numbers which can be written $\Sigma_{j=1}^{\infty} b_j/6^j$, where $b_j = 0$ or 5. (See Lemma 3.)

The set $C'$ has the property that if $x$ and $y$ are two points of $C'$, then $(x + y)/2$ is not a point of $C'$. Now assume that $K'$ is a closed subset of $C'$. Then if $x$ is a point of $C'$, there is a unique point $k_x$ of $K'$ which is closest to $x$ (in the usual metric). And the function $k: C' \to K'$ defined by $k(x) = k_x$ is continuous and onto (in fact, it is a retraction).

This completes the proof of the title theorem.

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## THE FUNDAMENTAL THEOREM OF ALGEBRA

Frode Terkelsen

The purpose of this note is to give an elementary and rather short proof of the fundamental theorem of algebra.

Theorem. *Every polynomial of positive degree has a root in the complex numbers.*

*Proof.* Let $P$ be a polynomial of positive degree, with coefficients in the complex field $\mathbb{C}$. Since $P$ is continuous with $|P(z)| \to \infty$ uniformly for $|z| \to \infty$, there exists $z_0 \in \mathbb{C}$ such that $|P(z_0)| \leqq |P(z)|$ for all $z \in \mathbb{C}$. By considering the polynomial $P(z + z_0)$ instead of $P(z)$, one may assume $z_0 = 0$. We want to show that $P(0) = 0$.

There exist an integer $n \geqq 1$ and $a, b \in \mathbb{C}$ with $b \neq 0$ such that

$$P(z) = a + bz^n + z^{n+1}Q(z),$$

where $Q$ is a polynomial. Suppose $P(0) = a \neq 0$. Choose an $n$th root $w \in \mathbb{C}$ of the number $-a/b$. There exists $t$ with $0 < t < 1$ such that $t|w^{n+1}Q(tw)| < |a|$. Now

$$\begin{aligned} P(tw) &= a + b(tw)^n + (tw)^{n+1}Q(tw) \\ &= (1 - t^n)a + (tw)^{n+1}Q(tw), \end{aligned}$$

because $bw^n = -a$, so

$$\begin{aligned} |P(tw)| &\leqq (1 - t^n)|a| + t^{n+1}|w^{n+1}Q(tw)| \\ &< (1 - t^n)|a| + t^n|a| = |a| = |P(0)|. \end{aligned}$$

This is a contradiction, proving that $P(0) = 0$.

The last part of the proof above can be extended, with minor modifications, to a proof of the minimum modulus theorem for analytic functions.

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## SETS THICK AND THIN

W. M. PRIESTLEY

Liouville numbers attract the attention of almost every student to whom they are properly introduced. The set of all such numbers has the surprising property of being bulky in the sense of category and yet sparse in the sense of measure. A search for proof of this fact leads us on a light excursion with side features of interest for a first course in real analysis.

This paper arose from conversations with J. G. Stampfli on how to solve a problem in [5].

In 1844, the year just prior to Cantor's birth, Liouville [3] considered the number $x = \Sigma\, 10^{-n!}$. The partial sums of this series are rational numbers which are easily seen to give $x$ the following property: for each $s$ in $Z^+$ there is a rational $m/n$, $n > 1$, so close to $x$ that

$$0 < |x - m/n| < 1/n^s. \tag{1}$$

Real numbers allowing some sequence of rationals to come so close so fast are named after Liouville: $x$ is a **Liouville number** if for each $s$ the inequality (1) admits one (and hence infinitely many) solutions $m/n$ with $n > 1$.

It is clear that no such number is rational, and Liouville gave an elementary argument showing that no such number is algebraic. In fact [1, pp. 105–107], if $x$ is irrational and algebraic of degree $s - 1$ then inequality (1) admits at most finitely many rational solutions. The term 'transcendental' was then used to denote a real number such as $\Sigma\, 10^{-n!}$ which is a root of no polynomial with integral coefficients.

Contemplation of the set $L$ of Liouville numbers evokes a natural question. Does $L$ constitute the bulk of the transcendentals and therefore the bulk of $R$, i.e., is $R - L$ sparse? Or should one perhaps think of the Liouville numbers as being *atypical* real numbers, in which case one expects $L$ itself to be sparse. As has been hinted above, the answer depends upon which definition of sparseness is taken: *$L$ is of measure zero, yet $R - L$ is of first category.*

To verify this assertion, it is time-saving first to modify it in a harmless way. The set $L$ is clearly invariant under addition by a fixed integer, as is $R - L$. Thus the sparseness, in either sense, of $L$ (resp., $R - L$) in $R$ will follow from the sparseness of $L \cap I$ (resp., $I - L$) in $I$, where $I$ is any fixed unit interval.

THEOREM 1. *The Liouville numbers form a set of measure zero.*

*Proof*: We prove instead that $L \cap I$ is of measure zero, where $I = [0, 1]$. If $x$ is in $L \cap I$ then (1) has a solution $m/n$, $n > 1$, for each $s$. It follows that for any particular $s$, say $s = 3$, inequality (1) has infinitely many solutions $m/n$. Thus $x$ lies in infinitely many of the sets $E_n$ where

$$E_n = \{0 < x < 1 \mid |x - m/n| < 1/n^3 \text{ for some } m \text{ in } Z\}.$$

A picture of $E_n$ shows that it consists of open intervals whose total length $|E_n|$ is $2/n^2$. Hence $\Sigma|E_n|$ converges.

Given $\varepsilon > 0$ choose $N$ in $Z$ such that $\Sigma_{n>N}|E_n| < \varepsilon$. Then $L \cap I$, being contained in $\bigcup_{n>N} E_n$, can be covered by a countable number of open intervals whose total length does not exceed $\varepsilon$. Hence $L \cap I$ is of measure zero.

COROLLARY 2. *There exist uncountably many transcendental numbers which are not Liouville numbers.*

*Proof*: Otherwise $R = L \cup L'$, where $L' = R - L$ is countable. This implies $R$ is of measure zero, a contradiction.

It is known (see [2] or [4]) that such familiar transcendentals as $e$, $\pi$, $\log 2$, and "Mahler's number" .123456789101112... are not in $L$.

To prove the second part of our assertion, that $R - L$ is of first category in $R$, it is convenient to prove instead that $I - L$ is of first category in $I$, where $I = [2, 3]$. To do this it is not much more trouble to answer a more general question that is of interest in its own right. This is the question *whether a given sequence of open intervals can be hit infinitely many times with integral multiples of an appropriately chosen real number.*

As an example, take $I_m = (m, m + 1/m^m)$ for $m$ in $Z^+$. Let

$$H = \{x \in I \mid nx \in I_m \text{ for infinitely many } m, n\}. \tag{2}$$

We might call $H$ the **hitting set** of the sequence $I_m$.

Is $H$ non-empty? We shall see that the hitting set of *any* sequence of open intervals is non-empty, provided only that the intervals **tend to infinity** in the sense that each neighborhood $(a, \infty)$ of infinity contains one. In fact, as proved in Theorem 3, $I - H$ is of first category in $I$.

The connection with our original assertion is that the set $H$ of (2) is a subset of $L \cap I$. To see this, let $x$ be in $H$ so that for infinitely many pairs $m, n$ of positive integers,

$$m < nx < m + 1/m^m,$$

$$m/n < x < m/n + 1/nm^m.$$

Since $x \geqq 2$ it follows that $m \geqq n$ so that

$$0 < |x - m/n| < 1/nm^m \leqq 1/n^m$$

for infinitely many rationals $m/n$. Hence $x$ is in $L \cap I$.

Since $L \cap I$ contains $H$, $I - H$ contains $I - L$. The following theorem then implies $I - L$ is of first category in $I$, completing the proof of our original assertion.

THEOREM 3. *Let $I$ be an interval in $(0, \infty)$, let $I_m$ be a sequence of non-empty open intervals tending to infinity, and let $H$ be defined as in formula (2). Then $I - H$ is of first category in $I$.*

*Proof*: For $x$ in $I$ let $h(x)$ be the number of intervals $I_m$ hit by the sequence $\{nx\}$. Thus $H = \{x \in I \mid h(x) = \infty\}$. The proof consists of two lemmas.

LEMMA A. *Let $V$ be an open interval contained in $I$ and assume $x \in V$ and $h(x) < \infty$. Then there is an $x'$ such that $x' \in V$ and $h(x') > h(x)$.*

*Proof of lemma*: Let $k = h(x)$ and let $I_{m_1}, I_{m_2}, \ldots, I_{m_k}$ be the intervals hit by $\{nx\}$. Since these intervals are open, there is a neighborhood $U$ of $x$ with $U \subset V$ such that $u \in U$ implies $\{nu\}$ also hits each of these $k$ intervals. Regardless of how small $U = (a, a + |U|)$ is, it follows that

$$\bigcup_{n \geqq 1} nU = \bigcup_{n \geqq 1} (na, na + n|U|)$$

contains a neighborhood of infinity, since the sets $nU$ become pairwise overlapping when $n$ exceeds $a/|U|$. Thus $\bigcup nU$ contains some $I_m$ lying beyond the $k$ intervals above. Choose $x'$ in $U$ such that $\{nx'\}$ hits $I_m$. Then $h(x') \geqq k + 1 > h(x)$.

LEMMA B. *$H_k = \{x \in I \mid h(x) < k\}$ is a closed, nowhere-dense subset of $I$.*

*Proof of lemma*: Assume $x \in I$ and $x \notin H_k$. Then $h(x) \geqq k$. As in the preceding proof there is a neighborhood $U$ of $x$ such that $u \in U$ implies $h(u) \geqq k$. Thus $U \cap H_k$ is empty, showing $H_k$ closed.

Let $V$ be an open interval in $I$. If $x$ is in $V$ then $k$ applications of Lemma A produce a point $x'$ in $V$ with $h(x') \geqq k$. Hence $H_k$ does not contain $V$ unless $V$ is empty. $H_k$ is then nowhere-dense, being

closed and containing no non-empty open interval. This proves Lemma B, from which Theorem 3 follows, since $I - H = \cup H_k$.

COROLLARY 4. *The non-Liouville numbers form a set of first category in R.*

Since $R$ is a complete metric space, $L$ is of second category by Baire's theorem. In particular, $L$ is uncountable.

It follows as well from Corollary 4 that *every real number x is the difference (hence also the sum) of two Liouville numbers.* This is equivalent to the claim that $S$ is non-empty, where

$$S = (x + L) \cap L, \qquad x + L = \{x + y \mid y \in L\}.$$

This claim is now obvious, since by DeMorgan's law $S' = (x + L') \cup L'$, the union of two sets of first category, showing that the complement of $S$ is of first category.

Theorem 3 has also the following delightful consequence, which is probably something of a folk theorem. (Cf. prob. 10, p. 38 of [5].)

COROLLARY 5. *Let f be a continuous real-valued function on* $(0, \infty)$ *such that* $f(nx) \to 0$ *for each x in some (non-empty) interval I. Then* $f(x) \to 0$ *as* $x \to \infty$.

*Proof*: If not, there is some $\varepsilon > 0$ and a sequence $\{x_m\}$ tending to infinity such that for each $m$,

$$|f(x_m)| > \varepsilon.$$

By continuity this inequality subsists on some open interval $I_m$ containing $x_m$. By Theorem 3 and Baire's theorem there is an $x$ in $I$ such that $\{nx\}$ hits infinitely many $I_m$. Thus $|f(nx)| > \varepsilon$ for infinitely many $n$, contradicting $f(nx) \to 0$.

The reader can easily derive a "power" version (considering $x^n$ rather than $nx$) of Corollary 5. In regard to this the following may be of interest. Let $I$ be a bounded interval contained in $(1, \infty)$ and let $f$ be continuous on $(1, \infty)$. Consider the following two conditions:

$$f(2^n x^n) \to 0, \qquad x \in I; \tag{3}$$

$$f(2^n x) \to 0, \qquad x \in I. \tag{4}$$

At first glance condition (3) appears weaker than (4); nevertheless, condition (3) implies $f(x) \to 0$ as $x \to \infty$ while (4) does not. The verification of this is left to the reader.

It is not hard to show that *the set H of Theorem* 3 *has measure zero provided*

$$|I_m| \to 0 \quad \textit{and} \quad \Sigma |I_m| \ln |I_m| > -\infty.$$

Thus a host of hitting sets furnish natural examples of sets of measure zero with complements of first category.

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# MATHEMATICAL EDUCATION

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Your attention is drawn to the existence of the new quarterly "Journal of Personalized Instruction," 29 Loyola Hall, Georgetown University, Washington, D.C. 20057. Our continuing backlog prompts us to ask potential contributors of articles on personalized systems of instruction to keep the above journal in mind as a possible publishing alternative. — *The Editors.*

## MATHEMATICS HISTORY: A UNIFYING DEVICE IN THE TEACHING OF MATHEMATICS

M. A. GOLBERG AND H. BOWMAN

**1. Introduction.** The recent introduction of the January semester into the academic calendars of many colleges and universities in the United States has provided an opportunity for the development of novel approaches to the teaching of standard subjects. In this article we describe one such course in mathematics history, which we believe to be unique.

Although originally planned and executed for the 1974 January semester at the University of Nevada, Las Vegas, we believe it can serve as a model for a new kind of educational experience for mathematics students.

**2. Mathematics through history.** The title of the course was "Some Recent Developments in Mathematics" and by recent we meant post 1800 with considerable attention being paid to the important transitional years of 1900–1930. Our basic idea was to teach mathematics through the perspective of historical development. The approach was to integrate the mathematics and the history into a cohesive unit so that neither dominated the other. We wished to create a course that was neither the "history of mathematics" nor "mathematics and history," but rather one in which the students (and professors) would learn mathematics in the time sequence in which it was created.

There are many reasons for wanting to do this. First, mathematics history is frequently a neglected or non-existent part of the mathematics curriculum. It is generally considered to be of little value and an academically inferior product when compared to the structure of mathematics itself. To some extent the criticism is valid, since until recently there have been few history books that were not merely of the story book, biographical type. Even those considered as classics such as Bell's *Men of Mathematics* are mostly collections of historical anecdotes containing little discussion of actual mathematics. Since prejudices tend to propagate themselves, what you think is of little value will tend not to be taught. As a consequence we continue to develop students who have little idea where the subject came from and many who have no idea where it is going.

Secondly, mathematics is most often presented to undergraduates (and graduates) as the disjoint union of algebra, analysis, topology, etc. Rarely does one see how one course interacts with another. The culmination of this can be seen in a prediction made by Paul Halmos several years ago in a letter to the *Notices of the American Mathematical Society*, where it was predicted that mathematics departments might fragment into departments of topology, algebra, and analysis, much like the schools of medicine. The fact is that many of the most brilliant accomplishments of modern mathematics have occurred through the synthesis of disparate subjects (for example the Index Theorem).

To counteract what was felt to be a weakness in mathematical education, we thought it should be

possible to devise material that would take a point of view of the historical development of mathematics and at the same time allow students to see the unity of mathematics both past and present. To this end we developed a course which took two significant areas of mathematics, algebra and analysis and attempted to weave them together through an historical approach. We felt that our initial approach was successful enough to use it as a model for the mathematics community to bring some unity to an often disjointed subject.

**3. Course description.** The course met for fifteen one hour and forty minute periods. There were six full time participants. They included two members of the mathematics faculty, three mathematics students, and the chairman of the engineering department who is also a mathematics student. Interested faculty and students were invited to come to the presentations and often did. The interests of the two faculty members were algebra and functional analysis, thus setting the direction of the course. The readings for the course included, *I Am a Mathematician* by Norbert Wiener, *The Works of Evariste Galois* by Evariste Galois, *The Theory of Functionals and Integrodifferential Equations* by Vito Volterra, and *Hilbert* by Constance Reid. Two films were shown, one about John von Neumann and the other about Courant entitled *Göttingen and New York*. Each participant delivered lectures in an area of interest. Because of the amount of research required, some of the lectures had to be assigned one or two months in advance of the course.

**4. Synopsis of the lectures.** To give the reader some idea of the subject matter that was covered, a brief synopsis of the lectures is included below. Needless to say, anyone considering using a similar format would most likely have a substantially different material content dependent on the interests of the students and faculty.

LECTURE 1. The first lecture was a short history of mathematical thought from ancient to modern times. Emphasis was based on the fact that mathematics has always received a major impetus from man's desire to understand the physical world.

It was also argued that there is no straight line progress in mathematics; old ideas have frequently been introduced to stimulate new results, the most important example being the rediscovery of the axiomatic approach at the turn of the century.

LECTURES (2–3). Lectures (2–3) surveyed the major developments in algebra. Among the references was the paper *Current Trends in Algebra* by Garrett Birkhoff.

This history of algebra was divided into three basic periods referred to as the Classical, the Transitional and the Modern. The Classical period extended from the Babylonians to about 1830. The chief concern was the solution of algebraic equations of one variable with coefficients in the real or complex fields. This period was terminated with the work of Gauss, Galois and Abel. The Transitional period, running roughly from 1830–1930, was characterized by the increasing abstraction and diversity of algebraic systems. The appearance of van der Waerden's textbook in 1930 placed algebra on an equal footing with analysis and firmly established the axiomatic method as a tool in the repertoire of mathematicians.

LECTURES (4–5). Lectures (4–5) developed some of the early results of category theory, using as a basis Eilenberg and Maclane's classic paper *General Theory of Natural Equivalences*. In particular it was shown how the subject grew out of a desire to understand certain "natural" transformations such as the construction of the double dual of a vector space.

LECTURES (6–8). Lectures (6–8) covered the history of the theory of differential equations. In contrast to algebra, existence of this subject was shown to be relatively short, stemming from the physical theories of Newton on planetary motion. Emphasis was placed on the similarities and differences in the development of analysis as compared to algebra. Whereas the motivation to consider algebraic problems was largely internal, the impetus to develop analysis was derived largely from the external world.

As it was for algebra, the history of differential equations was divided into three main periods: The Intuitive, the Transitional and the Modern. The Intuitive period ran roughly from the latter part of the

17th century to the third decade of the 19th and was characterized by the intense efforts to solve the equations of Newton. The Transitional period ended about 1900. It began with the simultaneous proof by Cauchy in France and Kowalewski in Russia of a very general existence theorem for solution of equations with analytic coefficients. These results were made possible as a consequence of the newly created complex variable theory of Cauchy and Gauss. In particular, the rigorization of the notion of limit and sequence by Cauchy enabled him to put the series method of the 18th century on a firm foundation. By the end of the 19th century period, Picard had shown how to weaken the conditions of Cauchy by using the method of successive approximations.

Some of Lie's ideas were also developed for Hamilton's equations of particle physics, showing that the problem of integrability could be rephrased purely as a question in algebra.

LECTURES (9–10). Lectures (9–10) dealt with the biography and principal work of Evariste Galois. Emphasis was placed on contrasting Galois' original constructions and the modern techniques of field theory which grew out of his work.

LECTURES (11–13). Lectures (11–13) were devoted to a discussion of integral equations, particularly the major results of Abel and Volterra. Emphasis was placed on Fredholm's theory as being an infinite analogue of Cramer's rule for the solution of $n$ linear algebraic equations in $n$ unknowns.

LECTURE (14). This talk centered on the work done in the development of calculus in the latter half of the 19th century and the early part of the 20th century. The significant results of Vito Volterra (1860–1940) were discussed in detail. It was shown how his ideas led to the development of such diverse areas as metric spaces, topology, and the theory of integral equations.

LECTURE (15). Lecture (15) dealt with the life and work of Norbert Wiener. A philosophical discussion centering around the need for axiomatization and interaction with the physical world for the progress of mathematics concluded the course.

At all times during the lectures needed mathematical concepts were developed in full, primarily as they appeared in the original literature. These constructions were frequently contrasted with more recent approaches to the same topic.

**5. Course assessment.** The course was given on an experimental basis. No such course was offered at U.N.L.V. and, to the best of our knowledge, was not offered in the regular curricula of other schools.

Mathematics is a vast enterprise. The AMS itself uses about 100 distinct divisions in its MOS classification scheme, each one of these a specialty. As such, one of the goals of the course was to find a basis on which people of diverse interests could participate in a single course. This objective was met to a large degree. If the course were to be taught on a regular basis for one semester, we envision having the participation of other students and faculty with interests ranging from foundations to numerical analysis.

Primarily we wanted to see if a history course could be made mathematically legitimate and useful in the teaching of mathematical concepts.

The preceding description of the course content indicates that this objective was met. A reason for this was the enthusiasm of the students. Students generally will work harder when they can choose the topics of most interest to them. A particularly interesting comment was made by the chairman of the engineering department, who was a participant in the course. He said this course enabled him to see for the first time why mathematicians appreciated mathematics for its own sake. This remark, coming from an engineer, was especially gratifying.

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## A NEW ONE-SEMESTER TERMINAL COURSE IN MATHEMATICS

L. R. SONS

Part of the charge to the faculty in a mathematics department is to keep the courses and programs of their curriculum under continual review. During the fall of 1971 the Undergraduate Committee in the mathematics department at Northern Illinois University looked deeply at our one-semester, three-hour course which was used to meet the all-university mathematics requirement by those students whose programs required no other mathematics. What was envisioned by the committee that year and has since evolved is a course which, to my knowledge, does not exist elsewhere at this writing and which I believe to be exceedingly valuable.

In the fall of 1971 the committee found the terminal course, about which I speak, to have set theory, logic, numeration systems, and the development of the number system as the major areas of content. Consequently there was a significant overlap in content between the course and our first course for students preparing to teach at the elementary level. (Indeed, historically the two courses at one time were identical.) The students in our terminal course viewed the content as irrelevant to their lives and saw the course merely as a hurdle they needed to jump in order to graduate.

The University did not see the terminal course as a general education course in the sense of a course aimed at gaining an appreciation for mathematics. Yes, it could aim at that to some extent, but primarily it should be a course to strengthen communication skills in mathematics. It should raise the level of mathematical literacy to allow for adequate participation in the daily life of our present society.

The committee needed answers to a number of questions: who *really* is to take this terminal course, what do such people need in so far as we can tell — how do these people encounter the mathematical sciences in their lives, what objectives should be set for such a course, and what content and method of teaching should be set for the course.

So who were (or are) these people? They are the students whose principal area of study has no required mathematics — the art major, the music major, the history major, to name some. They are students who possess a fear of mathematics — it has been a source of little success for them in the past. Many of them view it as something to learn by rote. They have had *no chance* to really appreciate mathematics — for these it is likely that appreciation may only come through doing some mathematics successfully. Finally, these students truly have little background in mathematics — most of them had one year of high school algebra at least *three* years ago!

What do such people need in so far as we can tell? That is not, of course, an easy question to answer. After much discussion and casting around, there are two sources with which we had substantial agreement: (1) *A Course in Basic Mathematics for Colleges* by the Committee on the Undergraduate Program in Mathematics (January, 1971) and (2) *Basic Skills Essential for Enlightened Citizens*, a report of the Committee on Basic Mathematical Competencies and Skills of the N.C.T.M. (reported later in THE MATHEMATICS TEACHER 65(1972) 671–677). Our discussion also made it clear to us that the students we were talking about had previously been exposed to some of the content we thought they should know but without their succeeding in learning it. Consequently a second (or third or ...) exposure to essentially the same material similarly organized was *not likely* to be successful. Spirit would be important in a course for these people. We needed a different approach, and the CUPM report mentioned gave us some ideas to try.

After the dust of deliberation settled we arrived at a three part course with approximately one third of the semester spent on each part: (1) "traditional" mathematical systems, (2) computers in society, and (3) statistics from a consumer's viewpoint. Our objectives would be:

1. to develop an awareness in the student of the nature and power of some of the elementary ideas in each of these divisions and also to realize some of the limitations of these approaches;

2. to expose the students to attitudes of mathematical thinking;

3. to develop the student's mathematical skills and competencies by means of an indirect approach (the skills and competencies as defined by the N.C.T.M. committee mentioned above).

In the part on "traditional" mathematical systems we do a little set theory, a little logic, a section on equations and algebraic systems (linear equations, groups, quadratic equations, linear inequalities) and a section on numeration systems. Our approach is to use throughout *the flow chart* and the systematic analysis it suggests. This brings an element of "newness" to the student even in "old" territory.

In the section on computing we look at how a computer can "solve" certain kinds of equations already discussed and some not yet discussed. We analyze some "practical word problems." We look at sequences — arithmetic, geometric, and Fibonacci. We discuss compound interest and payment schemes. Flow charts again are a key teaching tool — they make ideas such as iteration schemes relatively easy to follow! Truly we study basic mathematics in a new setting and at the same time hope to foster some understanding of the power and limitations of computers. The students learn how a computer receives a program, what a programming language is, and how a computer dispenses answers to program questions. *They* draw flow charts and write a few simple programs in BASIC. They see demonstrations of the computer and read simple computer output but do not themselves get on the machine (we have 600 students a semester in this course). They are told some of the ways computers are used in our society.

In the unit on statistics we consider the construction and use of statistical charts; measures of central tendency; misinterpretations arising from the faulty use of percentages, proportions, ranking indices; the normal distribution informally; and (informally also) the notion of bias in a sample and the idea of a random sample. We ask how the daily newspaper uses or abuses statistics. Again we are studying basic mathematics in a new setting and at the same time seeking to help the students become more intelligent consumers of statistical information.

Text material for this course was hard to find. There were very few books on the market which contained the "traditional" mathematical systems and even a dabbling in both computing and statistics — let alone anything near our vision! (I was (and am) appalled by books which contain ten pages on statistics whose primary aim seems to be to tell the student what the definitions are of mean, median, and mode, but not to even *try* to get them to know *why* anyone would want to consider three (or more) measures of central tendency!) We finally settled (for the first round) on use of two books and a bunch of mimeographed supplementary sheets and problems — the books were *The Nature of Modern Mathematics* by K. Smith, and (a paperback) *The Use and Abuse of Statistics* by W. Reichman. The Smith book was not highly satisfactory, but at least it contained rudiments of our first two parts of the course and enabled us to get the course off-the-ground with our supplementary material. Reichman's book is a beautiful volume with all the right spirit and depth — though we had to make up problems and the book has unfortunately *gone out of print.*

After a couple semesters of further developing the initial supplementary materials, one of our faculty wrote a set of notes for the course which aside from some slight changes has now become the sole text material for the course (these are duplicated by our office of printing and duplicating and sold at cost to the students). These notes should soon become a book.

How is it going? Well, we are now in the fifth full semester and students seem to be at least more receptive than they were to the old course — the unit on statistics gets high marks. The student enrollment increase in the course may mean the procrastination level for taking the course is declining. Testing evidence gathered so far suggests students *are* learning skills and have a more positive attitude toward their ability to do mathematics. Faculty enthusiasm is very good, and new faculty placed in the course seem to be able to gather the spirit, content, and approach reasonably. In short, we are pleased.

In the near future we hope to gather further testing evidence to support what we feel the course is accomplishing and may move to introduce some hand-calculator problems.

We feel the course has much to offer the student for whom it was designed. Successful completion of the course should lead such a student to improved mathematical skills and competencies, a greater appreciation for the mathematical sciences, and a greater ability to analyze all kinds of problems.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORTHERN ILLINOIS UNIVERSITY, DEKALB, IL 60115.

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## PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

### ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before January 31, 1977.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 2611. *Proposed by C. A. Nicol, University of South Carolina*

A long standing conjecture of D. H. Lehmer asserts that if $n \geq 2$ is a natural number and $\phi(n) | n-1$ then $n$ is prime.

Show that a natural number $n \geq 2$ is prime if and only if $\phi(n) | n-1$ and $n+1 | \sigma(n)$.

E 2612. *Proposed by Sidney Penner, Bronx Community College, CUNY*

How many diamonds can be packed in a Chinese checkerboard? This *board* consists of 2 order 13 triangular arrays of holes, overlapping in an order 5 hexagon, 121 holes in all. A *diamond* consists of four marbles that fill four adjacent holes.

E 2613. *Proposed by D. E. Knuth, Stanford University, and the Mayaguez Problems Group, University of Puerto Rico, (independently).*

Partition the real line $\boldsymbol{R}$ into a countable union of compact subsets.

E. 2614. *Proposed by Frank Siwiec, John Jay College of Criminal Justice, New York*

A set $A \subset \mathbb{R}^n$ is called a $g$-set if there is a countable family $\{U_n \mid n = 1, 2, \ldots\}$ of open sets containing $A$ with the property: For each open set $G \supset A$ there is a $U_n$ with $A \subset U_n \subset G$. Which subsets of $\mathbb{R}^n$ are $g$-sets?

E 2615. *Proposed by D. Rameswar Rao, Secunderabad, India*

Show that the system of Diophantine equations

$$x^2 + y^2 = u^2 + v^2, \qquad x^3 + y^3 = u^3 + v^3 \tag{1}$$

has no solutions in positive integers. Prove the same for the system

$$x^2 + y^2 = u^2 + v^2, \qquad x^5 + y^5 = u^5 + v^5. \tag{2}$$

E 2616.* *Proposed by Andrew Odlyzko and Lloyd Welch, Jet Propulsion Laboratory, Pasadena, California*

Let $a$ be a complex number with $|a| < 1$ and let $\varepsilon > 0$. Prove or disprove: There exists an algebraic integer $b$ such that $|a - b| < \varepsilon$ and all conjugates of $b$ lie in the annulus $|a| - \varepsilon < |z| < 1 + \varepsilon$.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### Acceleration with Constant Direction

E 2535 [1975, 520]. *Proposed by Murray S. Klamkin, University of Waterloo*

A body is projected in a uniform gravitational field and is subject to a resistance which is a function of its speed $|\mathbf{v}|$. If the acceleration $\mathbf{a}$ of the body always has a constant direction, no matter what the initial velocity $\mathbf{v}_0$, show that $\mathbf{a} = \mathbf{a}_0 e^{-kt}$ for some constant $k$.

*Solution by Harry Lass and Robert M. Georgevic (jointly), Jet Propulsion Laboratory, Pasadena, California.* We have $\mathbf{a} = \mathbf{g} - f(v)\mathbf{v} = \lambda(t)\mathbf{a}_0$ where $\mathbf{g}$ is the constant gravitational acceleration, $\mathbf{a}$ and $\mathbf{v}$ the acceleration and velocity of the body, $v = |\mathbf{v}|$, $t$ the time and $\mathbf{a}_0 = \mathbf{a}(0)$, $\lambda(0) = 1$. Differentiating with respect to time yields

$$(\lambda'(t) + f(v)\lambda(t))\mathbf{a}_0 = -f'(v)\frac{dv}{dt}\mathbf{v}.$$

For $\mathbf{v}$ not parallel to $\mathbf{a}_0$, it follows that $f'(v) = 0$ and $\lambda'(t) = -f(v)\lambda(t)$. Hence $f(v) = k$ (a constant), $\lambda(t) = e^{-kt}$, and $\mathbf{a} = e^{-kt}\mathbf{a}_0$.

Also solved by P. P. Kale & M. R. Modak (India), R. C. Lyness (England), L. E. Mattics, St. Olaf Problem Solving Group, Robert Weinstock, and the proposer.

### When $x^m = x$ Defines Boolean Rings

E 2536 [1975, 521]. *Proposed by Jacob Brandler, Brooklyn College*

If $x^6 = x$ for every element $x$ in the ring $R$, prove that $R$ is a Boolean ring. Generalize.

*Composite of solutions by Robert Gilmer, Florida State University and Thomas D. Kezlan, University of Missouri.* For an integer $m > 1$, call a ring $R$ an $m$-ring if $x^m = x$ for all $x \in R$. (We do not assume that $R$ has a unity.) The following theorem is a generalization of the result stated in the problem.

THEOREM. *In order that each $m$-ring be a 2-ring ($=$ Boolean ring) it is necessary and sufficient that $m$ be even and $m-1$ have no divisor of the form $2^p-1$, $p$ a prime.*

*Proof. Necessity.* If $m$ is odd then GF(3) is an $m$-ring but not a 2-ring. If $m$ is even and $2^p-1 \mid m-1$ for some prime $p$ then GF($2^p$) is an $m$-ring but not a 2-ring.

*Sufficiency.* Assume that $m$ satisfies the condition in the theorem and let $R$ be an $m$-ring. Since $m$ is even we have $x = x^m = (-x)^m = -x$ for all $x \in R$. To show that $R$ is a 2-ring, it suffices to show that for every $a \in R$, $a \neq 0$ the subring $A$ of $R$ generated by $a$ is a 2-ring. Clearly, $A$ is a finite commutative ring with unity $e = a^{m-1}$, $e \neq 0$ and $A$ has no non-zero nilpotent elements. Hence to show that $x^2 - x = 0$ in $A$ for all $x \in A$, it suffices to prove that $x^2 - x \in P$ for every prime ideal $P$ of $A$. Since $A/P$ is a finite integral domain of characteristic 2, it is in fact a finite field $F$ with, say, $2^t$ elements for some integer $t \geqq 1$. Since the multiplicative group of $F$ is cyclic of order $2^t - 1$ we must have $2^t - 1 \mid m - 1$. The hypothesis on $m$ implies that $t = 1$, $F = \text{GF}(2)$ and $x^2 - x \in P$ as asserted.

Also solved by Anders Bager (Denmark), Irl Bivens, Leonard Borucki & Edith Luchins, Philip Castevens, Stephen Currier, Jr., Lorraine Foster, Donald Fuller, M. G. Greening (Australia), Ralph Grimaldi, W. V. Grounds, Lee Hagglund, David Hammer, A. A. Jagers (Netherlands), Colonel Johnson, Jr., Jeff Lagarias, L. E. Mattics, M. R. Modak & S. G. Udpikar (India), Louise Moser & Theodore Tracewell, John O'Neill, M. J. Pelling (Nigeria), J. N. Ridley (South Africa), D. J. Samuelson, August Sardinas, Brian Wesselink, Jerry Wing, Edward Wong, and the proposer. Partial solution by Bhushan Kapoor (India).

*Editor's comment.* The idea of using the subring $A$ was suggested to us by Frederick Hoffman. This avoids the need of quoting the well-known result that every $m$-ring is commutative. For additional results see the solution of advanced problem 5972 [1976, 66].

**An Ambiguous Functional Equation**

E 2537 [1975, 521]. *Proposed by David Shelupsky, City College of C.U.N.Y.*

Find all continuous functions $f$ defined on $(0, \infty)$ for which $f(x_1y) - f(x_2y)$ is independent of $y$.

*Solution by Andrew Vogt, Georgetown University. Version* 1: $x_1$ and $x_2$ variable, $f$ real-valued. Since $f(xy) - f(y)$ is independent of $y$ we have $f(xy) - f(y) = f(x) - f(1)$. The function $g(x) = f(e^x) - f(1)$ satisfies $g(x+y) = g(x) + g(y)$, Cauchy's equation. Since $g$ is continuous we have $g(x) = \alpha x$. Hence $f(x) = \alpha \log x + \beta$ where $\alpha$ and $\beta$ are arbitrary constants.

*Version* 2: $x_1$ and $x_2$ distinct and fixed, $f$ real-valued. For the expression to be independent of $y$ we have $f(x_1y) - f(x_2y) = c$ or equivalently

$$f(ax) = f(x) + c$$

where $a = x_1/x_2 \neq 1$ and $c$ are constants. The function $g(x) = f(e^x) - cx/b$, where $b = \log a$, satisfies $g(x+b) = g(x)$. Thus

$$f(x) = g(\log x) + (c/b) \log x.$$

Conversely, it is easy to check that every function of the form

$$f(x) = g(\log x) + \alpha \log x,$$

where $\alpha$ is a constant and $g$ is continuous and periodic with period $b = \log(x_1/x_2)$, has the desired property.

Also solved by Stephen Berman, Irl Bivens, David Bloom, John Bryant & Robert Gilmer, Stephen Currier, Jr., Peter de Buda, Edward Dixon, Michael Ecker, Ron Evans, Donald Fuller, M. B. Gregory & E. O. Nelson, George Harrison, G. A. Heuer, Carl Hurd, Hans Kappus (Switzerland), Benjamin Klein, Jeff Lagarias, Harry Lass, Detlef Laugwitz (Germany), Joel Levy, R. M. Mathsen, L. E. Mattics, W. W. Meyer, M. R. Modak (India), Martin Muldoon (Canada), Ram Murty & Kumar Murty (Canada), David Neu, Richard Poppen, Daniel Rosenblum, Eric

Rosenthal, H. Schwerdtfeger (Canada), Kenzo Seo, Joseph Silverman, James Smith, Wolfe Snow, St. Olaf Problem Solving Group, Thomas Tucker, R. W. Wagner, Robert Young, and the proposer. Thomas Elsner submitted a partial solution.

**A Maximum Problem for the Triangle**

E 2538 [1975, 521]. *Proposed by Jack Garfunkel, Flushing, New York*

Let $ABC$ be a triangle. If $X$ is a point on side $BC$, let $AX$ meet the circumcircle of $ABC$ again at $X'$. Prove or disprove: If $XX'$ has maximum length, then $AX$ lies between the median and the internal angle bisector issuing from $A$.

*Solution by John Oman, University of Wisconsin at Oshkosh.* The statement is correct. The triangles $AXC$ and $BXX'$ are similar and so are triangles $ABX$ and $CX'X$. Therefore $XX' \cdot AC = BX' \cdot CX$ and $XX' \cdot AB = CX' \cdot BX$. Thus

$$(XX')^2 \cdot AB \cdot AC = (BX \cdot CX) \cdot (BX' \cdot CX'). \tag{1}$$

We may assume that $AC \geqq AB$. Let $M$ be the midpoint of $BC$ and let $N$ be the intersection of the angle bisector of $A$ with $BC$. The product $BX \cdot CX$ increases when $X$ moves from $B$ to $M$ and decreases when $X$ moves from $M$ to $C$. The product $BX' \cdot CX'$ increases when $X$ moves from $B$ to $N$ and decreases when $X$ moves from $N$ to $C$. Indeed, the angle $BX'C = \theta$ is independent of $X$ and $BX' \cdot CX' \sin\theta = 2 \cdot \text{area}\,(BX'C)$. Since $N$ lies on the segment $BM$ we conclude that both products $BX \cdot CX$ and $BX' \cdot CX'$ increase when $X$ moves from $B$ to $N$ and they both decrease when $X$ moves from $M$ to $C$. Hence (1) implies that the maximum of $XX'$ must occur for some $X$ on the segment $MN$.

Also solved by Aage Bondesen (Denmark), Ju. G. Dutkević (U.S.S.R), Michael Goldberg, Free Jamison, O. P. Lossers (Netherlands), R. C. Lyness (England), Juan Migliore, M. J. Pelling (Nigeria), David Robbins, R. W. Wagner, and Albert Walker (Canada).

**Sums of Reciprocals in a Finite Field**

E 2540 [1975, 659]. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Let $F$ be a finite field of order $q$, let $n$ be a divisor of $q-1$, and let $\alpha$ be a nonzero element of $F$. Evaluate

$$S(n, q: \alpha) = \Sigma(t^n - \alpha)^{-1},$$

the sum being over all $t \in F$ with $t^n \neq \alpha$.

*Solution by the proposer.* Let $d = (q-1)/n$. Since the multiplicative group of $F$ is cyclic of order $nd = q-1$ it is clear that a nonzero element $r \in R$ is an $n$th power in $F$ if and only if $r^d = 1$. In order to compute $S = S(n, q: \alpha)$ we shall distinguish three cases.

(1) Suppose $\alpha^d \neq 1$. Define $f(x) = \Pi(x - r)$, where $r$ ranges over all nonzero $n$th powers in $F$. Thus $f(x) = x^d - 1$. Let $f(x+\alpha) = \Pi(x - (r-\alpha)) = x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0$. Then $-c_1/c_0 = \Sigma(r-\alpha)^{-1}$, so $S = -\alpha^{-1} - nc_1/c_0$. But $f(x+\alpha) = (x+\alpha)^d - 1$, so $c_1 = d\alpha^{d-1}$ and $c_0 = \alpha^d - 1$. Since $nd = -1$ in $F$, we get $S = \alpha^{-1}(\alpha^d - 1)^{-1}$.

(2) Suppose $\alpha^d = 1$ and $d \geqq 2$. Define $f(x) = \Pi(x - r)$, where $r$ ranges over all nonzero $n$th powers in $F$ except $\alpha$. Let $f(x+\alpha) = x^{d-1} + \cdots + c_1 x + c_0$. Then as before $S = -\alpha^{-1} - nc_1/c_0$. From $f(x+\alpha) = ((x+\alpha)^d - 1)/x$ we have $c_0 = d\alpha^{d-1}$ and $c_1 = \binom{d}{2}\alpha^{d-2}$. This gives

$$S = -\alpha^{-1}\left(1 + \frac{n}{d}\binom{d}{2}\right).$$

In $F$ we have $nd = -1$ and if char $F \neq 2$

$$-\frac{n}{d}\binom{d}{2} = n^2\binom{d}{2} = \frac{n+1}{2}.$$

One can easily check that the last equality remains valid even when char $F = 2$. In that case $n$ is odd and $(n+1)/2$ should be first computed in $\mathbb{Z}$. Hence $S = \alpha^{-1}(n-1)/2$.

(3) Suppose $\alpha^d = 1$ and $d = 1$. Thus $\alpha = 1$, $n = q - 1$ and clearly $S = -1$.

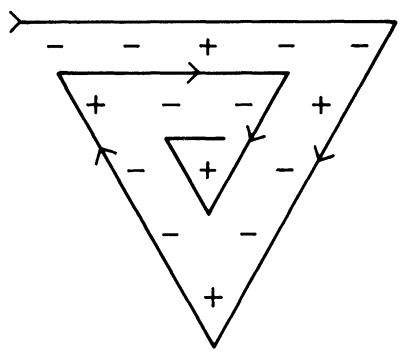
Also solved by L. Carlitz, A. Çharnow, Jesse Deutsch, Ron Evans, Lorraine Foster, Robert Gilmer, Jeff Lagarias, and M. R. Modak (India). Partial solution by S. G. Udpikar (India).

**Continuation of $--+--+\cdots$**

E 2541 [1975, 695]. *Proposed by E. T. H. Wang, Wilfred Laurier University*

A *Steinhaus triangle* is formed as follows: start with a row of $n$ plus and minus signs. Under each pair of like signs, a plus sign is written and under each pair of unlike signs, a minus sign is written. Continuing on, one finally obtains a triangle of $\frac{1}{2}n(n+1)$ plus and minus signs.

Prove that if the first row pattern of a Steinhaus triangle is $--+--+\cdots$ (i.e., two minuses followed by a plus), then the same pattern repeats itself when one traverses all of the entries in a clockwise spiral pattern as shown in the figure:




*Solution by Michael Joseph, Universidad de Costa Rica, San José.* Let $f(i,j) = 1$ or $0$ according to whether the $j$th sign in the $i$th row of the triangle is $-$ or $+$. We have

$$f(1,j) = \begin{cases} 0 & \text{if } 3 \mid j \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

because of the hypothesis concerning the first row of the triangle. We consider the values $f(i,j)$ as elements of the ring $\mathbb{Z}/2\mathbb{Z}$. By definition of Steinhaus triangles we have

$$f(i+1,j) = f(i,j) + f(i,j+1). \tag{2}$$

Define

$$g(i,j) = \begin{cases} 0 & \text{if } 3 \mid (j-i+1) \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

It is easy to verify that $g$ satisfies the recursive relation (2) and the boundary conditions (1). Hence $f(i,j) = g(i,j)$ for all $i$ and $j$.

We have to show that in any three consecutive signs of our spiral precisely one plus sign occurs. This follows from the fact that each of the triples

(i) $g(i,j-1), g(i,j), g(i,j+1)$;
(ii) $g(i,j), g(i,j+1), g(i+1,j)$;
(iii) $g(i-1,j+1), g(i,j), g(i+1,j-1)$;
(iv) $g(i+1,j), g(i,j), g(i-1,j)$

consists of one 0 and two 1's, which is immediate from (3).

Also solved by J. C. Binz (Switzerland), M. T. Bird, Irl Bivens, Philip Castevens, Charles Davis, Milton Eisner, William Gould, M. G. Greening (Australia), William Habakkuk, John Hall, Heiko Harborth (Germany), J. D. Hiscocks (Canada), Carl Hurd, George Konstantinow, L. Kuipers (Switzerland), Michael Mays, St. Olaf Problem Group, August Sardinas, James Singer, K. R. Pratap Singh (India), and Phil Tracy.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before January 31, 1977.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6108. *Proposed by Aleksander Ivić, Novi Sad, Yugoslavia*

Find all multiplicative functions $f(n)$ such that:

$$f(n^2)=\sum_{d\mid n}\mu^2(d)f\left(\frac{n}{d}\right) \quad \text{and} \tag{1}$$

$$f^2(n)=\sum_{d\mid n}f(d^2). \tag{2}$$

6109*. *Proposed by Stuart P. Lloyd, Bell Laboratories, Murray Hill, New Jersey*

Define functions $S_1(z), S_2(z), \cdots$, by $S_1(z)=z$, $S_{n+1}(z)=\phi(S_n(z))$, $n\geqq 1$, where $\phi(s)=s+s^2$. When $z$ is a positive integer, the series

$$\frac{1}{z}=\sum_{n=1}^{\infty}\frac{1}{S_n(z)+1} \tag{*}$$

is the nonterminating Sylvester series for the rational number $1/z$ [H. E. Salzer, *Further remarks on the approximation of numbers as sums of reciprocals*, this MONTHLY 55, (1948), 350–356]. Determine the region of convergence of (*) in the complex $z$-plane.

6110*. *Proposed by David M. Battany, Oceanside, California*

Let $p$ and $q$ be primes; not both even. Let $m$, $n$ and $v$ be integers; $m, n\geqq 2$; $v\geqq 0$. For each value of $v$ there exists at most one pair of powers $(p^m, q^n)$ such that $p^m-q^n=2^v$.

6111. *Proposed by Barthel W. Huff, Queen's University, Kingston, Ontario*

The following problem was encountered when studying certain stochastic processes. Evaluate

$$M(\lambda)=\lim_{n\to\infty}\left(-2^n\int_{-\infty}^{\infty}\left[(2\pi^3\lambda)^{-1/2}\int_0^{|x|}\exp\{-y^2/2\lambda\}dy\right]\left[\int_{-\infty}^{\infty}\exp\{-|u|^{\alpha}/2^n\}e^{-iux}du\right]dx\right),$$

where $0<\alpha<1$ and $\lambda>0$.

6112. *Proposed by Jan Mycielski, University of Colorado*

Let $f(x)$ be a differentiable function such that $f(0)=0$, $0<f(x)<x$ for $x>0$, and $f'(0)=1$. We put $f^0(x)=x$ and $f^{n+1}(x)=f(f^n(x))$ for $n=0,1,\ldots$. Find conditions under which the series $\sum_{n=0}^{\infty}f^n(1)$ converges (diverges).

6113. *Proposed by Claudia Simionescu, University of Brasov, Romania*

Let $X$ be a compact metric space, $F$ a real finitely additive set function not of bounded variation, and let $T_F$ be the set of Riemann-Stieltjes integrable functions. Then $T_F$ is of first category in $C(X)$. (Simionescu, *Remarque sur l'integrabilité ...*, Comunicările Acad. R.P.R. XIII (1973, # 12). Can this result be improved to: $T_F$ is nowhere dense?

## SOLUTIONS OF ADVANCED PROBLEMS

### A Banach Algebra

5124 [1963, 765]. *Proposed by Fred Suvorov*

Give an example of a Banach algebra such that each element is a sum of products, but which also contains an element which is not a product.

*Solution by R. J. Loy, University of Edinburgh, Scotland.* Take $\mathbb{C}^3$ with operations

$$(a, b, c) \cdot (a', b', c') = (aa', ab', ac'),$$

$$\|(a, b, c)\| = \max(|a|, |b|, |c|).$$

The element $(0, 1, 1)$ cannot be a product, but every element is the sum of two products.

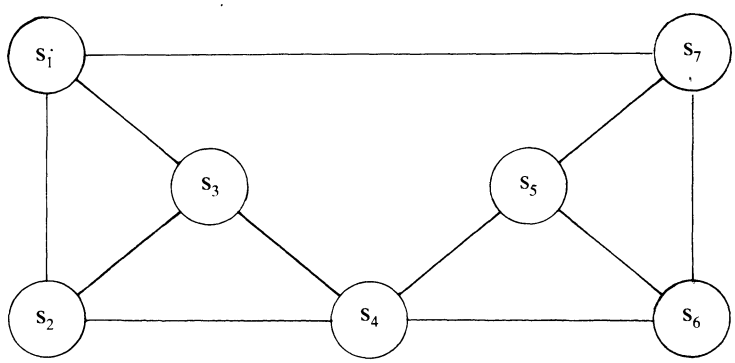
This example is non-commutative. The question of a commutative example remains unsolved.

### A Planar Graph

5385 [1966, 420]. *Proposed by Stanley Korn*

A set $S$ consists of the elements $s_1, \ldots, s_m$, which are related in pairs: $s_i * s_j$, $i \neq j$ (one element may be so related to more than one element, and $s_i * s_j$ is equivalent to $s_j * s_i$). $S$ is to be divided into $n$ nonoverlapping subsets, $n \geqq 1$, (which together contain all the elements of $S$) such that no two elements in the same subset are *-related. Show that this can always be done if every possible subset of $S$ consisting of $e$ elements, $e \geqq n$, contains not more than $(n-1)(e - \frac{1}{2}n)$ related pairs.

*Solution by James E. Desmond, Pensacola Junior College.* We propose a counterexample. Let $S = \{s_1, \ldots, s_7\}$ be a set with $m = 7$ distinct elements and let the *-relations between the elements of $S$ be denoted by the accompanying diagram in which a line segment joins a pair of elements of $S$ if and only if that particular pair is *-related.




Thus, for example, $s_1$ and $s_3$ are *-related, while $s_1$ and $s_5$ are not *-related.

Let $n = 3$. We shall show that every possible subset of $S$ consisting of $e$ elements, $e \geqq n = 3$, contains not more than $(n-1)(e - \frac{1}{2}n) = 2(e - 3/2) = 2e - 3$ related pairs.

We shall then show that this set $S$ cannot be divided into 3 nonoverlapping subsets such that no two elements in the same subset are *-related.

In order to verify that every possible subset of $S$ consisting of $e$ elements, $e \geqq n = 3$, contains not more than $2e - 3$ related pairs we must consider the cases $e = 7, 6, 5, 4, 3$ individually.

The only subset of $S$ with $e = 7$ elements is $S$ itself and we observe from the diagram that there are 11 related pairs in $S$. $11 \leqq 2e - 3 = 2 \cdot 7 - 3$.

For $e = 6$ we note that by deleting one element from $S$ we are eliminating, from the resulting subset, at least 3 of the possible 11 related pairs. So the number of related pairs in the resulting subset is not more than 8 and $8 \leqq 2e - 3 = 2 \cdot 6 - 3$.

For $e = 5$, by deleting two elements from $S$ we are eliminating, from the resulting subset, at least 5 of the possible 11 related pairs. So the number of related pairs in the resulting subset is not more than 6, where $6 \leqq 2e - 3 = 2 \cdot 5 - 3$.

For $e = 4$, by deleting three elements from $S$ we are eliminating, from the resulting subset, at least 6 of the possible 11 related pairs. So the number of related pairs in the resulting subset is not more than 5, where $5 \leqq 2e - 3 = 2 \cdot 4 - 3$.

For $e = 3$ there can be at most three related pairs. $3 \leqq 2e - 3 = 2 \cdot 3 - 3$.

We have verified that this set $S$ does satisfy the requirement that every possible subset of $S$ consisting of $e$ elements, $e \geqq n = 3$, contains not more than $2e - 3$ related pairs.

Now suppose that $S$ has been divided into $n = 3$ nonoverlapping subsets such that no two elements in the same subset are *–related. By the diagram, $s_1$, $s_2$ and $s_3$ are pairwise *–related and therefore must be in three different subsets. Also $s_2$, $s_3$ and $s_4$ are pairwise *–related and therefore must be in three different subsets. Since the three subsets are nonoverlapping, $s_1$ and $s_4$ must be in the same subset.

Again by the diagram, $s_4$, $s_5$ and $s_6$ are pairwise *–related and therefore must be in three different subsets. Also $s_5$, $s_6$ and $s_7$ are pairwise *–related and therefore must be in three different subsets. Since the three subsets are nonoverlapping, $s_4$ and $s_7$ must be in the same subset. Hence $s_1$, $s_4$ and $s_7$ are in the same subset, contradicting the fact that $s_1$ and $s_7$ are *–related. *Q.E.D.*

Also solved by Paul Erdös (Hungary), Hewitt Kenyon, John Mason, E. C. Milner (England), Patrick O'Neil, and the proposer. O'Neil notes that the four–color theorem would follow as a consequence if the problem were true. Milner notes that the statement of the problem implies that every graph would be 3–colorable.

### Arc Length when $f' = 0$ a.e.

6007 [1975, 84]. *Proposed by Rollin Sandberg, California State University at Fullerton*

Let $f$ be a non-decreasing, continuous function from $[0, a]$ onto $[0, b]$ such that $f'$ vanishes almost everywhere. Determine the length of this arc.

*Solution by R. A. Christianson, Grinnel College.* We proceed from basic principles to show that the length of the indicated arc is $a + b$. Given any partition $P = \{x_0, x_1, \ldots, x_k\}$ of $[0, a]$ with $0 = x_0 < x_1 < \cdots < x_k = a$, let

$$L(P) = \sum_{i=1}^{k} \sqrt{[f(x_i) - f(x_{i-1})]^2 + [x_i - x_{i-1}]^2}$$

be the length of the corresponding broken line segment. Now

$$L(P) \leqq \sum_{i=1}^{k} [(f(x_i) - f(x_{i-1})) + (x_i - x_{i-1})] = b + a.$$

Thus the length of the curve in question, which equals $\sup_P L(P)$, is at most $a + b$. Taking an arbitrary positive $\varepsilon$, we shall finish by finding a partition $P$ of $[0, a]$ such that $L(P) > a + b - \varepsilon$.

Let $D$ be the set of all $x$ in $[0, a]$ for which $f'(x) = 0$. For each $x$ in $D$, it is easy to see that there is an interval $(u_x, v_x)$ with rational endpoints containing $x$ such that if $u_x < u < x < v < v_x$, then $f(v) - f(u) < (v - u)\varepsilon/(2a)$. Since the measure $\mu(D) = a$, we may choose from the countable cover $\{(u_x, v_x) : x \in D\}$ a finite subcollection denoted by $\{(u_i, v_i) : i = 1, 2, \ldots, n\}$ such that

$$\mu[\cup\{(u_i, v_i) : i = 1, 2, \ldots, n\}] > a - \varepsilon/2.$$

We may assume that $0 < x_1 < x_2 < \cdots < x_n < a$ and, by shortening intervals where necessary, that $v_{i-1} \leqq u_i$, $0 < u_1$, and $v_n < a$. Consider now the partition

$$P = \{0 = v_0, u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_{n+1} = a\}.$$

We show that $L(P) > a + b - \varepsilon$. First observe that

$$\sum_{i=1}^{n+1} [f(u_i) - f(v_{i-1})] = b - \sum_{i=1}^{n} [f(v_i) - f(u_i)]$$

$$> b - \sum_{i=1}^{n} (v_i - u_i)\varepsilon/2a > b - \varepsilon/2.$$

Finally

$$L(P) \geqq \sum_{i=1}^{n+1} [f(u_i) - f(v_{i-1})] + \sum_{i=1}^{n} (v_i - u_i) > b - \frac{\varepsilon}{2} + a - \frac{\varepsilon}{2}.$$

Also solved by K. F. Anderson (Canada), Peter Borwein (Canada), L. E. Clarke (England), Thomas Foregger, A. A. Jagers (Netherlands), William Nuesslein, Nicholas Passell, M. J. Pelling (Nigeria), William Phillips & Peter Borwein (Canada), and the proposer.

*Editor's comment.* W. A. J. Luxemburg points out that this problem has appeared as # 5029 [1963, 674] proposed and solved by himself and A. Zaanen. Foregger finds the result in W. Rudin, *Real and Complex Analysis*, ch. 8, Exercise 10.

**A Vanishing Integral**

6008 [1975, 84]. *Proposed by P. B. Gilkey, University of California, Berkeley*

For $\xi = (x_1, x_2, x_3) \in \mathbb{R}^3$, set $A = A(\xi) = x_1A_1 + x_2A_2 + x_3A_3$, where

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix},$$

and let $\Gamma = \Gamma(\xi)$ be any positively oriented closed curve enclosing the eigenvalues of $A(\xi)$. Show that the integral

$$I(B) = \int_{\mathbb{R}^3} \oint_{\Gamma} \operatorname{tr}\{(\lambda - A)^{-1}[B(\lambda - A)^{-1}]^3\}\lambda \exp[-\lambda^2]\, d\lambda d\xi$$

vanishes for every $4 \times 4$ matrix $B$.

*Solution by M. J. Pelling, University of Belin, Nigeria.* The matrices $A_i$ are Dirac matrices, which anticommute and whose squares are 1. Thus if $R = +(x_1^2 + x_2^2 + x_3^2)^{1/2}$ then $A^2 = R^2$ and the eigenvalues of $A$ can only be $\pm R$. If $\lambda \pm R$ is a repeated factor of $|\lambda - A|$ then the conjugate $\lambda \mp R$ must occur with the same multiplicity, otherwise $R$ would be rationally expressible in terms of $x_1, x_2, x_3$. So $A$ has eigenvalues $R$, $R$, $-R$, $-R$ and if $f$ is a rational function $\operatorname{Tr} f(A) = 2f(R) + 2f(-R)$.

Next we have

$$\oint_{\Gamma} \operatorname{tr}\{(\lambda - A)^{-1}[B(\lambda - A)^{-1}]^3\}\lambda e^{-\lambda^2} d\lambda$$

$$= \oint_{\Gamma} \frac{\operatorname{tr}\{(\lambda + A)[B(\lambda + A)]^3\}}{(\lambda^2 - R^2)^4} \lambda e^{-\lambda^2} d\lambda$$

$$= \oint_\Gamma \frac{m_0\lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4}{(\lambda^2 - R^2)^4} \lambda e^{-\lambda^2} d\lambda$$

$$= 2\pi i \sum_{\{R,-R\}} \operatorname{Res} \left[ \frac{m_0\lambda^4 + m_1\lambda^3 + m_2\lambda^2 + m_3\lambda + m_4}{(\lambda^2 - R^2)^4} \lambda e^{-\lambda^2} \right]$$

$$= 4\pi i \operatorname*{Res}_{\{R\}} \left[ \frac{m_0\lambda^4 + m_2\lambda^2 + m_4}{(\lambda^2 - R^2)^4} \lambda e^{-\lambda^2} \right],$$

because the contributions to the two residues cancel for the $m_1$, $m_3$ terms and are equal for the $m_0$, $m_2$, $m_4$ terms. Making use of the trace law, $\operatorname{tr} CD = \operatorname{tr} DC$, we find $m_0 = \operatorname{tr} B^3$, $m_2 = R^2 \operatorname{tr} B^3 + 5 \operatorname{tr} B^2ABA$, $m_4 = R^2 \operatorname{tr} B^2ABA$.

Now $\operatorname{tr} B^2ABA$ *is quadratic in* $x_1, x_2, x_3$ and when the integral over $\mathbb{R}^3$ is performed, terms of the form $x_ix_jf(R)$, $i \neq j$, disappear and terms of the form $x_i^2 f(R)$ are equivalent to $\frac{1}{3} R^2 f(R)$ on grounds of symmetry. Thus $\operatorname{tr} B^2ABA$ may be replaced by $\frac{1}{3} R^2 \Sigma_i \operatorname{tr} B^2A_iBA_i$ whence

$$I(B) = 4\pi i (\operatorname{tr} B^3) \int_{\mathbb{R}^3} \operatorname*{Res}_{\{R\}} \left[ \frac{\lambda^4 + R^2\lambda^2}{(\lambda^2 - R^2)^4} \lambda e^{-\lambda^2} \right] d\xi$$

$$+ \frac{4\pi i}{3} \left( \sum_i \operatorname{tr} B^2A_iBA_i \right) \int_{\mathbb{R}^3} \operatorname*{Res}_{\{R\}} \left[ \frac{R^4 + 5R^2\lambda^2}{(\lambda^2 - R^2)^4} \lambda e^{-\lambda^2} \right] d\xi.$$

Denote these two integrals of residues respectively by $J_1$ and $J_2$. We show that $J_1 = J_2 = 0$. First, a direct computation shows that the residue in $J_2$ is $\frac{1}{4}(5R^2 - 2R^4)e^{-R^2}$ whence, using the formula

$$\int_0^\infty R^{2n} e^{-R^2} dR = \left( \frac{2n-1}{2} \right) \int_0^\infty R^{2n-2} e^{-R^2} dR, \qquad n \geq 1$$

we get

$$J_2 = \frac{1}{4} \int_{\mathbb{R}^3} (5R^2 - 2R^4) e^{-R^2} d\xi = \pi \int_0^\infty (5R^4 - 2R^6) e^{-R^2} dR = 0.$$

One may calculate $J_1$ in the same way, but it is easier to calculate $I(1) = 16\pi i(J_1 + J_2)$ from the original integral. This gives

$$I(1) = \int_{\mathbb{R}^3} \oint_\Gamma \{\operatorname{tr}(\lambda - A)^{-4}\} \lambda e^{-\lambda^2} d\lambda \, d\xi$$

$$= \int_{\mathbb{R}^3} \oint_\Gamma \left\{ \frac{2}{(\lambda - R)^4} + \frac{2}{(\lambda + R)^4} \right\} \lambda e^{-\lambda^2} d\lambda \, d\xi$$

$$= \int_{\mathbb{R}^3} 2\pi i \sum_{\{R,-R\}} \operatorname{Res} \left[ \left\{ \frac{2}{(\lambda - R)^4} + \frac{2}{(\lambda + R)^4} \right\} \lambda e^{-\lambda^2} \right] d\xi$$

$$= 8\pi i \int_{\mathbb{R}^3} \operatorname*{Res}_{\{R\}} \left[ \frac{\lambda e^{-\lambda^2}}{(\lambda - R)^4} \right] d\xi$$

$$= 8\pi i \int_{\mathbb{R}^3} \left( 4R^2 - 1 - \frac{4}{3} R^4 \right) e^{-R^2} d\xi$$

$$= 32\pi^2 i \int_0^\infty \left( 4R^4 - R^2 - \frac{4}{3} R^6 \right) e^{-R^2} dR$$

$$= 32\pi^2 i \left( 4 \cdot \frac{3}{2} - 1 - \frac{4}{3} \cdot \frac{5}{2} \cdot \frac{3}{2} \right) \int_0^\infty R^2 e^{-R^2} dR = 0.$$

Thus $J_1 + J_2 = 0$ and so $J_1 = J_2 = 0$. This implies $I(B) = 0$ for all $B$.

Also solved by D. Ž. Djoković, Clark Givens, and Thomas Seidman.

*Editor's note.* The proposer offers the more general problem of proving $I(B) = 0$ if $A_1, A_2, A_3$ are any $4 \times 4$ matrices for which $x_1A_1 + x_2A_2 + x_3A_3$ is non-singular and self adjoint, and $(x_1, x_2, x_3) \neq (0, 0, 0)$.

### A Non-Linear Isometry

6009 [1975, 84]. *Proposed by J. A. Goldstein, Tulane University*

Let $X$ be a finite dimensional topological vector space whose topology is given by a metric $d$. Let $T$ be a surjective isometry on $X$ such that $T0 = 0$. If $d$ is invariant, i.e., if $d(p, q) = d(p - q, 0)$ for all $p, q \in X$, so that $X$ is a Fréchet space, then $T$ is necessarily linear [Z. Charzynski, Studia Math. 13 (1953), 94–121]. Must $T$ still be linear if the assumption that $d$ is invariant is dropped? What if dim $X = 1$?

*Solution by Nicholas Passell, University of Florida.* Let $R$ be the real line with metric $d$ as follows:

(1) If $a, b \geqq 0$ $\quad d(a, b) = |b - a|$,
(2) If $a < 0, b \geqq 0$ $\quad d(a, b) = 2|a| + |b|$,
(3) If $a, b < 0$ $\quad d(a, b) = 2|b - a|$.

Then let $T$ be a map which takes $a \in (-\infty, 0)$ to $2|a|$, and $a \in (0, \infty)$ to $-a/2$. This is a non-linear isometry of the topological vector space $(R, d)$ fixing 0.

Also solved by S. Ditor (Canada), and by the proposer.

### A Combinatorial Identity

6010 [1975, 84]. *Proposed by L. Carlitz, Duke University*

Coefficients $c_{m,n}^{(k)}$ are defined by means of

$$(1+x)^m(1-x)^n = \sum_{k=0}^{m+n} c_{m,n}^{(k)} x^k \qquad (m \geqq 0, n \geqq 0).$$

Show that

$$\sum_{k=0}^{m+n} (c_{m,n}^{(k)})^2 = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}.$$

I. *Solution by H. K. Kuiken, Redhill, England.* Write $f(x) = (1+x)^m (1-x)^n$ and evaluate $f(x)f(1/x)$. It follows that

$$\sum_{k=0}^{m+n} (c_{m,n}^{(k)})^2 = (-1)^n c_{2m,2n}^{(m+n)} \tag{1}$$

$$= (-1)^m c_{2n,2m}^{(m+n)}. \tag{2}$$

If $m \geqq n$ we proceed using (1), otherwise using (2). Suppose $m \geqq n$. Then

$$\begin{aligned}(-1)^n c_{2m,2n}^{(m+n)} &= (-1)^n \sum_{i=0}^{2n} (-1)^i \binom{2m}{m+n-i}\binom{2n}{i} \\ &= (-1)^n \frac{(2m)!\,(2n)!}{\{(m+n)!\}^2} \sum_{i=0}^{2n} (-1)^i \binom{m+n}{i}\binom{m+n}{2n-i} \\ &= (-1)^n \frac{(2m)!\,(2n)!}{\{(m+n)!\}^2} \cdot c_{m+n,m+n}^{(2n)}.\end{aligned}$$

But $c_{m+n,m+n}^{(2n)}$ is the coefficient of $x^{2n}$ in $(1+x)^{m+n}(1-x)^{m+n}$, i.e., in $(1-x^2)^{m+n}$. Thus

$$c_{m+n,m+n}^{(2n)} = (-1)^n \binom{m+n}{n} = (-1)^n \frac{(m+n)!}{m!\,n!}$$

and the result follows.

If $m < n$ we proceed with (2) which leads to the same result.

II. *Solution by Irving Gerst, State University of New York at Stony Brook.* Let $f(x) = (1+x)^m(1-x)^n$ and denote the sum by $S$. Then (Parseval formula)

$$S = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})f(e^{-i\theta})d\theta = \frac{2^{2(m+n)-1}}{\pi}\int_0^{2\pi} \sin^{2n}(\theta/2)\cos^{2m}(\theta/2)\,d\theta.$$

Replacing $\theta/2$ by $\phi$ and noting the symmetry of the resulting integrand about $\pi/2$, we get

$$S = \frac{2^{2(m+n)+1}}{\pi}\int_0^{\pi/2} \sin^{2n}\phi\cos^{2m}\phi d\phi = \frac{2^{2(m+n)}}{\pi}\frac{\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2})}{\Gamma(m+n+1)}.$$

Since $\Gamma(n+\frac{1}{2}) = 2n!\sqrt{\pi}/2^{2n}n!$, the result follows.

Also solved by K. F. Andersen (Canada), David Anderson, Bruce Berndt, Paul Bruckman, David Carlson, Paul Chernoff, L. E. Clarke (England), S. Ditor (Canada), Ron Evans, Thomas Foregger, F. Gerish (England), M. G. Greening (Australia), Emil Grosswald, Richard Hager, Michael Hoffman, A. A. Jagers (Netherlands), M. S. Klamkin (Canada), L. Kuipers, Albert Leisinger, O. P. Lossers (Netherlands), R. C. Lyness (England), A. McD. Mercer (Canada), Ram Murty & Kumar Murty (Canada), Thomas Osler, M. J. Pelling (Nigeria), B. S. Popov (Yugoslavia), Otto Ruehr, E. B. Saff, Hermann Schmidt (Germany), A. Selby (Canada), Franklin Smith, Michael Somos, Allen Stenger, Phil Tracy, John Tung, L. van Hamme (Belgium), Hartmut Wellstein (Germany), and the proposer.

*Editor's comment.* (1) The proposer offers the following more general formula if $m + n = p + q = t$:

$$\sum_{k=0}^{t} c_{m,n}^{(k)} c_{p,q}^{(k)} = \begin{cases} 0 & \text{if } n+q \text{ is odd} \\ (-1)^{n+r}\dfrac{(2m+2n-2r)!\,(2r)!}{(m+n)!\,(m+n-r)!\,r!} & \text{if } n+q = 2r. \end{cases}$$

(2) R. C. Lyness observed that a question in the 1972 Mathematical Olympiad in Poland was to show that $(2m)!(2n)!/m!\,n!(m+n)!$ is an integer.

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## REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
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*Introduction to Methods of Optimization.* By Leon Cooper and David Steinberg. Saunders, Philadelphia, Pennsylvania. 1970, vii + 381 pp. $12.50. (Telegraphic Review, February 1971.)

This book seemed, on the basis of the Table of Contents and the general reputation of the authors, to be admirably suited for use as a text in optimization theory. On these grounds I selected the book for a new course at our school. I was forced to admit as much to a group of disgruntled students who asked me whether they should not be able to expect professors to read a book carefully before requiring members of a class to buy it.

Criticisms may be grouped under three headings. First, the authors' use of terminology not only

offends a generation raised on pedantic precision of expression, but it confuses even readers disposed to sacrifice correctness for convenience. Secondly, numerous assertions that are very clearly stated are no less clearly wrong. Finally, there are passages so murky that one is unable to decide whether the problem is with obscure exposition, an incorrect algorithm, or both.

The confused terminology is nicely illustrated in the authors' discussion of the Fibonacci search. Beginning with an initial interval $L_1 = [a_1, b_1]$ and the Fibonacci sequence $\{F_n\}$, we are told to form

$$\Delta_2 = L_1 \frac{F_{n-2}}{F_n}.$$

We are then advised to form the *points*

$$x_1 = a_1 + \Delta_2, \qquad x_2 = b_1 - \Delta_2,$$

and the *interval* $L_2 = L_1 - \Delta_2$. The same kind of terminology occurs on page 137 where instructions say to "subdivide the interval $[0, 1]$ into $\Delta x/2$ equally spaced intervals." Though no clue is given as to what $\Delta x$ is, subsequent reading suggests that the authors meant to say that $[0, 1]$ is to be divided into $N$ equal intervals, each of length $\Delta x/2$.

This spirit of being willing to figure out what the authors meant to say is essential to reading the book. In fairness, it is usually not too difficult to do (though students who found the $\varepsilon - \delta$ definition of continuity a bit rough are likely to find the definition of a weak relative minimum on page 88 to be positively lumpy), but one wonders why this additional burden must be placed on the undergraduate reader.

Turning to the matter of clearly incorrect statements, we offer for a starter the assertion on page 154 that to minimize $[f^2(x_1, x_2) + g^2(x_1, x_2)]$ is equivalent to finding a root of $f(x_1, x_2) = 0$, $g(x_1, x_2) = 0$. Errors like the one on page 98 (where the procedure outlined will miss the maximum of $f(x) = x - x(x-1)^{2/3}$ on $[0, 2]$) can be remedied by correctly qualifying the cases in which the method will work. Others, however, are beyond easy repair. Perhaps the best, or is it worst, such instance occurs on pages 112–113 where we find a discussion of how to locate extrema on the boundaries of a real valued function $f$ defined on a closed rectangle in $R^n$. The authors completely overlook the possibility that $f$ might be extreme at a point interior to a segment joining adjacent vertices, quite a *faux pas* in a book on optimization. And while giving honors for dubious "best examples," let us mention as most humorous the assertion on page 154. "It would be hard to overestimate the inefficiency of this procedure. For example, in a problem with 10 variables, with each variable divided into 20 subintervals (sic), we would have to carry out $10^{20}$ functional evaluations!" If one overlooks the fact that 20 subintervals give 21 points at which to evaluate $f$, the number of lattice points would be $20^{10}$. Hard as it might be to overestimate the inefficiency of the procedure, the authors managed to do so — by a factor of $5^{10}$.

Finally, we illustrate what we mean by passages too murky to be deciphered. On page 154 (a singularly unfortunate page) we read, "that 5000 pairs of random numbers were generated between $[0, 1]$ and $[-1, 0]$." The worst offense of this kind occurs in the "explanation" of Powell's method which is too long to quote here. Let it suffice to say that I was more than happy to award an A to the student who figured out with no other aid than this book how to make the method work.

I still believe that the book treats an appropriate collection of topics, and that the authors are able craftsmen. Unfortunately the latter opinion must be based on evidence other than what is presented here.

WAYNE ROBERTS, Macalester College

*Galois Theory.* By Ian Stewart. Halsted Press, New York, 1973. xxv + 226 pp. $11.00 (paper). (Telegraphic Review August–September 1973.)

Galois theory is a good topic for a second undergraduate course in abstract algebra. It gives the student an opportunity to use many of the ideas and tools he studied in a first course, including linear algebra, group theory, and ring theory. There are several introductory texts in abstract algebra that include Galois theory. However, if a student tries to use such a book just for the Galois theory, he finds it necessary to return frequently to earlier chapters for results he needs. This is why a self-contained textbook in Galois theory for an undergraduate course is a welcome addition to the literature.

I used the book by Stewart in a course for seniors who had one year of abstract and linear algebra at the level of *Elements of Linear and Abstract Algebra* by Paley and Weichsel. I was able to cover most of the book in one semester, omitting some of the proofs in the final chapters. I was well satisfied with it and I intend to use it again.

The book is written at a level appropriate for the students I had. It is not a watered-down treatment of field theory; for example, it is not restricted to fields of characteristic zero. However, the author limits himself to an exposition of Galois theory and does not include topics, such as transcendental extensions or inseparable extensions, that would make the book too long or too difficult. He does discuss in detail the impossibility of some geometric constructions, including the construction of regular polygons, and the solvability of equations. In one chapter he shows that there are fifth degree polynomials with rational coefficients that are not solvable by radicals, while in another chapter he develops the sufficient conditions for solvability and discusses the general polynomial. I ran out of time for a complete study of the latter chapter, but the course is still effective if one ends with the former chapter. The relevant group theory is included, but I found it advisable to shorten this by taking up only those topics that were actually needed.

The writing is excellent. The author does not hesitate to explain what he will be trying to do before he does it or to point out the meaning of what he has done. He has organized the book into short chapters, each one containing one major idea, so the reader is not overwhelmed by a long list of theorems. There are enough examples, including a detailed one illustrating the fundamental theorem of Galois theory. There is a good selection of problems, many with answers. One excellent feature is a set of true and false questions (with answers) at the end of most chapters. The students found these very helpful in testing their understanding of the book and several led to class discussions that were very valuable.

The book has some defects. There are a number of misprints that cause difficulty. The proof of the simplicity of the alternating group on $n$ elements, $n > 4$, is badly confused. The author apparently based this on the proof in Ledermann's group theory book, but in the process of transferring the proof to this book so many mistakes were made that it becomes incomprehensible. The proof of the theorem that the Galois group of a solvable extension is solvable (p. 144) has an error. The author states that under the Galois correspondence the Galois group of a normal intermediate field is a normal subgroup of the Galois group, whereas it is isomorphic to a quotient group of the Galois group.

In conclusion, I found this book to be excellent and I recommend it for use in a second course in abstract algebra at the undergraduate level.

ROBERT B. REISEL, Loyola University, Chicago

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
P = professional reading
L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(13), L. *Introductory College Mathematics*. Robert D. Hackworth, Joseph Howland. Saunders, 1976. *Real Number System*, 67 pp; *Geometry*, 72 pp; *Consumer Mathematics*, 67 pp; *Sets and Logic*, 77 pp; *Indirect Measurement*, 72 pp; *Algebra 1*, 65 pp; *Algebra 2*, 66 pp; *History of Real Numbers*, 62 pp; *Probability*, 80 pp; *Statistics*, 73 pp; *Numeration*, 60 pp; *Geometric Measures*, 70 pp; *Tables and Graphs*, 62 pp; *Metric Measure*, 65 pp; *Linear Programming*, 66 pp; *Computers*, 82 pp, $1.95 (P) each. A wide selection of topics for a very elementary introduction to mathematics. LLK

GENERAL, P. *American Mathematical Society Translations, Series 2, V. 104: Some Problems of Mathematics and Mechanics*. AMS, 1976, v + 265 pp, $30. Papers in celebration of the 70th birthday of M.A. Lavrent'ev; includes a lengthy personal and professional biography of Lavrent'ev. LAS

BASIC, T?(13: 1, 2). *Technical Mathematics*. Al H. Chew, Richard L. Little, Sherry B. Little. HM, 1976, xii + 397 pp, $11.95. Arithmetic, simple algebra, graphs, geometry, and trigonometry. Emphasizes applications. Frequent clumsy use of terminology. FLW

BASIC, T(13: 1, 2). *Business Mathematics*. George Kevorkian. Merrill, 1976, vi + 314 pp, $11.95. Reviews elementary algebra then considers the basic mathematics involved in accounting, marketing, and finance. The presentation attempts to "overcome the students' self-defeating concepts." FLW

BASIC, T(13: 1, 2). *An Elementary Approach to Mathematics*. Thomas Koshy. Goodyear, 1976, xvi + 588 pp, $14.95. For elementary education majors or students wanting a basic survey. Sets, logic, number systems, relations, groups, geometry, and very brief chapters on probability, statistics, and computers. FLW

BASIC, T(13: 1), S. *Basic Algebra*. Virginia Lee. Canfield Pr, 1976, x + 629 pp, $10.95 (P). Intended as a first text or review book in algebra. Topics: arithmetic, sets, rational and real numbers, equations, inequalities, exponents, polynomials, radicals, quadratic equations. SG

BASIC, T(13: 1). *Fundamental College Mathematics, From Patterns to Applications, Second Edition*. Ruric E. Wheeler. Brooks/Cole, 1976, xii + 379 pp, $12.95. A considerable revision of the first edition. Intended as a text for liberal arts students and elementary education majors, it will strike many teachers as too elementary and unrigorous for either. JD-B

BASIC, T(13). *From Arithmetic to Algebra*. Derek I. Bloomfield. Reston, 1976, xiii + 443 pp, $9.95 (P). Mostly arithmetic: whole numbers, prime numbers, fractions, decimals. Algebra: solution of linear equations. Chapters on ratio and proportion, measurement and units and tables and graphs apply some of the algebra. A final chapter on "Modern mathematics" discusses set theory. Diagnostic and achievement tests in each chapter. Lots of exercises. PJM

PRECALCULUS, T(13: 1). *Mathematics for Technical Education*. Dale Ewen, Michael A. Topper. P-H, 1976, xii + 414 pp, $13.95. A good choice of topics for a precalculus text. Good format, lots of word problems, and a Student Study Guide is available. LLK

PRECALCULUS, T(13: 1, 2). *Intermediate Algebra*. Melvin F. Janowitz. P-H, 1976, 436 pp, $10.50. A nicely organized algebra book. Chapters on sets, polynomials, fractions, first degree equations and inequalities, exponents and radicals, and quadratic equations and inequalities (including those involving square roots) are followed by a chapter on functions and relations. The text concludes with chapters on exponentials and logs, systems of equations, and sequences. Index, solutions to selected exercises. PJM

EDUCATION, T**(13: 1). *Algebraic and Arithmetic Structures: A Concrete Approach for Elementary School Teachers*. Max S. Bell, Karen C. Fuson, Richard A. Lesh. Free Pr, 1976, xxxv + 683 pp, $14.95. A nonbehavioristic "activities" approach utilizing "embodiments" or "concretizations" (rods, chips, etc.) and a large assortment of do-it-yourself projects. Divided into three parts: (a) illustrations of the usefulness of numbers and arithmetic operations; (b) practical algorithmic processes of arithmetic (includes discussion of metric system); (c) potpourri of enrichment material drawn from logic, sets, number theory, group theory, recreational mathematics. The topics presented and the methods used will make the transition from learning mathematics to teaching it to children a most natural development. LCL

EDUCATION, P. *Mathematics Tests Available in the United States*. James S. Braswell. NCTM, 32 pp, $1.50 (P). Updated version of booklet issued previously in 1959, 1968 and 1972. Includes complete addresses of sources and related evaluation services. Listings concentrate on school mathematics. LAS

EDUCATION, S(16-18), P. *Aufgabenarten bei objektivierten Lehr-und Testverfahren*. Hans Freibichler. Hermann Schroedel, 1976, 90 pp, DM 9,80 (P). A discussion of the evaluation of learning processes via various sorts of tests: short answer, multiple choice, essay, etc.. A rather formal but detailed discussion with an extensive bibliography. JAS

EDUCATION, P, L*, *The Intuitive Sources of Probabilistic Thinking in Children.* E. Fischbein. Reidel, 1975, xi + 204 pp, $29. Analysis of psychological studies of probabilistic thinking in children, leading to the conclusion: "Undeveloped probabilistic intuition is not able to follow the procedures of sophisticated reasoning...or evaluate the plausibility of the obtained result. ...[O]ur view is that this situation is explained by the quasi-systematic exclusion of statistical and probabilistic models of thinking from education." LAS

EDUCATION, P, *Mathematical Needs of School Leavers Entering Employment.* Inst Math Appl. *I:* 1974, v + 142 pp, £2.00 (P); *II:* 1975, ix + 110 pp, £2.50 (P). The proceedings of conferences held at Nottingham in 1974 and 1975 which discussed the problem of mathematical inadequacies amongst craft apprentices in the engineering industry and other users of mathematics in commerce. CEC

EDUCATION, P, *Intellectual Talent: Research and Development.* Ed: Daniel P. Keating. Johns Hopkins U Pr, 1976, xviii + 346 pp, $16.95; $3.95 (P). Proceedings of the sixth Hyman Blumberg symposium on research in early childhood education held in October 1974 at Johns Hopkins University. About half the papers deal with aspects of mathematical precocity. LAS

EDUCATION, S, *Computer Science for the Teacher.* James L. Poirot, David N. Groves. Sterling Swift, 1976, vii + 260 pp, $8.95 (P). A guide for secondary math teachers. Includes computer arithmetic, flowcharting, BASIC programming, computer games and advice on implementing a secondary level programming course and lending administrative assistance. Exercises. Bibliography. RWN

EDUCATION, P, *Individuum und Mathematikunterricht.* Hendrik Radatz. Hermann Schroedel, 1976, 173 pp, DM 18,80 (P). A study of the influence of the cognitive styles (roughly personal characteristics other than intelligence as measured by intelligence tests) of pupil and teacher on the learning of mathematics. Concentrates on impulsiveness and reflectiveness. JD-B

EDUCATION, T(14-15), *Elementary Mathematics, A Fundamentals and Techniques Approach.* Samuel R. Filippone, Michael Z. Williams. HM, 1976, xii + 370 pp, $11.95. Mathematical systems for the elementary teacher. Includes non-decimal bases, 2 x 2 matrices, topics from geometry. Exercises emphasize using formal properties of numbers. PJM

HISTORY, P, L, *Isaac Newton's Theory of the Moon's Motion (1702).* I. Bernard Cohen. Dawson, 1975, viii + 170 pp, $25. Facsimile texts of the Latin version plus three contemporary English translations preceded by an extensive (87 pp) introduction by I.B. Cohen. LAS

HISTORY, P, L, *Richard Dedekind et les Fondements des Mathématiques.* Pierre Dugac. Libr. Philo. J. VRIN (Paris), 1976, 334 pp, (P). A comprehensive analysis of the development of Dedekind's mathematical thought and its influence on his contemporaries. Half the volume consists of correspondence among Dedekind's mathematical contemporaries. LAS

FOUNDATIONS, T(14-17), S, P, L, *Introduction to the Theory of Fuzzy Subsets: Volume 1, Fundamental Theoretical Elements.* A. Kaufmann. Trans: D.L. Swanson. Acad Pr, 1975, xvi + 416 pp, $22.50. The "first systematic exposition" of the extensive theory developed by L.A. Zadeh. Includes fuzzy graphs, fuzzy relations, fuzzy logic, and some generalizations. FLW

FOUNDATIONS, P, L*, *Mind, Language and Reality, Philosophical Papers, V. 2.* Hilary Putnam. Cambridge U Pr, 1975, xvii + 457 pp, $25. 22 papers on the philosophy of language comprise the second half (V. 1: *Mathematics, Matter and Method,* TR, March 1976) of Putnam's most important published papers. LAS

FOUNDATIONS, P, *Logic and Probability in Quantum Mechanics.* Ed: Patrick Suppes. Reidel, 1976, xv + 541 pp, $39.50. 21 papers, half growing out of a 1972-74 Stanford seminar, grouped into sections on logic, on probability, and on completeness; most have previously appeared in Vol. 29 of *Synthese.* Concludes with a major subject-organized bibliography of nearly 1000 items on the history and philosophy of quantum mechanics. LAS

FOUNDATIONS, P*, L, *The Intentions of Intentionality and Other New Models for Modalities.* Jaakko Hintikka. Reidel, 1975, xviii + 262 pp, $17 (P); $34. Eleven recently published essays with thoughts about future directions and with suggestions for new interpretations and applications (especially to linguistics and the general theory of language) of what is more grandiosely known as "possible-worlds semantics." LCL

NUMBER THEORY, P, *Lecture Notes in Mathematics-502: Representations of Real Numbers by Infinite Series.* János Galambos. Springer-Verlag, 1976, vi + 146 pp, $7.40 (P). A 'representation' is a one-to-one correspondence between the real numbers $x$ (between 0 and 1) and sequences of functions $u_n(x)$ of a given form such that $x = \Sigma u_n(x)$ for all $x \in (0,1)$. This monograph surveys algorithms for the generation of such representations and the relation of these representations to various number-theoretic properties of $x$. LAS

NUMBER THEORY, P, *Class Groups and Picard Groups of Group Rings and Orders.* Irving Reiner. CBMS Reg. Conf. in Math., No. 26. AMS, 1976, 44 pp, $6.80 (P). An excellent introduction to the subject. Topics include explicit formulas for class groups, Mayer-Vietoris sequences, induction techniques, and survey of specific results. SG

ALGEBRA, P, *Equations in Free Semigroups.* Ju. I. Hmelevskii. Proc. of Steklov Inst. of Math., No. 107. AMS, 1976, iii + 270 pp, $50.40 (P). A study of word equations in finitely generated free semigroups. The author presents background material, constructs an algorithm for recognizing whether an equation in three unknowns has a solution, considers systems of equations in two unknowns, and studies coefficient free equations. SG

ALGEBRA, S(16-17), L. *Representations of the Rotation and Lorentz Groups, An Introduction.* M. Carmeli, S. Malin. Lect. Notes in Pure and Appl. Math., V. 16. Dekker, 1976, xi + 117 pp, $13.50 (P). A concise, well referenced, physics-oriented treatment that could well serve as seminar material for post-algebra undergraduates. Presumes little except the ability to read between the lines. LAS

ALGEBRA, S(18), P. *Relation Modules of Finite Groups.* Karl W. Gruenberg. CBMS Reg. Conf. in Math., No. 25. AMS, 1976, vi + 81 pp, $8.40 (P). Introduction to the application of integral representation theory to presentation theory of finite groups. Written for both ring theorists and group theorists. LCL

ALGEBRA, T(17-18: 1, 2), P, L. *Group Theory.* Eugene Schenkman. Krieger, 1975, xiv + 289 pp, $15.25. Reprint of the original 1965 edition, with corrections. Basic concepts and self-contained theory of abelian, nilpotent, and solvable groups. LCL

COMPLEX ANALYSIS, P. *Sur les sections analytiques de la courbe universelle de Teichmüller.* John Hamal Hubbard. Memoirs No. 166. AMS, 1976, ix + 137 pp, $7.60 (P). The author's dissertation on analytic sections of universal curves over Teichmüller space. JAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-505: Advances in Complex Function Theory.* Ed: W.E. Kirwan, L. Zalcman. Springer-Verlag, 1976, viii + 202 pp, $9.50 (P). A selection of papers presented at seminars and results obtained during the year-long emphasis on complex function theory at the University of Maryland in 1973-74. JAS

COMPLEX ANALYSIS, T(18: 1), P, L. *Banach Algebras and Several Complex Variables, Second Edition.* John Wermer. Grad. Texts in Math., V. 35. Springer-Verlag, 1976, ix + 161 pp, $14.80. An extended version of the first edition (TR, April 1972) with added sections on submanifolds of high dimension, generators, the fibers over a plane domain, and examples of hulls. JAS

COMPLEX ANALYSIS, T*(15-16: 1), L. *Elements of Complex Variables, Second Edition.* Louis L. Pennisi. HR&W, 1976, viii + 517 pp, $16.95. Differs from the earlier (1963) edition by the addition of topics from Riemann surfaces, analytic continuation, and the construction of harmonic functions. With these additions and minor editorial changes, the book remains a solid, readable first course. The extensive treatment of contour integration is particularly nice. TAV

DIFFERENTIAL EQUATIONS, P. *American Mathematical Society Translations, Series 2, V. 105: Partial Differential Equations.* AMS, 1976, v + 346 pp, $34.40. Proceedings of a symposium dedicated to S.L. Sobolev. LAS

DIFFERENTIAL EQUATIONS, P. *Numerical Solution of Partial Differential Equations-III, SYNSPADE 1975.* Ed: Bert Hubbard. Acad Pr, 1976, ix + 499 pp, $25. Proceedings of a quintennial symposium of the University of Maryland, May 1975. LAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-507: Abstract Non-Linear Wave Equations.* Michael Reed. Springer-Verlag, 1976, vi + 128 pp, $7.40 (P). An exposition of the "abstract" approach (via non-linear functional analysis) to non-linear wave equations. Author covers existence and properties of solutions and scattering theory. SG

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-501: Spline Functions.* Ed: K. Böhmer, G. Meinardus, W. Schempp. Springer-Verlag, 1976, vi + 421 pp, $14.40 (P). Proceedings of the Symposium at Oberwolfach in May 1975. JAS

NUMERICAL ANALYSIS, T(16-17: 1). *Einführung in die Numerische Mathematik.* Bruno Brosowski, Rainer Kress. Hochschultaschenbücher, B. 211. Bibliographisches Inst, 1976, 117 pp, (P). The second half of a rather sophisticated and difficult "introduction" to numerical analysis. The two chapters treat interpolation and numerical integration, and some optimization problems. JD-B

NUMERICAL ANALYSIS, P. *Studies in Spline Functions and Approximation Theory.* Samuel Karlin, *et al.* Acad Pr, 1976, xii + 500 pp, $19.50. This volume reports a series of research results, obtained by the authors mainly during September 1970 through June 1974 at the Weizmann Institute. Contributed papers cover aspects of the theory of best approximations and quadratures, the solution of certain extremal problems involving generalized Landau and Markov-type inequalities for derivative functionals, and a hierarchy of interpolation and convergence properties of spline functions. I-CH

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-506: Numerical Analysis.* G.A. Watson. Springer-Verlag, 1976, x + 201 pp, $9.50 (P). Proceedings of the conference at Dundee, Scotland in July 1975. JAS

FUNCTIONAL ANALYSIS, S(18), P. *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces.* Felix E. Browder. Proc. of Symp. in Pure Math., V. XVIII, Part 2. AMS, 1976, viii + 308 pp, $30. Belated publication of a book-length survey of nonlinear functional analysis presented at the April 1968 Chicago AMS meeting. No significant alterations have been made from the widely circulated 1968 mimeographed preprint, but minor errors were corrected. LAS

OPTIMIZATION, P. *Nonlinear Programming.* Ed: Richard W. Cottle, C.E. Lemke. SIAM-AMS Proc., V. IX. AMS, 1976, vii + 200 pp, $15.60. Proceedings of a symposium held in New York City, March 1975; begins with an historical survey by Harold Kuhn. LAS

OPTIMIZATION, P. *Combinatorial Programming: Methods and Applications.* Ed: B. Roy. Reidel, 1975, xvi + 386 pp, $36. Papers from a NATO Advanced Study Institute at Versailles, September 1974. LAS

ANALYSIS, P. *Existence and Regularity Almost Everywhere of Solutions to Elliptic Variational Problems with Constraints.* F.J. Almgren, Jr. Memoirs No. 165. AMS, 1976, viii + 199 pp, $8.40 (P). Includes the first proof of the existence of solutions to general minimal partitioning problems such as arise in capillarity and soap film phenomena. LAS

ANALYSIS, T(18), S, P. *Treatise on Analysis, V. II: Enlarged and Corrected Printing.* J. Dieudonné. Trans: I.G. Macdonald. Pure and Appl. Math., V. 10-II. Acad Pr, 1976, xv + 453 pp, $24. A translation of the 1974 second French edition; contains chapters 12-15 of the 25-chapter, four-volume treatise. The first edition was published in 1969 (TR, October 1970) and translated in 1970 (TR, August-September 1973). LAS

ANALYSIS, P. *Conuri Convexe de funatii continue pe Spatii Compacte.* Nicu Boboc, Gheorghe Bucur. Editura Academiei Romania, 1976, 198 pp, Lei 11 (P). A presentation of recent (within last 15 years) results, especially those of Choquet, concerning a compact space with a convex cone of lower semicontinuous functions on the space. JAS

ANALYSIS, S(18), P. *Topologische Gruppen.* Dieter Lutz. Bibliographisches Inst, 1976, 175 pp, (P). This presentation of the basic theory of topological groups grew out of lectures by the author at the University of Constance. Includes topological and algebraic preparation--based on beginning graduate courses in the same--but does not cover in depth representation theory or Lie group theory. No exercises. JAS

ANALYSIS, P. *Calculus of Variations and Control Theory.* Ed: David L. Russell. Acad Pr, 1976, xvi + 409 pp, $16. Proceedings of a Symposium conducted by the MRC at Madison, September 22-24, 1975. JAS

GEOMETRY, P. *Smooth Compactification of Locally Symmetric Varieties.* A. Ash, *et al.* Math Sci Pr, 1975, iv + 335 pp, $13.50 (P). A comprehensive presentation of the theory announced and outlined in Mumford's "A new approach to compactifying locally symmetric varieties" (Proceedings of the Tata Institute Colloquium, January 1973; Oxford Univ. Press, 1975), wherein compactifications of $D/\Gamma$ were considered for $D$ a bounded symmetric domain and $\Gamma$ a neat arithmetic subgroup of $\mathrm{Aut}(D)^\circ$. JAS

GEOMETRY, S*, P, L*. *Space Structures, Their Harmony and Counterpoint.* Arthur L. Loeb. A-W, 1976, xviii + 169 pp, $9.50 (P); $19.50. An aesthetically pleasing yet mathematically solid treatment of the "structure of structures", based largely on the Euler-Schlaefli relation and on Loeb's version of statistical symmetry. Addressed to philomorphs, those artists and scientists who share a common belief that structure touches on the very nature of things. LAS

GEOMETRY, T(15: 1). *The Foundations of Geometry and the Non-Euclidean Plane.* George E. Martin. Intext, 1972, xvi + 509 pp, $15. Absolute and non-Euclidean geometry via a transformation approach, incorporating interesting quotes, extensive historical commentary, mathematical elaborations and an emphasis on symbolism and terminology. Burdened by all this, the style becomes somewhat choppy and disjointed. JNC

GEOMETRY, P. *Topics in Differential Geometry.* Ed: Hanno Rund, William F. Forbes. Acad Pr, 1976, xi + 184 pp, $19.50. A volume of papers published as a memorial to Evan Tom Davies including a reprint of the obituary notice from *Bull. London Math. Soc.*, and an extensive bibliography of Davies' work. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-491: Invariants for Real-Generated Uniform Topological and Algebraic Categories.* Kevin A. Broughan. Springer-Verlag, 1975, x + 197 pp, $8.20 (P). A real-generated category is one whose objects are given a topological or uniform structure by functions into the reals. Examples: metric spaces, metrizable topological groups. Invariants for spaces in such categories are investigated. PJM

TOPOLOGY, P. *Lecture Notes in Mathematics-493: Foliated Bundles and Characteristic Classes.* Franz W. Kamber, Philippe Tondeur. Springer-Verlag, 1975, xiv + 208 pp, $9.50 (P). A foliation is a nice partition of a manifold. This monograph studies foliated principal bundles and their invariants. PJM

TOPOLOGY, P. *Parametrized Knot Theory.* Stanley Ocken. Memoirs No. 170. AMS, 1976, vi + 114 pp, $7.60 (P). A Kervaire knot is a map from $S^n$ to $S^{n+2}$. This monograph studies parametrized knots, where the parameters lie in a manifold $M$, i.e., maps of $S^n \times M$ into $S^{n+2} \times M$. The methods are those of algebraic topology, specifically Capell-Shaneson's homology surgery theory. PJM

TOPOLOGY, T(18), P. *3-Manifolds.* John Hempel. Annals of Math. Stud., No. 86. Princeton U Pr, 1976, xii + 195 pp, $13.50; $5 (P). A survey of results on the classification of 3-manifolds. Includes connections with group theory, a chapter on the Poincaré conjecture, and a chapter on open problems. Exercises scattered throughout text. A possible second graduate course in topology, if the first includes algebraic topology. PJM

TOPOLOGY, P. *Lecture Notes in Mathematics-510: Singularités $C^\infty$ en Présence de Symétrie.* Valentin Poènaru. Springer-Verlag, 1976, 174 pp, $8.20 (P). Symmetry here refers to the action of a compact Lie Group; the main body of the monograph is a theory of equivariant structural stability, i.e., stability for $C^\infty$ manifolds with Lie group actions, and equivariant maps between them. PJM

PROBABILITY, T(18: 1), P. *Queueing Systems, Volume II: Computer Applications.* Leonard Kleinrock. Wiley, 1976, xx + 549 pp, $24.95. Presumes Volume I (TR, March 1976) or other strong introduction to queueing theory. The computer applications in the title refers mostly to applications to computer systems, e.g., time sharing, multi-access systems, and computer communication systems. TAV

PROBABILITY, P*. *Principles of Random Walk, Second Edition.* Frank Spitzer. Grad. Texts in Math., V. 34. Springer-Verlag, 1976, xiii + 408 pp, $19.80. Minor editorial changes from the 1964 edition. While the recent achievements in random walks, notably Ornstein and Stone's extension of recurrent theory from the integers to the reals, do date the material of the text, the presentation here is lucid and the interplay of harmonic analysis and potential theory is demonstrated clearly. A worthwhile book to read. TAV

PROBABILITY, P. *Gaussian Processes, Function Theory, and the Inverse Spectral Problem.* H. Dym, H.P. McKean. Prob. and Math. Stat., V. 31. Acad Pr, 1976, xi + 333 pp, $35. A carefully written treatment of Gaussian processes with special attention paid to prediction and interpolation. Contains a substantial bibliography. TAV

PROBABILITY, T(16: 1), L. *Markov Chains, Theory and Applications.* Dean L. Isaacson, Richard W. Madsen. Wiley, 1976, x + 256 pp, $18.95. Intended for a course at the senior level, the book treats Markov chains from classical (e.g., Feller), algebraic and computer viewpoints. A good "short course" text written to be easily read. TAV

PROBABILITY, T(16-18: 1, 2), S, P. *Sums of Independent Random Variables.* V.V. Petrov. Trans: A.A. Brown. Ergebnisse der Math., B. 82. Springer-Verlag, 1975, x + 345 pp, $39.60. Infinitely divisible distributions, probabilistic inequalities, limit theorems, the laws of large numbers, and the law of the iterated logarithm. FLW

PROBABILITY, P. *The Concept of Probability in Psychological Experiments.* Ed: Carl-Axel S. Staël von Holstein. Reidel, 1974, xi + 153 pp, $19.50. Six papers (plus a selected bibliography) on interpretations of psychological experiments designed to study how people evaluate uncertainy and to see how well their behavior conforms to the formal axioms of probability. LAS

PROBABILITY, T(16), P, L. *Analysis of Queueing Systems.* J.A. White, J.W. Schmidt, G.K. Bennett. Acad Pr, 1975, x + 532 pp, $27. Emphasis is on queueing *systems*, not the theory of queues, thus the authors include discussions of data analysis, cost modeling and simulation to emphasize unified system approach. Substantial attention is paid to the parametric estimation necessary to apply the queueing theory. A welcome addition to the literature on queueing. TAV

PROBABILITY, T**(16-17: 1, 2), P*, L. *Introduction to Stochastic Processes.* Erhan Cinlar. P-H, 1975, x + 402 pp, $16.95. An exceedingly well conceived and clearly written text. Presumes an introductory course in probability. Emphasizes the behavior of sample paths, with a natural introduction to potentials via optimal stopping problems. Recommended reading for the theoretical as well as applied probabilist. TAV

PROBABILITY, P. *Les Processus Gaussiens et Les Distributions Aléatoires.* Ed Cent Nat Rech Sci, 1975, xv + 223 pp, 50,00 F (P). Proceedings of a colloquium held at Strasbourg in 1973. PJM

STATISTICS, S(16-18), P, L*. *Logic of Statistical Inference.* Ian Hacking. Cambridge U Pr, 1976, ix + 232 pp, $6.95 (P). First paperback edition of this 1965 monograph which reviews and contrasts competing philosophical bases for statistical reasoning. Includes a detailed rationale for Fisher's controversial 'fiducial' theory concerning degrees of confidence. LAS

STATISTICS, T(15-16: 2), P. *Statistical Quality Control Methods.* Irving W. Burr. Statistics, V. 16. Dekker, 1976, xii + 522 pp, $29.50. Intended as a text or for practicing engineers and statisticians. The approach is "how to", rather than mathematical. Numerous problems and examples. Each chapter contains an extensive bibliography. Reproduced from typed text. Note the price! TAV

STATISTICS, P. *Perspectives in Probability and Statistics.* Ed: J. Gani. Acad Pr, 1975, viii + 423 pp, $21.50. A volume of diverse papers dedicated to Maurice Bartlett, beginning with a listing of Bartlett's publications and an essay on his work on agricultural experiments. JAS

STATISTICS, T(15-16: 1, 2), S. *Statistical Theory, Third Edition.* Bernard W. Lindgren. Macmillan, 1976, xiii + 614 pp, $14.95. This edition of this solid text has a "good bit of rewritten material" and "significant changes in order." Discussion of decision theory is postponed so that consideration of standard tests can come earlier. Second edition reviewed here in October 1968. FLW

STATISTICS, T(13: 1). *Elementary Statistics, Second Edition.* Robert R. Johnson. Duxbury Pr, 1976, xviii + 638 pp, $13.95. Added sections on conditional probability, normal approximation to binomial, Mann-Whitney test. More than twice as many exercises as in the first edition (TR, June-July 1974). LCL

STATISTICS, T(16-17: 1), S, P, L. *Time Series Analysis and Forecasting: The Box-Jenkins Approach.* O.D. Anderson. Butterworths, 1976, vii + 182 pp, $13.95. The properties of the ARMA class of stable linear models, the Box-Jenkins method for forecasting, and a survey of more recent work.FLW

STATISTICS, S*(15-17), P, L*. *Probability, Statistics and Time: A Collection of Essays.* M.S. Bartlett. Chapman, 1975, viii + 148 pp, $12.75. An assembly of nine of Bartlett's relatively general essays from the past fifteen years. LAS

STATISTICS, S(16-18), P. *The Statistical Analysis of Spatial Pattern.* M.S. Bartlett. Chapman and Hall, 1975, x + 90 pp, $9.75. Considers the problem of analyzing departures from randomness when they are detected in relation to some stochastic model. Reviews the needed theory of stochastic processes, develops the theory for nearest-neighbor systems on a lattice, and give examples from biometrics. FLW

STATISTICS, S(18), P. *Normal Approximation and Asymptotic Expansions.* R.N. Bhattacharya, R. Ranga Rao. Wiley, 1976, xiv + 274 pp, $21.95. "Presents in a unified way various refinements of the classical central limit theorem for independent random vectors", including results of recent research. FLW

COMPUTER SCIENCE, T(16-17: 1), S, P, L. *The Computational Complexity of Algebraic and Numeric Problems.* A. Borodin, I. Munro. Am Elsev, 1975, x + 174 pp, $8.75 (P); $13.50. A study of the inherent complexity of solving algebraic problems. Applications include matrix multiplication, polynomial and rational function evaluation, Fast Fourier Transform. Includes serial and parallel computation. Most of the results are lower bounds. Most of the proofs are algebraic. Prerequisite: some modern algebra. Problems. Open Problems. RWN

COMPUTER SCIENCE, S(16-18), P, L. *Pattern Synthesis: Lectures in Pattern Recognition Volume 1.* U. Grenander. Appl. Math. Sci., V. 18. Springer-Verlag, 1976, vii + 509 pp, $14.80 (P). The first in a multivolume series presenting a general theory of patterns. This part studies "how patterns are formed from the generators by combination, identification, and deformation", and examines the logical structure of pattern formalism. FLW

COMPUTER SCIENCE, S(17-18), P. *Rekursive Funktionen in der Komputer-Theorie.* Rózsa Péter. Akademiai Kiado, 1976, 190 pp, $12. An account, without detailed proofs but with references to the original papers, of the author's work on recursiveness and theoretical problems arising from the actual use of computers. JD-B

COMPUTER SCIENCE, S(13-14), L. *Game Playing with Computers, Revised Second Edition.* Donald D. Spencer. Hayden, 1975, 312 pp, $14.95. A low-level survey of games, puzzles and magic squares that are amenable to computer implementation. Except for certain very simple examples, discussion is limited to an outline of rules: questions of strategy, and solution algorithms are not treated. LAS

COMPUTER SCIENCE, T(13: 1). *Information Processing, Second Edition.* Marilyn Bohl. SRA, 1976, viii + 438 pp, $8.95 (P). This business data-processing text has been updated to include recent developments such as large systems, minicomputers, structured programming, POS, data bases, and data communications. RWN

COMPUTER SCIENCE, T(15-17). *Digital Networks.* Janusz A. Brzozowski, Michael Yoeli. P-H, 1976, xiv + 398 pp, $18.95. A broad introduction to the theory and logical design of digital networks. Somewhat mathematical. No specific electronics prerequisite. Covers switches, gates, integrated circuits, Boolean expressions, sequential networks and error detection. RWN

COMPUTER SCIENCE, T(15: 1). *Software Tools.* Brian W. Kernighan, P.J. Plauger. A-W, 1976, 338 pp, $8.95 (P). Useful high-level programming techniques. Presumes programming experience; intends to enhance programming expertise. Discusses formatting, data transcribers, sorting, editing; demonstrates program design, readability, efficiency, reliability, portability. RWN

COMPUTER SCIENCE, T(15). *The Programmer's Introduction to SNOBOL.* W. Douglas Maurer. Am Elsev, 1976, x + 141 pp, $13.50; $8.75 (P). Presumes previous high-level experience. Moderate pace yet quite thorough. Pattern matching, recursion, etc.. Includes a chapter on systems programming. RWN

SYSTEMS THEORY, T(16-17: 1), P. *Linear Multivariable Systems.* W.A. Wolovich. Appl. Math. Sci., V. 1. Springer-Verlag, 1974, viii + 358 pp, $9.50 (P). Uses state-space approach to multivariable (multi-input/output) control system analysis and design, but also draws analogies with the more classical frequency domain approach. Methods apply to linear, time-invariant, finite dimensional systems. Prerequisites: classical control theory, differential equations. RBK

APPLICATIONS (ACTURARIAL SCIENCE), P. *Credibility, Theory and Applications.* Ed: P.M. Kahn. Acad Pr, 1975, xi + 414 pp, $17.50. Credibility is the actuarial term for the estimate of true expectation derived as a compromise between hypothesis and observation; a theory of credibility is essential to make possible the insurance of infrequently observed risks. This volume contains papers from an actuarial research conference held at Berkeley in September, 1974. LAS

APPLICATIONS (BIOLOGY), P, L. *Some Mathematical Questions in Biology, VII.* Ed: Simon A. Levin. Lect. on Math. in Life Sci., V. 8. AMS, 1976, viii + 182 pp, $20 (P). Six papers from the ninth annual symposium held at the January 1975 AAAS meeting in New York. Three papers deal with the critical role of population structure in ecology and evolution; the other three deal with neurobiology, including one containing speculation that neurobiology may be "totally non-mathematical." LAS

APPLICATIONS (ECONOMICS), S(16-18), P, L. *Lecture Notes in Economics and Mathematical Systems-112: Objectives and Multi-Objective Decision Making Under Uncertainty.* Jochen Wilhelm. Springer-Verlag, 1975, iv + 111 pp, $7.40 (P). A new approach to the foundations of decision theory. Uses the algebra of categories and discusses examples from management science. FLW

APPLICATIONS (ENGINEERING), P. *Walsh Functions and Their Applications.* K.G. Beauchamp. Acad Pr, 1975, xiii + 236 pp, $24.50. The first four chapters develop the mathematical basis for the use of Walsh and related functions (and discuss fast transform algorithms and computer programs for their implementation); the next two deal with sequency analysis and filtering; the last two treat applications in communications and in signal processing. Many references. DFA

APPLICATIONS (INFORMATION SCIENCE), P. *The Library and Information Science CumIndex.* Frederick G. Kilgour. R & D Pr, vi + 722 pp, $55. An alphabetical cumulation of over 90,000 back-of-the-book index entries from 96 books in library and information science. LAS

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# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

Professor Yousef Alavi, Western Michigan University, represented the Association at the inauguration of A.J. Diekema as President of Calvin College on March 2, 1976.

Professor S. R. Knox, Millsaps College, represented the Association at the inauguration of A. K. Lucas as President of the University of Southern Mississippi on March 19, 1976.

Professor W. E. Briggs, Dean of the College of Arts and Sciences of the University of Colorado at Boulder, has been elected President of the Council of Colleges of Arts and Sciences, a national organization of colleges of arts and sciences in state universities.

Professor Emeritus S. S. Cairns, University of Illinois at Urbana–Champaign, has been granted a Senior U.S. Scientist Award by the Alexander von Humboldt Foundation of West Germany. It will support six months of research at the University of Ulm, commencing October 1, 1976.

Dr. J. F. Hurley, Associate Professor at the University of Connecticut, was Visiting Fellow in the Institute of Advanced Studies, Research School of Physical Sciences, Australian National University, during the period March–June, after spending the 1975–76 academic year at the University of the Philippines under the National Science Foundation Scientists and Engineers in Economic Development Program.

Assistant Professor L. F. Klosinski, University of Santa Clara, has been promoted to Associate Professor.

Assistant Professor G. W. Pineau, University of Prince Edward Island, has been appointed Chairman of the Department of Mathematics.

Assistant Professor R. L. Tyler, Susquehanna University, has been promoted to Associate Professor and has been appointed Chairman of the Department of Mathematical Sciences.

## THE GREATER METROPOLITAN NEW YORK MATH FAIR

Fair Date: March 6, 1977

Papers Due: January 14, 1977

Eligibility: Students who have completed, or are currently taking, mathematics at the eleventh year level or higher in the public, private and parochial high schools in the New York City, Westchester, Putnam, Dutchess, or Rockland Counties are eligible to participate in the FAIR. Exceptional students who do not meet these requirements but wish to submit papers for consideration by the FAIR Committee are welcome to do so.

Purpose: The FAIR encourages a student to pursue, in depth, some phase of mathematics to which he is drawn.

Procedure: 1. Research a topic in mathematics. 2. Write a paper on this topic. 3. Give a talk (of no more than 15 minutes duration) on your paper to a group of judges.

Nature of Paper: The work need not be completely original, but the paper should reveal scholarship appropriate to the course level of the student. Teachers may suggest topics but the presentation should be voluntary and based entirely upon the student's own readings and investigations.

Further details and application forms may be obtained from: Dr. Theresa J. Barz, Secretary, MATH FAIR Committee, Dept. of Mathematics & Computer Science, St. John's University, Jamaica, New York 11439.

## AGREEMENT OF COOPERATION IN INFORMATION THEORY BETWEEN IEEE AND USSR

In the fall of 1974 the Working Group on Information Theory of the USSR and the IEEE Information Theory Group signed an agreement of cooperation in the field of information theory. Both groups expressed interest in extensive cooperative activities, as in recent years increasing contacts between members of each group have led to the solution of several outstanding problems in information theory.

The present agreement aims at developing the following cooperative activities: 1) scientist exchange for joint work; 2) meetings of working groups of scientists from both sides to conduct joint research and for scientific interchange; 3) joint publications; and 4) facilitation of exchange of literature and research results.

The cooperative activities started with a scientific workshop held in Moscow, December 14–20, 1975. From the IEEE side 15 scientists participated, 13 of whom are U.S. based and two are at European institutions. An NSF-grant provided for the roundtrip transportation of the entire IEEE delegation. In Moscow the IEEE delegates were guests of the USSR Academy of Sciences.

From the USSR side over thirty scientists participated in the workshop. The co-chairmen of the IEEE delegation were T. Cover and T. Kailath, whereas R. Dobrushin and M. Pinsker headed the USSR delegation. The papers presented by the IEEE and USSR participants will be published in a joint Proceedings.

At this 1st USSR–IEEE Workshop on Information Theory the cooperative agreement was reconfirmed and extended. The next activity will be a scientist exchange program of 12 man-months during the academic year 1976–77. Moreover, future workshops will be held alternately in the USA and the USSR. The next workshop is to take place in the US in 1977.

In Moscow, it was agreed that in the 1976–77 program the cooperative research will concentrate on the further development of the following topics: 1) source coding and coding for noisy channels; 2) complexity of coding and decoding schemes; 3) data transmission in large multiple-user and computer-communication networks; 4) statistical methods of information processing including stochastic processes and estimation and detection theory; and 5) information and communication theory for optical and quantum systems.

Anyone interested in obtaining more information about the IEEE–USSR cooperative program on information theory should contact Professor J. K. Wolf, Department of Electrical and Computer Engineering, University of Massachusetts, Amherst, MA 01002, USA.

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## MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

### FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual joint meeting of the Northern California Sections of MAA and SIAM was held at the University of California-Davis, on February 21, 1976, with 173 registrations.

The following papers were presented:

1. *Prime generating functions and congruences*, by Henry Alder, U.C. Davis, President-Elect, MAA.
2. *Galileo Galilei*, by George Polya, Stanford University.
3. *Aspects of the calculus of variations*, by M.M. Schiffer, Stanford University.
4. *The numerical range*, by Marvin Marcus, U.C. Santa Barbara.

In addition a new film, "Guessing and Proving," by George Polya, was presented by Professor Polya.

Chairperson Kenneth Rebman presided over the Business Meeting prior to a luncheon meeting for those interested in the particular problems of two-year institutions. Section officers were elected as follows: Chairperson: David Barnette, U.C. Davis; Vice-Chairperson: Jane Day, College of Notre Dame; Program Chairperson: Kenneth Rebman, Hayward State University; Secretary-Treasurer: Newman Fisher, San Francisco State University.

N. FISHER, *Secretary-Treasurer*

### APRIL MEETING OF THE IOWA SECTION

The 63rd regular meeting of the Iowa Section of the MAA was held at Clarke College, Dubuque, on April 10, 1976. Chairman Lawrence Hart presided. There were 50 in attendance, 34 of which were members of the section. For the invited address, sponsored jointly by MAA and the Iowa Academy of Science, there were 90 in attendance.

The program consisted of contributed papers, an invited address by Dr. H. O. Pollak, Governor Muller's report, and the business session. Mrs. Ellen Oliver, Westmar College, Lemars, was elected as Chairman-Elect.

The program, arranged by James Cornette, consisted of the following:

1. *A comparison theorem for Green's functions*, by M. C. Papenfuss, Dubuque.
2. *A finite Stone-Weierstrass theorem*, by S. E. Mosiman, Dubuque.
3. *Continuity of a two-variable rational function at the origin*, by George Bridgman, Waverly.
4. *Modularization of pre-calculus courses*, by B. E. Gillam, Des Moines.
5. *Geometry before Euclid*, by J. Mathews, Ames.

6. (Invited address) *Relationship between the applications of mathematics and the teaching of mathematics*, by Dr. H. O. Pollak, President: MAA, Director: Mathematics and Statistics Research Center, Bell Laboratories, Murray Hill, New Jersey.

7. *Mixed method numerical solutions of the advection equation*, by J. M. Leone and D. N. Yarger, Ames.

8. *Error analysis for Picard–Chebyshev iterations*, by D. R. Steele, Lamoni.

B. E. GILLAM, *Secretary–Treasurer*

## APRIL MEETING OF THE MARYLAND — DISTRICT OF COLUMBIA — VIRGINIA SECTION

The annual Spring meeting of the Maryland — District of Columbia — Virginia Section of the MAA was held April 24, 1976, at Bowie State College, Bowie, Maryland. Professor Ronald Davis, chairman of the Section, presided. Sixty-one persons attended.

During the morning there was a session of contributed papers, an invited speaker, and a business meeting. Following lunch there was another session of contributed papers and a second invited speaker. In the business meeting the following officers were elected: Chairman–Elect, Professor Orville Thomas, U.S. Naval Academy; Vice–Chairman for Programs, Professor J. M. Smith, George Mason University; Secretary, Professor Reuben Drake, Washington Technical Institute.

The invited speakers were Dr. Jay Alexander of the University of Maryland who spoke on "Matrices, eigenvalues, and complex projective space," and Dr. Henry Walbesser of the University of Maryland — Baltimore Campus who spoke on "Trends in School Mathematics."

The contributed papers presented were:

1. *A useful approximation to the error function*, by T. S. Schreiber, Fairfax, VA.

2. *A minimum covering scheme by neighborhoods for Boolean valued data*, by Jong C. Wang, Center for Energy Systems, General Electric.

3. *Playing with curves*, by Richard Holzager, American University.

4. *Reflection orbits in a tetrahedron*, by Michael Goldberg, Washington, DC.

5. *Automata in Semiotic–Dimensional Space*, by John Hays, Laurel Ridge Conservation Center.

JOHN M. SMITH, *Secretary*

## APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at Eastern New Mexico University, Portales, New Mexico, on April 2 and 3, 1976. Thirty-nine people registered their attendance.

The invited speaker was Dr. Carroll Newsom, Chairman of the Guggenheim Foundation and former editor of the MONTHLY.

The following papers were contributed:

1. *Optimization problem of Bellman*, by John Thomas, New Mexico State University.

2. *The partition function $P(n)$, nodulo $j$, for figurate values of $n$*, by J. G. MacCarthy, Deming, New Mexico.

3. *A generalized birthday problem*, by J. E. Nyman, University of Texas at El Paso.

4. *The matrix equation $X^n = A$*, by J. L. Winter, New Mexico Institute of Mining and Technology.

5. *Special multi–section partitions and ordered pairs of tableaux*, by R. M. Grassl, University of New Mexico.

6. *Values of circulants with integer entries*, by H. T. Laquer, University of New Mexico.

7. *Integer partitions specified by directed graphs*, by Keith Hafen, University of New Mexico.

8. *Integer partitions monotonic on a $2\times2\times2$ cube*, by A. P. Hillman, University of New Mexico.

9. *Permutations of the positive integers with restrictions on successive differences*, by W. Y. Velez, Sandia Laboratories.

10. *Permutations of the positive integers not containing long arithmetic progressions*, by R. C. Entringer, University of New Mexico.

11. *Problems of preparing elementary education majors*, by C. J. Searcy, New Mexico Highlands University.

W. G. Calton, Chairman of the Southwestern Section, acted as meeting coordinator and Chairman of the session on Contributed Papers.

A. SWIMMER, *Secretary*

### 1976 CONTRIBUTING MEMBERS AND SPECIAL GIFTS

The Association is deeply indebted to the generosity of 103 of its members who have elected to be Contributing Members, Sponsors, or Patrons in 1976 by making contributions beyond the normal dues. There are 94 Contributing Members in 1976. In addition, the following members are 1976 Sponsors and Patrons:

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The Association also acknowledges with deepest gratitude the following special gifts:

A gift of $1000 from H. M. Gehman.
An anonymous gift of $1000.

Beginning in 1977 the names of all Contributing Members who do not request that their gifts remain anonymous will be published in the October issue of the MONTHLY and the December issue of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL. The names of the 1976 Contributing Members have been omitted from this note only because these members have not been given the opportunity to approve this publication.

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## CALENDAR OF FUTURE MEETINGS

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

Fifty-seventh Summer Meeting, University of Washington, August 14–16, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, St. Francis College, Loretto, Pennsylvania, April 22–23, 1977.

FLORIDA, University of South Florida, Tampa, March 4–5, 1977.

ILLINOIS, Chicago Loop College, Chicago, May 6–7, 1977.

INDIANA, Manchester College, North Manchester, November 6, 1976.

INTERMOUNTAIN

IOWA, Drake University, Des Moines, April 22–23, 1977.

KANSAS, Tabor College, Hillsboro, March or April 1977.

KENTUCKY, early April. Deadline for papers 6 wks. bef. mtg.

LOUISIANA–MISSISSIPPI, University of New Orleans, Louisiana, February 25–26, 1977.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Virginia Commonwealth University, Richmond, Virginia, November 20, 1976.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Eastern Michigan University, Ypsilanti, May 6–7, 1977.

MISSOURI, University of Missouri, St. Louis, April 29–30, 1977.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 15–16, 1977.

NEW JERSEY, early November and early May.

NORTH CENTRAL, South Dakota State University, Brookings, October 22–23, 1976.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Marshall University, Huntington, West Virginia, October 22–23, 1976.

OKLAHOMA–ARKANSAS, Oral Roberts University, Tulsa, Oklahoma, April 1–2, 1977.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 wks. bef. mtg.

PHILADELPHIA, Montgomery County Community College, Blue Bell, Pennsylvania, November 20, 1976.

ROCKY MOUNTAIN, Metropolitan State College, Denver, Colorado, April 29–30, 1977.

SEAWAY, Broome County Community College, Binghamton, New York, November 5–6, 1976.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, Loyola Marymount University, Los Angeles, March 12, 1977.

SOUTHWESTERN, Phoenix College, Phoenix, Arizona, Spring 1977.

TEXAS, Baylor University, Waco, April 1–2, 1977.

WISCONSIN, University of Wisconsin, Oshkosh, Spring 1977.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 20–26, 1977.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, San Francisco, October 29–30, 1976.

AMERICAN MATHEMATICAL SOCIETY, St. Louis, Missouri, January 27–30, 1977.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of North Dakota, Grand Forks, June 13–16, 1977.

ASSOCIATION FOR COMPUTING MACHINERY, Houston, Texas, October 20–22, 1976.

ASSOCIATION FOR SYMBOLIC LOGIC, Chase Park Plaza Hotel, St. Louis, Missouri, January 27–28, 1977.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, Hamilton, Ontario, June 2, 1977 (tentative date).

FIBONACCI ASSOCIATION, University of Santa Clara, October 16, 1976 (Fall research conference).

INSTITUTE OF MATHEMATICAL STATISTICS, Seattle, Washington, August 14–18, 1977.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cincinnati, Ohio, April 20–23, 1977.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami Beach, November 3–5, 1976.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Commodore Perry Hotel, Toledo, Ohio, November 11–13, 1976.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Georgia Institute of Technology, Atlanta, October 18–20, 1976.

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## NOTICE

I am resigning as Editor effective January 1, 1977. Professor R. P. Boas will be my successor. Hence, effective immediately, all Main Articles should be sent to Professor R. P. Boas, Department of Mathematics, Northwestern University, Evanston, IL 60201, and all editorial correspondence should be sent to that address after January 1, 1977.

ALEX ROSENBERG, *Editor-in-Chief*

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# MATHEMATICIANS IN THE PRACTICE OF OPERATIONS RESEARCH

GORDON RAISBECK

**Abstract**

*Operations research* is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. The general procedure used in operations research is quite similar to that used in scientific research in general, the chief differences being the subject matter and the sense of urgency involved. Many mathematical tools are essential, but so also is a first-hand knowledge of the operations involved. The time and resources to carry out OR are usually limited, and results are judged primarily according to standards of practical usefulness rather than intellectual elegance. A mathematician willing to work under these constraints can be an effective member of an OR team. Many who were originally trained as mathematicians have later achieved a complete understanding of operations in some field of practical human endeavor, and combined this with their mathematical background to achieve distinction in operations research.

## I. INTRODUCTION

**A. To whom is this message addressed?** Several hundred listeners gathered in Washington in January 1975 to hear six lectures on operations research.* Most of them were graduate students and junior faculty members exploring the idea of becoming operations analysts. Some were teachers considering how to teach math to would-be operations analysts and other applied mathematicians. The last five lecturers gave technical discussions of problems and solutions, mathematical methods and applications. This paper is adopted from the initial introductory lecture. It describes some of the adaptations an academic mathematician must make if he would practice operations research, and some of the satisfactions he may reap if he does.

**B. What are my qualifications?** To give credible testimony, a witness should either be an expert with special training in weighing and evaluating evidence and observations, or he should limit himself to retelling his own experiences and observations. An expert qualified to talk about mathematicians in

* Short Course on Mathematics and Operations Research, presented under the auspices of the American Mathematical Society in Washington, D.C., on January 21 and 22, 1975.

the practice of operations research would have to be some kind of sociologist or social psychologist, which I am not. However, I can tell you about my own observations and experiences, first establishing the claim that I am a mathematician who has practiced operations research.

As a mathematician, I have earned a doctorate in mathematics and taught mathematics as a very junior staff member at two different universities. Although most of my work is found in journals of engineering and applied science, I have published research papers in the Bulletin of the American Mathematical Society [1] and the Canadian Journal of Mathematics [2], and expository papers in this MONTHLY [3] and the Mathematics Magazine [4]. My activities in operations research are not quite so clear cut, but for the last 15 years I have practiced the closely related discipline of systems engineering. For seven years I served an apprenticeship under the late George Kimball, who attained great distinction in operations research, and during that time and since I have enjoyed very close professional associations with other operations analysts. One of the principal examples to be given in this paper is from an operations research study which I supervised [5, 6].

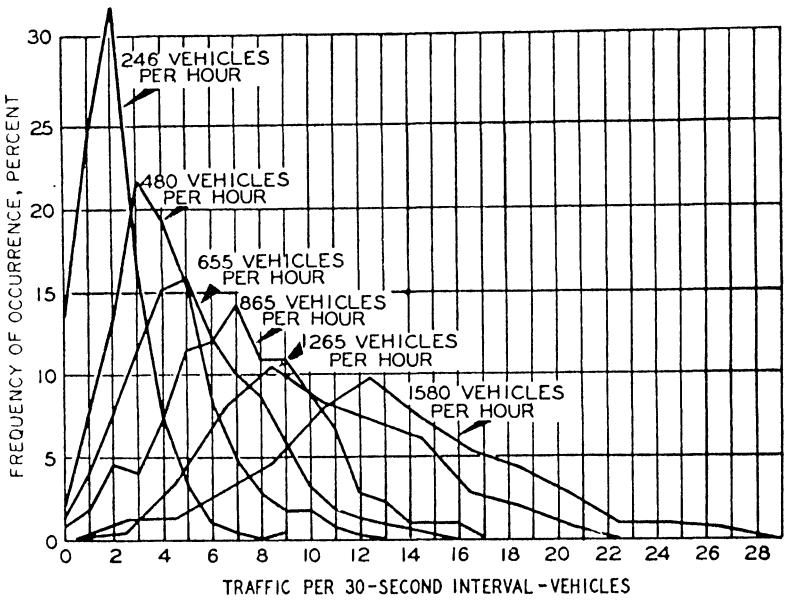
**C. What is operations research?** Authors and practitioners generally agree about what operations research is, but they differ in emphasis and detail, and would choose different words to express it. I take my text from the 1963 printing of Phillip Morse and George Kimball's "Methods of Operations Research" [7], first published as a Navy Operations Evaluation Group classified report in 1946.

> "Operations research is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.
>
> "... *operations research is a scientific method.* It is an organized activity with more or less definite methodology of attacking new problems and finding definite solutions. ... the term "*scientific method*" implies more than sporadic application and occasional use of a certain methodology; it implies recognized and organized activity amenable to application to a variety of problems....
>
> "... operations research is of service to *executive departments*.... [It is, therefore,] an applied science utilizing all known scientific techniques as tools in solving a specific problem.... As we shall see, operations research uses mathematics, but it is not a branch of mathematics.... Just as civil engineering uses the results of science in order to build a bridge, so operations research utilizes these various techniques as tools to help the executive. It is likely, however, that operations research should not be classed as a branch of engineering. .... The engineer is the consultant to the builder, the producer of equipment, whereas the operations research worker is the consultant to the user of equipment.
>
> "The next important word in the definition is *quantitative*. .... Certain aspects of practically every operation can be measured and compared quantitatively with similar aspects of other operations. It is these aspects which can be studied scientifically.
>
> "The task of the operations research worker is to present the quantitative aspect in intelligible form to point out, if possible, some of the non-quantitative aspects that may need consideration by the executive before he reaches his decisions. But the operations research worker does not and should not make the decision."

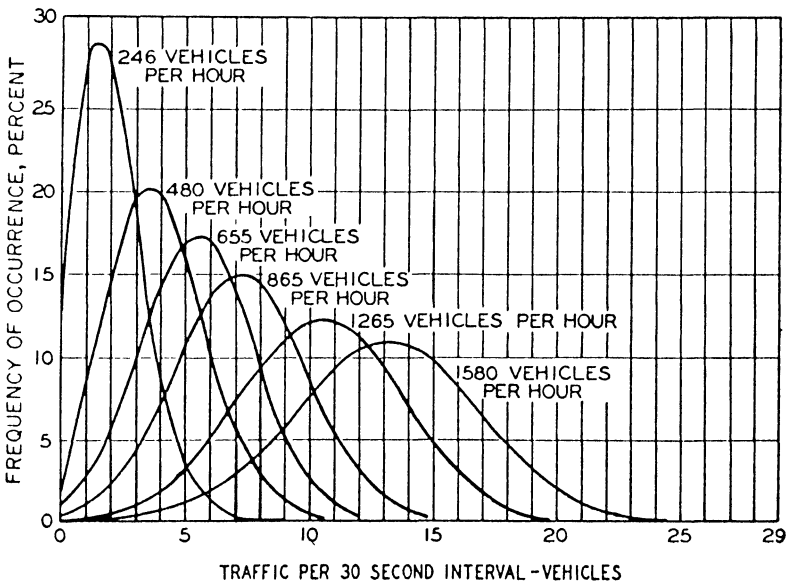
## II. THE ANATOMY OF AN OPERATIONS RESEARCH PROBLEM

**A. An Example.** As an example of a problem in operations research, I shall use "Traffic Delays at Toll Booths" by Leslie C. Edie [8]. This work was carried out more than 20 years ago, and was published in the Journal of the Operations Research Society of America in 1954. There is ample authority for choosing this as a typical example. The author was the first recipient of the Johns Hopkins University Lanchester Prize for the year's best paper on operations research. The paper was reprinted in its entirety by Churchman, Ackoff and Arnoff in their book "Introduction to Operations Research" [9] and again by McCloskey and Coppinger in their book "Operations Research for Management", Vol. 2 [10]. It is not my intention to review the content of this paper, which is already lucidly and engagingly written. My intention here is to examine the structure of the paper and of the technical study which underlay it.

*Perceiving the Need.* The general motivation for carrying out operations research studies on the operations of the Port of New York Authority is briefly but persuasively laid out in the first two paragraphs of the paper. This motivation is specialized in the first sentence of the third paragraph,




FREQUENCY DISTRIBUTION OF TRAFFIC ARRIVALS




THEORETICAL FREQUENCY DISTRIBUTION OF TRAFFIC ARRIVALS

FIG. 1

"The purpose of the police study was to determine whether the police staffing of the various facilities was sound and economical." The rest of the paragraph shows that this purpose is neither trivial nor frivolous. In these three paragraphs, the author shows the perception of a need which is to be satisfied by carrying out and implementing a piece of operations research.

It is possible to state a problem and present a solution without reference to the motivating need. It is hard to see the need as part of the novelty or original contribution in a paper, and easy to throw it away when referees press us to shorten. However, no piece of work is ever done for no reason at all, and the reason for doing it often adds a great deal to our understanding of how it is done. It also makes the paper more interesting.

*Formulating the Problem.* The next section of Edie's paper, comprising four paragraphs, ends with a statement of the general objectives of the study: "(1) To evaluate the grade of service given patrons and determine how it varies with the volume of traffic handled by toll lanes; (2) to establish the optimum standard of service; and (3) to develop a more precise method of controlling expenses and service while at the same time providing for well-spaced reliefs to the toll collector." The three preceding paragraphs limit the permissible solutions and the accompanying analysis to procedures, algorithms and schedules for manning toll booths and the experimental and analytic efforts required to investigate them. With this restriction, the three objectives define a problem or, rather, a set of interrelated problems. From the way they are stated and solved, I conjecture that the problem was not precisely formulated at the time that the study was initiated, and that the formulation in the published paper benefits from hindsight.

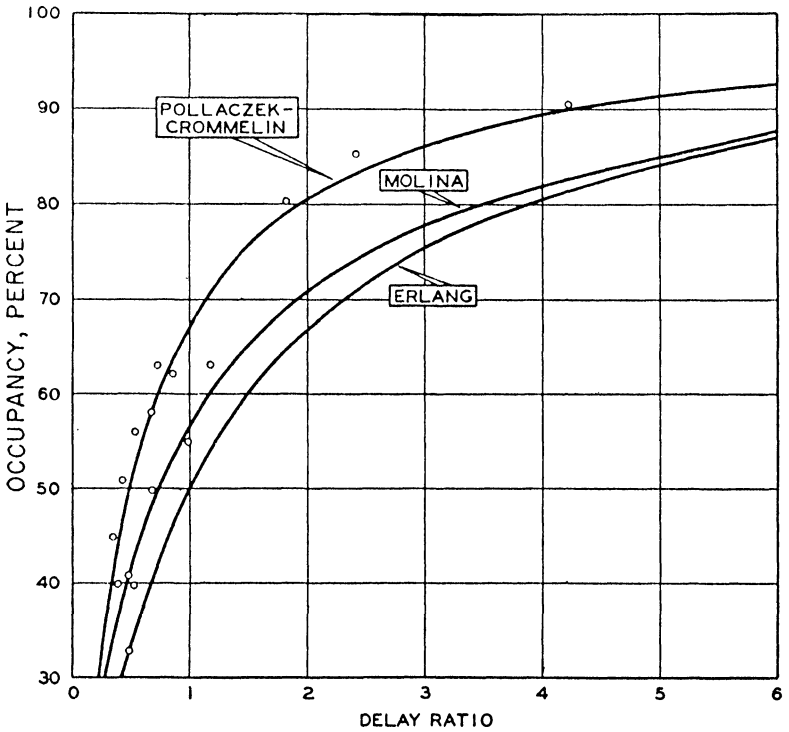
*Constructing a Model.* Most of the rest of the paper is taken up with accounts of observations concerning the flow of vehicular traffic through toll booths and attempts to fit these observations to empirical and theoretical models. A good model must be simple and perspicuous enough to be tractable, while at the same time faithfully representing the behavior of the system being modelled. The needs for fidelity and tractability ordinarily conflict, so some compromise is called for.

Edie's representation of the system is a queue. Idealization of the flow of traffic through toll booths as a mathematical queue is never explicitly stated, but early introduction of the theories of Erlang, Molina, Pollaczek and Crommelin and reliance upon them and related theories throughout the paper show that a queue model and none other is adopted. Most of the explicit discussion of the idealized representation of the process is concerned with what aspects of the queue model to focus attention on and what parameters should be observed and calculated.

The statistics of the arriving stream of traffic is examined with the help of figures like Figure 1. Here the upper chart shows the frequency distribution of traffic arrivals, that is, the frequency distribution of the number of cars arriving in a 30-second interval for several different mean arrival rates. The lower figure shows the theoretical frequency distribution supposing that the arriving traffic stream is governed by Poisson statistics. Clearly, there is a close resemblance between the experimental and theoretical distributions. This resemblance, supported by other considerations, leads the author to assume a Poisson arrival law as a satisfactory representation, rather than others he considered which showed a demonstrably poorer fit.

The amount of time required to serve a vehicle at the toll booth is rather hard to measure, but the implications of certain statistical assumptions about the service time can be inferred from a consideration of the occupancy delay (which I shall not attempt to define here). Figure 2 shows a comparison of actual points with three different theoretical occupancy delay curves for a single toll booth, showing clearly a better fit to the Pollaczek-Crommelin statistics than to either Molina or Erlang statistics. Unhappily, when the same test is applied to occupancy delay curves for several toll booths operating simultaneously, the empirical curves are a very poor fit to any among those theoretical distributions, and Edie must resort to empirical curve fitting.

Later, the distribution of number of vehicles in the waiting line, called the backup, and its maximum, the maximum backup, are introduced as measures of grade of service. The author uses




COMPARISON OF ACTUAL POINTS AND THEORETICAL OCCUPANCY DELAY CURVES FOR ONE TOLL BOOTH

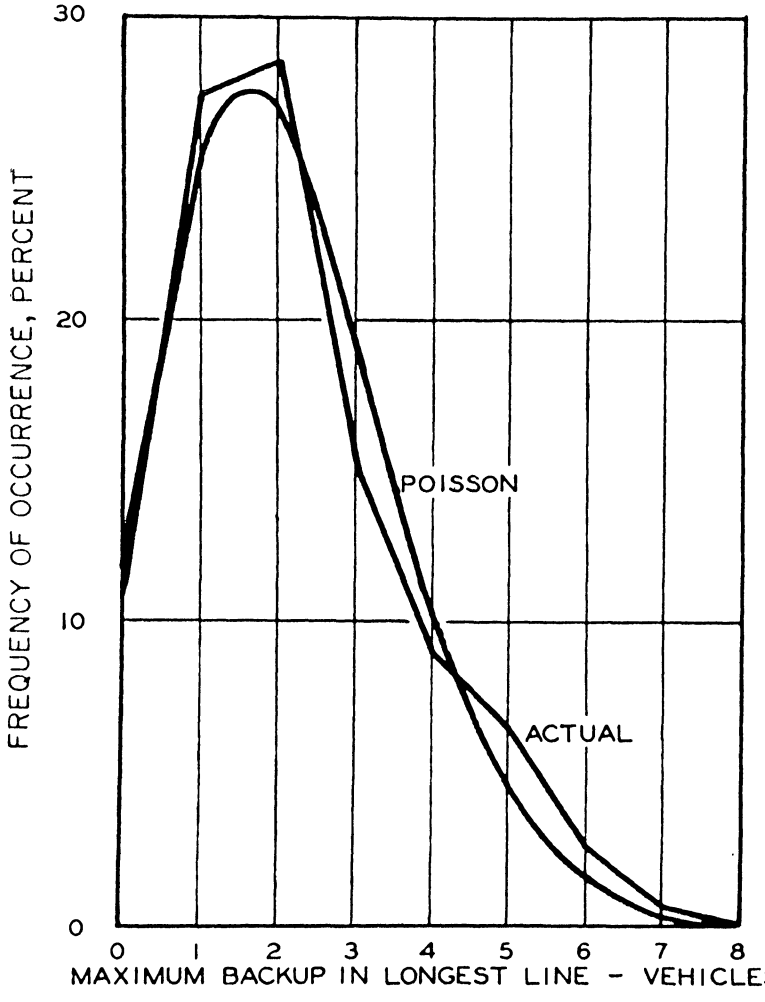
FIG. 2

figures like Figure 3, to show that the distribution of maximum backup in the longest waiting line closely conforms to a theoretical Poisson distribution. This closeness of fit guarantees a close relation between average delay and maximum backup and makes it unnecessary to differentiate strongly between those two criteria of performance.

With these and several other pieces of analysis, parameters are chosen for analysis and observation, parts of the traffic flow are described, the need for certain bodies of observation is recognized, and they are undertaken and analyzed. Thus, bit by bit, the general problem of describing the relationship between traffic flow and the manning of toll booths is specialized and made more precise.

Average delay is at the heart of the criterion of service which was adopted as standard, but the final service standard could not have been foreseen without carrying out some of the observations and analyses contained in this study. Average delay ranges over a continuum of values, whereas number of toll booths manned is an integer, usually rather small. In any solution where the number of toll booths manned is a function of the rate of arrival of vehicles, the average delay as a function of demand rate is a saw-tooth function with a discontinuity at every transition rate where an additional toll booth is manned. How should this jagged curve be matched to a desired bound on the average delay? For reasons which are persuasively argued in the paper, the midpoints of the discontinuities are matched to the nominal standard!

*Deriving a Solution and Testing the Solution and Model.* Once this decision has been made, the problem and its relation to all of the models, representations, analyses and observations are




ACTUAL AND THEORETICAL BACKUP FOR 615 VEHICLES PER HOUR IN THREE LEFT LANES AT LINCOLN TUNNEL

FIG. 3

sufficiently well defined so that a nearly unique solution can be stated in the form of the number of toll booths to be manned for each moment of each day in anticipation of a traffic demand based on observations adjusted for time of day, day of the week, season and long term trends. There was a momentary hesitation to determine whether manning close to the optimum could be met with a schedule which is reasonable in terms of distribution of working and relief hours through the day. This turned out to be no real constraint, although the detailed scheduling for an actual work force turned out to be non-trivial. This and other aspects of the solution were tested against the analytical and empirical representations, subject to practical constraints and boundary conditions; and the effects of departures from the ideal and approximations in measurement and analysis were considered. In terms of the objectives sought, the solution was found to be satisfactory.

*Establishing Controls over the Solution.* Part of the check of the validity of the overall solution and its component parts involved observing explicitly that theory and observation need not match

everywhere, but only over the range of values where the theory is to be exercised. It is important that the theory match observations for average delays of a few seconds up to 15 or 20 seconds, and with backups from a few vehicles to a dozen or a score, but it is not necessary for them to match otherwise. Whenever actual operation falls outside of these ranges, it will be for reasons which do not govern when Edie's solution is properly implemented. It is important, however, to state these controls over the solution so that the solution will not be misconstrued or misapplied by someone who has not laboriously followed it through from its beginnings.

*Implementation and Evaluation.* The author then reports that a trial was conducted on a sufficiently large scale to constitute an initial implementation of the solution, that this was followed by a full scale implementation, and that annual operating savings effected soon after completion amounted to more than ten times the cost of the study itself, with potential future annual savings, repeated each year, of more than twenty times the study cost. With this observation, the author brings us back to the originally perceived need, stating in quantitative terms the degree to which it has been met, and both the study and the author's paper are complete.

**B. Idealization.** Edie's work has now been sliced into eight slices:

- PERCEIVING THE NEED
- FORMULATING THE PROBLEM
- CONSTRUCTING A MODEL
- DERIVING A SOLUTION
- TESTING THE SOLUTION AND MODEL
- ESTABLISHING CONTROLS OVER THE SOLUTION
- IMPLEMENTATION
- EVALUATION

Actually, these phases of the activity flow continuously one into the other, and some of the boundaries are as vague as the boundaries which separate the colors in the visible spectrum. Some commentators might omit *perceiving the need* because they conceive of it as part of the boundary conditions rather than part of the operations research activity itself. I don't care whether you call *perceiving the need* part of the operations research activity or not, but I do believe that communicating an understanding of the need to the people working on the problem often increases their interest and sharpens their perception, and helps to guide them to a useful result. And, as I said before, reporting the need adds interest and understanding to an account of an OR activity.

*Formulating the problem* needs no definition, but it is a mistake to imagine that it is simple. Thirty years ago, already, Morse and Kimball said "it often occurs that the major contribution of the operations research worker is to decide what is the real problem." In my experience, their statement is as true now as it was then.

*Constructing a model* is a brief phase which I find unsatisfactory. Hillier and Lieberman [11] call this *constructing a mathematical model to represent the system under study.* In this context, *model* includes all kinds of logical and symbolic relations, including those expressed by deterministic and stochastic formulas, computer programs, and physical analogs. The two most important attributes by which the merits of a model are judged are fidelity and tractability.

It is clear what we mean by *deriving a solution.* We might have called the next stage *testing the solution.* Why did I call it *testing the solution and model*? If we always had the benefit of divine guidance, the model would be perfect in mimicking all significant relations among the realities it represents and ignoring all the insignificant ones. Unhappily, some of our models are less than perfect. We construct them to have those relations which we believe will be the most important. We exercise parts of the model to see if they exhibit the behavior we expect. Very often there are particular examples or special cases where we know *a priori* what to expect. When we test the corresponding newly generated solutions, we are testing not only the solutions but also the model.

Let me explain what is meant by *establishing controls over the solutions.* All of you must be familiar with problems in algebra or calculus which admit negative or imaginary roots when you know the true answer must be, say, a positive real number. In these cases, your mathematical representation allows a kind of behavior which is forbidden to the prototype. It is equally possible for the prototype to enjoy a kind of behavior which is forbidden to the mathematical model. Neither of these discrepancies need be fatal if we can define accurately the domain in which the prototype and the model do behave alike. Defining this domain, particularly with respect to the range of validity of solutions derived from the model, is what I've called *establishing controls over the solution.*

In Edie's paper on vehicular toll booths, he shows that scheduling of toll collectors is strongly constrained by the timing of mealtime and relief breaks. In our studies of air traffic control, which I shall talk about later, we found that many features are constrained by the legal and regulatory framework under which the Federal Aviation Administration and the common carrier airlines operate. Every real problem is likely to have some social, legal, organizational, economic, regulatory, or other constraints which do not fit into the simplest kind of models. It may happen that the constraints alone limit the solution to a single option, in which case there is nothing more to do. In the context of establishing controls over the solution, it very often pays to find out what kind of solutions they exclude. Sometimes the constraints limit the possible solutions to a small number of options, sometimes even to one, which really simplifies the problem.

*Implementation* is the process of translating the solution from the symbolic or mathematical terms appropriate to the model to the corresponding set of terms appropriate to the underlying reality. The important relationships between model and prototype were all established during the construction of the model, so it is usually easy to describe how the implementation should be carried out. However, actually making it happen may be quite difficult. Whereas we can manipulate the symbols and mathematical relationships in a model freely, it may be very difficult to cause the realities of a prototype to behave in the corresponding way. It may be expensive or dangerous, or people simply may refuse to do it because it is unconventional and they do not understand it.

*Evaluation* is carried out in the same frame of reference as *perceiving the need,* and brings us to the end of a complete circle.

Suppose you compare this idealized operations research activity with other kinds of activity in applied science and engineering, for instance; those described by Bright Wilson in his admirable book "An Introduction to Scientific Research" [12]. You will find them very much alike. On this subject, Morse and Kimball say:

> "The general procedure for operations research is, therefore, quite similar to that used in scientific research in general, the subject matter being the chief difference. Data concerning past operations of a given sort are collected. Variations and results are correlated statistically with variations and details of the operation to find the rough dependence of these results on the various operational parameters involved. Various measures of effectiveness of the operation as a whole are devised in order to discover which displays most fully the dependence of the outcome on these parameters and which measures the properties of the outcome that are most desirable. Research then concentrates on those parameters that produce the measures of effectiveness most critically. .... The research cannot be considered satisfactorily completed... until the dependence of the results on the various operational parameters are satisfactorily explained in terms of the elements involved.... At this point any of the techniques of science, mathematics, physics, or biology, may be used to aid in arriving at this explanation. Subsidiary experiments, detailed mathematical analyses and laboratory measurements are often needed... the problem is successfully completed only when the executive department understands the essential parts of the... analysis."
>
> "An important difference between operations research and other scientific work is the sense of urgency involved. In this field a preliminary analysis based on incomplete data may often be much more valuable than a more thorough study using adequate data simply because the crucial decisions cannot wait on the slower study...."

Perhaps the most important difference between applied science as described by Wilson, and

operations research as described by Morse and Kimball, is this: Wilson is concerned with mathematical models that *describe or predict* what would happen under specified conditions, with no particular concern over whether what happens is useful or useless, good or bad. Morse and Kimball are concerned also with mathematical models that *prescribe* what should be done to *improve* or optimize the operation of a process with reference to some criterion of value. Wilson's applied science is concerned primarily with the search for truth, while operations research is also concerned with exercising the truth in the pursuit of a practical goal or value. Thus, operations research is more like medicine than biology, more like engineering than physics.

## III. Mathematical Content of each Phase of an Operations Research Study

In Edie's paper on traffic delays on toll booths, there is quite a lot of mathematics, but it is not evenly distributed. For example, there is hardly any mathematics evident in the statement of the motivating need: "The purpose of the police study was to determine whether the police staffing of the various facilities was sound and economical." On the other hand, comparison of measured occupancy delay curves with theoretical points for Pollaczek-Crommelin, Molina and Erlang statistical models implies a substantial degree of mathematical sophistication. In my experience, most of the mathematics comes along somewhere near the middle of this sequence of eight steps, with comparatively less at the beginning and the end. Let us consider the mathematical content of each stage, one at a time (see Table I).

Table I: Mathematical Content of Various Stages of an Operations Research Study

| *Stage of an OR Study* | *Mathematical Content* |
|---|---|
| *Perceiving the Need* | *Only by Chance* |
| *Formulating the Problem* | *Some* |
| *Constructing a Model* | *Much* |
| *Deriving a Solution* | *Mathematics May Dominate* |
| *Testing Solution and Model* | *Mathematics May Dominate* |
| *Establishing Controls* | *Much* |
| *Implementation* | *Some* |
| *Evaluation* | *Only by Chance* |

**A. Perceiving the need.** Mathematics is only likely to be used in this phase if the need and the circumstances around it are commonly expressed in mathematical terms. This may happen, for instance, in studies of traffic congestion in telephone switching systems. When the subject matter concerns congestion at the toll booths of a vehicular tunnel, and in nearly all other operations research problems, mathematics does not figure in the recognition of a need, any more than biology figures in a sick person's recognition that he has a need for medical attention.

**B. Formulating the problem.** To the extent that a problem is formulated in quantitative terms, mathematics may enter. However, the terms and relations which are most important are those of decision-makers and their supporting executive departments who control the operations under study, and characteristically these are non-mathematical.

**C. Constructing a model.** In many operations research problems, there is a great deal of mathematics here. However, it is not all mathematics. A mathematical model like a Hilbert space or a Lobachevskian geometry may be mathematically satisfying by being logical and consistent, but a model in operations research must be related to the need which it will be used to satisfy and the problem it will be used to solve. We can assume that a mathematician will be well informed about the elements in his mathematical model and understands fully the relations among them. He must be just

as well informed about the elements and relations in the prototype, which functions in the real world, which the model is intended to represent.

**D. Deriving a solution from the model.** If the model is mathematical, as usual, mathematics is the principal tool for generating a solution. One can often gain insight, however, from experience, observations, and analogs that are outside of what is commonly called mathematics. Part of the solution may be carried out using physical models such as the electrical engineers' equivalent circuits, or computer models, which have a good deal more in them besides mathematics.

In my experience, most solutions are trivial, mathematically and otherwise. Although this may be a disappointment, it is inevitable in any problem solving enterprise. The frontier which separates problems whose solutions are well known and thoroughly understood from those which are unsolvable with today's state-of-the-art is narrow, and it is unlikely that a new problem falls close to this frontier. Usually you will find that the problem is totally unsolvable in the terms in which you originally posed it, or else that you know the solution as soon as you made the statement of the problem.

You should not confuse the intellectual content of a solution with its utility. "Take two aspirin and go to bed" is often the best medical advice. But it is important for the problem solver to know when the trivial answer is really the best one. Also, the investigator to whom a large body of problems is trivial always has an advantage over one whose knowledge and experience are more limited.

**E. Testing the solution and the model.** Here also, mathematics and mathematicians may dominate this phase of the OR study. The word *test* implies a decision point: if a test fails, one ordinarily must go back one or more steps to derive a new solution or possibly a new model.

**F. Establishing controls.** The model is unlikely to be a perfect representation of the real situation it represents. Only some of the features of reality will be represented, and even that representation may be valid only under restricted circumstances. Understanding the conditions under which the model and solution are valid requires mathematics to understand qualities of the solution and the mathematical model underlying it, and also an understanding of the qualities of the problem and the need underlying it, which are likely to be non-mathematical.

**G. Implementation.** Some mathematics may be required fully to exploit the qualities of the solution, but much of this phase may be non-mathematical. As in the formulation of the problem, the mathematical content of the implementation depends to a large extent on what is being implemented. In the toll booth traffic study, assigning toll collectors to particular toll booths according to a predetermined schedule is not very mathematical. On the other hand, the assignment of tape, disk, and core memories to various parts of a complex computerized operation may be a highly mathematical operation.

**H. Evaluation.** Is the need really filled? Ordinarily there is no mathematical content in answering this question. However, there are two special circumstances: first, if the need is expressed in mathematical terms and second, if the test of usefulness is in itself a significant scientific, engineering, or operations research problem. There is a large class of problems including many reliability, quality control, and war-game situations where the test of usefulness is difficult and expensive, so that a balance must be struck with a judicious undertesting of a solution which may be deliberately conservative. Who is going to test the usefulness of a program for hardening the sites for storage of nuclear weapons? How do you gather statistically significant data about improvements in aircraft safety when the total number of fatal accidents in U.S. scheduled common carrier aircraft is around 5 per year?

**I. On mathematical training recommended for operations analysts.** In describing the desirable training for a career in operations research, Hillier and Lieberman [**11**] identify three complementary types of academic training, of which the first includes the basic methodology of mathematics and

science, as well as such topics as linear algebra and matrix theory, probability theory, statistical inference, stochastic processes, computer science, microeconomics, accounting and business administration, organization theory, and the behavioral sciences. Morse and Kimball, on the other hand, state [7] that no particular correlation has been found between the particular scientific specialty in which the operations research worker was trained and his subsequent excellence in the field of operations research. If you examine both of these statements in the context in which they are presented, they are not inconsistent. A set of mathematical tools is necessary but not sufficient to carry out operations research.

As one sample of the mix between mathematical and extra-mathematical content, I scanned every page of each of the 66 papers and reports in the Proceedings of the Sixth IFORS International Conference on Operations Research held in the summer of 1972 [13], and found that 38 of these 66 papers contained not one single equation.

## IV. What a Mathematician Can Do

**A. What is a mathematician?** Even if we assume that we all know what we mean by *mathematics* and by *doing mathematics*, there are two ways to define a mathematician. First, we could define a mathematician as anyone who is doing mathematics, but only while he is doing mathematics; or, second, we could define a mathematician as anyone who is capable of and spends a significant part of his time doing mathematics, whether or not he is actually engaged in doing it. But these two definitions do have a feature in common: they define a mathematician in one way or another as a person who does mathematics.

It is actually quite difficult to define what mathematics is and to distinguish between mathematicians and non-mathematicians by applying criteria to the content of the work that they do. These classes are really organizationally and socially determined. They have their own private vocabulary, and their own in-jokes about whose pupil is married to what other mathematician's daughter, and the point of accumulation of what national group is an essential singularity. The people who teach and do research in departments of mathematics in universities proclaim themselves to be mathematicians. By general agreement, they apply the term to a number of others, usually people who have earned advanced degrees in mathematics at universities, who publish papers in mathematical journals, and who have an intellectual life style enough like that of university mathematics professors so that they do not cause mutual discomfort when they mingle. Mathematics is what editors will publish in mathematical journals, what university professors will accept as doctoral dissertations, what mathematicians will go to colloquia and symposia to discuss. Sometimes mistakes are made: we "know" today that Weierstrass was a mathematician and that the work which he produced is mathematics, but until he reached middle age the truth of these statements must have escaped some of his prestigious contemporaries, for he was denied the freedom of the community of university scholars and access to journals for publication.

It might be better for the profession if mathematicians adopted a more inclusive view of what mathematics really is. This means, of course, that we must abandon easier criteria and judge performance by deeper and more universal standards. As in other fields, much of the every day work in operations research is hardly more than ballast. Much of the good work, however, would reflect credit to the mathematical profession as well as to that of operations research.

Prestigious members of the mathematical profession have often looked down their noses at activity which they think is not pure enough. A common target for their scorn is what they call "applied mathematics." A number of universities like Stanford, Brown and Columbia have found it desirable at one time or another to set up separate departments of pure mathematics and of applied mathematics. Without tracing the rise and fall of this idea in each case, I offer my opinion that their experience is a good argument for an inclusive rather than an exclusive concept of what mathematics is.

**B. Roles for mathematicians.** There is a continuous spectrum of degrees of involvement of mathematicians in operations research problems, but we can distinguish three levels.

1. The mathematician as a *protagonist*, leading the investigation and making significant decisions about how it will be carried forward.

2. The mathematician as a *participant*, interacting freely with others who are also contributing to the effort.

3. The mathematician as a *resource*, but not necessarily interacting with other participants in a way significantly involving the subject matter of the OR study.

The examples which follow show how a mathematician can function in each of these roles.

**C. An example of the mathematician as a protagonist.** As an example of work done by a mathematician as protagonist, I shall briefly cite the high points of a project on torpedo simulation models carried out by my colleague, Dr. James Dobbie. I shall not start the description at the beginning nor carry it to the end, but will jump in to the middle. However, following what you have just read, you will recognize this as part of the problem formulation, model building, and problem solving stages of a larger operations research problem. Dr. Dobbie was principal investigator in charge of the whole job*, and also carried out the particular piece of work described next.

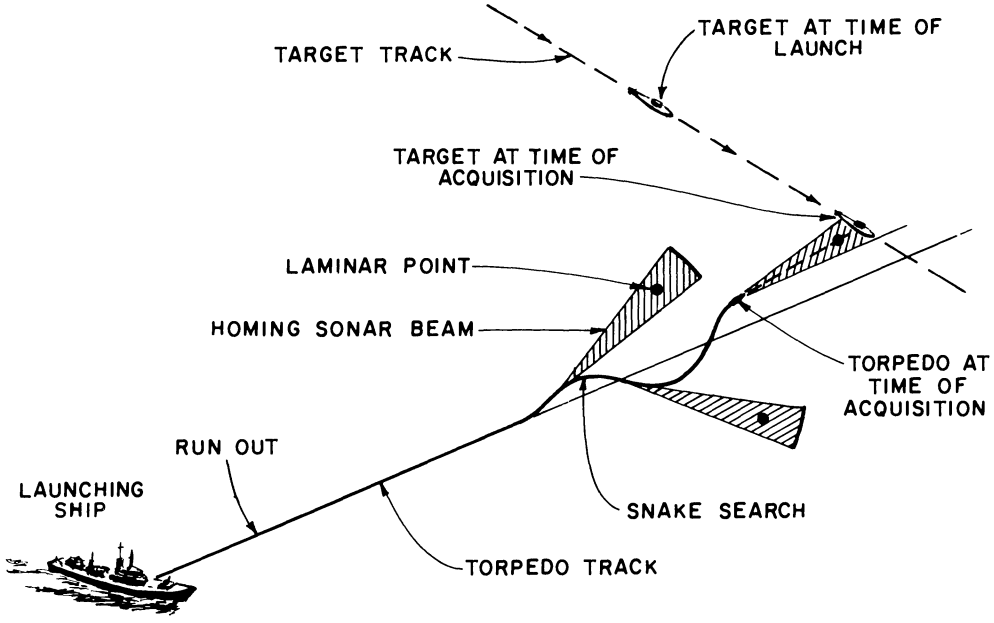
The Naval Underwater Ordnance Station in Newport, shortly before the name was changed to the Naval Underwater Weapons Station, asked us to help them develop methods of estimating the fire control requirements of weapon systems and to measure the performances of existing and proposed fire control systems. We started with the MK46 torpedo.

The MK46 is an unguided acoustic homing torpedo launched from a ship or aircraft. Its method of operation is as follows: (See Figure 4) a (supposed) target is detected and tracked by the launching ship's sonar, aircraft sonobuoys, or otherwise. At some moment the torpedo is launched along a track that depends on the search pattern that is set in the torpedo. After the torpedo has run out a predetermined distance a self-contained sonar is turned on, and the torpedo goes into the preset search pattern. In the *snake search* mode the torpedo follows a snake-like course to search out a wide swath on both sides of the mean path, which is a continuation of the initial track. The mean path is computed to achieve interception of the target (on its predicted path) by a point, called the *laminar point*, that is near the center of the search beam. The laminar point, rather than the torpedo, is used as the guide point, since the objective of this part of the operation is to detect the target with the sonar beam, rather than to hit the target with the torpedo. Another search pattern is a *circular search*, in which the torpedo runs in a small circle and the sonar sweeps out a large circular area around it. In this case the torpedo is launched on a track that is computed to put the torpedo on the expected track of the target in a leading position, or is delivered there by rocket, aircraft, or helicopter.

*Acquisition* is said to occur when the sonar of the torpedo detects the target and switches to the homing mode. The torpedo then is directed continuously toward the target by the homing sonar. The torpedo now overtakes the target, and is designed to strike, detonate, and kill the target. However, the homing sonar may lose the target, or the torpedo may run out of fuel, or the torpedo may pass close to the target but not close enough so that detonation of the warhead will kill the target. The distance of closest approach is called *miss distance.*

We undertook to estimate requirements and measure performance with the use of a simulation model that simulates the operations involved, from tracking to kill, and computes various measures of performance, such as average miss distance, probability of acquisition, kill probability, etc. A preliminary study showed that the number of essentially different sources of error that could affect the outcome significantly was too large to permit the use of Monte Carlo simulation; averages would have to be computed analytically to obtain short running times. The models that we developed are analytic

* Contracts N000140–66C–0003 and N000140–68–C–0278 with the Naval Underwater Research and Engineering Station, Newport, R.I. (now Naval Underwater Systems Center, Newport, R.I.).




THE OPERATION OF AN ACOUSTIC HOMING TORPEDO

FIG. 4

models in which a sequence of equations is used to describe the various operations and compute the measures of performance. A high speed computer is used to do the bookkeeping, sequence the operations, and compute the measures of performance. Models were developed for snake search following over-the-side launch from a torpedo tube, and for circle search following delivery by rocket or by air drop. Running times are fractions of a second.

When our client learned that our models would be analytic, *they* decided to develop a Monte Carlo simulation model. Running in real time, this model required 10 to 15 minutes per run and yielded a yes-no answer to the question of acquisition. Progress of the run was displayed on an oscilloscope, and acquisition was indicated by a flashing light.

The development of two models simultaneously was beneficial, since the results obtained from them could be compared. In this way several important discoveries were made. For example, consistently higher acquisition probabilities obtained with our model were traced to a difference in lead distances for the laminar point. The value used by the client was obtained from the operators handbook, while our value was computed (from parameters that would be known) to maximize the acquisition probability.

Gradually the client's skepticism about our analytic model changed to enthusiastic endorsement. It was used to obtain results, while the computer-embodiment of the Monte Carlo simulation was used to entertain visitors. Our models were used in the design of a new fire-control system.

For those of you unfamiliar with the term, let me explain what is meant by a *Monte-Carlo simulation.* The term is used in somewhat different ways by different authors, but in this case a sufficiently general definition is the following. A Monte-Carlo simulation model is one in which a stochastic variable is represented by a sample, each member of which may be called a particular case. For each particular case a deterministic result is calculated. Desired estimates and probability distributions for parameters such as performance measures are inferred from statistics of the results of

the numerous particular cases. The name "Monte-Carlo" arises because usually a random number generator functionally equivalent to a roulette wheel is used to generate a statistically representative sample of special cases.

The strength of Monte-Carlo methods is that the shape of the probability distribution is ignored in the process except in generating a sample. Thus we need not be restricted by assuming tractable distributions — Gaussian, symmetric, or what-not — but can have any distribution, even one which is experimentally determined.

One weakness of Monte-Carlo methods is their lack of high precision. Any Monte-Carlo computation inescapably involves an analog, the representation of a probability distribution by a particular sample. You are aware of the limitations of that famous analog device, the slide rule, which must be made ten times larger to gain a factor of ten in precision. Well, because of the way that variances propagate, the number of cases in a Monte-Carlo simulation must be increased one hundred-fold to gain a factor of ten in precision. Other shortcomings of Monte-Carlo are described in Reference [17].

In the history of Mathematics, the best-known early example of Monte-Carlo simulation is called the Buffon needle problem. It describes a way in which the value of $\pi$ can be estimated numerically by dropping a needle with uniformly distributed random orientation on a grid of equally spaced parallel lines and counting the intersections. The Monte-Carlo feature arises from the presumption that throwing the needle in the air will generate the desired uniform angular distribution.

After our success with the MK46 torpedo our client asked us to construct similar models for the MK48 torpedo. The MK48 torpedo is attached to the launching platform by a wire which unreels from both ends as the torpedo runs out. (Why both ends? The wire is too fragile to drag through the water without breaking, but must lie at rest relative to the surrounding water.) As the torpedo runs out, the launching ship continues to observe the target, and transmits steering signals down the wire to the torpedo. The description of this post-launch control via the wire and the effects of wire breakage are difficult to do analytically. Also, unlike the unguided MK46 torpedo, which should only be launched when a fairly reliable target track has been generated, the MK48 with its post-launch control is intended to be used when the target track is inferred from sonar relative bearing observations only. The limitation of tracking data to bearings only complicates the target motion analysis and makes it difficult to compute the effects of non-linear target motion. Therefore, modeling requirements and measures of performance for the MK48 torpedo are much more complex than for the MK46, and we were forced to develop new methods of simulation.

These difficulties are overcome in our models by the use of a deterministic computer simulation, embedded in the analytic model, that computes the mean track of the torpedo with all bias errors in tracking, target motion analysis, fire-control solution, and post-launch control. Analytic methods then are used to compute the distribution of miss distances, probability of acquisition, etc. Monte Carlo simulation, while not used in the main model, is used to obtain distributions of miss distances that are produced by some error sources that could not be obtained analytically. Thus the computer is used in three ways: in Monte-Carlo simulation to obtain input distributions, in deterministic simulation to compute the mean torpedo track, and to make the computations.

The combination of analytic methods with deterministic computer simulation and Monte-Carlo simulation to compute acquisition and kill is accepted as a useful technique, and seems to me to qualify, by any standard, as a characteristically mathematical contribution. It has been put to use, and is effective and valuable. Thus we have an example of a mathematical contribution made by a mathematician to an OR problem of which he was in charge.

**D. Example of the mathematician as a participant.** And now I would like to go on to another example of the kind of contribution that a mathematician makes to an operations research study. In the following example, the mathematician was my colleague, Dr. Bernard Koopman, and his role, at least in the context of this particular example, was that of a participant, as defined above. The

operations research study was an analysis of air traffic control system capacity which was carried out under my direction from 1970 to 1973, under contract to the Federal Aviation Administration [**5**, **6**, **14**]. As you will see, there are many similarities between this study and Edie's study of toll booths in vehicular tunnels.

At the time when this study was carried out, there was no accepted definition of or means to measure the capacity of an air traffic control system to function at an acceptable level of service in response to the challenge of a stated traffic demand. The purpose of this study was to establish measures of the system's effectiveness in performing its functions and to examine its operation, properties, and reactions to various conditions and requirements, so that the effects of proposed changes in equipment, methods of operation, or imposed demands could be foreseen and expressed in terms of those measures of effectiveness.

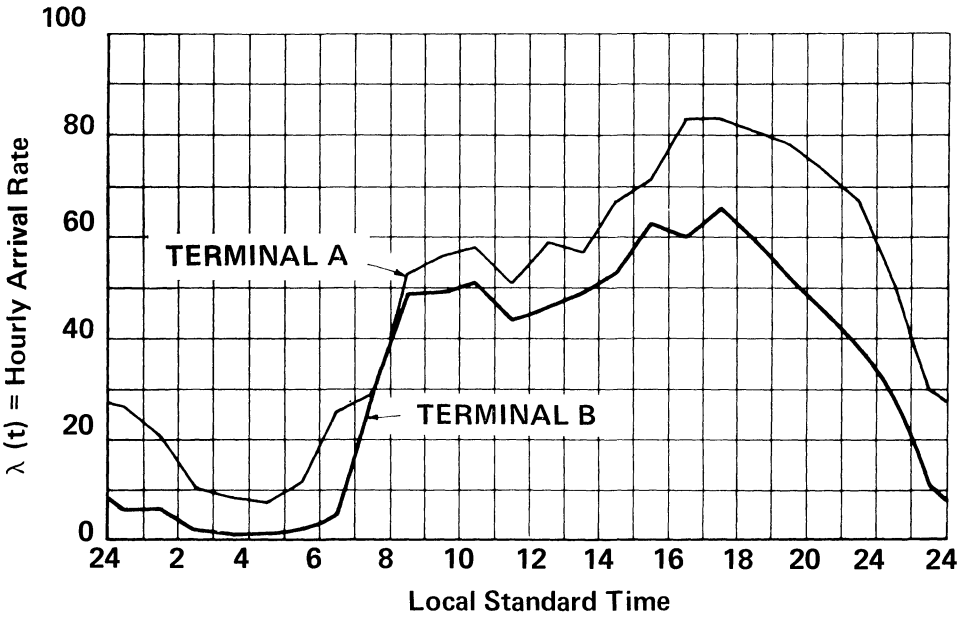
When an aircraft lands or takes off from an airport, it usually occupies the runway for about a minute, more or less. It is extremely unsafe to have two aircraft using the same runway at the same time, but with careful control it is quite safe to have a stream of aircraft use one runway in quick succession. At our most heavily used airports, the maximum number of landings and take-offs is usually limited by the availability of runways. This situation is quite naturally modelled by a queue in which a single server (the runway) accepts and services in sequence a number of users (the aircraft).

Queuing theory has a long history, and new and original work is being done right up to the present day. We believed it was desirable to have a highly qualified person investigate what could be done with applications of up-to-date developments in queuing theory. We were particularly interested, of course, in those aspects of the queue which would have to be measured in order to define capacity and quality of service. It has been known for a long time that many aspects of the behavior of the queue are quite sensitive to the statistics of the demand and the service time.

Inasmuch as requests for landing and take-off originate from a large number of autonomous aircraft, it is easy to imagine that these demands approximate, over a short period of time, a Poisson series. It is also easy to verify this conjecture. On the other hand, the service time, that is the sum of the aircraft's runway occupancy time and the additional time required for the control system to avoid inadvertent overlap, has a distribution which does not obviously fit any very simple analytic model. An exponential distribution of service times, whose modal value is zero, is obviously absurd. It is much more natural to assume that every aircraft occupies the runway for exactly the same time, leading to the assumption of a singular distribution of constant service times. However, we know that runway occupancy times are not all exactly equal and that fluctuations about the mean service time do influence queue characteristics in many circumstances. We must therefore wonder whether the assumption of constant service time is too restrictive.

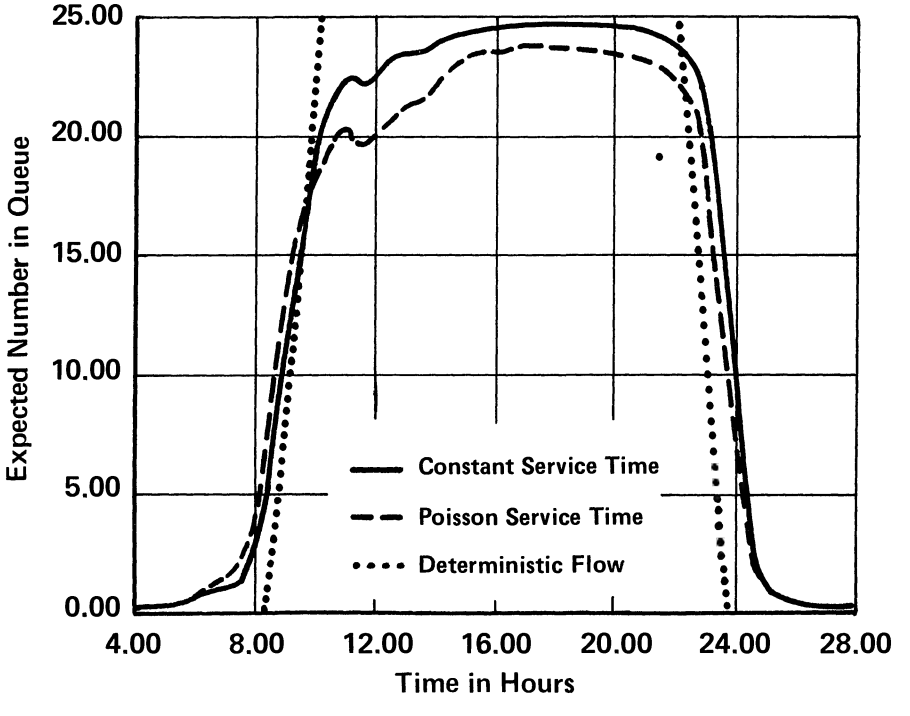
Before conjecturing or analyzing this problem to death, Dr. Koopman set about to examine a few special cases. First, he made the observation that a comparison of behavior with constant service times and behavior with exponentially distributed service times should give an indication of the effect of service times statistics in general. This is because a constant service time has no randomness whatsoever and an exponential distribution has more randomness than could conceivably occur in any real life situation. Secondly, he made use of the observation that the dominant criterion of performance is delay, whose expected value is proportional to the expected number in the queue. We therefore set to work to examine the expected number in the queue under various demand and service assumptions.

For demand curves, he used averages of measured values at two actual air terminals (Figure 5). We assumed three different average service rates, 45 per hour, 55 per hour, and 70 per hour, and two different service time distribution assumptions, *constant* service time and a Poisson or *exponential distribution* of service times. Figure 6 shows the expected number in the queue for the demand curve of terminal A when the service rate is 45 per hour. The solid line shows the expected number in the queue if the service time is constant and the dashed line shows the expected number in the queue if the service time has an exponential distribution. Clearly, these two curves are very close. All curves




HOURLY ARRIVAL RATES – TERMINALS A AND B

FIG. 5



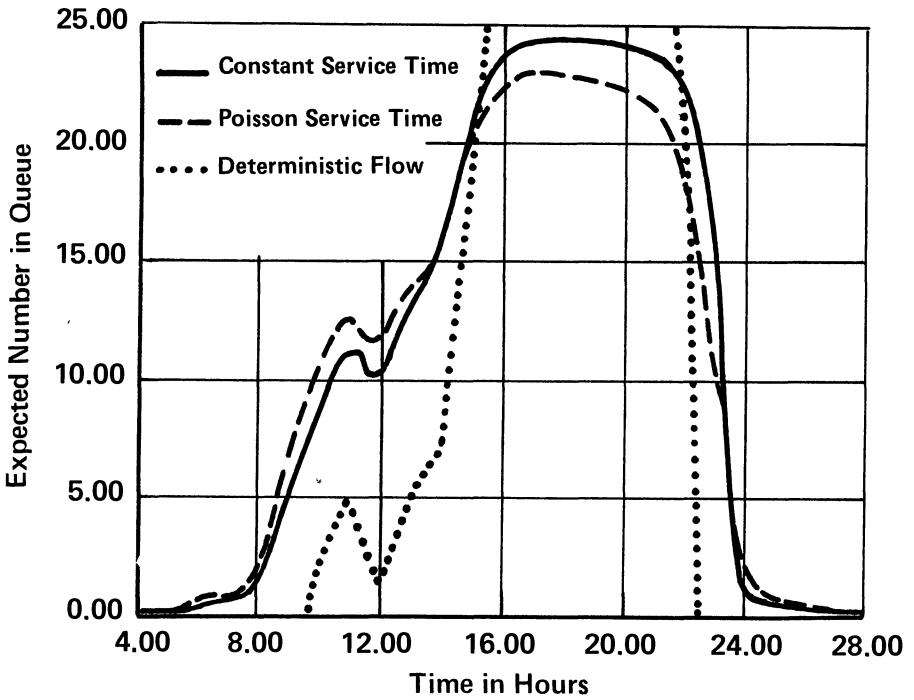

EXPECTED NUMBER IN QUEUE
TERMINAL "A", $\mu$ = 45 PER HOUR = 1/c

FIG. 6

saturate at or below 25, because we assumed for the purposes of this example that only 25 aircraft could be stacked in the terminal area, and that further demands would be refused when the queue length reached 25.

Actually, there is a third curve shown on this figure labeled deterministic flow. This is the result you get if you assume the demand and the service are in the form of a continuous flow, like a stream of fluid, instead of discrete units of one aircraft each.

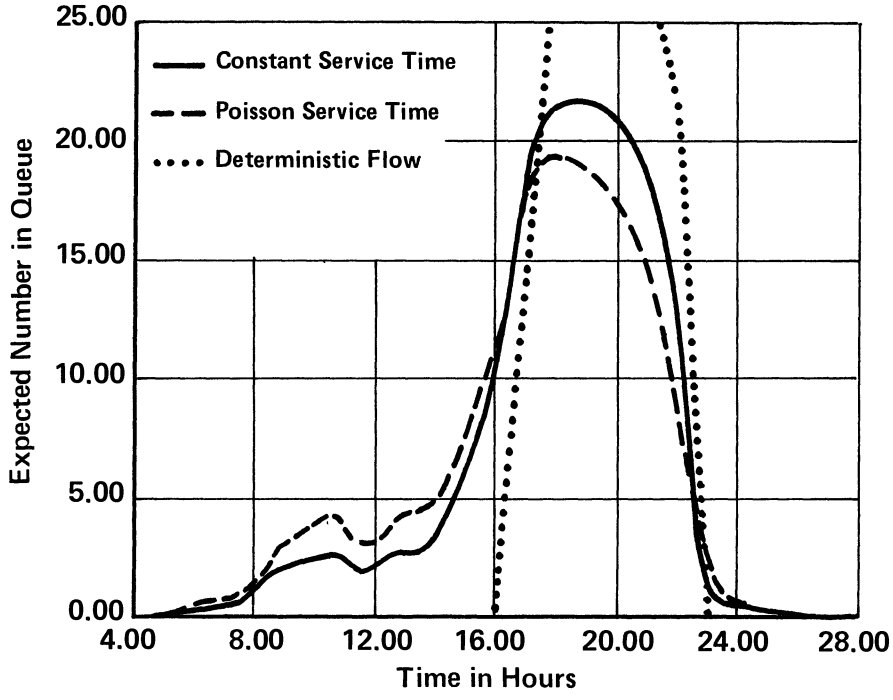
In this example, the expected queue length is nearly the same with constant service time and with exponentially distributed service time, and the results with deterministic flow are not far different. Figure 7 shows the result when the average service rate is 55 per hour, and Figure 8 shows the result when the service rate is 70 per hour. In every example, the expected queue length is about the same for constant and exponentially distributed service times. The results using the demand curve labeled *terminal B* were similar.




EXPECTED NUMBER IN QUEUE
TERMINAL "A", $\mu$ = 55 PER HOUR = 1/c

FIG. 7

With these examples in front of you, it is not difficult to work out the explanation. Whenever the average demand is substantially greater than the service rate, the queue builds up toward saturation at a rate proportional to the difference. And whenever the demand is substantially less than the average service rate, the queue shrinks toward zero in the same way. Only when the demand and the average service rate are nearly the same for an extended period do the service time statistics have much effect on the expected number in the queue. But in the context of air traffic control, the demand curve has a very strong diurnal fluctuation. In the examples used in this study, which are taken from actual airport statistics, the busy hour demand was about ten times as great as the least busy hour demand at terminal A, and thirty times as great at the terminal B. A further examination of the actual constraints




EXPECTED NUMBER IN QUEUE
TERMINAL "A", $\mu$ = 70 PER HOUR = 1/c

FIG. 8

in air traffic system operation convinced us that it is very unlikely that the demand rate would hover for any extended period of time at a level approximately equaling the service rate. In fact, the system must be designed so that the service rate exceeds the busy hour demand rate in good weather high visibility conditions. Only under poor environmental conditions may the service rate be depressed below the demand rate, in which case the queue builds up as we have seen. This results in severe traffic congestion during the busy period, which contributes a substantial amount to the average delay.

We therefore arrived at the conclusion that the average delay and other criteria of performance in which we were most interested are quite sensitive to the relationship between the expected demand and mean service rate and quite insensitive to the statistics of service times. In more homey terms, the results are driven by fluctuations in the mean service time and the demand, and not by short term statistical fluctuations in the service time.

The same kind of mathematical analyses can be used to study the queues which develop in an automatic dial telephone switching system. The mathematics is the same, but the result is different. In the telephone switching system, service time is measured in milliseconds and maximum acceptable delay is a small fraction of a second, four orders of magnitude smaller than the corresponding parameters in an air traffic control system. Consequently, the queue is in near equilibrium almost all of the time. If the mean demand rate exceeds the service rate in a telephone switching system, some kind of saturation sets in within seconds. Further considerations of average delay are then ignored in favor of seeking a strategy to avoid complete disaster. For example, service may be denied at the source to some proportion of users or some classes of calls until the threat of saturation and blockage of the switching system is over. On the other hand, in a telephone switching system the question of how close

the average busy hour demand may approach the service rate from below is quite meaningful, and is quite sensitive to demand and service statistics and queue discipline.

This contrast between air traffic queues and telephone service queues illustrates what I mean by *establishing controls over the solution.* In the context of the eight stages of a problem I described before, the whole point of Koopman's effort was to show that the validity of a particular class of methods and solutions is *not* limited by the fit between the real and assumed service time distributions.

**E. The mathematician as a resource.** Most of the people who do any operations research at all have at least studied calculus and a little bit of probability. To the extent that they were taught these in college by mathematicians, mathematicians have already played a major role as a resource. The spectrum goes all the way from there to the staff mathematician who will undertake to optimize a stated objective function under constraints or numerically integrate a differential equation, treating it as a completely abstract statement and ignoring any non-mathematical content. You can imagine how mathematicians operate in this role, and I need not give you any specific examples.

**F. But even if we believe a mathematician can do it, will he?** It is obvious that there are a great many places in an operations research study where mathematical and non-mathematical comprehension are joined.

| WHAT IS DONE / WHO DOES IT | What Requires a Mathematician – "MATHEMATICS" | IN BETWEEN | What Does Not Require a Mathematician – "NON-MATHEMATICS" |
|---|---|---|---|
| MATHEMATICIANS | EAGERLY | WILLINGLY (Sometimes) | RELUCTANTLY (Too Often) |
| HYBRIDS | WILLINGLY | WILLINGLY | WILLINGLY |
| NON-MATHEMATICIANS | RELUCTANTLY (Sometimes) CAUTIOUSLY (Sometimes) WITH CHEERFUL ABANDON (Watch Out!) | WILLINGLY | EAGERLY |

FIG. 9

Figure 9 shows a $3 \times 3$ matrix whose columns are labeled *what is done*, either mathematics, in between, or non-mathematics, and whose rows are labeled *who does it*, either mathematicians, hybrids, or non-mathematicians.

The blocks on the main diagonal of this matrix are easy to fill in and understand. In general, mathematicians are eager to do mathematics, non-mathematicians are eager to do non-mathematics, and hybrids are quite willing to do work which is in between or which combines the two. The squares adjacent to the main diagonal, particularly those to the lower left, present no real difficulty either. Hybrids are usually willing to do mathematics and willing to do non-mathematics as well as doing things that are mixtures or in between. Non-mathematicians are usually willing to do things which have some mathematical overtones, and some mathematicians are sometimes willing to do work which has a non-mathematical flavor.

It is at the two ends of the minor diagonal that trouble brews. Non-mathematicians are sometimes properly reluctant to do mathematics, sometimes venture to do mathematics but cautiously, and sometimes undertake it with cheerful abandon. In the latter case, watch out! If good mathematical

techniques and judgment are really required, letting a non-mathematician do it may be a prelude to disaster.

At the other end of the diagonal, it has always seemed to me that most mathematicians are far too reluctant to do non-mathematical work, or even work which partakes of mathematics but has a non-mathematical flavor. Why is this?

Mathematics is one of the instruments we use to enhance our perception, understanding and enjoyment of the universe in which we live. It is capable of generating much aesthetic satisfaction, but rarely exists in that context alone. More often we use it to shed light on the relationships among ideas, concepts and physical realities. Now some of our colleagues behave as though mathematical descriptions were noble when inspired by ideas and concepts, products of the human mind, but ignoble when inspired by observations and physical realities. Some histories of mathematics reinforce this view, but the biographies* of great mathematicians of the past refute it. Bernoulli allowed himself to be inspired by pendulums and fluid flow, Gauss by optics and magnetism, and practically every well-known mathematician of the last few centuries has done likewise. This is not to say that there are no aesthetic value judgments: there is good math and there is bad math, and there are many reasons why math may be good and why it may be bad. However, I don't think that mathematics must be removed from physical reality to qualify as good, and if you do, this conviction is likely to disqualify you from doing operations research.

## V. Summary

Let me recapitulate my opinions thus: Operations research is much like problem-solving in other branches of applied science and engineering, the chief differences being the subject matter and the sense of urgency. Mathematical tools and people who know how to use them are essential, but so is a thorough understanding of the operations under analysis. OR is normally carried out with limited time and resources, and results are judged primarily by their usefulness. A mathematician willing to work under these constraints can be an effective member of an OR team. If he is also able to achieve a reasonable understanding of operations in some field of practical human endeavor, and is able to combine this with his mathematical background, he may be able to achieve distinction in operations research.

Or, to say it less pompously

— Will knowing mathematics make me a better operations analyst?
— Yes.
— Will being a mathematician make me a better operations analyst?
— It helps, but only if you invest a lot of hard work.
— Will being an operations analyst make me a happier mathematician?
— For many, it has.

**VI. Acknowledgments.** Permission of Leslie C. Edie to reproduce Figures 1, 2, and 3, and to quote from Reference [8], and from Philip M. Morse, James M. Dobbie and Bernard O. Koopman to quote extensively from their work, is gratefully acknowledged. I thank Alan Goldman, Raymond H. Burros, and Daniel H. Wagner for their review and comments, many of which have been incorporated in the revised text, and Daniel H. Wagner for an opportunity to read his paper *Mathematicians in Operations Research Consulting* [17] in manuscript.

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* Most, but not all. G. H. Hardy disagrees explicitly [15], but his view is so much in the minority that serious commentators like James R. Newman openly brand Hardy's statement as nonsense [16].

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## THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

A. P. HILLMAN, G. L. ALEXANDERSON, L. F. KLOSINSKI

The following results of the thirty-sixth William Lowell Putnam Competition, held on December 6, 1975, have been determined in accordance with the regulations governing the competition. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology,** Pasadena, California. The members of the team were Christopher L. Henley, Frank M. Liang, and James B. Shearer; a prize of one hundred dollars was awarded to each of these students.

The second prize, four hundred dollars, was awarded to the Department of Mathematics of the **University of Chicago,** Chicago, Illinois. The members of the team were Franklin T. Adams, Peter L. Dordal, and Andrew M. McLennan; each was awarded a prize of seventy-five dollars.

The third prize, three hundred dollars, was awarded to the Department of Mathematics of the **Massachusetts Institute of Technology,** Cambridge, Massachusetts. The members of the team were David J. Anick, Sheldon H. Katz, and Richard E. Stone; each was awarded a prize of fifty dollars.

The fourth prize, two hundred dollars, was awarded to the Department of Mathematics of **Princeton University,** Princeton, New Jersey. The members of the team were Robert L. Anderson, Eric S. Lander, and Karl C. Rubin; each was awarded a prize of fifty dollars.

The fifth prize, one hundred dollars, was awarded to the Department of Mathematics of **Harvard**

**University,** Cambridge, Massachusetts. The members of the team were Sholom Feldblum, Thomas G. Goodwillie, and Alan S. Grenadir; each was awarded a prize of fifty dollars.

The five highest ranking individual contestants, in alphabetical order, were **Franklin T. Adams,** University of Chicago; **David J. Anick,** Massachusetts Institute of Technology; **Ernest S. Davis,** Massachusetts Institute of Technology; **Thomas G. Goodwillie,** Harvard University; and **Christopher L. Henley,** California Institute of Technology. Each of these students has been designated as a Putnam Fellow by the Mathematical Association of America and was awarded a prize of two hundred and fifty dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were *H. Turner Laquer,* University of New Mexico; *Joseph E. Quinsey,* University of Waterloo; *Karl C. Rubin,* Princeton University; *James B. Shearer,* California Institute of Technology; and *Brian C. White,* Yale University. Each of these students was awarded a prize of one hundred dollars.

The following teams, named in alphabetical order, won honorable mention: *University of British Columbia* with team members Paul A. Hildebrand, Allan D. Jepson, and Arnold D. Satanove; *Carleton College* with team members Max L. Benson, Albert T. Borchers, and Andrew A. Rich; *University of Illinois, Urbana,* with team members Bruce E. Hajek, Allen Moy, and Daniel D. Sleator; *Michigan State University* with team members Mark P. Merriman, Karl W. Pettis, and Ian H. Redmount; and *University of Waterloo* with team members Richard P. Anstee, Marks S. Brader, Gregory J. Fee.

Honorable mention was given to the following thirty individuals, named in alphabetical order: *J. Eric Brosius,* Pennsylvania State University; *F. Michael Christ,* Harvey Mudd College; *Roy E. De Meo,* Massachusetts Institute of Technology; *Andrew Z. Fire,* University of California, Berkeley; *James A. Gardner,* University of Waterloo; *Ian P. Goulden,* University of Waterloo; *Philip. I. Harrington,* Washington University; *Karl W. Heuer,* Concordia College; *David C. Hobby,* University of Washington; *Sheldon H. Katz,* Massachusetts Institute of Technology; *Donald T. Kersey,* McMaster University; *Nathaniel S. Kuhn,* Harvard University; *Frank J. Lhota,* Wayne State University; *Alan D. Listoe,* University of Saskatchewan; *Russell D. Lyons,* Case Western Reserve University; *Andrew M. McLennan,* University of Chicago; *Ross E. Millikan,* University of California, Berkeley; *Stephen W. Modzelewski,* Harvard University; *David R. Morrison,* Princeton University; *Allen Moy,* University of Illinois, Urbana; *N. Christopher Phillips,* University of California, Berkeley; *Nicholas S. Robins,* University of California, Berkeley; *Adam N. Rosenberg,* Princeton University; *David J. Rusin,* Princeton University; *Mark W. Saaltink,* University of Victoria; *Roger S. Schlafly,* Princeton University; *Matthew R. Smith,* University of Waterloo; *Daniel J. Velleman,* Dartmouth College; *Paul A. Vojta,* University of Minnesota, Minneapolis; *Norman J. Wildberger,* University of Toronto.

The other individuals who achieved ranks among the top one hundred, in alphabetical order of their schools, were: Bowdoin College, *Richard M. Crew*; Brown University, *Joseph H. Silverman*; University of British Columbia, *Allan D. Jepson, Arnold D. Satanove*; California Institute of Technology, *Robert W. Cox, Frank M. Liang, Stephen R. Roe, Douglas B. Tyler*; University of California (Berkeley), *Steven T. Tschantz*; University of California (Davis), *Stephen R. Peck*; Carleton College, *Albert T. Borchers, Andrew A. Rich*; University of Chicago, *Robert M. Beals, Peter L. Dordal, Ngaiming L. Mok*; Drexel University, *Dennis M. DeTurck*; Georgia Institute of Technology, *Michael E. Hoffman*; Grinnell College, *Dale R. Worley*; Harvard University, *Richard I. Anders, Robert F. Coleman, Peter G. Doyle, Sholom Feldblum, Tony G. Horowitz, Douglas W. Oman, David R. Richman, Vladislav G. Rutenburg, James P. Sethna*; Holy Cross College, *Gary B. Page*; Indiana University, *Philip H. Dybvig, Arvind N. Srivastava*; Iowa State University, *David L. Gordon*; University of Maryland (College Park), *Robert B. Bundy*; Massachusetts Institute of Technology, *Dean G. Sturtevant*; McGill University, *Gerald M. Cohen*; Michigan State University, *Mark P. Merriman, Karl W. Pettis*; University of North Carolina, *Wesley H. Presler*; Oberlin College, *Spencer W. Thomas*; Pomona College, *Eric V. Level*; Princeton University, *Michael J. Barall, Alan S. Geller,*

*Eric S. Lander, Jeffrey N. Rottman*; Purdue University, *Max C. Ng*; Rice University, *C. Robin Graham, John W. Myre*; San Diego State University, *John O. Lamping*; Stanford University, *Nick E. Baxter, John P. Dawson, Bruce A. Fast, Scott E. Kim*; Union College, *James B. Saxe*; United States Naval Academy, *Steven J. Raher*; Wabash College, *Thomas M. Sellke*; Washington University, *Tim J. Steger, Philip N. Strenski*; University of Waterloo, *Mark S. Brader, Gregory J. Fee, Peter F. Schneider, Douglas R. Stinson*; University of Windsor, *Bradley J. Lucier*; Worcester Polytechnic Institute, *John A. Major*; Yale University, *David V. Feldman.*

There were 2203 individual contestants from 355 colleges and universities in Canada and the United States in the competition of December 6, 1975. Teams were entered by 285 institutions.

The Questions Committee, consisting of G. D. Chakerian, J. D. E. Konhauser, and J. I. Richards (Chairman), prepared the problems listed below and were most prominent among those suggesting solutions.

## PROBLEMS, PART A

A–1. Supposing that an integer $n$ is the sum of two triangular numbers,

$$n = \frac{a^2+a}{2} + \frac{b^2+b}{2},$$

write $4n+1$ as the sum of two squares, $4n+1 = x^2+y^2$, and show how $x$ and $y$ can be expressed in terms of $a$ and $b$.

Show that, conversely, if $4n+1 = x^2+y^2$, then $n$ is the sum of two triangular numbers.

[Of course, $a$, $b$, $x$, $y$ are understood to be integers.]

A–2. For which ordered pairs of *real* numbers $b$, $c$ do both roots of the quadratic equation

$$z^2 + bz + c = 0$$

lie inside the unit disk $\{|z|<1\}$ in the complex plane?

Draw a reasonably accurate picture (i.e., 'graph') of the region in the real $bc$-plane for which the above condition holds. Identify precisely the boundary curves of this region.

A–3. Let $a$, $b$, $c$ be constants with $0<a<b<c$. At what points of the set

$$\{x^b + y^b + z^b = 1,\ x \geqq 0,\ y \geqq 0,\ z \geqq 0\}$$

in three-dimensional space $R^3$ does the function $f(x, y, z) = x^a + y^b + z^c$ assume its maximum and minimum values?

A–4. Let $n = 2m$, where $m$ is an odd integer greater than 1. Let $\theta = e^{2\pi i/n}$. Express $(1-\theta)^{-1}$ explicitly as a polynomial in $\theta$,

$$a_k\theta^k + a_{k-1}\theta^{k-1} + \cdots + a_1\theta + a_0,$$

with *integer* coefficients $a_i$.

[Note that $\theta$ is a primitive $n$-th root of unity, and thus it satisfies all of the identities which hold for such roots.]

A–5. On some interval $I$ of the real line, let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation

$$y'' = f(x)y,$$

where $f(x)$ is a continuous real-valued function. Suppose that $y_1(x)>0$ and $y_2(x)>0$ on $I$. Show that there exists a positive constant $c$ such that, on $I$, the function

$$z(x) = c\sqrt{y_1(x)y_2(x)}$$

satisfies the equation

$$z'' + \frac{1}{z^3} = f(x)z.$$

State clearly the manner in which $c$ depends on $y_1(x)$ and $y_2(x)$.

A–6. Let $P_1$, $P_2$, $P_3$ be the vertices of an acute-angled triangle situated in three-dimensional space. Show that it is always possible to locate two additional points $P_4$ and $P_5$ in such a way that no three of the points are collinear and so that the line through any two of the five points is perpendicular to the plane determined by the other three.

In writing your answer, state clearly the locations at which you place the points $P_4$ and $P_5$.

## PROBLEMS, PART B

B–1. In the additive group of ordered pairs of integers $(m, n)$ [with addition defined componentwise: $(m, n)+(m', n') = (m + m', n + n')$] consider the subgroup $H$ generated by the three elements

$$(3, 8), \qquad (4, -1), \qquad (5, 4).$$

Then $H$ has another set of generators of the form

$$(1, b), \qquad (0, a)$$

for some integers $a$, $b$ with $a > 0$. Find $a$.

[Elements $g_1, \ldots, g_k$ are said to *generate* a subgroup $H$ if (i) each $g_i \in H$, and (ii) every $h \in H$ can be written as a sum $h = n_1g_1 + \cdots + n_kg_k$ where the $n_i$ are integers (and where, for example, $3g_1 - 2g_2$ means $g_1 + g_1 + g_1 - g_2 - g_2$).]

B–2. In three-dimensional Euclidean space, define a *slab* to be the open set of points lying between two parallel planes. The distance between the planes is called the *thickness* of the slab. Given an infinite sequence $S_1, S_2, \ldots$ of slabs of thicknesses $d_1, d_2, \ldots$, respectively, such that $\Sigma_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.

B–3. Let $s_k(a_1, \ldots, a_n)$ denote the $k$-th elementary symmetric function of $a_1, \ldots, a_n$. With $k$ held fixed, find the supremum (or least upper bound) $M_k$ of

$$s_k(a_1, \ldots, a_n)/[s_1(a_1, \ldots, a_n)]^k$$

for arbitrary $n \geqq k$ and arbitrary $n$-tuples $a_1, \ldots, a_n$ of positive real numbers.

[The symmetric function $s_k(a_1, \ldots, a_n)$ is the sum of all $k$-fold products of the variables $a_1, \ldots, a_n$. Thus, for example:

$$s_1(a_1, \ldots, a_n) = a_1 + a_2 + \ldots + a_n;$$

$$s_3(a_1, a_2, a_3, a_4) = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4.$$

It should be remarked that the supremum $M_k$ is never attained; it is approached arbitrarily closely when, for fixed $k$, the number $n$ of variables increases without bound, and the values $a_i > 0$ are suitably chosen.]

B–4. Does there exist a subset $B$ of the unit circle $x^2 + y^2 = 1$ such that (i) $B$ is topologically closed, and (ii) $B$ contains exactly one point from each pair of diametrically opposite points on the circle?

[A set $B$ is topologically *closed* if it contains the limit of every convergent sequence of points in $B$.]

B–5. Let $f_0(x) = e^x$ and $f_{n+1}(x) = xf_n'(x)$ for $n = 0, 1, 2, \ldots$. Show that

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^e.$$

B–6. Show that if $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n$, then

(a) $n(n+1)^{1/n} < n + s_n$ for $n > 1$, and

(b) $(n-1)n^{-1/(n-1)} < n - s_n$ for $n > 2$.

## SOLUTIONS

In the 12–tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, $n_i$ for $10 \geqq i \geqq 0$ is the number of students, among the top 212 contestants, who received $i$ points for the problem and $n_{-1}$ is the number not submitting solutions.

A–1. $(152, 7, 6, 18, 3, 7, 0, 0, 0, 0, 1, 18)$

Let $n = [(a^2+a)/2]+[(b^2+b)/2]$, with $a$ and $b$ integers. Then

$$4n+1 = 2a^2+2a+2b^2+2b+1 = (a+b+1)^2+(a-b)^2.$$

Conversely, let $4n+1 = x^2+y^2$, with $x$ and $y$ integers. Then exactly one of $x$ and $y$ is odd and so $a = (x+y-1)/2$ and $b = (x-y-1)/2$ are integers. One easily verifies that

$$[(a^2+a)/2]+[(b^2+b)/2] = (x^2+y^2-1)/4 = n.$$

A–2. (100, 6, 2, 0, 0, 5, 10, 3, 1, 27, 43, 15)

The desired region is the inside of the triangle with vertices $(0,-1)$, $(2,1)$, $(-2,1)$. The boundary segments lie on the lines

$$L_1: c = 1, L_2: c-b+1 = 0, L_3: c+b+1 = 0.$$

To see this, we let $f(z) = z^2+bz+c$ and denote its zeros by $r$ and $s$. Then $-b = r+s$ and $c = rs$. Also

$$(r+1)(s+1) = rs+r+s+1 = c-b+1 = f(-1),$$

$$(r-1)(s-1) = rs-r-s+1 = c+b+1 = f(1).$$

On or below $L_2$, at least one zero is real and not greater than $-1$; this follows either from $(r+1)\cdot(s+1) \leqq 0$ or from $f(-1) \leqq 0$ and the fact that the graph of $y = f(x)$, for $x$ real, is an upward opening parabola. Similarly, on or below $L_3$ one zero is real and at least 1. On or above $L_1$, at least one zero has absolute value greater than or equal to 1. Hence the desired points $(b, c)$ must be inside the described triangle.

Conversely, if $(b, c)$ is inside the triangle, $|c|<1$ and so $|r|<1$ or $|s|<1$ or both. If the zeros are complex, they are conjugates and $|r| = |s|$; then $|r| = |s| < 1$ follows from $|c|<1$. If the zeros are real, $|c|<1$ implies that at least one zero is in $(-1,1)$. Then $(r+1)(s+1) = f(-1) > 0$ and $(r-1)\cdot(s-1) = f(1) > 0$ imply that the other zero is also in $(-1,1)$.

For full credit, the region had to be depicted.

A–3. (10, 6, 2, 2, 0, 8, 6, 7, 7, 23, 29, 112)

Let $h(x) = x^a - x^b$ and $k(z) = z^c - z^b$. The desired points also give the maximum and minimum of the function

$$g(x,z) = (x^a+y^b+z^c)-(x^b+y^b+z^b) = h(x)+k(z)$$

on the domain obtained by projection of the solid domain on the $xz$-plane. For all points under consideration, both $x$ and $z$ are in $[0,1]$. Examining its derivative, one sees that $h(x)$ increases from 0 at $x = 0$ to a maximum at $x_0 = (a/b)^{1/(b-a)}$ and then decreases to 0 at $x = 1$. (This uses the hypothesis $0 < a < b$.) Similarly, $k(z)$ decreases from 0 at $z = 0$ to a minimum at $z_0 = (b/c)^{1/(c-b)}$ and then increases to 0 at $z = 1$. Since $(1, z_0)$ and $(x_0, 1)$ are not in the domain of $g(x, z)$, the function $f$ achieves its maximum only at $(x, y, z) = (x_0, [1-x_0^b]^{1/b}, 0)$ and achieves its minimum only at $(0, [1-z_0^b]^{1/b}, z_0)$.

A–4. (23, 3, 1, 0, 0, 2, 0, 0, 0, 0, 74, 109)

Let $n = 4k+2$ with $k > 0$. Then

$$0 = \theta^n - 1 = \theta^{4k+2} - 1 = (\theta^{2k+1}-1)(\theta^{2k+1}+1),$$

$$0 = (\theta^{2k+1}-1)(\theta+1)(\theta^{2k}-\theta^{2k-1}+\theta^{2k-2}-\cdots-\theta+1).$$

Since $\theta$ is a primitive $n$th root of unity with $n > 2k+1$ and $n > 2$,

$$(\theta^{2k+1}-1)(\theta+1) \neq 0.$$

Hence

(A) $$\theta^{2k} - \theta^{2k-1} + \theta^{2k-2} - \cdots + \theta^2 - \theta + 1 = 0,$$

$$1 = \theta - \theta^2 + \theta^3 - \cdots - \theta^{2k} = (1-\theta)(\theta + \theta^3 + \theta^5 + \cdots + \theta^{2k-1}),$$

$$(1-\theta)^{-1} = \theta + \theta^3 + \cdots + \theta^{2k-1} \text{ [where } 2k-1 = (n-4)/2\text{]}.$$

Another solution is $(1-\theta)^{-1} = 1 + \theta^2 + \theta^4 + \cdots + \theta^{2k}$ as one sees from (A).

A–5. (22, 13, 2, 2, 2, 3, 3, 3, 2, 2, 21, 137)

The answer for $c$ is $\sqrt{2/w}$, where $w$ is the wronskian $y_1y_2' - y_2y_1'$ (and will be seen below to be constant).

Let $c^2 = 2k$. Then $z^2/2 = ky_1y_2$. Differentiating twice, one has

$$zz' = k(y_1y_2' + y_2y_1'),\ zz'' + (z')^2 = k(y_1y_2'' + y_2y_1'' + 2y_1'y_2').$$

Since $y_1'' = fy_1$ and $y_2'' = fy_2$, this implies

$$zz'' + (z')^2 = 2k(fy_1y_2 + y_1'y_2') = f(2ky_1y_2) + 2ky_1'y_2' = fz^2 + 2ky_1'y_2'.$$

Now

$$z^3z'' + (zz')^2 = fz^4 + 2kz^2y_1'y_2',$$

$$z^3z'' + k^2(y_1y_2' + y_2y_1')^2 = fz^4 + 4k^2(y_1'y_2')^2,$$

$$z^3z'' + k^2(y_1y_2' - y_2y_1')^2 = fz^4,$$

(1) $$z^3z'' - fz^4 = -k^2(y_1y_2' - y_2y_1') = -k^2w^2 = -c^4w^2/4.$$

Since $w' = (y_1y_2' - y_2y_1')' = y_1y_2'' - y_2y_1'' = y_1(fy_2) - y_2(fy_1) = 0$, $w$ is a constant. Solving $c^4w^2/4 = 1$ for $c$ gives $c = \sqrt{2/w}$; for this $c$, (1) implies $z'' - fz = -z^{-3}$ or $z'' + z^{-3} = fz$.

A–6. (3, 3, 5, 6, 3, 6, 6, 26, 8, 29, 26, 91)

Let $\lambda$ denote the line through the desired points $P_4$ and $P_5$. Let $\pi$ be the plane of $P_1$, $P_2$, and $P_3$ and let $H$ be the intersection of $\lambda$ with $\pi$.

Let $v_u$ be the vector $\mathbf{HP}_u$ and $|v_u|$ be its magnitude. We wish to have the dot product

(1) $$d = \mathbf{P}_h\mathbf{P}_k \cdot \mathbf{P}_i\mathbf{P}_j = (v_k - v_h)\cdot(v_j - v_i) = v_k \cdot v_j - v_k \cdot v_i - v_h \cdot v_j + v_h \cdot v_i$$

zero for all choices of $h$, $k$, $i$, $j$ as distinct indices in $\{1, 2, 3, 4, 5\}$.

Since $\lambda$ is to be perpendicular to $\pi$, we must have

(2) $$v_h \cdot v_i = 0 \text{ for } h \in \{4, 5\} \text{ and } i \in \{1, 2, 3\}.$$

If $h, k \in \{4, 5\}$ and $i, j \in \{1, 2, 3\}$, (2) implies that the dot product $d$ of (1) is zero. If $h \in \{4, 5\}$ and $k, i, j \in \{1, 2, 3\}$, (2) implies that the $d$ of (1) becomes

(3) $$d = v_k \cdot v_j - v_k \cdot v_i = v_k \cdot (v_j - v_i) = \mathbf{HP}_k \cdot \mathbf{P}_i\mathbf{P}_j.$$

Clearly the $d$ of (3) are zero simultaneously if and only if $H$ is the orthocenter (i.e., intersection of altitudes) of $\Delta P_1P_2P_3$. With this choice of $H$, the vanishing of the $d$ of (3) implies

(4) $$v_2 \cdot v_3 = v_1 \cdot v_3 = v_1 \cdot v_2.$$

Now let $h, i \in \{4, 5\}$ and $k, j \in \{1, 2, 3\}$. Then (2) implies

(5) $$d = v_k \cdot v_j + v_4 \cdot v_5.$$

Assuming (4), one sees that all the $d$'s of (5) will be zero if $v_4 \cdot v_5 = -v_1 \cdot v_2$. The hypothesis that

$\Delta P_1P_2P_3$ is acute-angled tells us that $H$ is inside the triangle. Then at least one (actually, all) of the angles $\measuredangle P_1HP_2$, $\measuredangle P_2HP_3$, $\measuredangle P_3HP_1$ must be obtuse and so the equal dot products of (4) must be negative. Hence $v_4 \cdot v_5$ must be positive; this means that $P_4$ and $P_5$ must be on the same half-line of $\lambda$ determined by $H$.

Now the location of $P_4$ and $P_5$ can be given. Let $H$ be the orthocenter of $\Delta P_1P_2P_3$ and $\mu$ be either half-line perpendicular to plane $P_1P_2P_3$ at $H$. Then $P_4$ may be any point on $\mu$ such that $|v_4|$ is neither zero nor $(-v_1 \cdot v_2)^{1/2}$ and $P_5$ must be the unique point on $\mu$ with $|v_5| = -v_1 \cdot v_2/|v_4|$. Then each $d$ of (1) is zero and no three of the $P_i$ are collinear.

B-1. (12, 6, 96, 6, 6, 46, 10, 3, 7, 5, 11, 4)

The answer is $a = 7$. Also one must have $b \equiv 5 \pmod 7$.

*Proof*: The subgroup $H$ must contain $4(3,8) - 3(4,-1) = (0,35)$, $4(5,4) - 5(4,-1) = (0,21)$, and then $2(0,21) - (0,35) = (0,7)$. Now $(0,7)$ and $(1,b)$ will generate $H$ iff $(1,b)$ is in $H$ and there exist integers $u$, $v$, and $w$ such that

$$(3,8) = 3(1,b) + u(0,7),\ (4,-1) = 4(1,b) + v(0,7),\ (5,4) = 5(1,b) + w(0,7).$$

These hold iff $8 = 3b + 7u$, $-1 = 4b + 7v$, and $4 = 5b + 7w$. With $b = 5 + 7k$, $k$ any integer, the desired coefficients $u$, $v$, and $w$ exist in the form $u = -1 - 3k$, $v = -3 - 4k$, $w = -3 - 5k$. It now suffices to let $k = 0$ and to note that $(1,5) = (4,-1) - (3,8) + 2(0,7)$ is in $H$.

B-2. (79, 41, 4, 4, 1, 0, 1, 0, 3, 10, 31, 38)

Let $\Sigma d_i = d$ and let $S$ be a sphere of radius $r > d/2$. The area of $S$ contained in slab $S_i$ is at most $2\pi d_i$. It follows that the area of $S$ contained in the union of the slabs $S_i$ is at most $2\pi d < 4\pi r =$ (area of $S$). Hence there are points of $S$ that are not in any of the slabs.

The problem may also be done using volumes of intersection of the slabs with an appropriately chosen sphere.

B-3. (43, 3, 6, 3, 1, 5, 2, 23, 18, 3, 24, 81)

In the expansion of $s_1^k = (a_1 + a_2 + \cdots + a_n)^k$, every term of $s_k$ appears with $k!$ as coefficient and the other coefficients are nonnegative. Hence $s_k/s_1^k \leqq 1/k!$

If we let each $a_i = 1$,

$$\frac{s_k}{s_1^k} = \binom{n}{k} \Big/ n^k = \frac{n(n-1)\cdots(n-k+1)}{k!n^k} = \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right),$$

which approaches $1/k!$ as $k$ is held fixed and $n$ goes to infinity. These facts show that the supremum $M_k$ is $1/k!$.

B-4. (73, 46, 15, 5, 1, 5, 2, 14, 4, 0, 32, 15)

No. Since the mapping with $(x, y) \to (-x, -y)$ is a homeomorphism of the unit circle on itself, the complement $-B$ of such a subset $B$ would also be closed. Thus the existence of such a $B$ would make $C$ the union $-B \cup B$ of disjoint nonempty closed subsets; this would contradict the fact that $C$ is connected.

B-5. (14, 11, 17, 7, 2, 0, 4, 0, 1, 1, 27, 128)

Since $f_0(x) = \Sigma_{k=0}^{\infty} x^k/k!$, one easily shows by mathematical induction that $f_n(x) = \Sigma_{k=0}^{\infty}(k^n x^k/k!)$. Then, since all terms are positive, one has

$$\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{k^n}{k!\,n!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k^n}{n!} = \sum_{k=0}^{\infty} \frac{e^k}{k!} = e^e.$$

B-6. $(2, 0, 0, 0, 0, 1, 0, 0, 0, 0, 48, 161)$

Both parts are done easily using the Inequality on the Means. For (a), one has

$$\frac{n+s_n}{n} = \frac{(1+1)+(1+\frac{1}{2})+\cdots+(1+(1/n))}{n} > \sqrt[n]{(1+1)(1+\tfrac{1}{2})\cdots(1+(1/n))} = \sqrt[n]{2\cdot\tfrac{3}{2}\cdot\tfrac{4}{3}\cdots(n+1)/n}$$
$$= (n+1)^{1/n}$$

and so $n + s_n > n(n+1)^{1/n}$.

For (b), one has

$$\frac{n-s_n}{n-1} = \frac{(1-\frac{1}{2})+(1-\frac{1}{3})+\cdots+(1-(1/n))}{n-1} > \sqrt[n-1]{(1-\tfrac{1}{2})(1-\tfrac{1}{3})\cdots(1-(1/n))} = \sqrt[n-1]{\tfrac{1}{2}\cdot\tfrac{2}{3}\cdots(n-1)/n}$$
$$= n^{-1/(n-1)}$$

and so $n - s_n > (n-1)n^{-1/(n-1)}$.

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"No bubble is so iridescent or floats longer
than that blown by the successful teacher."
William Osler

## LEE LORCH AT FISK: A TRIBUTE

VIVIENNE MAYES

At the first Convocation held at Fisk University in September, 1950, Lee Lorch was introduced as the new chairman of the Department of Mathematics. Such a perfunctory ritual would have utterly failed to register in my student memory had not the president of Fisk, the late Charles S. Johnson, accompanied the introduction with an unusual story. He told how Dr. Lorch had lost his professorship at Penn State University because he had sublet his apartment to a Black family. This story aroused admiration in all of the students at the service. We were Black. And this story of a white sacrificing for a Black family in 1950 was unforgettable. But Lee Lorch had only begun to influence us. He remained at Fisk for five years, and through his personal example of teaching and humanity, he succeeded in redirecting the career goals and affecting the lives of many of his students.

Shortly before that first semester's classes began, I approached the new chairman for employment as a grader in the math department. I was a junior, and I was undecided between majoring in math or chemistry. Dr. Lorch offered me the job of grading for a freshman level class in Elementary Analysis that he would be teaching, and he suggested I attend the first few meetings of the class to get a feel of what he expected the students to do in their work. After the first class meeting, it was clear to me that I would be attending each meeting of the class for the entire semester.

Dr. Lorch conducted this class as he did all of his other courses. He believed that the students could *understand* the material, not just learn to do it. He was interested in teaching them the *why* of mathematics in addition to the *how*. He also watched the reactions of his class intently, and always maintained good rapport. At the first sign that the class was lost, he would stop, repeat, and give more specific examples. He would motivate the students to participate in class discussions, and would often, through his questioning, draw out of the class the proof of some general result. The students saw that he expected them to learn the material, and they felt compelled to live up to his expectations.

My duties soon included tutoring as well as grading. As a result, I was probably the chief beneficiary of the course. At the end of the semester, I was confident of my knowledge of freshman mathematics, but more than that I was beginning to become aware of what mathematics was really all about. Needless to say, by the end of the first semester, I had decided — mainly because of Dr. Lorch — to major in mathematics.

After graduation, I stayed on at Fisk as one of the graduate students in his department. On the graduate level, we became acquainted with Lee Lorch, the research mathematician. Several of his more than thirty-five papers were published while he was at Fisk. As he brought his articles to our attention, we developed interest in the resource value of the journal literature. Always questioning us about details which were omitted in the texts and articles, he required us to read mathematics well.

Dr. Lorch's ability to nurture the bud of mathematical understanding was appreciated by all of his students, who readily understand these remarks by Etta Zuber Falconer as she recalls the senior seminar which all of us attended:

> "Somehow under his prodding, I gradually became aware of the beauty and power of mathematics...Dr. Lorch truly loved mathematics, and it was impossible to listen to him without acquiring some of this love."

In the early Fifties, the idea of encouraging Blacks, and especially females, to prepare for academic careers was unheard of. Since Black colleges were so few in number, it was not economically sound to plan on teaching appointments or even to pursue advanced academic degrees. In those days, we were counselled to prepare for health professions, the ministry, or public schoolteaching, the few careers which offered an opportunity for livelihood.

However, as one of his colleagues at Fisk, Evelyn Boyd Granville, noted, when Lee Lorch assumed the chair,

> "he began immediately to upgrade curriculum, establish bridges with the departments 'across town,' and encourage attendance of faculty at national mathematical meetings. He encouraged students to study to qualify for graduate school, and recruited Black students from other colleges to attend the graduate program in mathematics at Fisk."

Of the students who were in his department for at least two of the five years he was at Fisk, he influenced one-fourth of them to pursue and earn the master's degree in pure mathematics. Moreover, one tenth of the students continued to the doctorate. Each known doctoral recipient credits Lee Lorch as the greatest influence in his choice of career. Speaking for us all, Dr. Falconer comments:

> "Dr. Lorch encouraged me in the pursuit of a mathematical career at a time when it was unpopular for both Blacks and women. He was well before his time in promoting the rights of all people. He refused to see a color line or sex line that limited ability, and for this reason he has profoundly affected the development of many persons who would otherwise have carried out the low expectations of the time."

Today, there are students at predominantly Black colleges and at several predominantly white universities who are studying under former Lorch students from Fisk. Not even Lorch himself in 1950 could have guessed that students of all colors and ethnic groups in the Sixties and Seventies would benefit so much from his efforts at Fisk in the Fifties.

At the same time Dr. Lorch was encouraging students to pursue graduate studies, he was equally vocal about job opportunities for mathematicians in government, business and industry. Corneila Haddox Edwards, Marjorie Scott Bethea and Sally Duval Richmond are Senior Computer Programmers with TRW Systems, Inc., the National Security Agency, and Computer Sciences Corporation, respectively; Larkin Teasley is the chief actuary and Senior Vice President of the Golden State Mutual Life Insurance Company, and Timothy Donaldson is Governor of the Central Bank of the Bahamas. All of these students acknowledge that Lee Lorch exerted a strong influence in their career choice.

Aside from his duties as educator and chairman, Dr. Lorch remained dedicated to the broader ends and personal ideals which made knowing him so rewarding. Dr. Falconer recalls how she was welcomed into his home on campus by his family:

> "Dr. Lorch was the first white person I had met and one of the few I have since encountered in which the relationship was completely devoid of the color barrier. We treasured this man who sometimes ran across the campus, and then slowed to a walk, deep in thought."

However, Dr. Granville recalls that he was not equally welcomed by the Nashville community.

> "He and his family, his wife and his daughter, were quite vocal in the community about human dignity. His endeavors at integration led to animosity from the white community and from many Blacks on campus. His life at Fisk was stormy. Despite many unpleasantries he persevered in being vocal about his beliefs about the equality of mankind."

Dr. Lorch left Fisk in 1955 to assume a post at Philander-Smith College in Arkansas. During this time he and his wife actively participated in the explosive Little Rock school integration case. From there he went to the University of Alberta and then to his current appointment at York University, Toronto.

His perspicacity in recognizing talent in its early stages enabled him to help students whom he had taught for only short periods of time. For example, he left Fisk at the end of Gloria Conyers Hewitt's sophomore year. Yet two years later, without her knowledge, he recommended her to two universities and encouraged them to seek her application to pursue graduate studies. As Gloria Hewitt expresses it: "As he (Lorch) had taught me for such a short time, it amazes me, to this day, that he felt I could handle graduate school in mathematics...the thought of entering graduate school in mathematics had never crossed my mind; I never knew it had crossed his until I heard of his recommendations."

Throughout his career he has continued to be interested in the academic activities of all of the students he taught at Fisk. He has been a continuing influence in their further growth and development. All of us who knew him share with Dr. Falconer in feeling a warmness towards him "that is borne of knowing that he is a truly great man who has dedicated his life to the power and beauty, not only of mathematics, but of the human race."

Recently, Dr. Lorch has been honored for his past contributions and his continuing influence on the education of many Black Americans. In the spring of 1975, the administration of Fisk invited him to serve on a committee with a charge to study the present academic programs of the university and to make recommendations for their improvement. Such an invitation acknowledges the outstanding contribution which he made to the university as a former department chairman. In the fall of the same year, the Mathematics Department of Howard University awarded him a plaque "In appreciation for exemplary courage and personal sacrifice in the struggle for Human Rights...(and) for his singular contribution to the Education of Black Mathematicians."

Charles Costley echoes the sentiments of all who were strongly influenced by Lee Lorch when he observes that "Probably the most important lesson I learned from Dr. Lorch and his late wife Grace, is that my career and success as a mathematician should be cherished and are important — but that decency is more important than success."

**Acknowledgments:** The author is indebted to Aline Rivers and Robert Poole, the registrar and alumni secretary at Fisk, who provided the names and addresses of Fisk math majors from 1952 to 1957; to Doctors Etta Zuber Falconer, Evelyn Boyd Granville, Gloria Conyers Hewitt and Charles Costley for permission to use their comments on Lee Lorch; and to a former student of the author's, Patrick Conoley, M.D., for his suggestions regarding the organization of this paper.

**Summary of the Available Data:** A survey of the spring commencement programs of Fisk University indicated that for the five year period, 1952–57, a total of forty-eight degrees (thirty-nine

Bachelor of Arts and nine Master of Arts) in mathematics was conferred. The present status of the recipients of these degrees follows: two are deceased; two are disabled; three are computer programmers; one is a Fellow of the Society of Actuaries, and one who received the M.A. from the University of Minnesota is a banker; one with an M.A. is on the physics faculty of a college; and seven are known to be high school math teachers. Five have earned a Ph.D. (four of these are college professors and the fifth is one of the disabled). These five are the first graduates in the history of Fisk to continue to the doctorate in mathematics. Moreover, a sixth person is at the dissertation level. Fourteen are known to be in areas which are not math related. At this time, the author has no information on the remaining twelve of the forty-eight graduates.

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## A FAST SWITCH

SAMUEL COPPAGE AND J. T. SCHWARTZ

It has for many years been an article of faith at our Institute that the roots of mathematics lie in its applications, out of which grow such gorgeous efflorescences as topology, functional analysis, and so forth. In past years the Institute's work, and witness to this principle, has generally been in the area in which classical applied mathematics makes contact with partial differential equations, fluid mechanics, elasticity theory, etc. In the last few years, during which computer science has grown considerably at NYU, we have seen the way in which this same principle applies to the 'new applied mathematics'; and specifically to computer science, which interacts in significant, lively ways with areas of mathematics that were traditionally considered very 'pure': mathematical logic, combinatorics, abstract algebra. I may say that computer science often makes contact with particularly deep mathematical issues; for example, quite a few beautiful quantitative generalizations of Gödel's theorem have developed out of computer-oriented studies of 'Computational Complexity' (see Meyer and Stockmeyer [**9** and **10**], Fisher and Rabin [**4**], Chaitin [**3**]. In the short time available today I will however not attempt to touch upon these deep matters, but shall confine myself to discussing a simple, initially very applied problem, but one which leads very directly and naturally to 'abstract' mathematical analysis. The problem with which we start is that of switching telephones.

Over the years, many interesting combinatorial constructions have developed out of the problem of connecting a telephone dialer to his intended 'dialee'. In the original telephone system, installed in Boston, a telephone subscriber merely rented a pair of receivers from Mr. Bell; a long wire was then strung between the two locations for which service was desired. This rudimentary technique is acceptable if one only wishes to dial a few numbers, but in general it will require $\binom{N}{2}$, or approximately $\frac{1}{2}N^2$ lines to service $N$ points.

It is fortunate that rising telephone use has been accompanied by advances in switching technique which make unnecessary the maze of trillions of wires which Mr. Bell's original system, applied in modern conditions, would have required. One simple idea is to serve multiple users with a single 'trunk line', approximately as in Fig.1.

This solution connects $N$ callers to $N$ callees and requires only $2N$ lines and $2N$ switches. Its drawback is that it allows only one call to be made at a time. Once a single call is initiated, other 'callers' get a 'spurious busy signal' — spurious because what is busy is not the phone that is being called, but rather the trunk line itself. Such a network is said to have a *bottleneck* or to be a *blocking network.*

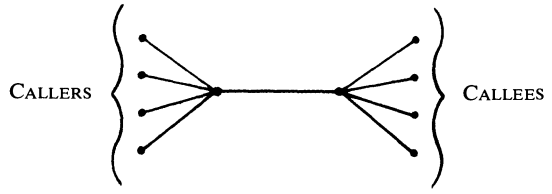



FIG. 1. Use of a 'trunk line'.

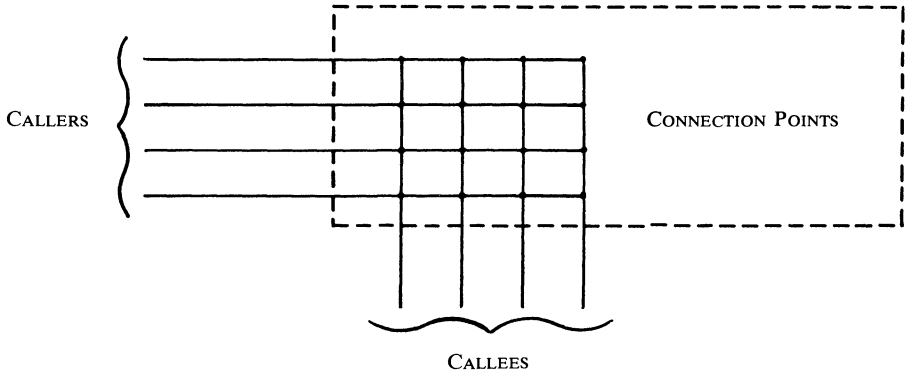



FIG. 2. An $N \times N$ 'full crossbar' switch.

A considerably more expensive technique, but one which is not subject to this objection, uses a 'full crossbar' switch to connect incoming and outgoing lines; this requires a full $N \times N$ matrix of switches but still only $2N$ lines. The pattern of connections is shown in Fig.2.

The pattern of connections makes it unnecessary to generate a busy signal unless the very phone that a caller is attempting to reach is busy. Such a network is said to be *nonblocking*. However, we might ask if this is the minimal switch with this property. A short calculation based on Stirling's formula indicates that it is not. An $N \times N$ crossbar has $N^2$ elements, each of which has two settings, i.e., 'open' and 'closed'. There are thus $2^{N^2}$ possible switch settings. But there are only $N!$ possible permutations in which all $N$ callers could call $N$ callees without duplication. By Stirling's formula,

$$N! \approx \sqrt{2\pi N}(N/e)^N \approx 2^{N \log_2 N - N \log_2 e}$$

which is clearly much less than $2^{N^2}$ for large $N$. This gives clear indication of possible improvement; specifically, it encourages us to look for a pattern in which approximately $N \log_2 N$ rather than $N^2$ switches can be used to connect $N$ callers to $N$ callees in a strictly nonblocking fashion. A much more efficient, but quite nonobvious pattern of connections, which lies close to the theoretical $N(\log_2 N - \log_2 e)$ optimum makes use of switches arranged in columns or stages. We shall have a look at one such pattern, and shall see that its workings are best comprehended group-theoretically.

**'2 × 2 Interchange switches' and their composition.** A nonblocking connection between *two* callers can be established using a single $2 \times 2$ switch (see Fig.3).

The possible switch settings for a switch of this kind are by definition as follows (see Fig.4): connect 'straight through' i.e., top to top and bottom to bottom, or connect 'with interchange', i.e., top to bottom and bottom to top. This switch realizes both of the two possible 1–1 maps of two objects $A$, $B$ onto two other objects 1, 2.

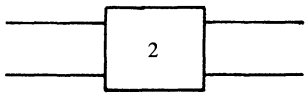



FIG. 3. A $2 \times 2$ interchange switch.

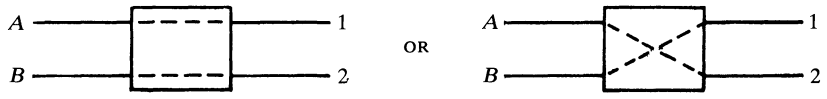



FIG. 4. The settings of a $2 \times 2$ interchange switch.

Now observe that using six $2 \times 2$ switches we can construct a '$4 \times 4$' interchange switch, i.e., a network capable of connecting four incoming lines to four outgoing lines in any one-to-one pattern. The necessary connections are shown in Fig.5.

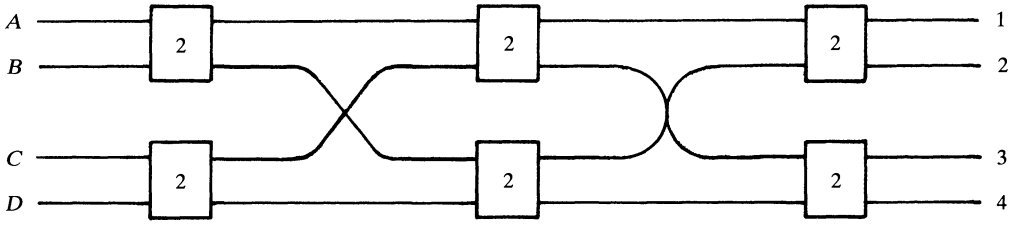



FIG. 5. A $4 \times 4$ interchange realized using six $2 \times 2$ interchange switches.

We shall see in our subsequent analysis that this network does have the property claimed for it. But it is not at all obvious that it does! The reader who wishes to test our claim is invited to work out the switch setting that connects $A$, $B$, $C$, $D$ to 4, 3, 2, 1.

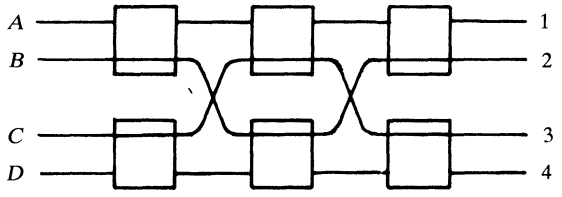



FIG. 5a. A fully connected $4 \times 4$ interchange.

If a switching network has the property that given a collection of calls already in progress, and assuming that a line is idle, we never have to change current switch settings in order to accommodate a new call, it is said to be *strictly nonblocking*. The networks we have described are nonblocking, but not strictly nonblocking. For example, if $A, B, C, D$ are connected to $1, 2, 3, 4$ in the following pattern (see Fig.5a), and $A$ and $D$ complete their calls, then even though line 4 is no longer busy, $A$ cannot complete a call to 4 without resetting some of the switches used by $B$ and $C$ which are still in progress. Note that the 'full crossbar' switch is strictly nonblocking.

We shall find it convenient to represent the pattern of connections shown in Figure 5 by the

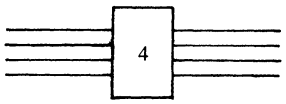



FIG. 6. 'Macrosymbol' for a $4 \times 4$ interchange switch.

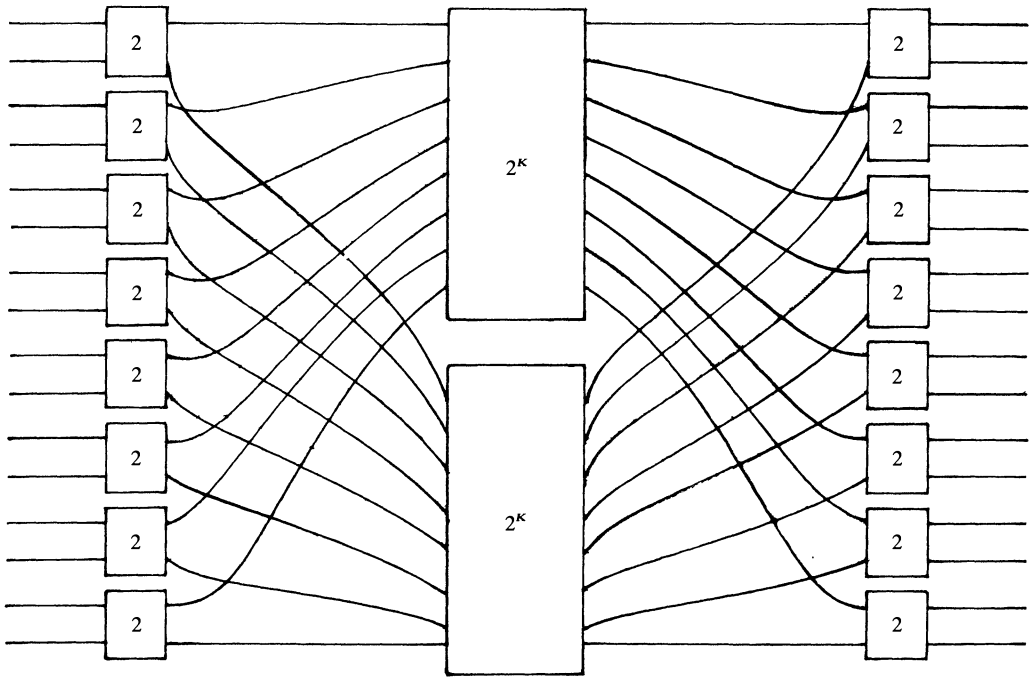



FIG. 7. A $2^{k+1} \times 2^{k+1}$ interchange switch formed from $2^{k+1}$ $2 \times 2$ switches and two $2^k \times 2^k$ switches.

'macrosymbol' shown in Fig.6. Subsequently, as we build up composite interchange switches with 8, 16, 32,... inputs and outputs, similar macro symbols will be introduced for them also.

Inductively generalizing the pattern shown in Figure 5, we can, following Clos, construct a $2^{k+1} \times 2^{k+1}$ interchange switch using $2^{k+1}$ $2 \times 2$ interchange switches and two $2^k \times 2^k$ interchange switches. The connections to be used are shown in Fig.7.

The pattern of connections shown in Fig.7 can be described as follows. Let the $2^{k+1}$ incoming lines on the left be numbered $1, \bar{1}, 2, \bar{2}, \ldots, K, \bar{K}$, where $K = 2^k$; i.e., the upper incoming wire connected to the $j$th $2 \times 2$ interchange is labeled $j$ and the corresponding lower wire is labeled $\bar{j}$. (We shall also find it convenient to refer to $j$ and $\bar{j}$ as 'barred' and 'unbarred' indices respectively.) Let the outgoing wires of the column of two-way interchange switches on the left be numbered in the same fashion. Then in Fig. 7 we have connected the $j$th output wire to the $j$th input position on the upper $2^k \times 2^k$ interchange switch, and the $\bar{j}$th output wire to the $j$th input position on the lower $2^k \times 2^k$ interchange switch. The same pattern of labels is used symmetrically for the connections to the right-hand column of $2 \times 2$ switches.

Before going on to analyze the action of the $2^{k+1} \times 2^{k+1}$ interchange switch, let us first count the number of elementary 2-way interchange switches which it contains. If we consider the switch shown in Figure 7 to be made up of $2 \times 2$ switches arranged in rows and columns, we see that the number of

columns of 2-way switches increases by 2 when we go from a $2^k \times 2^k$ to a $2^{k+1} \times 2^{k+1}$ switch, while the number of rows doubles. Thus a $2^{k+1} \times 2^{k+1}$ switch contains $2k+1$ columns of elementary 2-way switches, each column being of height $2^k$. Thus the total number of 2-way switches in a single $2^{k+1} \times 2^{k+1}$ switch is $(2k+1)2^k$. If we designate the number of inputs to the switch by $2K = 2^{k+1}$ we see that the total number of elementary 2-way components required for a $2^{k+1} \times 2^{k+1}$ switch is approximately $K \log_2 2K$ as opposed to $4K^2$ required by the $2K \times 2K$ 'full crossbar' solution to the nonblocking switching problem. Notice that the number of switch settings for a switch with $2^k$ inputs is

$$2^{(2k-1)2^{k-1}} \approx 2(K/2)^K$$

and Stirling's approximation for $K! \approx 2\sqrt{K}(K/e)^K$; thus the $2^k \times 2^k$ switch is not terribly far from optimal.

But does the switch shown in Figure 7 really have the nonblocking property we hope for? To justify the networks described on the preceding pages we must prove the following:

THEOREM. *By setting the individual* $2 \times 2$ *switches of a* $2^{k+1} \times 2^{k+1}$ *interchange switch properly,* $2^{k+1}$ *callers may be connected to a like number of callees in any permutation.*

To prepare for the proof we first observe the following: Both settings of a $2 \times 2$ switch realize a permutation of the two input lines to the two output lines. Inspecting Figure 7, we see that the left column of $2 \times 2$ switches permutes the $2^{k+1}$ input lines into the $2^{k+1}$ lines entering the pair of $2^k \times 2^k$ switches at the center of the figure. By induction, each of these $2^k \times 2^k$ switches permutes the $2^k$ lines coming into it and connects them in some permuted fashion to its outgoing lines. Then the $2 \cdot 2^k = 2^{k+1}$ output lines of the two central $2^k \times 2^k$ switches are input to the $2^k \times 2^k$ switches comprising the right-hand column. Each of the $2 \times 2$ switches in the right-hand column permutes a pair of input lines. Thus the switch shown in Figure 7 effects the product of three permutations, say, $I_2 U I_1$.

The last sentence of the preceding paragraph contains the key to the rest of our analysis. Suppose that the original input lines are numbered $1, \bar{1}, 2, \bar{2}, \ldots, K, \bar{K}$ as above, and that the lines emerging from the left-hand column of $2 \times 2$ switches, the lines entering the right-hand column of $2 \times 2$ switches, and the final output lines are numbered similarly. Then both the left-hand and right-hand columns of switches effect permutations of their inputs which preserve every one of the two-element sets $\{1, \bar{1}\}, \{2, \bar{2}\}, \ldots, \{K, \bar{K}\}$. The collection of all such permutations forms a subgroup of the symmetric group $S_{2K}$ on $2K$ objects. Call this subgroup $\mathscr{I}$. Next observe that by induction on $k$ it follows that the two central $2^k \times 2^k$ switches in Figure 7 can be used to permute their incoming lines to their outgoing lines according to any permutation which preserves the two sets $\{1, \ldots, N\}$ and $\{\bar{1}, \ldots, \bar{N}\}$. Denote this subgroup of $S_{2K}$ by the letter $\mathscr{U}$. Then the switch shown in Figure 7 can realize any permutation of the form $I_2 U I_1$, with $I_1, I_2 \in \mathscr{I}$ and $U \in \mathscr{U}$, and our target theorem amounts simply to the following assertion: Any $P \in \mathscr{S}_{2K}$ can be written in this form. Since each $I \in \mathscr{I}$ is a product of disjoint transpositions, we have $I^2 = e$ (the identity permutation) for all $I \in \mathscr{I}$. Thus $P = I_2 U I_1$ will follow if we can show that there exist $U \in \mathscr{U}$ and $I_1, I_2 \in \mathscr{I}$ such that

$$(**) \qquad UI_1 = I_2^{-1} P = I_2 P.$$

To this end, the following lemma is convenient.

LEMMA. *A permutation* $Q$ *has the form* $UI_1$ *(with* $U \in \mathscr{U}$ *and* $I_2 \in \mathscr{I}$*) if and only if there exists no pair of the form* $\{N, \bar{N}\}$ *which maps onto an 'unbarred' pair* $\{N_1, N_2\}$*, i.e., onto a pair whose elements both lie outside the set* $\{\bar{1}, \ldots, \bar{K}\}$.

*Proof*: We leave it to the reader to show that every permutation of the form $Q = UI_1$ has the property stated and confine ourselves to showing that if $Q$ maps each pair $\{N, \bar{N}\}$ onto a pair which is not unbarred, then $Q$ can be written as $UI_1$. For this, consider a $Q$ with this property, and a pair

$\{N, \bar{N}\} \subset \{1, \bar{1}, \ldots, K, \bar{K}\}$ such that $Q\{N, \bar{N}\} \subset \{1, 2, \ldots, K\}$. The total number of elements being permuted is $2K$, and there are $K$ 'barred' and $K$ 'unbarred' elements. Thus if no $Q\{N, \bar{N}\}$ is 'unbarred', then each $Q\{N, \bar{N}\}$ must contain one barred element and one unbarred element, so we may assume that each set $Q\{N, \bar{N}\}$ has the form $\{N_1, \bar{N}_2\}$. Then for each $N$ such that $QN = N_1$ and $Q\bar{N} = \bar{N}_2$ we define

$$I_1 N = N \qquad \text{and} \qquad I_1 \bar{N} = \bar{N},$$

and for each $N$ such that $QN = \bar{N}_2$ and $Q\bar{N} = N_1$ we define

$$I_1 N = \bar{N} \qquad \text{and} \qquad I_1 \bar{N} = N.$$

Clearly $I_1 \in \mathcal{I}$ and in all cases we have

$$QI_1 N = N_1, \qquad QI_1 \bar{N} = \bar{N}_2.$$

This shows we can find $I_1$ such that $QI_1 = U$ preserves the sets $\{1, \ldots, K\}$ and $\{\bar{1}, \ldots, \bar{K}\}$, i.e., such that each of these sets is mapped onto itself by $QI_1$. Thus $U \in \mathcal{U}$; hence $Q = UI_1^{-1} = UI_1$. Q.E.D.

In view of the last lemma, the proof of our target theorem will be completed once we prove the following

LEMMA. *Given any permutation $P$, there always exists some $I_2 \in \mathcal{I}$ such that, for all $1 \leqq N \leqq K$, $I_2 P\{N, \bar{N}\} = \{I_2 PN, I_2 P\bar{N}\}$ contains one barred and one unbarred element.*

*Proof*: Denote the set $\{1, \bar{1}, \ldots, K, \bar{K}\}$ by the symbol $\mathcal{M}$. Since $P$ is a permutation, the pairs $P\{N, \bar{N}\} = \{PN, P\bar{N}\}$, $1 \leqq N \leqq K$, constitute a disjoint covering $\mathcal{C}$ of $\mathcal{M}$. Let $X \to \tilde{X}$ be the map which sends each unbarred element into the corresponding barred element and vice versa; thus $\tilde{N} = \bar{N}$, $\tilde{\bar{N}} = N$, and $X = \tilde{\tilde{X}}$ for all $X \in \mathcal{M}$.

Now arrange the pairs of the covering as follows: choose some initial pair, and write it as $\{X_1, \tilde{X}_2\}$. It may be that $X_2 = X_1$, in which case the pair $\{X_2, \tilde{X}_2\} = \{X_1, \tilde{X}_1\}$ is said to constitute a 'cycle of length 1.' If not, then $X_2$ belongs to some other pair $\{X_2, \tilde{X}_3\}$. If $X_3 = X_1$, then

$$\{X_1, \tilde{X}_2\}\{X_2, \tilde{X}_3\} = \{X_1, \tilde{X}_2\}\{X_2, \tilde{X}_1\}$$

constitutes a cycle of length 2, etc. Proceeding in this way, we will succeed in arranging the elements of the covering $\mathcal{C}$ of $\mathcal{M}$ as a sequence of cycles

$$\{X_1, \tilde{X}_2\}\{X_2, \tilde{X}_3\} \cdots .$$

It is clear that when $X_j$ appears first in some pair of such a cycle, then $\tilde{X}_j$ appears second in some pair of the same cycle.

Now define $I_2$ as follows: For all $1 \leqq j \leqq K$, if $X_j$ is barred, let $I_2$ interchange $X_j$ and $\tilde{X}_j$; otherwise let $I_2 X_j = X_j$, $I_2 \tilde{X}_j = \tilde{X}_j$. Then clearly every $I_2 X_j$ is unbarred and every $I_2 \tilde{X}_{j+1}$ is barred. Since for each $N$ there exists some $j$ such that $I_2 P\{N, \bar{N}\} = I_2\{X_j, \tilde{X}_{j+1}\}$, it is clear $I_2 P$ maps every pair $\{N, \bar{N}\}$ into a pair one of whose elements is barred while the other is unbarred. Q.E.D.

The proofs presented in the preceding pages justify and are summarized by the following algorithm for decomposing a permutation $P$ of $2^{k+1}$ elements into a product $I_1 UI_2$:

(1) Arrange the set of pairs $P\{N, \bar{N}\}$ as a list of cycles $\{X_1, \tilde{X}_2\}$, $\{X_2, \tilde{X}_3\}$, ....

(2) For each barred quantity $X_j$ which is first in its pair within the list, let $I_2$ interchange $X_j$ and $\tilde{X}_j$ then $I_2 P$ maps each pair $\{N, \bar{N}\}$ into a mixed pair. (The permutation $I_2$ determines the switch settings for all $2 \times 2$ switches in the right-hand column of a $2^{k+1} \times 2^{k+1}$ switch.)

(3) For each unbarred $N$ such that $I_2 PN$ is barred, let $I_1$ interchange $N$ and $\bar{N}$. ($I_1$ determines the switch settings of the $2 \times 2$ switches in the left-hand column of the same $2^{k+1} \times 2^{k+1}$ interchange switch.)

(4) Now put $U = I_2 PI_1$. By our preceding proof, $U$ preserves $\{1, 2, \ldots, K\}$ and $\{\bar{1}, \bar{2}, \ldots, \bar{K}\}$ so we

can use steps 1–4 of the algorithm recursively to determine the switch settings for each of the central pair of $2^k \times 2^k$ interchange switches shown in Figure 7.

**Some few remarks on history and current practice; Additional readings.** It must now be said that the actual development of telephone network construction has not proceeded in quite the way that the preceding discussion might suggest; nor do typical contemporary switching networks or practices conform entirely to our discussion. The first interconnection stations were merely manual switchboards at which jacks could be plugged into sockets to effect a particular connection. When telephone service over fairly long distances became feasible, 'trunk lines' were installed and calls to a distant point required that an operator at one end should use one of several trunks at his disposal to connect a caller to another operator near his callee, who then completed the desired connection. This basic concept of local switchboards, now called central offices, connected to a global pattern of trunks, has remained in use. Contemporary central offices are neither strictly nonblocking nor even nonblocking; instead, the telephone company relies on a probabilistic approach to telephone traffic, originally developed by Erlang in terms of a concept of statistical equilibrium, and later justified using the theory of stationary processes. The pragmatic basis of this approach is the fact that it is highly unlikely that a large proportion of telephone subscribers should simultaneously decide to initiate calls. For accounts of the modern statistical theory of telephone networks, see Beneš [1] and Syski [11]. An authoritative discussion of the early history of the telephone can be found in Kingsbury [5]. For an account of the use of networks for sorting and merging see Liu [8] and Knuth [6]. Liu also discusses switching networks like those we have considered, and discusses the similar switching networks with numbers of inputs differing from $2^k$ which can in some circumstances be more economical than those we have described.

The combinatorial results reviewed in this paper can be subsumed in a more general theory which brings together telephone networks, permutation group theory, and the 'matching theorem' of combinatorics. An excellent and quite accessible account of this theory is given in Beneš [2].

A concluding word concerning the title of this paper is in order. The switching networks that we have described are not especially 'fast' in the normal engineer's sense of requiring signals to pass only through some very small number of elementary switches. (From this point of view, the more expensive full crossbar switch, which passes signals only through one elementary switch, is better.) Rather, we have used the phrase 'a fast switch' in the colloquial sense of 'a permutation or other transposition whose effects are, at first sight, hard to follow'.

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After a talk delivered by J. T. Schwartz at the Florida Regional Meeting of the MAA, Bradenton, Florida, March 8, 1975. The authors would like to thank the National Science Foundation, Office of Computing Activities, for support during the period of preparation of this article, Grant NSF–GJ–1202X3.

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## ADDENDUM TO "ANATOMY OF THE ORDINARY DIFFERENTIAL EQUATION"

(This MONTHLY, 82 (1975) 971–984)

W. T. REID

Two recent results on differential equations in abstract spaces are noted, in view of their significance for problems mentioned in Section 4 of the author's cited paper.

**1.** A. N. Godunov [*Peano's theorem in Banach spaces*, Functional Anal. Appl., July, 1975, pp. 53–55; translation of Funktsional. Anal. i Prilozhen., Vol. 9, No. 1, pp. 59–60, January-March, 1975], has shown that all the Banach spaces in which Peano's theorem holds are finite dimensional.

**2.** W. J. Knight [*Counterexample to a theorem on differential equations in Hilbert space*, Proc. Amer. Math. Soc., Vol. 51, No. 2, September, 1975, pp. 378–380], has constructed a weakly continuous function $f: [0,1] \to l^2$ such that $F$ defined by the Bochner integral $F(t) = \int_0^t f(s)ds$ is not strongly differentiable at $t = 0$, and hence $u'(t) = f(t)$, $u(0) = 0$, is a counterexample to a result stated by F. Browder [Ann. of Math. (2), 80 (1964), pp. 485–523], which has been referenced in many places, including the paper to which this is an Addendum.

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# MATHEMATICAL NOTES

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## A NOTE ON URYSOHN'S METRIZATION THEOREM

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Since knowledge of all of the properties of a base for a given topological space yields knowledge of all of the properties of the space itself, it is not surprising that we may insure the metrizability of a space by placing conditions on a base for the space. Urysohn's achievement fifty years ago was in discovering that, in the presence of a separation axiom, countability of the base alone would suffice.

Throughout this note let $X$ be regular and $T_1$.

THEOREM 1. *If $X$ has a countable base, then $X$ is metrizable.*

Since there are metrizable spaces which do not have a countable base (such as $\boldsymbol{R}$ with the discrete topology), a major problem was that of finding a base condition equivalent to metrizability; this problem was solved in the early 1950's by a well-known theorem of Nagata, Smirnov, and Bing.

THEOREM 2. *$X$ has a $\sigma$-locally finite base if and only if $X$ is metrizable.*

A collection $\mathcal{U}$ of subsets of $X$ is *locally finite* if each point of $X$ has a neighborhood meeting just finitely many members of $\mathcal{U}$; a base $\mathcal{B}$ is *$\sigma$-locally finite* if it can be written as the union of countably many locally finite collections. Since each countable base is $\sigma$-locally finite, Theorem 1 follows easily from Theorem 2.

The purpose of this note is to show that Theorem 2 can also be used to prove a stronger variant of Theorem 1. Let us say that a base $\mathcal{B}$ is *locally countable* if each point of $X$ has a neighborhood meeting just countably many members of $\mathcal{B}$. Note that each countable base is locally countable and although $\boldsymbol{R}$ with the discrete topology does not have a countable base, it does have a locally countable base.

We use a lemma due to Smirnov and found on page 228 of [2]. Let $\mathcal{V}$ be a collection of sets. Let us say that two members $V$ and $V'$ of $\mathcal{V}$ are equivalent if there exist $V_1, \ldots, V_i$ in $\mathcal{V}$ such that each of the sets

$$V \cap V_1,\ V_1 \cap V_2, \ldots, V_{i-1} \cap V_i, \quad \text{and} \quad V_i \cap V'$$

is non-empty. The resulting equivalence classes are called *components* of $\mathcal{V}$.

LEMMA. *Let $\mathcal{V}$ be a collection of sets such that each member of $\mathcal{V}$ meets just countably many members of $\mathcal{V}$. Then each component of $\mathcal{V}$ is countable.*

THEOREM 3. *If $X$ has a locally countable base, then $X$ is metrizable.*

*Proof.* Let $\mathcal{B}$ be a locally countable base for $X$. For each $x$ in $X$, let $B_x$ in $\mathcal{B}$ contain $x$ and meet just countably many members of $\mathcal{B}$. Let $\mathcal{V} = \{B_x \mid x \in X\}$ and let $\{\mathcal{V}_t \mid t \in T\}$ consist of the components of $\mathcal{V}$. By the lemma we know that each $\mathcal{V}_t$ is countable so we can let $\mathcal{V}_t = \{B(t, m) \mid m \in \boldsymbol{N}\}$; also for each $m$ in $\boldsymbol{N}$ let $\{B(t, m, n) \mid n \in \boldsymbol{N}\}$ consist of the members of $\mathcal{B}$ which $B(t, m)$ meets. For all $t$ in $T$, $m$ in $\boldsymbol{N}$, and $n$ in $\boldsymbol{N}$, let $U(t, m, n) = B(t, m) \cap B(t, m, n)$. Finally, let

$$\mathcal{U} = \bigcup_{m \in \boldsymbol{N}} \bigcup_{n \in \boldsymbol{N}} \{U(t, m, n) \mid t \in T\}.$$

To see that $\mathcal{U}$ is a base for $X$, let $p$ be an element of an open set $W$. Since $B_p$ is in $\mathcal{V}$, let $t_0$ in $T$ and $m_0$ in $\boldsymbol{N}$ be such that $B_p = B(t_0, m_0)$. Since $\mathcal{B}$ is a base for $X$, let $B_0$ in $\mathcal{B}$ be such that $p \in B_0 \subseteq W$. Since $B(t_0, m_0)$ meets $B_0$, let $n_0$ in $\boldsymbol{N}$ be such that $B_0 = B(t_0, m_0, n_0)$. Then $U(t_0, m_0, n_0)$ is a member of $\mathcal{U}$ containing $p$ and included in $W$.

Finally we show that for each $m$ and $n$ the collection $\{U(t, m, n) \mid t \in T\}$ is locally finite. Let $t_0$ be the unique member of $T$ such that $p$ is in $\cup \mathcal{V}_{t_0}$. Then $\cup \mathcal{V}_{t_0}$ is a neighborhood of $p$ which meets only the member $U(t_0, m, n)$ of $\{U(t, m, n) \mid t \in T\}$.

It is easy to give an example of a metric space $M$ which does not have a locally countable base. Since this example is presented in detail elsewhere ([3], Example 140), we only sketch it briefly. The set $M$ is taken to be the plane, and the distance between two points is defined to be the smallest distance one must travel to go from one point to the other, with the restriction that one may travel only along straight lines radiating out from the origin. Let $p$ be the origin, let $\mathcal{B}$ be any base for $M$, and suppose that $U$ were an open neighborhood of $p$ meeting only countably many members of $\mathcal{B}$. Let $n$ in $\boldsymbol{N}$ be such that the circle $C$ of radius $1/n$ with center at $p$ is included in $U$. For each $x$ on (the perimeter of) $C$ let $B_x$ be a member of $\mathcal{B}$ containing $x$ and included in the straight line passing through $x$ and $p$. Then $U$ meets each of the uncountably many members of $\{B_x \mid x \in C\}$, a contradiction.

The notion of a space with a locally countable base is not to be confused with that of a space which is "locally second countable." For let $W$ be the space of ordinals less than the first uncountable ordinal, with the interval topology. Then each point of $W$ has a neighborhood which is second countable, yet $W$ is not metrizable so it could not have a locally countable base.

REMARKS. The "separation axiom" assumed by Urysohn was actually normality, rather than regularity; the next year Tychonoff proved that regular spaces having a countable base are normal. (See Remark 15.7 of [4].) Theorem 3 appears in a more general form as Theorem 1.2 of [1]. The proof of Theorem 3 also shows that if $X$ is connected and has a locally countable base, then $X$ has a countable base.

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## PRIMITIVE ROOTS SATISFYING A COPRIME CONDITION

MIRIAM HAUSMAN

**1. Introduction.** Let $p$ be an odd prime. It has been conjectured [1] that there exists an integer $g \leqq p-1$ which is a primitive root modulo $p$ and which is relatively prime to $p-1$.

In this note we show that this conjecture is true for $p$ sufficiently large. Although the method used here does provide an explicit formula for a lower bound for the number of integers from 1 to $p-1$ which satisfy the above conditions, the value of $p$ from which point on the conjecture is true, is numerically so large as to be inaccessible even by computer means. Thus checking of the conjecture in the finite range will probably require some other methods and may even be more difficult than the infinite segment handled in this note.

**2. Notation and definitions.** A character modulo $m$ is a complex valued function $\chi(n)$, not identically zero, defined for all integers $n$ such that

1. If $(n, m) \neq 1$, then $\chi(n) = 0$.
2. $\chi(n)$ is completely multiplicative, $\chi(n_1 n_2) = \chi(n_1)\chi(n_2)$.
3. If $n_1 \equiv n_2 \pmod{m}$, then $\chi(n_1) = \chi(n_2)$.

The principal character modulo $m$, $\chi_0(n)$ is defined as

$$\chi_0(n) = \begin{cases} 1 & \text{if } (n, m) = 1 \\ 0 & \text{if } (n, m) \neq 1. \end{cases}$$

For any prime $p \geqq 2$, and for every positive $d \mid p-1$ there are exactly $\phi(d)$ characters of order $d$. These will be denoted by $\chi^{(d)} = \{\chi_1, \chi_2, \ldots, \chi_{\phi(d)}\}$, where

$$\chi_i^d = \chi_0 \quad i = 1, \ldots, \phi(d)$$

and are defined as

$$\chi_i(n) = \begin{cases} 0 & \text{if } (n, p) \neq 1 \\ \rho^v & \text{for } n \equiv g^v \pmod{p}, \end{cases} \tag{2.1}$$

where $\rho$ is any one of the $\phi(d)$ primitive $d$th roots of unity and $g$ is any fixed primitive root mod $p$.

**3. Preliminary lemmas and theorems.** The proof of the main theorem relies on the following known results which are stated here without proofs, [2], [3].

LEMMA 3.1. $\Sigma_{\delta|n}|\mu(\delta)| = 2^{\nu(n)}$, *where* $\mu(n)$ *is the Moebius function, and* $\nu(n)$ *denotes the number of distinct prime factors of the integer* $n$.

LEMMA 3.2. *For any* $\varepsilon > 0$, *there exists an* $n_0 = n_0(\varepsilon)$ *such that for all* $n \geqq n_0$

$$d(n) < 2^{(1+\varepsilon)\log n/\log\log n} = n^{(1+\varepsilon)\log 2/\log\log n}.$$

LEMMA 3.3.

$$\underline{\lim}\, \phi(n)\log\log n/n = e^{-\gamma}$$

*where* $\gamma$ *is Euler's constant.*

The following theorem gives us a good estimate for $\Sigma_{n\leqq x}\chi(n)$ when $x$ is small relative to the fixed modulus $p$.

THEOREM 3.1. (Polya's theorem). *For a fixed modulus* $p$ *and* $\chi \neq \chi_0$

$$\left|\sum_{n\leqq x}\chi(n)\right| \leqq \sqrt{p}\log p.$$

In order to test for primitive roots we shall use the following result:

THEOREM 3.2. *For a fixed modulus* $p$, *and* $n$ *an integer,* $0 < n < p$

$$\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\chi(n) = \begin{cases} (p-1)/\phi(p-1) & \text{if } n \text{ is a primitive root modulo } p \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

*Proof*: As in [3], we note that because of the presence of $\mu(d)$, only square free $d$ will appear in (3.1). Using (2.1) it is easily verified that

$$\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\chi(n) = \sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\rho(d)}\rho^{v}$$

$$= \prod_{q|p-1}\left(1+\frac{\mu(q)}{\phi(q)}\sum_{\rho(q)}\rho^{v}\right),$$

where this product is over the prime factors $q$ of $p-1$. If $n$ is a primitive root modulo $p$, $(v, p-1) = 1$ $(v, q) = 1$ and $\Sigma_{\rho(q)}\rho^{v} = -1$.

Thus

$$\prod_{q|p-1}\left(1+\frac{\mu(q)}{\phi(q)}\sum_{\rho(q)}\rho^{v}\right) = \sum_{q|p-1}\left(1+\frac{1}{q-1}\right) = \frac{p-1}{\phi(p-1)}.$$

If $n$ is not a primitive root modulo $p$, it is a $q$th power residue for some $q \mid p-1$, and the corresponding factor

$$1+\frac{\mu(q)}{\phi(q)}\sum_{\rho(q)}\rho^{v} = 0.$$

**4. Main result.**

THEOREM 4.1. *Let* $p$ *be an odd prime. Then for* $p$ *sufficiently large there exists an integer* $g \leqq p-1$ *which is a primitive root modulo* $p$ *and which is relatively prime to* $p-1$.

*Proof.* Using the result of Theorem 3.2, we set up the following function.

$$(4.1)\qquad f(m)=\frac{\phi(p-1)}{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\chi(m)=\begin{cases}1 & \text{if } m \text{ is a primitive root modulo } p\\ 0 & \text{otherwise.}\end{cases}$$

The number $N_p$ of primitive roots modulo $p$ among the integers $1,2,\ldots,p-1$ which are also relatively prime to $p-1$ is

$$N_p=\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1} f(m).$$

The theorem will be true if we show that $N_p>0$ for $p$ sufficiently large.

Substituting (4.1) into the expression for $N_p$ yields

$$(4.2)\qquad \begin{aligned}N_p&=\frac{\phi(p-1)}{p-1}\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\chi(m)\\&=\frac{\phi(p-1)}{p-1}\sum_{d|p-1}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\Bigg(\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1}\chi(m)\Bigg).\end{aligned}$$

Breaking off the term when $d=1$, (4.2) becomes

$$N_p=\frac{\phi^2(p-1)}{p-1}+\frac{\phi(p-1)}{p-1}\sum_{\substack{d|p-1\\d>1}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\Bigg(\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1}\chi(m)\Bigg).$$

Let

$$(4.3)\qquad E_p=\sum_{\substack{d|p-1\\d>1}}\frac{\mu(d)}{\phi(d)}\sum_{\chi\in\chi^{(d)}}\Bigg(\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1}\chi(m)\Bigg).$$

The inner sum of $E_p$ is estimated as follows where for a fixed $d$, let $\chi\neq\chi_0$

$$\begin{aligned}\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1}\chi(m)&=\sum_{m=1}^{p-1}\chi(m)\sum_{\delta|(m,p-1)}\mu(\delta)=\sum_{\delta|p-1}\mu(\delta)\sum_{\substack{m=1\\m\equiv 0(\mathrm{mod}\,\delta)}}^{p-1}\chi(m)\\&=\sum_{\delta|p-1}\mu(\delta)\chi(\delta)\sum_{t=1}^{(p-1)/\delta}\chi(t)\end{aligned}$$

since $m\equiv 0 \pmod{\delta}$ is equivalent to $m=\delta t$. Thus

$$(4.4)\qquad \Bigg|\sum_{\substack{m=1\\(m,p-1)=1}}^{p-1}\chi(m)\Bigg|\leqq\sum_{\delta|p-1}|\mu(\delta)|\Bigg|\sum_{t=1}^{(p-1)/\delta}\chi(t)\Bigg|\leqq 2^{\nu(p-1)}\sqrt{p}\log p$$

by Lemma 3.1 and Theorem 3.1.

Inserting (4.4) in the expression (4.3)

$$\begin{aligned}|E_p|&\leqq\sum_{\substack{d|p-1\\d>1}}\frac{|\mu(d)|}{\phi(d)}\sum_{\chi^{(d)}}2^{\nu(p-1)}\sqrt{p}\log p\\&\leqq 2^{\nu(p-1)}\sqrt{p}\log p\sum_{\substack{d|p-1\\d>1}}\frac{|\mu(d)|}{\phi(d)}\sum_{\chi^{(d)}}1\\&\leqq 4^{\nu(p-1)}\sqrt{p}\log p.\end{aligned}$$

The following lower bound is thus obtained for $N_p$,

$$N_p \geqq \frac{\phi(p-1)}{p-1}(\phi(p-1) - 4^{\nu(p-1)}\sqrt{p}\log p)$$

and $N_p \geqq 0$ if

$$\phi(p-1) > 4^{\nu(p-1)}\sqrt{p}\log p. \tag{4.5}$$

The results of Lemmas 3.2 and 3.3 say that for $n$ sufficiently large there exist constants $c_1$ and $c_2$ such that

$$\phi(n) > c_1 \frac{n}{\log\log n}$$

and, since $2^{\nu(n)} \leqq d(n)$,

$$2^{\nu(n)} < 2^{c_2 \log n/\log\log n}$$

In order to prove that (4.5) is true for $p$ sufficiently large, it is thus sufficient to show that

$$c_1 \frac{p-1}{\log\log p - 1} > 4^{c_2 \log(p-1)/\log\log(p-1)}\sqrt{p}\log p$$

or

$$c_1 \frac{p-1}{\log\log p - 1} > (p-1)^{c_2 \log 4/\log\log(p-1)}\sqrt{p}\log p,$$

which is clearly true for $p$ large.

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## PROBABILISTIC GENERALIZATION OF A NUMBER-THEORETICAL INEQUALITY

I. Z. RUZSA

**1.** Let $a_1, \ldots, a_m$ be natural numbers, and let

$$T(a_1, \ldots, a_m)$$

denote the density of natural numbers, divisible by no $a_i$. (The density of a sequence $A$ is defined as

$$\lim_{x\to\infty} \frac{A(x)}{x},$$

where $A(x)$ denotes the number of elements of $A$ which are $\leqq x$.) This sequence has a density, since it is periodic with the period $M = [a_1, \ldots, a_m]$. Using the sieve-formula we get

$$T(a_1, \ldots, a_m) = 1 - \sum \frac{1}{a_i} + \sum_{i<j} \frac{1}{[a_i, a_j]} - \cdots + (-1)^m \frac{1}{[a_1, \ldots, a_m]}.$$

F. A. Behrend [1] proved

THEOREM 1. *For all natural numbers* $a_1, \ldots, a_m, b_1, \ldots, b_n$ *we have*

$$T(a_1, \ldots, a_m, b_1, \ldots, b_n) \geqq T(a_1, \ldots, a_m)\, T(b_1, \ldots, b_n).$$

This is a very natural statement: it means that the event

"$x$ is divisible by no $a_i$"

becomes more (or at least not less) probable under the condition

"$x$ is divisible by no $b_j$,"

but as far as I know no natural proof has been published. — For another proof, due to C. A. Rogers, see H. Halberstam and K. F. Roth [2], chapter V.

**2.** First we give a probabilistic interpretation. Let the probability space consist of the residue classes modulo

$$N = [a_1, \ldots, a_m\ b_1, \ldots, b_n],$$

each class having probability $1/N$. Let

$$N = p_1^{t_1} \cdots p_k^{t_k},$$

and if $w$ is a residue class, let $\xi_i(w)$ denote the (common) exponent of $p_i$ in the numbers $(a, N)$, $a \in w$. The $\xi_i$'s are obviously independent. They do not determine the residue class uniquely, but they do determine whether the elements of $w$ are or are not divisible by an $a_i$ or a $b_j$.

We can express the event "the numbers in $w$ are divisible by no $a_i$" as

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k) \in A,$$

where $x = (x_1, \ldots, x_k) \in A$ if and only if

$$p_1^{[x_1]} \cdots p_k^{[x_k]}$$

is divisible by no $a_i$.

$A$ is a *cut*, that is, $x \in A$, $y \leqq x \Rightarrow y \in A$, where $x \leqq y$ means $x_1 \leqq y_1, \ldots, x_k \leqq y_k$.

Similarly we can define another cut $B$ with the aid of the $b_j$'s. A natural way to get Theorem 1 is to prove the following generalization.

THEOREM 2. *Let* $A$, $B \subset R^k$ *be cuts and* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_k)$ *a random variable with independent components. Then*

$$P(\boldsymbol{\xi} \in A \cap B) \geqq P(\boldsymbol{\xi} \in A)\, P(\boldsymbol{\xi} \in B).$$

The idea of Theorem 2 is due to D. J. Kleitman [3], who proved a special case (in a rather different form).

**3.** Denote the distribution function of $\xi_i$ by $D_i$. Theorem 2 can be formulated as

$$\int \chi_A \chi_B\, dD_1(x_1) \cdots dD_k(x_k) \geqq \left(\int \chi_A\right) \left(\int \chi_B\right),$$

where $\chi_A$, $\chi_B$ are the characteristic functions of the sets $A$ and $B$.

Call a function $f: R^k \to R$ *decreasing*, if

$$x \leqq y \Rightarrow f(x) \geqq f(y).$$

Since $A$ and $B$ are cuts, their characteristic functions are decreasing. Theorem 2 is easier to prove in the following, little more general form.

THEOREM 2*. *If $f, g: R^k \to R$ are both decreasing and the $D_i$'s are distribution functions, then*

$$\int fg\, dD_1(x_1) \cdots dD_k(x_k) \geqq \left(\int f\right)\left(\int g\right),$$

*if the integrals are absolutely convergent.*

In terms of probability this means

$$E(f(\xi)g(\xi)) \geqq E(f(\xi))E(g(\xi)),$$

where $E$ denotes expectation.

**4.** Now we prove our Theorem 2* by induction on $k$. In case $k = 1$, since both $f$ and $g$ are decreasing, we have

$$h(x, y) = (f(x) - f(y))(g(x) - g(y)) \geqq 0$$

for all $x, y$. (Observe that this is not true for $k \geqq 2$.) Therefore

$$\int h(x, y)\, dD(x)\, dD(y) \geqq 0;$$

after an easy computation we get the required result.

Suppose we know the result for $k - 1$. Let

$$F(x_k) = \int f dD_1(x_1) \cdots dD_{k-1}(x_{k-1}),$$

$$G(x_k) = \int g\, dD_1(x_1) \cdots dD_{k-1}(x_{k-1}).$$

By the induction hypothesis we have

$$\int fg\, dD_1(x_1) \cdots dD_{k-1}(x_{k-1}) \geqq F(x_k)\, G(x_k).$$

On the other hand, also $F$ and $G$ are decreasing. Hence

$$\begin{aligned}\int fg\, dD_1(x_1) \cdots dD_k(x_k) &\geqq \int F(x_k)\, G(x_k)\, dD_k(x_k) \geqq \int F(x_k)\, dD_k(x_k) \int G(x_k)\, dD_k(x_k) \\ &= \int f dD_1(x_1) \cdots dD_k(x_k) \int g\, dD_1(x_1) \cdots dD_k(x_k),\end{aligned}$$

and the proof is completed.

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# RESEARCH PROBLEMS

EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)*

## SOME COVERING AND PACKING PROBLEMS

H. GROEMER

In $n$-dimensional euclidean space $R^n$ let $\mathbf{K}=\{K_i\}$ be a countable class of closed convex sets $K_1, K_2, \cdots$. The diameter of $K_i$ will be denoted by $d(K_i)$ and its ($n$-dimensional) volume by $v(K_i)$. The density of $\mathbf{K}$ is defined by

$$\lim_{r\to\infty} \frac{1}{v(S_r)} \sum_{i=1}^{\infty} v(K_i \cap S_r),$$

where $S_r$ is a solid sphere in $R^n$ centered at the origin of the coordinate system and of radius $r$. We say that $\mathbf{K}$ permits a covering of $R^n$ (or packing in $R^n$) if there exist rigid motions $\sigma_i$ so that $R^n \subset \bigcup_{i=1}^{\infty} \sigma_i K_i$ (or $\operatorname{int} \sigma_i K_i \cap \operatorname{int} \sigma_j K_j = \emptyset$ for $i \neq j$). If the set of diameters $\{d(K_i)\}$ is bounded this will be conveniently expressed by saying that $\mathbf{K}$ is bounded.

In answer to a question of G. D. Chakerian [1] it has been shown in [3] that any such bounded $\mathbf{K}$ permits a covering of $R^n$ if and only if $\Sigma_{i=1}^{\infty} v(K_i) = \infty$. Moreover, if this condition is satisfied, it is shown that $\mathbf{K}$ permits such a covering of density at most $2 \cdot 3^{n-1} n^n$ (i.e., the resulting covering $\{\sigma_i K_i\}$ has this density).

This result suggests immediately the following two problems.

(a) *What is the smallest constant $c_n$ such that every bounded class $\{K_i\}$ with $\Sigma_{i=1}^{\infty} v(K_i) = \infty$ permits coverings of $R^n$ of densities arbitrarily close to $c_n$?*

(b) *What is the largest constant $k_n$ such that every bounded class $\{K_i\}$ with $\Sigma\, v(K_i) = \infty$ permits packings in $R^n$ of densities arbitrarily close to $k_n$?*

The result cited before shows that

$$c_n \leqq 2 \cdot 3^{n-1} n^n,$$

and it follows from the work of Kosiński [4] on the "potato bag problem" that

$$k_n \geqq 1/3^{n-1} n!.$$

The constants $2 \cdot 3^{n-1} n^n$ and $1/3^{n-1} n!$ are probably very far from the true values of $c_n$ and $k_n$, even if $n = 2$. The proofs of these estimates are obtained by considering first the analogous problems for boxes (i.e., isometric images of intervals $\{(x_1, x_2, \ldots, x_n)\colon a_i \leqq x_i \leqq b_i\}$). If $c_n'$ and $k_n'$ are defined similarly to $c_n$ and $k_n$, but with the restriction to classes of boxes, one finds that $c_n' \leqq 2 \cdot 3^{n-1}$ and $d_n' \geqq 1/3^{n-1}$. Again, these estimates are probably very crude. In fact, it might be possible that $c_n' = 1$ and $d_n' = 1$. In other words, the following question may have a positive answer.

*Does every bounded class of boxes whose total volume is infinite permit a covering of $R^n$ (or a packing in $R^n$) of density* 1?

Under the additional assumption that the volumes of the boxes are bounded away from 0 it has been proved in [3] that every such class does indeed permit coverings of density 1, and the same method of proof can be used to establish the corresponding result for packings.

Another variation of the above problems (a) and (b) concerns coverings and packings with regard to a convex subset $C$ of $R^n$, rather than the whole of $R^n$. For example, the covering problem in the case when $C$ is a unit cube and the diameters $d(K_i)$ are not greater than 1 is treated in [3]. For this kind of problem it is typical that conditions involving only the volumes of $C$ and of the sets $K_i$ are not sufficient. However, one can formulate the analogous problem where $C$ is replaced by a complete $n$-dimensional spherical surface (with obvious changes in the pertinent definitions). In other words, we may ask the following question:

*Let $\{K_i\}$ be a class of spherically convex subsets on the $n$-dimensional unit sphere $S^n$. What is the smallest constant $g_n$ with the property that $\{K_i\}$ permits a covering of $S^n$ whenever $\Sigma v(K_i) \geqq g_n$? Similarly, what is the largest constant $h_n$ such that $\{K_i\}$ permits a packing in $S^n$ whenever $\Sigma v(K_i) \leqq h_n$?*

Finally, we mention that the original problems (a) and (b) become much more complicated if the class $\{K_i\}$ is not bounded, even if each individual $K_i$ is still assumed to be bounded. For example, the condition $\Sigma v(K_i) = \infty$ is no longer sufficient in order that $\{K_i\}$ permits a covering of $R^n$, and satisfactory necessary and sufficient conditions are only known for $n = 2$ (see [1] and [2] for further discussions on this matter). Even more complicated is the situation when the sets $K_i$ themselves are no longer assumed to be bounded. Already for the plane $R^2$ we do not know of any non-trivial characterizations of those classes of closed (and possibly unbounded) convex sets that permit coverings of $R^2$ or packings in $R^2$.

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## CLASSROOM NOTES

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### THE DERIVATIVE RECONSIDERED

LEON W. COHEN

Consider a function $f$ and an increasing function $g$ on $\mathbb{R}$. Define

$$(df/dg)_x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)}. \tag{1}$$

If $g(x) = x$ this is the classical definition. Recall Lebesgue's theorem stating that if $f$ is an increasing function, $df/dx$ is finite almost everywhere. Let the $g$-length of an interval $(a, b)$ be $g(b) - g(a)$ and let $\mu_g$ be the measure generated by the $g$-lengths of intervals. It is reasonable to ask if, for an increasing function $f$, $df/dg$ is finite almost everywhere in the measure $\mu_g$. The answer is "no."

Example: Let

$$f(x) = -1 \quad \text{for} \quad x < 0, \quad = 0 \quad \text{for} \quad x = 0, \quad = 1 \quad \text{for} \quad x > 0,$$

$$g(x) = -1 + x \quad \text{for} \quad x < 0, \quad = 1 + x \quad \text{for} \quad x \geqq 0.$$

Then $f$ and $g$ are increasing functions on $R$,

$$\lim_{h \downarrow 0} \frac{f(h) - f(0)}{g(h) - g(0)} = \lim_{h \downarrow 0} \frac{1}{h} = +\infty, \quad \lim_{h \uparrow 0} \frac{f(h) - f(0)}{g(h) - g(0)} = \frac{1}{2}.$$

Hence $(df/dg)_0$ does not exist. Since the $g$-length of $(-h, h)$ is $g(h) - g(-h) = 2 + 2h$, $\mu_g(\{0\}) = 2$. Thus the generalized Lebesgue theorem does not hold under (1).

But interest in the derivative arose from its interpretation as instantaneous velocity. From this point of view definition (1) has a defect. It requires that the position $f(x)$ be known precisely at time $x$, which is scarcely possible experimentally. It is more likely that positions at times $x'$ somewhat earlier than $x$ and at times $x''$ somewhat later than $x$ can be observed. Then the average velocities

$$\frac{f(x'') - f(x')}{x'' - x'}$$

can be computed and their behavior examined for small time intervals $(x', x'')$. This suggests the definition

$$(df/dg)_x = a$$

if for $\varepsilon > 0$ there is some $\delta > 0$ such that

$$\left| \frac{f(x'') - f(x')}{g(x'') - g(x')} - a \right| < \varepsilon \quad \text{if} \quad x - \delta < x' < x < x'' < x + \delta. \tag{2}$$

For $g(x) = x$ we have the velocity situation. If $(df/dg)x$ is finite under definition (1), it has the same value under (2). In the above example under definition (2)

$$(df/dg)_0 = \frac{f(0+) - f(0-)}{g(0+) - g(0-)} = 1, \quad (df/dg)_x = 0 \quad \text{for} \quad x \neq 0.$$

Now the generalized Lebesgue theorem holds under (2). Further, the fundamental theorem of the calculus holds in the following form. If $f$ is an increasing function which is absolutely continuous relative to an increasing function $g$, then

$$f(b+) - f(a-) = \int_a^b (df/dg)_x d\mu_g.$$

It may be noted that this fills a gap left by the Radon-Nikodym theorem. The full story, which is long, will appear elsewhere.

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## ON A CHARACTERIZATION OF COMPACTNESS FOR $T_1$ SPACES

JAMES E. JOSEPH

If $X$ and $Y$ are topological spaces, a function $g: X \to Y$ has a *closed graph* if the graph of $g$ is a closed subset of the product space $X \times Y$. It is known that if a space $Y$ is compact, functions with closed graphs into $Y$ are continuous (see [1], p. 228). Not so well known (see [2], Corollary 2, and [5],

p. 435) is that a converse is true for $T_1$ spaces, $Y$, i.e., if $Y$ is $T_1$ and all functions with closed graphs into $Y$ are continuous, then $Y$ is compact. In this paper, a short proof of this result is given which makes use of the notion of a complete accumulation point.

A point $x$ is a *complete accumulation point* of a subset $K$ of a topological space iff for each open set $V$ about $x$ the sets $K$ and $V \cap K$ have the same cardinality ($|K| = |V \cap K|$). It is known that a space is compact iff every infinite subset has a complete accumulation point (see [3], p. 163).

THEOREM. *If every function with a closed graph into a $T_1$ space $Y$ is continuous, the space $Y$ is compact.*

*Proof.* Let $Y$ be a $T_1$ space and let $X$ be an infinite subset of $Y$ without a complete accumulation point. Fix $x_0 \in X$ and let $\mathcal{T} = \{A \subset X: x_0 \in X - A \text{ or } |X - A| < |X|\}$. Using the facts for cardinals that $|\varnothing| < |X|$, $|A| \leqq |B|$ when $A \subset B$, and $|A \cup B| < |C|$ if $|A| < |C|$ and $|B| < |C|$ when $C$ is infinite, it is readily checked that $\mathcal{T}$ is a topology on $X$. Let $g$ be the identity function from $X$ into $Y$. To establish the theorem by contradiction, we shall show that $g$ has a closed graph and is not continuous. To see that $g$ is not continuous, let $V$ be open in $Y$ about $x_0$ such that $|V \cap X| < |X|$. Then $x_0 \in g^{-1}(V)$ and $|X - g^{-1}(V)| = |X|$, so $g^{-1}(V) \notin \mathcal{T}$. The function $g$ has a closed graph; let $D = \{(x, x): x \in X\}$ be the graph. We must prove that if $x \neq y$ and $(x, y) \in X \times Y$, then $(x, y)$ does not belong to the closure of $D$. Since $Y$ is $T_1$, if $x \neq x_0$, $\{x\} \times (Y - \{x\})$ is an open set about $(x, y)$ in $X \times Y$ not intersecting $D$. If $x = x_0$, there is an open set $V$ about $y$ in $Y$ with $|V \cap X| < |X|$; since $Y$ is $T_1$, we may choose $V$ such that $x \in X - V$. $(X - V) \times (V - \{x\})$ is then an open set about $(x, y)$ in $X \times Y$ not intersecting $D$. This completes the proof.

That some restriction on the space $Y$ in the theorem is necessary is to be expected since (see [4], p. 198) if $Y$ is the image of a function with a closed graph, then $Y$ is $T_1$. We leave it to the interested reader to construct an example of a non-compact topological space having the property that every function into the space with a closed graph is continuous.

The author thanks the referees of this paper for their valuable suggestions.

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## A NOTE ON THE ERDÖS-FELLER-POLLARD THEOREM

MARK A. PINSKY

In the main limit theorem for countable Markov chains [1] the convergence proof depends on the following theorem:

*Let $\{u_n\}_{n \geqq 0}$, $\{f_n\}_{n \geqq 1}$ be sequences of real numbers subject to*

$$f_n \geqq 0, \sum_1^\infty f_n = 1, \sum_1^\infty n f_n = m < \infty; \tag{1}$$

$$u_0 = 1, \; u_n = f_n + f_{n-1} u_1 + \cdots + f_1 u_{n-1}, \; (n \geqq 1); \tag{2}$$

$$\text{the g.c.d. of } \{n: f_n > 0\} \text{ is } 1. \tag{3}$$

*Then* $\lim_{n \to \infty} u_n = 1/m$.

In view of this fact, we can write

$$u_n - 1/m = \frac{1}{2\pi}\int_{-\pi}^{\pi} [1/(1 - F(e^{i\theta}) - 1/(m(1 - e^{i\theta}))]e^{-in\theta}d\theta, \tag{10}$$

where the integrand is an integrable function on $[-\pi, \pi]$. We have written $u_n - 1/m$ as the $n$th Fourier coefficient of this function. But according to the classical Riemann-Lebesgue lemma, the Fourier coefficients of an integrable function tend to 0. This completes the proof of the theorem.

Professor S. Wainger has remarked that the condition (4) implies the stronger result: $\Sigma_1^\infty 1/n \mid u_n - 1/m \mid < \infty$. This follows from a theorem of Hardy and Littlewood [4, p. 286, Theorem 8.7].

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## MATHEMATICAL EDUCATION

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### HAND-HELD CALCULATORS IN THE FRESHMAN MATHEMATICS CLASSROOM

JOAN LEITZEL AND BERT WAITS

At The Ohio State University we are faced with a situation shared by many colleges and universities. We are an "open door" university with the consequence that this past autumn quarter (1974) we faced approximately 4500 students who were not prepared to begin our precalculus courses. In fact, there has been about a 100% increase in our remedial enrollment in the past 10 years, even though our math placement exams and beginning course placement algorithms have not changed during that period [1].

The content of our remedial course is a standard review of elementary and intermediate high school algebra and coordinate geometry. It is offered in both a regular pace, one-quarter, or reduced pace, two-quarter, format. In past years we used a television lecture technique to provide basic instruction for our immense remedial audience. However, this past autumn quarter, we were able to teach our regular pace remedial group (a 5-credit-hour course of about 2600 students) using experienced mathematics faculty presenting live lectures three times per week to groups of about 300 students. Small recitation classes, meeting twice each week, were conducted by graduate student teaching assistants.

The enrollment in our remedial course includes typically a large number of students from diverse backgrounds, with equally diverse abilities, with poor attitudes toward the study of mathematics, with poor study habits and, to a large extent, poor academic motivation. A majority of the students who

place in our regular pace remedial course have been exposed to two or three years of high school algebra and geometry. We find that a one-quarter review course is sufficient to prepare most of them for a standard precalculus course.

The persons in our department who lectured and coordinated our regular pace course autumn quarter were Harry Allen, Franklin Demana, Judith Elkins, Joan Leitzel and Robert Wright. These teachers had certain commitments: the course should not be merely a rerun of the typical high school course where most of these students had, in reality, only limited success; the course should be based on problem solving and, to a large extent, on computation and manipulation; problems should be inherently interesting and challenging. Consideration of requiring a hand-held calculator for this course arose naturally in the hope that it could be used as an effective pedagogical device to gain (and hold) student interest, to relieve the teacher of teaching arithmetic, and to permit the confronting of more realistic problems.

The immediate problem was to find a high-quality calculator in the \$15.00 to \$20.00 range. After spending a few weeks considering several calculators, we made arrangements with Texas Instruments Corporation to manufacture, for our experimental use, the Exactra 19 calculator, a four-function, six-digit, floating-decimal, algebraic-entry type calculator with a built-in, repeat-operation capability. These calculators were sold to our students through local bookstores at a list price of \$16.30, excluding batteries. The purchase of a calculator was required of the student in the same sense that the purchase of a textbook was required. It was used by the student for classwork and homework and during all quizzes and examinations given during the quarter.

We found no text exactly suitable for our needs. The text finally adopted, although well-written and readable, did not focus on computation and did not provide sufficient problems appropriate for the calculator. Each regular pace lecturer wrote weekly supplementary problem sets designed to investigate topics such as area, radicals, rules of exponents, graphing, limits, and slope through a basically computational approach. The problem sets were designed to encourage students to explore and experiment.

Our autumn quarter efforts indicate considerable promise. The calculator provides a sound method for empirical experimentation in approaching written problems; indeed, many students understand for the first time what these problems are asking. The dependence of area and volume on length of side can be vividly demonstrated, as can other functional relationships. The meaning of the graph of a function can be understood by plotting a very large number of points. Students can be encouraged to approximate solutions to equations and inequalities and to check exact solutions to equations. The notion of limit can be first experienced in a computational way.

We found that in using calculators students raised questions about arithmetic properties of numbers that would have been of little interest to them otherwise. The student who did not understand the mathematics underlying his computations could not use his calculator successfully; moreover, the desire to use the calculator seemed often to motivate this understanding. The question of whether a person who uses a hand-held calculator to do computations is somehow less educated than a person who does computations mentally we will leave for others to decide.

Student attendance was good throughout the quarter and student attitudes remained positive regarding the use of the calculator. A summary of autumn quarter student evaluation results for both our reduced pace and regular pace courses is given in the accompanying chart. About 2250 students completed and returned the evaluation form. We have evidence that suggests the group this year using calculators performed equivalently to the group last year on problems testing basic algebraic skills. In addition, the calculator group did very well on problems requiring some understanding and putting together of ideas.

For spring quarter, we are decreasing the lecture time and increasing the recitation time in the course. We are also reducing the size of the lecture sections to about 150 students. The recitation time will be used primarily for supervised problem-solving in a laboratory spirit. In future quarters, we shall move to a comparably priced, 8-digit calculator. We are also naturally exploring the routine use of

| Statement | Percent Response: Strongly Agree | Agree | Disagree | Strongly Disagree |
|---|---|---|---|---|
| 1. Having a calculator was helpful in this course. | 56 | 35 | 6 | 3 |
| 2. I am glad I took this course before taking a higher level math course. | 56 | 33 | 7 | 4 |
| 3. My present attitude toward mathematics is favorable. | 15 | 44 | 24 | 17 |
| 4. This course improved my attitude toward mathematics. | 10 | 36 | 33 | 21 |

calculators in the follow-up precalculus courses. Our hope is that soon the prices of full feature scientific calculators (transcendental functions, $x^y$, etc.) will be within the reach of most students with majors in mathematics, engineering and the sciences. At such time we believe the traditional precalculus curriculum will need to be significantly revised.

We cannot suggest that the calculator will solve all of the difficult instructional problems in the teaching of mathematics at the beginning college level. However, it does seem to have considerable potential for alleviating some problems inherent in remedial courses.

**Added in proof:** In 1975–76 we taught the regular pace sections of the course described above with two lecture meetings (150 students each) and three problem sessions (25 students each) a week. We will continue this format in autumn quarter, 1976, and also introduce a lower level course for those students who formerly have been placed into the reduced pace sections. This new course will also require calculators. Fortunately the present state of technology permits students to purchase suitable four-function calculators (capable of handling the arithmetic of signed numbers) for less than $10.00.

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## PROBABILITY AND STATISTICS: EXPERIMENTAL RESULTS OF A RADICALLY DIFFERENT TEACHING METHOD

Julian L. Simon, David T. Atkinson and Carolyn Shevokas

**Introduction.** With the Monte Carlo method, students from high school to graduate school can quickly acquire the ability to handle probabilistic problems of daily living or scientific research. And the students understand what they are doing, with little danger of the formula-grabbing which too often afflicts conventional analytic methods.

The Illinois procedure for teaching the Monte Carlo method has been used since 1965 with success: (a) for teaching research methods to graduate students in several fields at the University of Illinois who have already had one or several conventional statistics courses, but who nevertheless find themselves insufficiently equipped to handle the statistical problems in research projects; (b) with undergraduates in research methods courses; (c) with undergraduates as part of conventional statistics courses; and (d) with high school students down to age 13 or 14, in the U. S. (Simon and Holmes, 1969), in Israel and in Puerto Rico.

The results seem successful to the teachers and to the students, as evidenced by the teachers' judgments and the students' answers to informal questionnaires. But such "soft" evidence is

insufficient to convince skeptics — which is perhaps as it ought to be. Harder evidence is therefore needed. To supply that evidence is the task of this paper.

We first recapitulate the method and its logic. Then we describe three experiments that test the value of the method in a variety of class settings.

The Monte Carlo method is not offered as a successor to analytic methods. Rather, it can be an underpinning for analytic teaching to help students understand analytic methods better. But it is also a workable and easily-taught alternative for students who will never study conventional analytic methods to the point of practical mastery — and this includes most students at all educational levels. It may be especially useful for the introduction to statistics of mathematically-disadvantaged students. (But please do not infer from this that the method is intellectually inferior; the method is logically acceptable and intuitively instructive for all students.)

It must be emphasized that the Monte Carlo method as described here really is intended as an alternative to conventional analytic methods in actual problem-solving practice. This method is *not* a pedagogical device for improving the teaching of conventional methods. This is quite different than the past use of the Monte Carlo method to help teach sampling theory, the binomial theorem and the central limit theorem. The point that is usually hardest to convey to teachers of statistics is that the method suggested here really is a complete break with conventional thinking, rather than a supplement to it or an aid in teaching it. That is, the simple Monte Carlo method described here is complete in itself for handling most — perhaps all — problems in probability and statistics.

The Monte Carlo method always provides a logically acceptable solution. But more specifically with respect to statistical hypothesis testing, the Monte Carlo tests based on a randomization logic have properties that statisticians are now finding attractive because they are more robust than traditional parametric tests. (For the test of differences in means between two groups based on Fisher's randomization test, see Dwass 1957, and Chung and Fraser, 1958; for a variety of other tests see Simon, 1969, Chapters 23 and 24.) Hence the Monte Carlo test is often a better scientific choice than the conventional test — in addition to its pedagogical advantages.

To illustrate the method, here is a sample question and examination answer by a high school student (one who qualified for an experimental course) after just six hours of classroom instruction:

> John tells you that with his old method of shooting foul shots in basketball his average (over a long period of time) was .6. Then he tried a new method of shooting and scored successes with nine of his first ten shots. Should he conclude that the new method is really better than the old method?
>
> *Student A.S.*:
> (a) Take twelve hearts to represents hits in shooting and eight spades to be misses; this is John's old probability in scoring.
> (b) Shuffle, draw a card and record "hit" or "miss", replace it and shuffle.
> (c) Repeat ten times altogether for one trial.
> (d) See how many times nine hits or more come up on ten tries.
>
> 1. Hit, hit, miss, miss, h, h, h, h, h, m, 7/10 hits
> 2. 7/10
> 3. 8/10
> 4. 4/10
> 5. 6/10
> 6. 7/10
> 7. 7/10
> 8. 7/10
> 9. 6/10
> 10. 9/10
> 11. 5/10
> 12. 7/10
> 13. 8/10
> 14. 6/10
> 15. 8/10
>
> Only 1 time in 15 times will 9/10 shots be made by the old .6 chance, so it seems probable here that John's new method helped.

The Monte Carlo method is not explained by the instructor. Rather it is discovered by the students. With a bit of guidance the students invent, from scratch, the procedures for solution. For example, at the beginning of the first class the instructor may ask, "What are the chances that if you

have four children three of them will be girls?" A few students do some calculations without success (in a naive class); the rest fidget. Then the students say that they don't know how to get the answer. The instructor presses the class to think of *some* way to come up with an answer. Someone suggests in jest that everyone in the class should go out and have four children. The instructor chooses to take this seriously. He says that this is a very good suggestion, though it has some obvious drawbacks. Someone suggests substituting a coin for a birth. This raises the issue of whether it is reasonable to approximate a 106:100 event with a 50–50 coin, and what is reasonable under various conditions. The instructor points out that the class still has no answer. Someone suggests that each student throw four coins. Someone else amends this by saying that four flips of one coin are just as good. The instructor questions whether the two methods are equivalent, and the class eventually agrees that they are. Finally, each student performs a trial, the data are collected, and an estimate is made. Someone wonders how good the estimate is. Someone else suggests that the experiment be conducted several more times to see how much variation there is.

The meaning of the concept "chances" comes up in the discussion, and "probability" is defined pragmatically. By this process of self-discovery, students develop useful operating definitions of other necessary concepts such as "universe," "trial," "estimate," and so on. And together they invent — after false starts and class corrections — sound approaches to easy and not-so-easy problems in probability and statistics. For example, with a bit of guidance, an average university class can be brought to re-invent such devices as a Monte Carlo version of Fisher's randomization test. In an earlier report (Simon, 1969, Chapters 23 and 24) the flexibility and range of the Monte Carlo method is shown in problems ranging from permutations to correlation to randomization tests.

In this manner the students learn more than how to do problems. They gain the excitement of true intellectual discovery. And they come to understand something of the nature of mathematics and its creation.

Though the experience of shuffling cards and counting tabled random numbers is educational at first, it tends to be a nuisance after awhile, and a deterrent to the use of the method. Furthermore, in some problems, the sample size required for the desired accuracy makes such hand methods onerous if not impossible. Therefore, a computer program, SIMPLE, has been developed that will perform the necessary operations rapidly and yet can be used immediately by a person with absolutely no computer experience. The SIMPLE program is also designed to be used as the method of choice for sophisticated statisticians in many sorts of applications. This program is described in Simon and Weidenfeld (1974), and materials are available upon request.

A *systematic* Monte Carlo method is taught at the University of Illinois; this is an important difference from some examples in the literature of *ad hoc* Monte Carlo problem solution, e.g. Zelinka, 1973. The student is taught to work in a series of discrete steps. The first step is the construction of the universe whose behavior one is interested in. The second step (or set of steps) is the drawing of a sample from that universe. The third step is the computation of the statistic of interest, and, in inferential statistics, comparison of the experimental statistic to the "observed" or "bench-mark" statistic. The fourth step is the repetition of the sampling procedure a large number of times. And then the fifth step is the calculation of the proportion of "successes" to experimental trials, which estimates the probability of the event in which one is interested.

**The experiments.** Three controlled experimental tests of the pedagogical efficiency of the Monte Carlo method have now been completed.

*The University of Illinois Experiment*: The experimental situation was a one-semester elementary statistics class of 25 mostly economics and business undergraduate majors in 1973 at the University of Illinois. The course, taught by Simon, was primarily a conventional statistics course, using a conventional text (Spurr and Bonini, rev. ed., 1973); the Monte Carlo method was taught only as a supplement, with no reading on it other than the simulation chapter in Spurr and Bonini and the Zelinka article (1973) and suggested reading in Simon (1969, Chapters 23–25). All problems that were

treated by the Monte Carlo method in class were also demonstrated by analytic methods, whereas many problems were solved by analytic methods that were *not* treated in class by the Monte Carlo method. Therefore, analytic methods had a very large advantage over the Monte Carlo method in student time and attention, both in reading and in class.

Among the ten questions on the final exam (of which the student had to answer 8), there were four that the student could *choose* whether to answer by analytic methods or by Monte Carlo; the question given earlier is an example of these four questions. The choices of method by the students on the optional-method question give an indication of the usefulness of the Monte Carlo method.

Some additional conditions relevant to the experiment: The students could bring books and notes. (A closed-book exam, where formulae had to be remembered, would disadvantage analytic methods relative to Monte Carlo methods.) And the four optional-method problems were extremely simple ones for the use of analytic methods. (Complex problems would tend to improve the relative performance of the Monte Carlo method, because complexity is its comparative advantage.)

The results were as follows:

1. Almost every student used the Monte Carlo method for some question. This is the most exciting result of the experiment, because it suggests that the method has *some* usefulness to *almost everyone.* In total, more than half of the answers used the Monte Carlo method (44 of 86).

2. Almost every student did *some* questions by analytic methods. This implies that teaching the Monte Carlo method does not prevent the learning and use of analytic methods; that is, Monte Carlo does not drive out analytic methods. This is also a gratifying result.

3. There is a slightly-greater propensity for students who did better on the examination as a whole to do a larger proportion of problems by the Monte Carlo method. But the relationship is certainly not strong, which suggests that the Monte Carlo method is useful both to the good and to the less-good students. (And the lack of strong relationship also implies that we need not worry that the students who got better scores on the exam did so *because* they used the Monte Carlo more extensively and were therefore graded more easily.)

4. On *each question* some students used analytic methods and others used Monte Carlo methods. This shows that the Monte Carlo method is not specialized to some sorts of problems in the minds and practices of the students.

5. The average grades that the students received were higher on the questions answered with the Monte Carlo method than on those questions answered with analytic methods — 9.1 versus 7.5 on a scale of 10.

*Polk Community College Experiment*: At Polk Community College, Winter Haven, Florida in 1974, Shevokas taught separate classes of General Mathematics, a $6\frac{1}{2}$ week 17 class-hours unit in probability and statistics, in three ways: conventional analytic method, Monte Carlo method with computer, Monte Carlo method without computer. The enrollments were 19, 39, and 13, respectively. Beforehand, the groups were given a cooperative Arithmetic Achievement Test (Educational Testing Service, 1962) and two attitude-toward-mathematics tests (Aiken and Dreger, 1961; McCallon and Brown, 1971; sample item: "The feeling I have toward math is a good feeling"). The differences in results among groups were not statistically significant, so we can safely consider that the groups were similar to start with.

Only a mini-computer was available, and hence the types of programs that could be offered were not satisfactory. And the computer group had less time to learn probability because of the time devoted to learning about the computer programs. For these and other reasons we would have liked to confine our attention to the non-computer aspects of the experiment, but we include the with-computer group to increase Monte Carlo sample size.

The conventional analytic group was assigned two conventional chapters on probability and statistics in a basic text (Meserve and Sobel, 1973); the Monte Carlo group was given duplicated reading materials prepared by Shevokas.

1. All students were given the same seven-question exam on completion of the probability unit; a typical question was: "Suppose a machine produces bolts, 10% of which are defective. Find the probability that a box of three bolts contains as least one defective bolt." The mean scores were: conventional, 35.8; Monte Carlo no-computer, 58.5; Monte Carlo with-computer, 50.8, on a basis of 100. While one could wish for higher scores altogether, the Monte Carlo groups did better. The difference between Monte Carlo and conventional groups is statistically significant, but even more important, it is of an educationally significant magnitude; the Monte Carlo no-computer group got 62% higher scores than the conventional group.

2. The two attitude-toward-mathematics scales were again administered afterwards. The Monte Carlo groups showed more favorable attitudes than the conventional group, with the non-computer Monte Carlo group being most favorable; considering the two scales together, the post-scores differ significantly among the groups. Perhaps most interesting, the mean changes from "before" to "after" for the Aiken-Dreger and McCallon-Brown scales were: conventional, $-5\%$ and $-9\%$; Monte Carlo with-computer, 0% and $-8\%$; Monte Carlo no-computer, $+22\%$ and $+8\%$. To put it more concretely, five of 19 conventional-group students had an improved attitude, 13 a worsened attitude (one tie); among the Monte Carlo no-computer group, 8 students had improved attitudes, 5 worsened. (The attitudes of the Monte Carlo with-computer group were apparently harmed by their need to spend extra hours on campus to use the computer.)

3. It is an important result that despite an initially-cool attitude toward the no-computer Monte Carlo method by the teacher, she came to enjoy teaching the Monte Carlo method much more than the conventional method, because the students reacted to the Monte Carlo work in an interested and enthusiastic manner.

*Olivet Nazarene College Experiment*: At Olivet Nazarene (four year) College, Kankakee, Illinois, during the second half of each semester in 1974–1975 one class in Mathematics for General Education was taught probability and statistics by Atkinson in a conventional analytic fashion, while a second class was taught the Monte Carlo method. Class size was 21 students in each section the first semester; in the second semester there were 37 and 34 students, respectively, in the Monte Carlo and conventional sections. As in the case of the Polk Junior College experiment, students in this course generally have low skills and little interest in mathematics. Comparable duplicated reading materials prepared by Atkinson on the conventional and Monte Carlo methods were distributed to the respective classes.

1. In the first semester two pre-exam quizzes were given to each group, whereas three quizzes were given in the second semester. These quizzes each contained 1, 2 or 3 probability or statistical problems. On each quiz the Monte Carlo section did better than the conventional group, achieving class mean scores as much as twice as high as the conventional group.

2. The first part of the final exam the first semester was "conceptual." It asked the student to analyze problem data and describe the population, the hypotheses, and so on. The conventional group did better, getting a mean score of 47.9 compared to 40.8 for the Monte Carlo group ($t = 1.35$). The analogous first part of the second semester's final exam was a 20-question multiple-choice test on the concepts of hypothesis testing. This time the Monte Carlo group did better, 60.3 to 51.8 ($t = 2.06$).

3. The most important measure of performance was the second part of the final exam containing, respectively, three and four problems in the first and second semesters. Mean scores were: Semester 1: Monte Carlo, 69.5; conventional, 59.4 ($t = 1.7$). Semester 2: Monte Carlo, 67.6; conventional, 56.6 ($t = 2.06$). Inspection of second-semester tests showed that the Monte Carlo group did better on each and every question.

4. If one considers questions and answers only as "right" or "wrong," in the second semester 45.9% of the Monte Carlo students answered at least two questions correctly, whereas among the conventional group only 26.5% got two or more questions right. And the Monte Carlo group got 34.4% of the total questions right whereas the conventional group got 19.8% of the questions right. Comparative scoring of Monte Carlo and analytic answers requires some judgment. But the fact that

the teachers in the Polk and Olivet Nazarene experiments (though not in the Illinois experiment) were not initially in favor of the non-computer Monte Carlo method provides some protection.

5. The Monte Carlo section had less mathematical ability than the conventional section in both semesters; the Monte Carlo groups had lower mean scores on the ACT math test, two quizzes and the midterm exam on the algebra material taught in the first half of the semesters, some of the differences being statistically significant. Hence the better performance shown by the Monte Carlo groups on the probability and statistics material was despite a lower endowment of mathematics ability.

6. A twenty-question attitude-toward-mathematics scale similar to the Aiken-Dreger scale was given before and after the probability-statistics unit. In both semesters the Monte Carlo groups began with less favorable attitudes. But by the end of the experiment the Monte Carlo groups' attitudes toward math were more favorable than those of the conventional groups.

7. An attitude-toward-probability-and-statistics scale was given after the probability-statistics instruction. In the first semester, eight of ten questions were answered more favorably among the Monte Carlo group by substantively large and statistically significant differences; the other two differences were tiny, and the questions referred to future plans rather than attitudes. In the second semester, 15 of 17 attitudes were more favorable in the Monte Carlo group, most of the differences being large and all with $t_c \geqq 1$; two other questions were very slightly more favorable in the conventional group, with $t \leqq .23$.

8. The teacher's subjective evaluation, as in other classes where the Monte Carlo has been taught, is that the students seem relatively interested in and enthusiastic about the material, with a great deal of class discussion. This made for an enjoyable experience for the teacher, despite initial doubts about the value of the Monte Carlo method.

**Conclusions.** Taken as a whole, the evidence shows that the Monte Carlo method is a tool that students can and will use to arrive at correct answers to probabilistic-statistical problems. Therefore, it would seem to make sense to teach students to do standard probabilistic questions with the Monte Carlo method. In a conventional university probability or statistics course, this implies teaching the Monte Carlo method along with the analytic methods. In high school or college situations in which the student will not get a course or even a long section on probability and statistics, this implies teaching a block of 6–10 hours of the Monte Carlo method in the basic mathematics course so that the student will have at least some tools at his disposal.

If one has to make a pedagogical choice between analytic and Monte Carlo methods, it would seem that Monte Carlo is the method of choice on a "cost/benefit" basis — that is, it yields more usable output per unit of learning input. But luckily one does not usually have to make such a choice, because there can be plenty of time in the conventional elementary course for the Monte Carlo method to be treated along with the analytic method. And in a situation where the Monte Carlo method and *only* the Monte Carlo method might be taught — say a high school and junior college — the conventional method usually has no real opportunity at present to receive the attention that it must for students to acquire a usable tool, and hence the conventional approach is not a real alternative to the Monte Carlo method.

Lest this be unclear or seem to equivocate: Where there is limited time, or where students will not be able to grasp conventional methods firmly, we advocate teaching the Monte Carlo approach, and perhaps that only. Where there is more time, and where students will be able to well learn conventional methods, we advocate (a) teaching Monte Carlo methods at the very beginning as an introduction to statistical thinking and practice; and (b) afterwards teaching the Monte Carlo method with the conventional method as alternatives to the same problems, to help students learn analytic methods and to give them an alternative tool for their use.

Teaching the Monte Carlo method also has additional pedagogical advantages. It produces (in fact, demands) a high level of class participation and teacher-student interaction. This makes the class time lively and enjoyable. The method also leads students to discover for themselves the intuitive meaning

of fundamental concepts such as independence. And it increases their readiness to challenge the validity of the underlying data, which they must receive in raw form for the Monte Carlo method rather than in the defect-hiding summary form as, for example, "a population with $\mu = 100$ and $\sigma = 10$," the sort of language in which conventional problems are usually stated.

The advantage of the Monte Carlo Method seems to stem from its greater simplicity in a fundamental intuitive sense due to having fewer "working parts," and because the student never needs to take anything on faith, especially the sort of faith that is necessary with analytic methods that work by way of the central limit theorem ("It is shown in advanced texts that...).

Do we not owe it to our students and ourselves to at least give the Monte Carlo a hearing and a try?

**Acknowledgment.** Kenneth Travers supervised Atkinson's and Shevokas's theses at the College of Education of the University of Illinois, from which their results are drawn; we are grateful for Travers's important contribution to this work. We also appreciate helpful comments in an earlier draft from Bob Bohrer.

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# PROBLEMS AND SOLUTIONS

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

## ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before February 28, 1977.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 2617. *Proposed by Eugene Ehrhart, Institut de Recherches Mathématiques Avancées, Strasbourg, France*

A convex body is cut by three parallel planes. If the three sections thus produced have the same area, show that the portion of the body lying between the two outside planes is a cylinder. Does the same conclusion follow if instead we are given that the three sections have the same perimeter?

The theorem of Brunn-Minkowski (see T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Chelsea, New York, 1948, §41, p. 71) is relevant for this problem.

E 2618.* *Proposed by Amy J. Phelps, Student, West Side Middle School, Rocky Face, Georgia*

The numbers of the form $\frac{1}{2}k(k+1)$, $k^2$, $\frac{1}{2}k(3k-1)$, where $k$ is a positive integer, are called triangular, square, and pentagonal numbers, respectively. Find all triangular-square-pentagonal numbers.

E 2619.* *Proposed by Thomas C. Brown, Simon Fraser University, Burnaby, B. C., Canada*

Let $a_1 = 1$ and $a_{n+1} = a_n + [\sqrt{a_n}]$ for $n = 1, 2, \ldots$. Show that $a_n$ is a square if and only if $n = 2^k + k - 2$ for some positive integer $k$.

E 2620. *Proposed by Albert Mullin, Ft. Hood, Texas, and Derek Zave, Sperry UNIVAC (independently).*

Let $\Gamma$ be the graph consisting of the vertices and edges of one of the five regular polyhedra. Suppose all edges of $\Gamma$ are one-ohm resistors. Compute the resistance between any two of the most remote vertices of $\Gamma$.

Answer the same question when $\Gamma$ is the graph of the $n$-dimensional cube.

E 2621. *Proposed by Barry Powell, Kirkland, Washington*

Prove that $x^n + 1 = y^{n+1}$ has no solutions in positive integers $x, y, n$ $(n \geqq 2)$ with $(x, n+1) = 1$.

E 2622. *Proposed by S. Zaidman, University of Montreal*

Let $f: [a, b] \to R$ be a continuous function which is twice differentiable in $(a, b)$ and satisfies $f(a) = f(b) = 0$. Prove that

$$\int_a^b |f(x)|dx \leqq \frac{1}{12} M(b-a)^3,$$

where $M = \sup|f''(x)|$ for $x \in (a, b)$.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### A Brick Packing Problem

E 2524 [1975, 300]. *Proposed by Thomas H. Foregger, University of Wisconsin*

Show that 41 $1 \times 2 \times 4$ bricks can be placed in a $7 \times 7 \times 7$ box. Is there a packing of 42 such bricks into this box?

*Part I. Solution by the proposer.* It is easy to pack 12 bricks into a $7 \times 7 \times 2$ box. One can pack 29 bricks into a $7 \times 7 \times 5$ box as follows

| *level* 1 | | | | | | | *level* 5 | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | 6 | 8 | 1 | 1 | 1 | 1 | | | | 22 | 22 | 23 | 23 |
| 7 | 7 | 8 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 24 | 24 | 25 |
| 9 | 9 | 11 | 11 | 13 | 13 | | 3 | 3 | 3 | 3 | | | 25 |
| 10 | 10 | 12 | 12 | 14 | 2 | 2 | 4 | 4 | 4 | 4 | | 26 | 26 |
| 15 | 15 | 16 | 16 | 14 | 2 | 2 | 4 | 4 | 4 | 4 | | 27 | 27 |
| 17 | 17 | 18 | 18 | 19 | 2 | 2 | 5 | 5 | 5 | 5 | | 28 | 28 |
| 20 | 20 | 21 | 21 | 19 | 2 | 2 | 5 | 5 | 5 | 5 | | 29 | 29 |

The bricks are numbered by integers 1–29. The two levels shown determine the position of each brick.

*Part* II. *Solution by Michael Mather.*

THEOREM. *It is possible to pack a cube of side* $2n+1$ *with* $2n(2n+1)$ *bricks, each of size* $1 \times 2 \times (n+1)$, *if and only if* $n$ *is even, or* $n = 1$.

*Proof.* It is simple to find a packing for $n = 1$, or $n$ even. Let $n$ be odd and $\geqq 3$. Color the central cell in the box black. Color each of the $6n$ other cells "collinear" with the central cell red. Color each of the other $12n^2$ cells "coplanar" with the central cell white. There are $12n^2 + 6n + 1$ colored cells.

Assume that the $2n(2n+1)$ bricks are packed in the box. Each brick meets the colored cells in one of four ways:

two white cells;

one red cell, $n+1$ white cells;

two red cells, $2n$ white cells;

the black cell, $n+1$ red cells, and $n$ white cells.

Thus each brick not containing the black cell contains $nr + 2$ colored cells, where $r = 0, 1$ or 2 is the number of red cells that it contains.

Volume considerations reveal that there are $2n + 1$ vacant cells. Further, since $n$ is odd, each $1 \times (2n + 1) \times (2n + 1)$ plane in the box contains an odd number of vacant cells. Thus each such plane contains exactly one vacant cell. Since any two cells colored red or black lie in a plane, at most one such cell can be vacant.

There are three cases to consider:

(i) If no red or black cell is vacant, the bricks occupy $2(n + 1) + 2(4n^2 + 2n - 1) + (5n - 1)n = 13n^2 + 5n$ colored cells.

(ii) If a red cell is vacant, the bricks occupy $2(n + 1) + 2(4n^2 + 2n - 1) + (5n - 2)n = 13n^2 + 4n$ colored cells.

(iii) If the black cell is vacant, the bricks occupy $2(4n^2 + 2n) + 6n^2 = 14n^2 + 4n$ colored cells.

In each case, we have counted more than the $12n^2 + 6n + 1$ colored cells in the box. This contradiction shows that the $2n(2n + 1)$ bricks will not fit.

Both parts solved by Spottswood Burwell. The first part also solved by Anders Vretblad (Sweden), and the second part also solved by Chip Ashley and by Allen Schwenk.

*Editor's Comment.* The theorem above was communicated by L. E. Mattics, but his proof contained some gaps. Mather's proof was for the case $n = 3$ and it was easy to generalize it.

Ashley asks: Is it possible to pack $\frac{1}{2}n(n + 1)(2n + 1)$ bricks of size $1 \times 2 \times 4$ into a cube of side $2n + 1$? He claims that this is impossible for $n \equiv 1$ or 2 (mod 4).

### A Known Unsolved Problem in Disguise

E 2539* [1975, 521]. *Proposed by Andrew J. Vince, Woods Hole, Massachusetts*

Let $F_n$ denote the $n$th Fibonacci number: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Prove or disprove: If $m^2 \mid F_n$, then $m \mid n$.

*Comments by Donald W. Robinson, Brigham Young University, and David E. Penney & Carl Pomerance, University of Georgia.* For every positive integer $m$ there exists an integer $n$ such that $m \mid F_n$ and let $m'$ be the smallest such integer. The following facts are known (see the references below):

(1) $m \mid F_n \Leftrightarrow m' \mid n$;

(2) If $p$ is a prime then $p'$ divides $p - 1$ or $p + 1$;

(3) $(2^s)' = 2^{s-2} \cdot 3$ for $s \geq 3$;

(4) If $(p^2)' \neq p'$ where $p$ is an odd prime, then $(p^s)' = p^{s-1} \cdot p'$ for $s \geq 1$;

(5) $(m \vee n)' = m' \vee n'$ where $m \vee n = \mathrm{LCM}(m, n)$.

Let (A) and (B) be the following statements:

(A) $m^2 \mid F_n \Rightarrow m \mid n$ for all $m, n$;

(B) $(p^2)' \neq p'$ for all primes $p$.

We claim that (A) $\Leftrightarrow$ (B). Assume that (A) holds and that $(p^2)' = p'$ for some odd prime $p$. Then $p^2 \mid F_{p'}$ and by (A) $p \mid p'$. This is a contradiction since by (2) $p' \mid (p - 1)$ or $p' \mid (p + 1)$. This proves that (A) $\Rightarrow$ (B).

Conversely, assume that (B) holds and that $m^2 \mid F_n$. Write $m = 2^s p_1^{e_1} \cdots p_k^{e_k}$ where $s \geq 0$, $e_i \geq 1$ and $p_1, \ldots, p_k$ are distinct odd primes. Then $(m^2)' \mid n$ by (1) and, using (5), we have that each of

$$(2^{2s})', (p_1^{2e_1})', \ldots, (p_k^{2e_k})' \mid n. \tag{6}$$

By (B) and (4) we deduce that

$$(p_i^{2e_i})' \mid n \Rightarrow p_i^{e_i} \mid n.$$

Using (3) it is easy to show that

$$(2^{2^s})' \mid n \Rightarrow 2^s \mid n.$$

Hence it follows from (6) that $m \mid n$. Therefore (B) $\Rightarrow$ (A).

(B) is a well-known question due to D. D. Wall [5] who has verified the result for $p < 10{,}000$. J. H. Halton [2] claims that (B) has been verified for $p \leqq 28{,}837$. Penney and Pomerance inform us that they have verified it (using a computer) for $p \leqq 177{,}409$.

The same sort of comments are made by the following contributors: M. T. Bird, A. A. Jagers (Netherlands), G. S. Lessels & M. J. Pelling (Nigeria), Ram Murty & Kumar Murty (Canada), Barry Powell, Martin Schechter & Peter Borwein (Canada), and the proposer.

Gregory Wulczyn gives infinitely many $m$ for (A) is true. Nevertheless most commenters seem to expect that there is a counterexample.

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3. D. W. Robinson, *ibid.* 1 (1963), 29–36.
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**A Bigger Group Does the Job**

E 2542 [1975, 660]. *Proposed by Ronald J. Evans, University of Wisconsin*

Let $G$ be the group generated by the transformations $T$ and $S$ on the extended complex plane, where $zT = -1/z$ and $zS = z + 2i$. Suppose that $z_0$ is fixed by some non-identity transformation in $G$. Prove that $z_0$ must lie on the extended imaginary axis.

*Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands (revised by the editor).* The matrices

$$A = \begin{pmatrix} a & bi \\ ci & d \end{pmatrix}; \quad a, b, c, d \in \mathbb{Z},$$

where either

(i) $a \equiv d \equiv 1$, $b \equiv c \equiv 0 \pmod 2$ and $ad + bc = 1$; or

(ii) $a \equiv d \equiv 0$, $b \equiv c \equiv 1 \pmod 2$ and $ad + bc = -1$,

form a group $H$. To each

$$A = \begin{pmatrix} a & bi \\ ci & d \end{pmatrix} \in H$$

we associate a fractional linear transformation $\hat{A}$ of the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by the formula $z\hat{A} = (az + ci)/(biz + d)$. The map $A \mapsto \hat{A}$ is a homomorphism of $H$ into the group of all fractional linear transformations. We denote by $\hat{H}$ the image of $H$. Since $S, T \in \hat{H}$, we see that $G$ is a subgroup of $\hat{H}$ and it suffices to prove the statement for $\hat{H}$. Thus, let

$$A \in H, A = \begin{pmatrix} a & bi \\ ci & d \end{pmatrix} \neq \pm I.$$

We have to show that the solutions of the equation

$$z\hat{A} = z \tag{1}$$

lie on the extended imaginary axis. If $b = 0$ and $a = d$ then (1) has only the solution $\infty$. If $b = 0$ and $a \neq d$ then (1) has solutions $\infty$ and $ci/(d-a)$. If $b \neq 0$ then (1) can be written as $biz^2 + (d-a)z - ci = 0$. For $iz = w$ we get the equation $bw^2 + (d-a)w + c = 0$. It remains to show that

$$\Delta = (d-a)^2 - 4bc = (a+d)^2 - 4(ad+bc) \geq 0.$$

This is clear in case (ii). In case (i) we have $\Delta = (a+d)^2 - 4$ and since $a \equiv d \equiv 1 \pmod 2$ it suffices to show that $a + d \neq 0$. But $ad = 1 - bc \equiv 1 \pmod 4$ implies that $a \equiv d \pmod 4$ and hence $a + d \equiv 2 \pmod 4$. Therefore $a + d \neq 0$.

Also solved by Paul Bruckman, Thomas Foregger & Jeff Lagarias, M. G. Greening (Australia), Benjamin Klein, L. E. Mattics, and the proposer.

*Editor's comment.* Klein proves that the statement remains valid when $S$ is defined by $zS = z + it$ where $t \geq 2$ is an integer. He conjectures that the requirement that $t$ be an integer may be omitted.

## $\sigma(n)/n$ versus $\tau(n)$

E 2543 [1975, 660]. *Proposed by the late C. W. Anderson, University of California, Berkeley*

For the natural number $n$, let $\tau(n)$ be the number of divisors, and $\sigma(n)$ the sum of these divisors. Show that there exists a constant $k > 0$ such that, if $x = \sigma(n)/n$ is sufficiently large, then

$$\tau(n) > 2^{\exp(kx)}.$$

Show also that there exist $n$ for which $\tau(n)$ is arbitrarily large, but for which $\sigma(n)/n$ is arbitrarily close to unity.

*Solution by the proposer (revised by the editor).* Let $n$ be a natural number and $n = \Pi p^{n(p)}$ its prime factorization, where $p$ runs through all the primes. The number of primes $p$ such that $n(p) \geq 1$ is finite and we denote it by $\nu(n)$. We have

$$x(n) = \sigma(n)/n = \prod_{p|n} \frac{p^{n(p)+1}-1}{(p-1)p^{n(p)}} \leq \prod_{p|n} \left(1-\frac{1}{p}\right)^{-1} \leq \prod_{i=1}^{\nu(n)} \left(1-\frac{1}{p_i}\right)^{-1},$$

where $p_1, p_2, \ldots$ is the sequence of all primes arranged in increasing order.

A theorem of F. Mertens (see G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1965 p. 251) asserts that

$$\prod_{p \leq m} \left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log m} \qquad (m \to \infty),$$

where $p$ is prime and $\gamma$ is the Euler's constant. Hence

$$\prod_{i=1}^{\nu(n)} \left(1-\frac{1}{p_i}\right)^{-1} \sim e^{\gamma} \log p_{\nu(n)} \sim e^{\gamma} \log \nu(n) \qquad (n \to \infty).$$

The fact that $\log p_{\nu(n)} \sim \log \nu(n)$ $(n \to \infty)$ follows from the Prime Number Theorem

$$\pi(m) \sim m/\log m \qquad (m \to \infty),$$

where $\pi(m)$ is the number of primes not exceeding $m$.

Hence, if $\varepsilon > 0$ is given, there exists a natural number $N(\varepsilon)$ such that

$$\prod_{i=1}^{\nu(n)} \left(1-\frac{1}{p_i}\right)^{-1} < (1+\varepsilon)e^{\gamma} \log \nu(n), \qquad n > N(\varepsilon).$$

Thus, if $k(\varepsilon) = e^{-\gamma}/(1+\varepsilon)$ then

$$\log \nu(n) > k(\varepsilon)x(n),\ \nu(n) > \exp(k(\varepsilon)x(n)),\ \tau(n) = \prod_{p|n}(n(p)+1) \geqq 2^{\nu(n)} > 2^{\exp(k(\varepsilon)x(n))}$$

hold for $n > N(\varepsilon)$. It remains to notice that if $x(n)$ is sufficiently large then $n > N(\varepsilon)$.

The second assertion is verified by taking $n_i = p_i^i$ $(i = 1, 2, \ldots)$ since

$$\sigma(n_i)/n_i \to 1 \qquad (i \to \infty),$$

$$\tau(n_i) = i + 1.$$

Also solved by Peter Borwein & Martin Schechter (Canada), David Bressoud, J. C. Lagarias, Eric Lander, Joel Levy, and L. E. Mattics. Partially solved (second part bnly) by Thomas Elsner, Carl Hurd, and George Pfeiffer.

**Fibonaccian Juxtaposition**

E 2544 [1975, 660]. *Proposed by Harvey Cohn, City College of CUNY.*

Consider the sequence of words formed by "Fibonaccian juxtaposition": $w_1 = 0$, $w_2 = 1$, $w_{n+2} = w_n w_{n+1}$ for $n \geqq 1$. (Thus $w_3 = 01$, $w_4 = 101$, $w_5 = 01101$, etc.) Form the sequence $S$ by

$$S = w_1 w_2 w_3 w_4 \cdots = 010110101101 \cdots.$$

Now let $\gamma = \frac{1}{2}(5^{1/2} - 1)$ be the reciprocal of "golden mean" and define $t_n = [n\gamma] - [(n-1)\gamma]$ for $n = 1, 2, \ldots$, where the square brackets denote the greatest integer function. Form the sequence $T = t_1 t_2 t_3 \cdots$. Show that the sequences $S$ and $T$ are identical.

Generalize.

*Solution by Greg Kermayer, student, City College of CUNY.* We treat a more general problem. Let $S$ be the set of all finite sequences of 0's and 1's. For $x \in S$ we let $|x|$ be the *length* of $x$ and $x^n = xx \cdots x$ with $n$ $x$'s. For $x, y \in S$ we shall write $x \to y$ (resp. $y \leftarrow x$) if there exist $z \in S$ such that $x = zy$ (resp. $x = yz$). It is clear that $x \to \varnothing$ and $\varnothing \leftarrow x$ for all $x \in S$, where $\varnothing \in S$ is the empty sequence.

We fix an integer $m \geqq 1$ and let $\theta = \frac{1}{2}(\sqrt{m^2+4} - m)$. Then we define $a_n = 0$ or $1$ $(n \geqq 0)$ by $a_n = [(n+1)\theta] - [n\theta]$.

We define $x_n \in S$ $(n \geqq 0)$ by $x_0 = 0$, $x_1 = 0^{m-1}1$ and $x_{n+1} = x_n^{m-1} x_{n-1} x_n$ $(n \geqq 1)$. The integers $u_n = |x_n|$ satisfy $u_0 = 1$, $u_1 = m$ and

$$u_{n+1} = mu_n + u_{n-1}, \qquad (n \geqq 0), \tag{1}$$

where we put $u_{-1} = 0$.

Finally define $b_n = 0$ or $1$ $(n \geqq 0)$ by $x_0 x_1 x_2 \cdots = b_0 b_1 b_2 \cdots$.

THEOREM. $a_n = b_n$ $(n \geqq 0)$.

If $m = 1$ then $\theta = \gamma$, $a_n = t_{n+1}$, $x_n = w_{n+1}$ and we obtain the result stated in the problem.

The theorem will be proved in several steps.

$$\text{(i)} \qquad \sum_{i=0}^{n-1} u_i = \frac{1}{m}(u_n + u_{n-1} - 1) \qquad (n \geqq 0).$$

*Proof.* Since $u_{-1} = 0$, $u_0 = 1$, (i) is valid for $n = 0$. The general case is proved by induction on $n$ using (1).

$$\text{(ii)} \qquad x_{n-1}x_n \to x_1 x_2 \cdots x_{n-1} \qquad (n \geqq 1).$$

*Proof.* For $n = 1$ this reduces to $x_0 x_1 \to \varnothing$. For $n = 2$ we have $x_1 x_2 = x_1^m x_0 x_1 \to x_1$. Assume (ii) holds for $n \leqq k$ where $k \geqq 2$. Then $x_{k-1}x_k \to x_1 x_2 \cdots x_{k-1}$ and hence

$$x_{k-1}x_k x_{k+1} \to x_1 x_2 \cdots x_{k-2} x_{k-1} x_k^{m-1} x_{k-1} x_k. \tag{2}$$

Using $x_{k-1}x_k \to x_{k-2}x_{k-1}$ it is easy to prove (by induction on $m$) that $x_{k-2}x_{k-1}x_k^{m-1} \to x_{k-2}x_{k-1}$. Hence, from (2) and the induction hypothesis

$$x_{k-1}x_kx_{k+1} \to (x_{k-2}x_{k-1})(x_{k-1}x_k) \to x_1x_2\cdots x_k. \tag{3}$$

Since by (i)

$$|x_kx_{k+1}| = u_k + u_{k+1} > |x_1\cdots x_k| = \frac{1}{m}(u_k + u_{k+1} - 1) - 1,$$

it is clear that (ii) for $n = k+1$ follows from (3).

$$x_0x_1\cdots x_n = b_0b_1\cdots b_{u_n}x_1\cdots x_{n-1} \qquad (n \geqq 1). \tag{iii}$$

*Proof.* This follows from (ii), the definition of the $b_i$'s and the equality of lengths of the two sides of (iii).

$$\sum_{i=1}^{u_n} b_i = u_{n-1} \qquad (n \geqq 1). \tag{iv}$$

*Proof.* Let $v_n$ be the sum of the terms of the sequence $x_n$. It is clear from the definition of $x_n$ that $v_n$ satisfy the recursive relation (1). Since $v_0 = 0 = u_{-1}$ and $v_1 = 1 = u_0$ we must have $v_n = u_{n-1}$ $(n \geqq 0)$. Now (iv) follows from (iii).

$$u_{n-1} = u_n\theta + (-\theta)^{n+1} \qquad (n \geqq 0). \tag{v}$$

*Proof.* Since $\theta(\theta + m) = 1$, (v) is valid for $n = 0, 1$. The general case follows by induction using (1).

$$b_{u_n+r} = b_r \quad \text{for} \quad 0 < r < u_{n+1} - u_n. \tag{vi}$$

*Proof.* These inequalities imply that $n \geqq 1$ and we can use (iii). Hence

$$x_0x_1\cdots x_{n+1} = b_0b_1\cdots b_{u_n}x_1x_2\cdots x_{n-1}x_n^{m-1}x_{n-1}x_n.$$

Since $x_1x_2\cdots x_{n-1}x_n^{m-1} \leftarrow x_1x_2\cdots x_n(x_n^{m-1}x_{n-1}x_n) = x_1x_2\cdots x_{n+1}$ it follows that (vi) is valid for those $r > 0$ which satisfy

$$r \leqq |x_1\cdots x_{n-1}x_n^{m-1}| = \frac{1}{m}(u_n + u_{n-1} - 1) - 1 + (m-1)u_n$$

$$= \frac{1}{m}(u_n + u_{n-1} - 1) - u_{n-1} + (u_{n+1} - u_n - 1)$$

$$= \frac{1}{m}(u_{n-1} + u_{n-2} - 1) + (u_{n+1} - u_n - 1).$$

$$(r+1)\theta - 1 < \sum_{i=0}^{r} b_i < (r+1)\theta \qquad (r \geqq 0). \tag{vii}$$

*Proof.* Write $r = u_{n_1} + \cdots + u_{n_k}$ where $n_1 \geqq \cdots \geqq n_k \geqq 0$ and $u_{n_i} + \cdots + u_{n_k} < u_{n_i+1}$ for each $i$. If $p \geqq 1$ is an integer then this implies that the number of indices $i$ such that $n_i = p$ does not exceed $m$ and if $p = 0$ it is $\leqq m - 1$. By (vi) and (iv) we have

$$\sum_{i=0}^{r} b_i = u_{n_1-1} + \cdots + u_{n_k-1}.$$

By (v) we have

$$\sum_{i=0}^{r} b_i < (u_{n_1} + \cdots + u_{n_k})\theta + m\sum_{s=1}^{\infty} \theta^{2s} = r\theta + m\theta^2/(1-\theta^2) = (r+1)\theta$$

and

$$\sum_{i=0}^{r} b_i > (u_{n_1} + \cdots + u_{n_k})\theta - (m-1)\theta - m\sum_{s=1}^{\infty} \theta^{2s+1}$$
$$= (r+1)\theta - m\theta - m\theta^3/(1-\theta^2)$$
$$= (r+1)\theta - m\theta - \theta^2 = (r+1)\theta - 1.$$

(viii) $$(r+1)\theta - 1 < \sum_{i=0}^{r} a_i < (r+1)\theta \qquad (r \geqq 0).$$

*Proof.* We have

$$\sum_{i=0}^{r} a_i = \sum_{i=0}^{r} ([(i+1)\theta] - [i\theta]) = [(r+1)\theta].$$

The assertion of the theorem follows from (vii) and (viii).

Also solved by Donald Fuller, L. E. Mattics, and Kenneth Stolarsky.

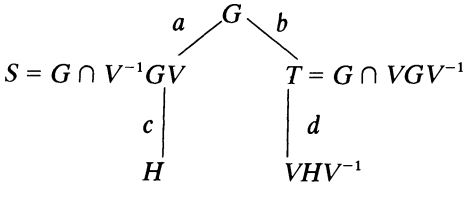
*Editor's comment.* C. D. Olds points out that J. V. Uspensky has made an extensive investigation of the sequences $t_m = [(m+1)\gamma + b] - [m\gamma]$ (Math. Reviews 8 (1947) 5, 6, 443). E. Rosenthal mentions that the result occurs in D. E. Knuth, *The Art of Computer Programming*, vol. 1 in the solution of Exercise 1.2.8–36 (pp. 85, 493). Knuth also refers to this sequence in his solution to Problem E 2307 [1971, 792; 1972, 773]. Stolarsky supplies a copy of his manuscript, *Beatty Sequences, Continued Fractions, and Certain Shift Operators*, submitted to the Canad. Math. Bull., which contains a thorough investigation of these sequences and an extensive bibliography.

### Indices of Subgroups of $SL_n(Z)$

E 2545 [1975, 660]. *Proposed by Ron Evans, University of Wisconsin*

Let $V$ be an invertible $n \times n$ matrix with rational entries and let $G$ denote the group of all $n \times n$ matrices with integral entries and determinant 1. Prove that if $H$ and $VHV^{-1}$ are subgroups of $G$ of finite index $p, q$ respectively, then $p = q$.

*Solution by L. E. Mattics, University of South Alabama.* Since $VHV^{-1} \subset G$, i.e., $H \subset V^{-1}GV$ we have a diagram




where $a, b, c, d$ are the corresponding indices. Since $p = ac$, $q = bd$, and $c = d$ is obvious, it remains to show that $a = b$. By a well-known classical theorem one can write $V = ADB$, where $A, B$ are integral matrices of determinant $\pm 1$ and $D$ is a diagonal rational matrix. Then

$$S = B^{-1}(G \cap D^{-1}GD)B, \qquad T = A(G \cap DGD^{-1})A^{-1}$$

and consequently $a = (G : (G \cap D^{-1}GD))$ and $b = (G : (G \cap DGD^{-1}))$. Now $a = b$ follows from the fact that the matrix transposition induces an anti-automorphism of $G$ which maps $G \cap D^{-1}GD$ onto $G \cap DGD^{-1}$.

Also solved by Olga Taussky & Hans Zassenhaus, and the proposer.

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate signed sheets and should be mailed before February 28, 1977.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6114. *Proposed by R. M. Norton, College of Charleston*

Let $Z_1 = XY$ where $X$ and $Y$ are independent standard normal random variables, and $Z_1$ and $Z_2$ be independent and identically distributed. Derive the density function $f(x)$ of $Z_1 + Z_2$.

6115. *Proposed by L. Franklin Kemp, Jr., Tulsa, Oklahoma*

Let $F_1(x_1), F_2(x_2), \ldots, F_n(x_n)$ be $n$ probability distribution functions (d.f.'s). If $H(x_1, x_2, \cdots, x_n)$ is any $n$-dimensional d.f. with marginals $F_i(x_i)$, then the solution to Problem 5894 [1974, 413] showed that $\min_i F_i(x_i)$ is an $n$-dimensional d.f. such that $H(x_1, x_2, \cdots, x_n) \leqq \min_i F_i(x_i)$. What is the $n$-dimensional d.f. that bounds any $H(x_1, x_2, \ldots, x_n)$ from below?

6116. *Proposed by S. H. Cox, Jr., Universidad de Puerto Rico*

Let $A$ be an integral domain satisfying the condition: For every nonzero ideal $I$ of $A$ there is an epimorphism $A \to A'$ of rings such that $I$ and $A'$ are isomorphic $A$-modules. For example, a principal ideal domain satisfies the condition with $A \to A'$ the identity $A = A'$. Show that each domain satisfying the condition is a principal ideal domain. (Warning: ring epimorphisms need not be surjective.)

6117*. *Proposed by M. J. Pelling, University of Belin, Nigeria*

A well-known theorem asserts that given entire functions $f(z)$, $g(z)$ with no common zero, then there exist entire functions $a(z)$, $b(z)$ such that $af + bg = 1$ identically.

(i) Show that it is always possible to choose $a(z)$ zero-free.

(ii) Is it always possible to choose both $a(z)$ and $b(z)$ to be zero-free?

6118*. *Proposed by M. J. Pelling, University of Belin, Nigeria*

(i) Show there is no nonconstant solution to $e^{f(z)} + e^{g(z)} = 1$ in entire functions $f(z)$ and $g(z)$.

(ii) Is there a nonconstant solution to $e^{f(z)} + e^{g(z)} + e^{h(z)} = 1$ in entire functions $f$, $g$, $h$?

6119*. *Proposed by M. J. Pelling, University of Belin, Nigeria*

Are there any algebraic number fields $A$ with the property that $A = A_1 + A_2$ (qua abelian groups), where $A_1$, $A_2$ are proper subfields of $A$? (cf. Problem 5993 [1974, 911]).

## SOLUTIONS OF ADVANCED PROBLEMS

### An Integer Sequence from the Harmonic Series

5989* [1974, 910]. *Proposed by R. H. C. Newton, University College of North Wales*

Prove or disprove: The $k$th term of the infinite sequence $\{n_k\} = \{1, 4, 11, 31, \ldots\}$ defined by

$$h(n_k - 1) \leqq h < h(n_k) \qquad (k = 1, 2, \ldots),$$

where $h(n) = 1 + 1/2 + 1/3 + \cdots + 1/n$, is given by the integer nearest to $\exp(k - \gamma)$. Here $\gamma = 0.577\ldots$ is Euler's constant.

*Comment by R. P. Boas, Northwestern University.* The proposition proposed in the problem has been established by computation for $k \leqq 1000$ (for $k \leqq 20$, by Wrench, published in Boas and Wrench, *Partial sums of the harmonic series*, this MONTHLY 78 (1971), 864–870; for $k = 21, 22$, by H. P. Robinson, in a supplement to N. J. A. Sloane, *A Handbook of Integer Sequences*, Academic Press, 1973; for $k \leqq 200$ by Boas (unpublished) and for $k \leqq 1000$ by R. Spira (unpublished)). It is known (L. Comtet, Problem 5346 [1967, 209]) that $n_k$ is either $m_k$ or $m_k + 1$, where $m_k = [e^{k-\gamma}]$.

To simplify further discussion I introduce a function $\omega(k)$ ("Comtet function") such that the first case occurs if the fractional part of $e^{k-\gamma}$ is less than $\omega(k)$; the second, if the fractional part of $e^{k-\gamma}$ is greater than $\omega(k)$. The problem asks, then, whether $\omega(k) = \frac{1}{2}$ is a Comtet function. It was shown by Boas and Wrench (*op. cit.*; note a correction: in their Theorem 1, last line, read $m$ for $n$) that there is a Comtet function between $\frac{1}{2} - m_k^{-1}$ and $\frac{1}{2} + m_k^{-1}$. The same result was established independently by Stephen M. Zemyan in a partial solution submitted for this problem. Zemyan also showed that if we write $h(n_k - 1) + \alpha_k/n_k = k$, $0 \leqq \alpha_k < 1$, then we may take $\omega(k) = \frac{1}{2}$ provided that $1/181 \leqq \alpha_k \leqq 225/226$; his calculations are too elaborate to be reproduced here. In a paper to appear in Math. of Comp., Boas shows that there is a Comtet function between $\frac{1}{2} + 1/(24m_k) - 1/(47m_k^2)$ and $\frac{1}{2} + 1/(24m_k) - 1/(49m_k^2)$; and as $k \to \infty$ there is a Comtet function of the form $\frac{1}{2} + 1/(24m_k) - (1/(48m_k^2))(1 + o(1))$. According to Spira, the nearest miss for $k \leqq 1000$ is at $k = 565$ where the fractional part of $e^{k-\gamma}$ is slightly over 0.50001. Since a counterexample to the proposition for some $k > 1000$ would require the fractional part of $e^{k-\gamma}$ to be closer to $\frac{1}{2}$ than about $2 \times 10^{-436}$, there seems to be little hope of finding one by computation. It is perhaps worth noting that if the fractional part of $e^{k-\gamma}$ were ever exactly $\frac{1}{2}$, we would have $\gamma = k + \log 2$ and we would then know that $\gamma$ is transcendental. Incidentally, $\gamma$ is now known to more than 7000 decimal places (W. A. Beyer and M. S. Waterman, *Error analysis of a computation of Euler's constant*, Math. of Comp. 28 (1974), 599–604). For further comments on the problem see E 2484 [1975, 760–761].

**Fields, the Sum of Two Proper Subfields**

5993 [1974, 911]. *Proposed by V. Dlab, Carleton University, E. Formanek, Institute for Advanced Study, and C. M. Ringel, Universität Bonn, West Germany*

Let $F$ be a countable field, let $K = F(X_1, X_2, \ldots)$ be the rational function field over $F$ in countably many indeterminates, and let $A$ be an algebraic closure of $K$. Show that $A$, as an abelian group under addition, is the sum of two proper subfields.

*Solution by M. J. Pelling, University of Benin, Nigeria.* The solution is contained in the following theorem: *Let $B$ be any field having countably infinite degree of transcendence over its prime field $P$. Then, as an abelian group under addition, $B = B_1 + B_2$ where $B_1$, $B_2$ are proper subfields of $B$.*

*Proof.* $B$ is a purely algebraic extension of $P(x_0, x_1, \ldots)$ where the $x_i$ are a countably infinite algebraically independent set of indeterminates over $P$; every element of $B$ is an algebraic function of a finite number of the $x_i$. Let $\{b_1, b_2, \ldots\}$ be an enumeration of $B$ and for all $n$ let $k(n)$ be an integer such that $b_1, b_2, \ldots, b_n$ do not involve $x_i$ for $i > k(n)$.

Define $F_0 = G_0 = P$ and suppose $F_n$, $G_n$ have been defined so that $b_1, b_2, \ldots, b_n$ are in $F_n + G_n$ where $F_n$, $G_n$ are subfields of $B$ whose elements do not involve $x_0$ or $x_i$ for $i > q$ for some $q = q(n)$. Let $p = \max(q, k(n+1)) + 1$ and set $F_{n+1} = F_n(b_{n+1} + x_p)$, $G_{n+1} = G_n(x_p)$. Since $x_p$ is an indeterminate over $F_n(x_0, b_{n+1})$, so is $y = b_{n+1} + x_p$. Hence if $x_0$ is in $F_n(y)$ then it would be in $F_n$ which is impossible. Similarly $x_0$ is not in $G_{n+1}$. Thus $F_{n+1}$, $G_{n+1}$ have the same properties as $F_n$, $G_n$.

Put $B_1 = \bigcup_{n=0}^{\infty} F_n$ and $B_2 = \bigcup_{n=0}^{\infty} G_n$. These are proper subfields of $B$ since neither contains $x_0$, and $B = B_1 + B_2$. Also $B_2$ is purely transcendental over $P$.

**Equal Sum Partitions in an Abelian Group**

6011 [1975, 85]. *Proposed by M. Slater, the University, Bristol, England*

According to a question in the 1973 William Lowell Putnam competition, B–1, this MONTHLY, 81 (1974) 1090, the group $Z$ of integers has the following property $X$:

For any $n$, suppose $A$ is a list of $(2n+1)$ terms in $Z$, such that on removal of any one term, the remainder can be divided into two batches of $n$ terms having equal sums. Then all the terms of $A$ are equal.

Determine exactly what abelian groups $G$ have property $X$.

*Solution by G. A. Heuer and C. V. Heuer, Concordia College.* An abelian group has property $X$ iff all elements of finite order in $G$ have order a power of 2. We shall prove the following generalization:

Say that a list $A$ of $pn+1$ terms of $G$ has property $Y(p)$ if, whenever one term is removed, the remaining terms can be divided into $p$ batches of $n$ terms having equal sums. Say that $G$ has property $X(p)$ in case every list $A$ in $G$ having property $Y(p)$ has all terms equal.

THEOREM. *An abelian group $G$ has property $X(p)$ iff all elements of finite order in $G$ have order a power of $p$.*

*Proof.* Suppose $G$ has an element of finite order not a power of $p$. Then $G$ has an element $b$ of prime order $q \neq p$. Choose $r$, $1 \leq r < p$, such that $qr \equiv 1 \pmod{p}$. Let $A$ be a list of $pq+1$ elements of which $qr$ are $b$ and the remainder are 0. If a 0 is deleted, divide the remaining terms into $p$ batches with each batch consisting of $q$ $b$'s or $q$ 0's. If a $b$ is deleted, divide the remainder into $p$ identical batches each having $m = (qr-1)/p$ $b$'s and $(p-m)$ 0's. Thus we see that $G$ does not have property $X(p)$.

Suppose, conversely, that all elements of $G$ having finite order have order a power of $p$. Then $p$ is necessarily prime. To show that $G$ has property $X(p)$ we can assume without loss of generality that $G$ is finitely generated, and hence is a direct product of a finite number, say $k$, of cyclic groups, each having order a power of $p$ or infinity. By looking at elements of $G$ as $k$-tuples we see that a counterexample to $X(p)$ in $G$ would imply such a counterexample in some component. By an obvious modification of the published solution to the Putnam problem (this MONTHLY, 1974, p. 1093), the infinite cyclic group has property $X(p)$, so it remains only to show that cyclic groups of order $p^e$ have it.

Thus, suppose $G$ is cyclic of order $p^e$. If $A = (a_1, a_2, \ldots, a_{pn+1})$ is a list in $G$ having property $Y(p)$, then any two terms of $A$ differ by $p$ times an element of $G$, so the case $e = 1$ is obvious. If $e > 1$ and $B = (b_1, b_2, \ldots, b_{pn+1})$ with $b_i = a_i - a_1$, then $B$ has property $Y(p)$ and each $b_i = pc_i$ for some $c_i$ in $G$. Then $C = (c_1, c_2, \ldots, c_{pn+1})$ is a list in a subgroup of order $p^{e-1}$ having property $Y(p)$, so by induction $G$ has property $X(p)$.

Also solved by R. B. J. T. Allenby (England), Albert Borchers, Stephen Bronn, Paul Chernoff, S. Z. Ditor (Canada), D. Ž. Djoković (Canada), Thomas Forreger, A. A. Jagers (Netherlands), O. P. Lossers (Netherlands), John O'Neill, S. J., and the proposer.

**Infinite Product over a Set of Primes**

6012 [1975, 183]. *Proposed by Daniel Shanks, Naval Ship Research and Development*

Prove the following infinite products over certain sets of primes $p$:

$$\prod_{p=14k\pm1}\left(1+\frac{3p^2}{(p^2-1)^2}\right)=\frac{840}{817}, \qquad \prod_{p=18k\pm1}\left(1+\frac{3p^2}{(p^2-1)^2}\right)=\frac{40}{39}.$$

*Solution by Irving Gerst, State University of New York at Stony Brook.* Consider the $L$-functions $L(s,\chi)$ and $L(s,\chi^2)$ where $\chi(n)$ is a character modulo 7 defined as $\chi(3)=\omega$, $\omega$ being a primitive cube root of unity. Then since $\chi(\pm1)=\chi^2(\pm1)=1$, and $\chi(n)\chi^2(n)=1$, $\chi(n)+\chi^2(n)=-1$ for $n=2,3,4,5$, we have for $s>1$,

$$L(s,\chi)L(s,\chi^2)=\frac{1}{\prod_p(1-\chi(p)p^{-s})(1-\chi^2(p)p^{-s})}$$

$$=\frac{1}{\prod'(1-p^{-s})^2\cdot\prod''(1+p^{-s}+p^{-2s})},$$

where $\prod'$ is taken over all primes $p\equiv\pm1 \pmod 7$ and $\prod''$ over all $p\equiv 2,3,4,5 \pmod 7$. Thus

$$(1-7^{-s})\zeta(s)L(s,\chi)L(s,\chi^2)=\frac{1}{\prod'(1-p^{-s})^3\cdot\prod''(1-p^{-3s})}$$

$$=(1-7^{-3s})\zeta(3s)\cdot\prod'\frac{1-p^{-3s}}{(1-p^{-s})^3}.$$

Since $(1-p^{-3s})/(1-p^{-s})^3=(1+p^{-s}+p^{-2s})/(1-p^{-s})^2=1+3p^s/(p^s-1)^2$, we see that, when $s=2$, this last product $\prod'$ is the first of the required products, call it $P_1$. Substituting the known values $\zeta(2)=\pi^2/6$ and $\zeta(6)=\pi^6/945$, we get

$$P_1=\frac{3\cdot5\cdot7^5}{2\cdot19\cdot43\pi^4}L(2,\chi)L(2,\chi^2).$$

Now

$$L(2,\chi)=\frac{1}{7^2}\left\{\sum_{n=0}^{\infty}\frac{1}{(n+1/7)^2}+\sum_{n=0}^{\infty}\frac{1}{(n+6/7)^2}\right.$$

$$+\omega\left(\sum_{n=0}^{\infty}\frac{1}{(n+3/7)^2}+\sum_{n=0}^{\infty}\frac{1}{(n+4/7)^2}\right)$$

$$\left.+\omega^2\left(\sum_{n=0}^{\infty}\frac{1}{(n+2/7)^2}+\sum_{n=0}^{\infty}\frac{1}{(n+5/7)^2}\right)\right\}.$$

Taking note of the known expression

$$\pi^2\csc^2\pi x=\sum_{n=-\infty}^{\infty}\frac{1}{(n+x)^2}=\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}+\sum_{n=0}^{\infty}\frac{1}{(n+1-x)^2},$$

we get $L(2,\chi)=\pi^2(r_1+r_3\omega+r_2\omega^2)/7^2$ where $r_j=\csc^2(j\pi/7)$. Then, since $L(2,\chi^2)=\bar{L}(2,\chi)$, we have $L(2,\chi^2)=\pi^2(r_1+r_3\omega^2+r_2\omega)/7^2$, so that

$$L(2,\chi)L(2,\chi^2)=\frac{\pi^4}{7^4}\left\{\sum_{i=1}^{3}r_i^2-\sum_{1\le i<j\le3}r_ir_j\right\}$$

$$=\frac{\pi^4}{7^4}\left\{\left(\sum_{i=1}^{3}r_i\right)^2-3\sum_{1\le i<j\le3}r_ir_j\right\}.$$

But we can readily find a polynomial, $p(z)$, over the rationals whose roots are precisely the $r_i$. For the $r_i$ are obtained by applying the transformation $4z^{-1}=-(x-1/x)^2$ to the roots $x_i$ of $(x^{14}-1)/(x^2-1)=0$. The latter equation is equivalent to $1+\Sigma_{k=1}^{3}(x^{2k}+x^{-2k})=0$, and since $x^2+x^{-2}=2-4z^{-1}$, the usual procedure for handling reciprocal equations may be applied to get $p(z)=7z^3-56z^2+112z-64$. Hence $\Sigma r_i=8$, $\Sigma r_ir_j=16$, giving $L(2,\chi)L(2,\chi^2)=16\pi^4/7^4$ and $P_1=840/817$.

The second product, call it $P_2$, is obtained similarly. Define the character $\chi$ modulo 9 by $\chi(2)=\omega$.

Then

$$P_2 = \frac{(1-3^{-2})\zeta(2)L(2,\chi)L(2,\chi^2)}{(1-3^{-6})\zeta(6)} = \frac{5\cdot 3^6}{2\cdot 13\pi^4}\, L(2,\chi)L(2,\chi^2).$$

Here $L(2,\chi) = \pi^2(s_1 + \omega s_2 + \omega^2 s_3)/9^2$ where $s_1 = \csc^2(\pi/9)$, $s_2 = \csc^2(2\pi/9)$ and $s_3 = \csc^2(4\pi/9)$. Again $L(2,\chi^2) = \bar{L}(2,\chi)$, and so

$$L(2,\chi)L(2,\chi^2) = \frac{\pi^2}{9^4}\left\{\left(\sum_{i=1}^{3} s_i\right)^2 - 3\sum_{1\le i<j\le 3} s_i s_j\right\}.$$

In this case, the $s_i$ are roots of a polynomial $q(z)$ obtained by applying the previous transformation to $(x^{18}-1)/(x^6-1) = 0$. We find $q(z) = 3z^3 - 36z^2 + 96z - 64$, from which $\Sigma s_i = 12$, $\Sigma s_i s_j = 32$, giving $L(2,\chi)L(2,\chi^2) = 16\pi^4/3^7$ and $P_2 = 40/39$.

Also solved by Jeff Lagarias, and by the proposer.

**Integrals of Haar Functions**

6013 [1975, 183]. *Proposed by J. R. Higgins, Cambridgeshire College of Arts and Technology, England*

Let $\{h_r(t)\}_{r=1}^{\infty}$ be the orthonormal Haar functions, defined by

$$h_1(t) \equiv 1, \qquad t \in [0,1]$$

$$h_r(t) = \begin{cases} 2^{m/2} \operatorname{sgn}\sin(2^{m+1}\pi t), & t \in ((k-1)/2^m, (k/2^m)) \\ 0 & \text{elsewhere in } [0,1]. \end{cases}$$

where $r = 2^m + k$, $m = 0, 1, \ldots,$ and $k = 1, \ldots, 2^m$.

Let $p$ be any odd positive integer and $q$ any positive integer such that $p < 2^q$. Put

$$I(p,q,m,k) = \int_0^{p/2^q} h_r(t)\, dt.$$

Show that

$$\sum_{m=0}^{q-1}\sum_{k=1}^{2^m} [I(p,q,m,k)]^2 = \frac{p}{2^q}\left(1 - \frac{p}{2^q}\right).$$

*Solution by L. E. Mattics, University of South Alabama.* Referring to Problem 5960 [1975, 859], we have

$$I(p,q,m,k)^2 = \begin{cases} 2^m I(p,q,m+1)^2 & \text{if } p/2^q \in ((k-1)/2^m, (k/2^m)) \\ 0 & \text{otherwise.} \end{cases}$$

The present result is immediate.

Also solved by C. W. Onneweer, and the proposer.

**Uncountable Sets with All Closed Subsets Countable**

6014 [1975, 183]. *Proposed by Clark H. Kimberling, University of Evansville*

Does there exist an uncountable set of real numbers all of whose closed subsets are countable?

*Solution by T. E. Gantner, University of Dayton.* Yes. In 1908, with the aid of the axiom of choice, F. Bernstein constructed a set $B$ of real numbers such that every uncountable closed set in the real line

intersects both $B$ and the complement $B'$ of $B$. Thus, every closed subset of $B$ or $B'$ must be countable; both $B$ and $B'$ are uncountable because they are non-measurable. For a recent exposition of such sets, see John C. Oxtoby, *Measure and Category*, Springer-Verlag, New York, 1971, pp. 23–24.

Also solved by Paul Humke, H. Kestelman (England), D. A. Moran, John Morgan II, Darwin Peek, Harold Schneider, Frank Siwiec, William Weiss (Canada), Albert Wilansky, and the proposer.

*Editor's comments.* Morgan and Peek note the similarity of this problem to 5126 [1964, 808], and 5704 [1971, 88]. Wilansky notes that the result was also given by Kuratowski and Sierpinski, in Fundamenta Math., 8 (1926) p. 193. Weiss points out the implicit assumption that the closed subsets are closed in the reals.

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## REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
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COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

*We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.*

*Introduction to Ordinary Differential Equations.* By Shepley L. Ross. John Wiley, New York, 1974. ix + 432 pp. $13.95. (Telegraphic Review, May 1975.)

*Introduction to Differential Equations.* By R. C. Buck and Ellen F. Buck. Houghton-Mifflin, Boston, Massachusetts, 1976. vii + 356 pp. $13.50.

*A First Course in Differential Equations.* By Frank G. Hagin. Prentice–Hall, Englewood Cliffs, New Jersey, 1975. ix + 342 pp. $13.50. (Telegraphic Review, October 1975.)

Each of these books is appropriate for use in a one semester introductory course in ordinary differential equations. The students who take such a course at my university come from a variety of disciplines but primarily from chemistry, physics and mathematics. All have had at least two semesters of calculus, but very few have had linear algebra. The emphasis of this course is on techniques and applications; little on proofs of theorems. The topics usually covered are first order equations, higher order equations (with emphasis on second order), series solutions, Laplace transforms, applications, and some work on systems usually done with transforms.

The text by Ross has been used here for quite some time and I have to admit to a certain bias; I like this book. The author states concepts clearly, illustrates them with many examples, and provides ample routine exercises to reinforce the student's understanding of these concepts. There are some minor criticisms which will be alluded to later.

Generally speaking the exposition is good in all three. Hagin's writing is informal and should appeal to the students. In my opinion he uses his examples to motivate much better than the others. The language in Buck and Buck is modern, especially in the chapter on linearity where concepts are stated in terms of operators on linear spaces.

Each of the authors begins with an introduction to modeling. Buck and Buck have done a marvelous job. They illustrate the modeling concept with problems on the motion of a pendulum, the flight of a rocket and the concentration of toxic substances in a fluid. I especially liked the idea of returning to this subject at the end of the text. More often than not, authors of textbooks in mathematics merely stop at the conclusion of the last subject in the book without attempting to reiterate the objectives of the text or to point the way for further investigations by the reader. Buck and Buck is the rare exception.

In chapter two on techniques of solutions, Buck and Buck deal with only a subcollection of exact

differential equations, namely separable equations and linear first order equations. I think the authors justify their omission when they remark that the integrals often used would not be able to be evaluated. Any general explanation of exact differential equations does require some knowledge of the partial derivative and the differential of a function of more than one variable. Hagin has a brief section in which he discusses these ideas. At the conclusion of the chapter both Buck and Buck and Hagin make a strong case for considering numerical solutions. Hagin actually includes a computer program for the improved Euler technique for solving a first order initial value problem.

All seem to have an adequate number of applications. Both Buck and Buck and Hagin integrate the applications into the exercises following the discussions of the techniques. Ross saves the applications of first order equations for a separate chapter. (My students remarked that most of these involve separable differential equations.)

All of the authors discuss standard topics in the sections on second order linear equations. Ross states the essential theorems and illustrates them with a number of examples. Hagin does an excellent job in explaining the ideas with examples. For instance, his introduction of the Wronskian is especially good. All discuss the concepts, such as linear independence of functions, which are required from linear algebra. Buck and Buck spend much more time talking about topics such as linear spaces, linear operators and the basis of a linear space. As mentioned earlier, their results concerning solutions of an equation are written in terms of differential operators.

Neither Ross nor Hagin do enough in the way of motivating the student to understand Euler's formula, $e^{ix} = \cos x + i \sin x$. Buck, however, does develop this result in an earlier section on complex-valued series. The method of undetermined coefficients is discussed from a variety of viewpoints. Hagin uses the idea of an annihilator; Buck and Buck state theorems each of which covers a different possibility for a solution (i.e., polynomial, exponential or trigonometric); Ross defines a collection of "UC" functions and outlines a process by which a particular solution may be found. He remarks, "We frankly admit that this outline of procedure may seem unnecessarily complicated. Once it is understood, however, it frees one from the need of considering separately all of the special cases which it covers." Even so, a simple example to indicate first how the method works and then perhaps a simple special case such as $y'' - y = Ce^x$ to show there are exceptions, might have better justified his procedure.

All three texts use the harmonic oscillator as an illustration of second order linear equations. Ross does the most thorough job. He discusses Hooke's law in detail, then explains the physical significance of each term in the differential equation.

Both Ross and Hagin include a traditional development of power series solutions. They discuss solutions near a regular singular point as well as the solution of Bessel's equation. Buck and Buck treat series solutions as an approximation method but include no discussion of solutions near regular singular points.

Laplace transforms are discussed quite extensively in both Ross and Hagin. Buck and Buck relegate this topic to the appendix. Their brief outline does, however, cover most of the elementary concepts although the instructor might have to supplement the exercises. The material in Ross is more than adequate for the one-semester course which I teach. Hagin extends the use of Laplace transforms to integral equations, partial differential equations and the unit impulse.

Each text contains a lengthy treatment of linear systems and each devotes space to developing the required rudiments of matrices.

The chapters on numerical techniques vary. Hagin and Buck and Buck spend much time on these topics. As mentioned earlier, Hagin even includes programs written in "Basic" and "Fortran." Ross, on the other hand, limits his discussion to solutions of first order linear equations. If the emphasis in a course is numerical then probably Hagin or Buck and Buck are more suitable as texts.

All three books can satisfy a variety of tastes and any one of these should be considered as a text for a course of the type I have described.

J. RICHARD MORRIS, Virginia Commonwealth University

## TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, L*. *100 Numerical Games*. Pierre Berloquin. Scribner, 1976, viii + 152 pp, $7.95. Actually a collection of challenging popular puzzles by the French equivalent of Martin Gardner. Complete solutions are provided. LAS

GENERAL, P**, L. *Mathematical Sciences in Canada*. Klaus P. Beltzner, A. John Coleman, Gordon D. Edwards. Science Council of Canada, 1976, 339 pp, $7.80 (P). An impressive and frequently eloquent statement--part report, part analysis, part exhortation--of the total role of mathematics in (Canadian) society, based on questionnaires, invited briefs, seminars and university site visits. Major conclusions: the mathematical community must increase communication with the users of mathematics. Contains much of unique value to anyone preparing students for careers in the mathematical sciences. LAS

GENERAL, P**, L*. *Mathematical Developments Arising from Hilbert Problems*. Ed: Felix E. Browder. Proc. of Symp. in Pure Math., V. 28. AMS, 1976, xii + 628 pp, $37.60. 23 papers on modern themes related to Hilberts 23 problems (not in one-to-one correspondence) from the May, 1974 conference at DeKalb. Includes a reprint of the English translation of Hilbert's original problems address, a collection of contemporary problems solicited and edited by Dieudonné, and photographs of the conference speakers. LAS

GENERAL, T*(13: 1, 2). *Fundamentals of Mathematics*. William M. Setek, Jr.. Glencoe Pr, 1976, 520 pp, $12.95. Mathematics for liberal arts students. Light, easy style. Excellent illustrations, examples, problems. Wide range of topics including logic, statistics, systems of numeration. LH

BASIC, T*(13: 1), S. *Mathematics for Electronics*. Charles M. Thomson. P-H, 1976, xx + 634 pp, $12.95. Operations on numbers and polynomials, quadratic formula, trigonometry, exponents, logs, vectors--all introduced as necessary tools for working with electronics (current, voltage, resistance, in d.c. and a.c. circuits). The applied context should make this content exciting and meaningful to students in technical programs. LCL

BASIC, T(13: 1). *Beginning Algebra*. Ignacio Bello, Jack R. Britton. Saunders, 1976, xi + 435 pp, $11.95. Elementary. Motivates students with newspaper sportswear advertisements. Workbook accompanies text. Self-tests at end of each chapter. LH

PRECALCULUS, T(13: 1). *Precalculus Mathematics for Technical Students*. Panayotis D. Mavrommatis, Philip Reichmeider. P-H, 1976, viii + 408 pp, $13.50. Includes trigonometry, logarithms, complex numbers and progressions, with exercises illustrating a particularly broad range of technical applications. PJC

PRECALCULUS, T*(13: 1). *Precalculus Algebra and Trigonometry*. Daniel D. Benice. P-H, 1976, xiii + 365 pp, $10.95. A precalculus text in the traditional style which makes a concerted effort to emphasize those skills which cause problems in calculus. Also includes introductions to probability and calculus. An adequate supply of examples and exercises. CEC

EDUCATION, P. *Beiträge zur Mathematikdidaktik: Festschrift für Wilhelm Oehl*. Heinrich Winter, Erich Wittmann. Hermann Schroedel, 1976, 220 pp, DM 49,80. Eleven articles on aspects of the teaching and learning of mathematics especially at the primary and secondary level, together with an overview of Oehl's life and work. JAS

EDUCATION, S(15-17), P, L. *Mathematics Learning in Early Childhood: Thirty-Seventh Yearbook*. Ed: Joseph N. Payne. NCTM, 1975, vi + 297 pp, $12. Considers the cognitive development of children age 3-8, how they learn mathematics and the related curriculum and research. Stresses problem solving, relation of mathematics to child's real world and a developmental point of view. Theory supported by many, many classroom activities. PSJ

EDUCATION, T*(15-16: 1), L. *Teaching Secondary School Mathematics*. Stephen Krulik, Ingrid B. Weise. Saunders, 1975, x + 243 pp, $13.95. Short and sweet. A good blend of the history of mathematics teaching, current trends and teaching suggestions. Concise examples followed by well-chosen references for further study. "Problems for Investigation" in each chapter provide an excellent study guide. Recommended for those who want a solid base in the textbook, yet ample time for supplementary work. PSJ

HISTORY, P. *Die Werke von Jakob Bernoulli, Band 3*. B.L. van der Waerden. Birkhauser, 1975, x + 585 pp, $40. This third volume contains texts with critical commentary of Bernoulli's papers and notes on probability. A number of these texts are of handwritten notes not previously available. The complete series, five volumes in all, will contain a reprinting of the 1744 volume of Bernoulli's works, a fourth volume of additional notes and a fifth volume of personal letters and other documents relating to Bernoulli as a person. JAS

HISTORY, P*. *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry.* Wilbur Richard Knorr. Texts and Stud. in Hist. of Logic and Philo., V. 15. Reidel, 1975, ix + 374 pp, $49. Turns inside out the prevalent thesis of a pre-Euclidean "foundations crisis" as a factor in the reformation of geometry. Claims that, rather than the dialecticians (such as Zeno) provoking the mathematicians to construct a valid theory of incommensurables, that theory came first and afforded the logicians their model of deductive reasoning. The author tries carefully to sort out the sequence of Greek mathematical work leading up to the *Elements* but does not even allude to the possibility of pre-Greek roots (see, e.g., A. Seidenberg, *Archive Hist. Exact. Sci.*, 14:4 (1975) 263-295). No topical index. The price is simply outrageous. PJC

HISTORY, P, L. *The New Elements of Mathematics.* Charles S. Peirce. Ed: Carolyn Eisele. Humanities Pr, 1976. *V. I: Arithmetic,* xl + 260 pp, $29.50; *V. II: Algebra and Geometry,* xxxi + 672 pp, $76. The first two of four planned volumes on Peirce's mathematical papers. (His other works have already been published in *Collected Papers,* Vols. I-VIII.) Each volume contains an extensive critical introduction by the editor. LAS

COMBINATORICS, P. *Parallelisms of Complete Designs.* Peter J. Cameron. London Math. Soc. Lect. Notes, No. 23. Cambridge U Pr, 1976, 144 pp, $9.95 (P). A study of parallelism in various combinatorial structures. Discusses the parallelogram property, Steiner systems, connections with graph theory, automorphism groups of parallelisms. No exercises, but some open questions are raised. SG

COMBINATORICS, S, P, L. *On Numbers and Games.* J.H. Conway. Acad Pr, 1976, ix + 238 pp, $15.75. A highly original, parallel development of a comprehensive "surreal" (see Knuth: *Surreal Numbers*) theory of numbers (integers, rationals, reals, infinitesimals, ordinals, etc.) and a very general theory of positional games (e.g., Nim, Kayles, Hackenbush). Conway's fertile imagination, unconventional notation and whimsical vocabulary produce an exotic mix of mathematics and mystery. LAS

NUMBER THEORY, T**(16-17: 1, 2), S, L*. *Introduction to Analytic Number Theory.* Tom M. Apostol. Springer-Verlag, 1976, xii + 338 pp, $14.80. A beautiful introduction to analytic number theory for the strong undergraduate or beginning graduate student. No previous number theory is assumed. Includes a good historical introduction, Dirichlet's Theorem, the Prime Number Theorem, excellent problems, a bibliography and much more. CEC

LINEAR ALGEBRA, P. *Generalized Inverses and Applications.* Ed: M. Zuhair Nashed. Acad Pr, 1976, xiv + 1054 pp, $34. 13 lengthy survey papers plus a 250 page annotated bibliography from an October, 1973 seminar at the Mathematics Research Center in Madison. LAS

LINEAR ALGEBRA, T(14-15: 1), S, L. *Linear Algebra with Geometric Applications.* Larry E. Mansfield. Pure and Appl. Math., V. 34. Dekker, 1976, ix + 496 pp, $14.50. Introductory text with theoretical flavor--expansive exposition with careful attention to geometrical interpretations (in Euclidean space). Includes chapters on second degree curves and surfaces, and canonical forms. Large problem sets (computational and theoretical) with unusually complete appendix of answers and suggestions to all problems. No mention of applications outside mathematics. LCL

LINEAR ALGEBRA, T(16-17: 2), P. *Introduction to Vectors and Tensors.* Ray M. Bowen, C.-C. Wang. Plenum Pr, 1976. *V. 1, Linear and Multilinear Algebra,* xiii + 250 pp, $19.50; *V. 2, Vector and Tensor Analysis,* xiii + 197 pp, $19.50. A textbook intended for engineers and scientists. Its flavor and level of difficulty are suggested by the following sample of topics treated in the second volume: Euclidean manifolds, hypersurfaces in a Euclidean manifold, vector fields and differential forms, integration of fields on Euclidean manifolds. JD-B

ALGEBRA, T(18: 2), P. *Character Theory of Finite Groups.* I. Martin Isaacs. Pure and Appl. Math., V. 69. Acad Pr, 1976, xii + 303 pp, $29.50. Properties of characters with application to the structure of finite groups. Emphasis on characters over complex numbers; final chapters on recent, rather specialized, results. Problem sets with hints. LCL

ALGEBRA, T(18: 1, 2), P, L. *Automata, Languages, and Machines, Volume B.* Samuel Eilenberg. Pure and Appl. Math., V. 59B. Acad Pr, 1976, xiii + 387 pp, $22.50. A lot of relatively new algebra is developed and then applied to the study of sets recognized by finite state automata and sequential functions. Most of this volume is independent of Volume A (TR, February 1975). Two chapters by Bret Tilson dealing with complexity theory are included. Exercises. CEC

FINITE MATHEMATICS, T*(14-16: 1), S, P, L*. *Finite Markov Chains.* John G. Kemeny, J. Laurie Snell. Springer-Verlag, 1976, ix + 210 pp, $14.80. The 1960 Van Nostrand classic reprinted here in Springer-Verlag's series of Undergraduate Texts. Basic ideas and applications; assumes minimal mathematical background--e.g., restricting attention to finite chains avoids infinite series and eigenvalues. LCL

CALCULUS, T(13: 2). *Freshman Calculus, Second Edition.* Robert A. Bonic, *et al.* Heath, 1976, x + 435 pp, $14.95. Moderate revision of an interesting text (first edition TR, January 1972). Limits and continuity have been added in Chapter 2 and the definite integral as a limit of sums has been added in Chapter 5. More exercises in applied mathematics included. LLK

CALCULUS, S(13). *Interface: Calculus and the Computer.* David A. Smith. HM, 1976, xv + 260 pp, $6.95 (P). Can be used in a supplemental computer lab attached to a beginning calculus course. Develops standard concepts in calculus from the computational point of view. Seems to presume that the reader will make the jump from discussion given to coding in a programming language, since little actual code is given in the text. Good historical references. Some flow-charts. Exercises. Bibliography. Appendices on FORTRAN, BASIC, and PL/1. RJA

CALCULUS, T*(13: 2), L. *Calculus with Applications and Computing, V. I.* Peter Lax, Samuel Burstein, Anneli Lax. Springer-Verlag, 1976, xi + 513 pp, $14.80. A revolutionary text by master expositors presenting a streamlined selection of high-priority classical topics refreshingly enriched by lengthy development of certain applications: probability and diffusion, mechanics, vibrating systems, population dynamics, chemical reactions. Calculator and computer based numerical methods are "applied lovingly" throughout the text. LAS

DIFFERENTIAL EQUATIONS, T*(17: 1, 2), S, P, L. *Ordinary Differential Equations in the Complex Domain.* Einar Hille. Wiley, 1976, xi + 484 pp, $27.95. A well-written text presenting the behavior of solutions of ordinary differential equations in the complex domain. The author has incorporated material not available in other books with full and extensive treatment of the standard topics. Many historical comments, lots of problems and exhaustive lists of references enhance this book. CEC

DIFFERENTIAL EQUATIONS, T*(15: 1), S, L*. *Introduction to Differential Equations.* R. Creighton Buck, Ellen F. Buck. HM, 1976, x + 418 pp, $13.50. An introduction to the techniques of solving differential equations which has a significant emphasis on modeling. This is a solid one semester course which includes standard solution techniques, numerical methods, approximation methods and systems of differential equations. CEC

NUMERICAL ANALYSIS, S(16-18), P. *Procédures Algol en Analyse Numérique, Tome 1.* J. Kuntzmann. Ed Cent Nat Rech Scient, 1975, 324 pp, 40,00 F. Collection of procedures written in ALGOL that provide computer solutions to problems commonly encountered in numerical analysis. Provides short topical discussion prior to each procedure and an example of its use afterwards. RJA

NUMERICAL ANALYSIS, P. *Numerical Methods for Differential Systems: Recent Developments in Algorithms, Software, and Applications.* L. Lapidus, W.E. Schiesser. Acad Pr, 1976, xi + 291 pp, $13.50. Algorithms, some available with computer codes; mostly for stiff ODE systems, some PDE. Some applications presented, mostly in chemistry. Panel discussion from 1975 symposium in Boston. Index. JAS

FUNCTIONAL ANALYSIS, T(17), L. *An Invitation to C*-Algebras.* William Arveson. Grad. Texts in Math., V. 39. Springer-Verlag, 1976, x + 106 pp, $12.80. A C*-algebra is a Banach algebra with involution. This monograph studies C*-algebras which can be represented as algebras of compact operators. PJM

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-512: Spaces of Analytic Functions.* Ed: O.B. Bekken, B.K. Øksendal, A. Stray. Springer-Verlag, 1976, viii + 204 pp, $9.50 (P). Papers from the seminar on functional analysis and function theory held at Kristiansand, Norway, June, 1975; most are on algebras and spaces of analytic functions in one or several complex variables. JAS

OPTIMIZATION, T(15-16: 2), S. *Mathematics of Organization.* Mircea Malita, Corneliu Zidăroiu. Abacus Pr (Distr. in U.S. by ISBS), 1974, 383 pp, $30. Basic deterministic and statistical methods in operations research. Begins each chapter with practical problems and ends with numerical examples. Applications rather than rigorous development. Includes nonlinear, dynamic programming, queueing theory. Expensive. LH

OPTIMIZATION, T(18: 1), S, P. *Deterministic and Stochastic Optimal Control.* Wendell H. Fleming, Raymond W. Rishel. Appl. of Math., No. 1. Springer-Verlag, 1975, 222 pp, $24.80. Stochastic differential equations: dynamic programming methods for control of Markov diffusion processes. Relationship between dynamic programming and Pontryagin's principle. Discontinuous feedback controls, linear regulator problem. The Kalman-Bucy filter. LH

ANALYSIS, T(15-16: 2), L. *Introduction to Mathematical Physics.* Charlie Harper. P-H, 1976, xvi + 301 pp, $16.50. A concise presentation of the mathematical topics needed by physics majors: vector analysis, operators and matrices, functions of a complex variable, residues, differential equations, special functions, Fourier series and transforms, tensor analysis. SG

ANALYSIS, T(15-16: 1, 2). *Advanced Calculus for Applications, Second Edition.* Francis B. Hildebrand. P-H, 1976, xiii + 733 pp, $15.95. To the 1962 edition have been added: some proofs of theorems, a section on the method of variation of parameters for partial differential equations, some text in other sections, a chapter "Applications of Analytic Function Theory", 250 problems. Still usable in a wide variety of courses. DFA

ALGEBRAIC GEOMETRY, P. *Introduction to Algebraic Geometry Through Affine Algebraic Groups.* Alain Robert. Pure and Appl. Math., No. 44. Queen's U, 1976, 298 pp, (P). Regarded by the author as an introduction to the books of Borel and Humphreys on algebraic groups. Concerned primarily with algebraic groups which are used to motivate the necessary geometry. Covers affine groups, representation theorem, quotients, complete varieties, triangular and solvable groups, Cartan subgroups, outline of classification of simple groups. SG

GEOMETRY, S*, L*. *Polyhedra, A Visual Approach.* Anthony Pugh. U of Calif Pr, 1976, x + 118 pp, $4.95 (P). Descriptions, illustrations, and properties of regular and semiregular polyhedra, including geodesic and Kepler-Poinsot polyhedra and a chapter on joining of polyhedra. Several model-making techniques are described; the reader is referred to other works for planar nets. PJC

GEOMETRY, S*, L*. *An Introduction to Tensegrity.* Anthony Pugh. U of Calif Pr, 1976, x + 121 pp, $4.95 (P). Exploratory work on tensegrity, a word and concept treated by R.B. Fuller (from tensional integrity) which refers to a balance of tensile and compression forces. The physical structures described here are modelled on polyhedra; they are formed of rigid struts suspended from one another by tendons. Excellent practical tips on building models. PJC

GEOMETRY, S*(14-18), L*. *Geodesic Math and How to Use It.* Hugh Kenner. U of Calif Pr, 1976, xi + 172 pp, $5.95 (P). Clear and comprehensive guide to calculations for domes--by far the best yet. Geodesics are treated as extensions of tensegrity structures, and (for the first time) spherical co-ordinates are used. Contains 40 pages of tables plus some HP-65 calculator programs. No index.PJC

GEOMETRY, P. *Foundations of Geometry, Selected Proceedings of a Conference.* Ed: P. Scherk. U of Toronto Pr, 1976, x + 336 pp, $10 (P). Reports of the work of 13 mathematicians who participated in a conference on the foundations of geometry, held at University of Toronto from July 17 to August 18, 1974. RJA

TOPOLOGY, P. *Lecture Notes in Mathematics-509: Contact Manifolds in Riemannian Geometry.* David E. Blair. Springer-Verlag, 1976, vi + 146 pp, $7.40 (P). A contact manifold is an odd-dimensional manifold with a nice 1-form. Author studies geometric consequences of existence of a contact structure. PJM

TOPOLOGY, P. *Localization and H-Spaces.* Martin Arkowitz. Lect. Notes Ser., No. 44. Aarhus U, 1976, iii + 143 pp, $2 (P). An abelian group is a Z-module. Its localization at p is a Z/pZ-module. Given a topological space, its localization is a space whose homotopy groups are localizations of $\pi_n(X)$. This monograph studies localization of groups and spaces with applications to H-spaces. PJM

TOPOLOGY, P. *A Differential Geometric Study on Strongly Pseudo-Convex Manifolds.* Noboru Tanaka. Kinokuniya Books, 1975, 158 pp, $7.50 (P). Cohomology groups of nice submanifolds of complex manifolds, calculated by geometric methods. PJM

TOPOLOGY, T(18), L. *Rings of Continuous Functions.* Leonard Gillman, Meyer Jerison. Grad. Texts in Math., V. 43. Springer-Verlag, 1976, xiii + 300 pp, $14.80. Reprint of the 1960 Van Nostrand original edition, changed only by inclusion of a brief new preface noting more recent volumes treating subsequent developments. LAS

TOPOLOGY, T(18: 1), P. *Lecture Notes in Mathematics-517: Proximal Flows.* Shmuel Glasner. Springer-Verlag, 1976, viii + 153 pp, $7.40 (P). From a course at Maryland in 1973-74. Considers Ellis's "algebraic" theory of topological dynamics and Furstenberg's theory of boundaries for Lie groups and of harmonic functions, and the interactions of these theories. Descriptive material, references to sources open each chapter. DFA

PROBABILITY, S*(15-18), P*, L*. *Theory of Probability: A Critical Introductory Treatment, V. 2.* Bruno de Finetti. Trans: Antonio Machi, Adrian Smith. Wiley, 1975, xviii + 375 pp, $29.50. Second of two volumes (TR, V. 1, December 1974) expositing a foundation of subjective probability whose cornerstone is this: "Probability does not exist." This volume treats stochastic processes, multi-dimensional phenomena and statistical inference. "Destined ultimately", claims D.V. Lindley in the preface, "to be recognized as one of the great books of the world." LAS

STATISTICS, T(13-14: 1, 2), *Beginning Statistics.* R. Lowell Wine. Winthrop Pub, 1976, viii + 432 pp, $13.95. Presupposes only high school algebra. No discussion of subjective probabilities. Includes a chapter on correlation analysis. FLW

STATISTICS, T(17-18: 1, 2), P. *Tratat de Statistică Matematică, V. I: Selectie si Estimatie.* Gheorghe Mihoc, Virgil Craiu. Editura Academiei Romania, 1976, 451 pp, Lei 30. A thorough and advanced treatment of theoretical statistics. JAS

STATISTICS, T(13: 1), *Understanding Statistics, Second Edition.* William Mendenhall, Lyman Ott. Duxbury Pr, 1976, xi + 387 pp, $12.95. (First Edition TR, March 1974.) More attractive format; larger, easier to read print. Additional chapter on the binomal experiment; otherwise unchanged. LCL

STATISTICS, T(13: 1), *Elementary Statistical Method.* James Lumsden. U of W Australia Pr, 1974, xiv + 166 pp, $10.50 (P). Presupposes only high school algebra. Little discussion of probability. Factor analysis is discussed in a brief addendum. FLW

STATISTICS, T(14-15: 1), *Einführung in die Statistik.* Paul R. Lohnes, William W. Cooley. Hermann Schroedel, 1976, 328 pp, DM 44,80 (P). A translation of the 1968 Wiley text *Introduction to Statistical Procedures: With Computer Exercises.* JD-B

STATISTICS, T(13-15: 1, 2), S, L. *Basic Statistics, Fifth Edition.* Dick A. Leabo. Irwin, 1976, xxii + 707 pp, $14.95. For undergraduate or graduate students in business administration and economics. Includes consideration of index numbers, time series, multiple regression analysis, and Bayesian decision theory. FLW

STATISTICS, S*(13-16), L*. *Statistics: A Guide to Business & Economics.* Ed: Judith M. Tanur, *et al.* Holden-Day, 1976, xii + 132 pp, $3.50 (P). The first of several planned specialized sequels to the widely-used 1972 compendium *Statistics: A Guide to the Unknown*, this one provides a dozen brief, concrete, nonmathematical essays on the use of statistics in counting and estimating economic and social data. LAS

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Loren Haskins, Carleton; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Frank L. Wolf, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

Assistant Professor D. F. Addis, Texas Christian University, has been promoted to Associate Professor.

Dr. R. A. Alo, Carnegie-Mellon University, has been appointed Professor and Head of the Department of Mathematics at Lamar University.

Assistant Professor R. H. Franke, Naval Postgraduate School, has been promoted to Associate Professor.

Professor J. P. King, Lehigh University, received the $3000 Donald B. and Dorothy L. Stabler Foundation Award for excellence in teaching. It was presented at Lehigh's 29th Annual Faculty Dinner in May 1976.

Associate Professor Wayne Woodworth, Chairman of the Mathematics Department at Drake University, has been promoted to Professor. He will be Visiting Associate Professor in the Department of Mathematical Sciences at Clemson University, 1976–77.

Professor Emeritus William L. Ayres, Southern Methodist University, died on January 24, 1976, at the age of 70. He was a member of the Association for fifty-three years.

Mr. Bruce B. Clark, IBM, Gaithersburg, Maryland, died in August 1975 at the age of 54. He was a member of the Association for twenty-seven years.

Professor Emeritus J. Stuart McNair, State University of New York, Plattsburgh, died on February 13, 1976, at the age of 69. He was a member of the Association for forty-two years.

Mr. Philip A. Rognlie, University of North Dakota, died on December 11, 1975, at the age of 67. He was a member of the Association for twenty-seven years.

Dr. Paul McCartney Swingle, University of Miami, died on March 29, 1976, at the age of 76. He was a member of the Association for thirty-seven years.

## EUROPEAN MEETING OF STATISTICIANS 1977 (Preliminary announcement)

The 1977 European Meeting of Statisticians will be held at the *Katholieke Universiteit te Leuven*, Belgium, August 22–26, 1977. This conference is sponsored jointly by the Institute of Mathematical Statistics and the Bernoulli Society for Mathematical Statistics and Probability. This meeting will be similar in format to the previous European Meetings of Statisticians, and will feature both invited and contributed papers. The program committee consists of R. L. Plackett (chairman), F. Eicker, A. N. Shiryaev and E. C. van der Meulen. The chairman of the local arrangements committee is J. L. Teugels. More detailed announcements will follow in the near future. For information write to: EMS 1977 Program Committee, Department of Mathematics, Katholieke Universiteit te Leuven, Celestijnenlaan, 200B, 3030 Heverlee, Belgium.

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# MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## MARCH MEETING OF THE OKLAHOMA-ARKANSAS SECTION

The thirty-eighth annual meeting of the Oklahoma-Arkansas Section of the MAA was held at Hendrix College, Conway, Arkansas on March 26 - 27, 1976. There were 67 members and 41 others in attendance, making a total of 108. Professor James Yates of Central State University presided over the meeting.

At the dinner held on Friday evening at the Ramada Inn, Mr. M. G. Tiefenback of Hendrix College received an award for scoring the highest on the William Lowell Putnam Competition among the participants from Oklahoma and Arkansas. The invited address entitled "The Graph of a Group as Seen by a Nonexpert" was given on Friday afternoon by Professor J. W. Keesee of the University of Arkansas at Fayetteville. Friday evening, Professor R. C. Buck of the University of Wisconsin, Madison, gave the fourth N. A. Court Lecture. The title was "A Role for Speculation."

The following officers were elected: Chairman, Professor Temple Fay, Hendrix College; Past-Chairman, Professor James Yates, Central State University; First Vice-Chairman, Professor Verbal Snook, Oral Roberts University; Second Vice-Chairman, Leon Semkoff, Claremore Junior College; and, Secretary-Treasurer, E. K. McLachlan, Oklahoma State University. The following were appointed to continue as co-chairmen of the High School Mathematics Contest: For Arkansas, Professor Claude Duplissey of the University of Arkansas at Little Rock, and for Oklahoma, Professor Thomas Cairns of the University of Tulsa. Other parts of the program were a panel discussion entitled "Credit for the CLEP examinations in Mathematics," which was led by Professor James Scroggs of the University of Arkansas at Fayetteville, and an expository talk on "What is Mathematical Geophysics?" given by Professor J. B. Bednar of the University of Tulsa.

In Oklahoma 61 schools involving 2694 students and in Arkansas 56 schools involving 1886 students participated in the high school contest.

The following papers were presented:

1. *Characterizations of compact and countably compact spaces*, by L. L. Herrington, University of Arkansas at Pine Bluff.
2. *S-closed spaces*, by Travis Thompson, University of Arkansas at Fayetteville.
3. *Functions with strongly-closed graphs*, by P. E. Long, University of Arkansas at Fayetteville.
4. *Uniform continuity, neighborhood spaces and completion*, by A. C. Cochran, University of Arkansas at Fayetteville.
5. *A brief proof of Cauchy's theorem*, by R. L. Rea, Hendrix College.
6. *Two new proofs for the countability of the rationals*, by Gary Samuelson and Ken Sewell, Harding College.
7. *Two more new proofs for the countability of the rationals*, by Phil Elam and Alan Whitten, Harding College.
8. *Probabilities in proofreading: a generalization*, by David Bonner, Hendrix College.
9. *Moments for the median in Cauchy distributions*, by R. B. Deal, Oklahoma University Medical Center.
10. *Johnson $S_n$-distribution and its application to guidance system centering error*, by Don Pate, Federal Aviation Administration.
11. *An equal-sum assignment of consecutive integers to the vertices of an N-cube*, by D. Y. Hsu, University of Arkansas at Little Rock.
12. *A proof for Euler's theorem*, by C. H. Harbison, University of Arkansas at Little Rock.
13. *On the countability of the Cartesian product of two countable sets*, by Sherry Bradke, University of Central Arkansas.
14. *On ternary operations induced by semi-group operations*, by Janet Dillahunty, Hendrix College.
15. *On local principal ideal rings*, by David Kircher, Oklahoma Baptist University.
16. *From numerology to number theory*, by Dean Minasian, Oral Roberts University.
17. *The limiting function of a curve stich*, by W. R. Orton III, Hendrix College.
18. *Smoothability in Banach spaces*, by D. C. Kemp, Oklahoma State University.
19. *The algebra of pseudotopologies*, by C. V. Riecke, Cameron University.
20. *Some nowhere equicontinuous homeomorphisms*, by Ki Woong Kim, Oklahoma State University.
21. *On subspaces of products of two- and three-point topological spaces*, by M. G. Tiefenback, Hendrix College.
22. *Some operators on arithmetic functions*, by David Husnian, Central State University.
23. *On difunctional and circular relations*, by Alma Elizabeth Posey, Hendrix College.
24. *Utilization of a computer in analysis of chi-square distribution*, by Delmar Clark, Oklahoma Baptist University.
25. *Mathematics and the liberal arts*, by Verbal Snook, Oral Roberts University.
26. *Comparison of a personalized system of instruction and the traditional method in college mathematics*, by J. E. Smith, Claremore Junior College.
27. *On the well-ordering principle*, by Timothy Ford, Oral Roberts University.
28. *Some exponent properties in higher dimensional real Carley-Dickson algebras*, by H. C. Brown, Harding College.
29. *A characterization of the derivations on the residue field of a ramified V-ring which lift*, by T. F. Peter, University of Arkansas at Little Rock.
30. *Structure of subsemi-group generated by two mutually prime positive numbers*, by Tetsundo Sekiguchi, University of Arkansas at Fayetteville.
31. *Some embedding theorems for groups*, by T. K. Teague, Hendrix College.

E. K. MCLACHLAN, *Secretary-Treasurer*

## APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The spring meeting of the Allegheny Mountain Section of the MAA was held at West Virginia University in

Morgantown, West Virginia, on April 23 and 24, 1976, with Dr. I. D. Peters as host. Approximately 125 members registered at the meeting.

The Friday evening portion of the program consisted of an address by Norman Zabusky, University of Pittsburgh, entitled "Synergetic Mathematics and Nonlinear Waves." This was followed by a panel on "The Role of Computers in the Mathematics Curriculum" moderated by Richard McDermot, Allegheny College, Meadville, PA. The panel members included J. S. Frame, Michigan State University; Benjamin Haytock, Allegheny College; Thomas Porsching, University of Pittsburgh; and E. G. Whitehead, University of Pittsburgh.

Fifteen minute talks by faculty included:

*Trisections and the sine*, by J. C. Eaves, W. V. U.
*Zeeman's catastrophe machine*, by Carl Irwin, W. V. U.
*The Wedderburn-Malcev theorem and the gospel of inertial subalgebras*, by Nicholas Ford, Penn State University, Fayette.
*Reflecting harmonic functions of two variables*, by Oleg Carleton, W. V. U.
*Murasaki and Bell and Catalan in* 1000 *A. D.*, by H. W. Gould, W. V. U.
*The game of Go*, by J. B. Kim, W. V. U.

Student presentations included:

*On the intersection of Sylow subgroups in a finite simple group*, by Suzy Stewart, Allegheny College.
*An application of Boolean algebra*, by Alfred Kabana, Duquesne University.
*The Stone-representation theorem in Boolean algebras*, by Charles Wills III, Duquesne University.
*The girth of the unit ball in Hilbert space*, by Kate Boaz, Allegheny College.
*How to compute large numbers*, by Greg Stump, I. U. P.
$\pi$ *to* 10,000 *places*, by John Svedman, W. V. U.
*Comparison of linear programming algorithms*, by G. E. M. Pope, W. V. U.
*On teaching a computer to learn to play games*, by George Bradley, Allegheny College.
*Cryptanalysis*, by Dorothy Divers, Allegheny College.

Saturday morning there were three invited addresses:

1. *Factors of the binomial circulant determinant*, by J. S. Frame, Michigan State University.
2. *The equal temperament musical scale and some related geometry*, by I. J. Schoenberg, University of Pittsburgh.
3. *Permutations with restricted positions and Hamiltonian paths in graphs*, by David Roselle, Virginia Polytechnic Institute and State University, Secretary, MAA.

A feature of the Saturday morning program was a panel discussion moderated by Charles Cable, Allegheny College, on "Enrollment Trends, Curriculum Innovations and Problems Facing Mathematics Departments". The panelists were Dave Brown, Bethany College; Mary McCammon, Pennsylvania State University; and Richard Moore, Carnegie Mellon University.

Professor James Derr, Chairman of the Section, presided at the Business Meeting. The secretary's report included the list of Putnam competition winners. The top four students from the section in the Putnam competition were J. E. Brosius, Penn State; Susan Stewart, Allegheny College; Lisa Mantini and Kathryn J. Blackmond, of the University of Pittsburg. A one year subscription to the *Mathematics Magazine* will be awarded to these students. A report on the High School Mathematics Contest was given by Professor Francis Hall of Penn State University, Fayette Campus, and Professor I. D. Peters of West Virginia University, for Western Pennsylvania and West Virginia, respectively. The Sectional Governor, Earle Myers of the University of Pittsburg, reported on the Speakers Bureau, the Winter Meeting of the MAA, and the Preparation of Students, Grades K-12. A report was also given by the Secretary of the MAA, Dr. David Roselle. Dr. M. R. Woodard, Indiana University of Pennsylvania, was commended for his years of service as secretary-treasurer.

Article III of the By-Laws was amended and the change was sent to the Board of Governors of the MAA for approval.

Officers elected were Frank Kocher, Pennsylvania State University, Second Vice-Chairman; and J. W. Milsom, Butler County Community College, Secretary-Treasurer. Continuing as Chairman is James Derr of West Virginia University; Albert Rabenstein of Washington and Jefferson College moved up to First Vice-Chairman.

J. W. MILSOM, *Secretary-Treasurer*

## APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirty-fifth annual meeting of the Metropolitan New York Section of the MAA was held at Hofstra University on April 25, 1976. One hundred persons registered.

Professor Gerald Freilich of Queens College, CUNY, Chairman of the Section, presided at the morning session which began with the business meeting. Professor William Zlot of New York University, Sectional Governor, reported on major items before the Board of Governors in the past year, and committee chairmen gave their reports as follows: Professor Helen Siner, Staten Island Community College, CUNY, reported on the Section Committee for Two-Year Colleges; Professor Godfrey Isaacs, Lehman College, CUNY, reported on the Mathematics Services Committee; Professor Howard Kleiman, Queensborough Community College, CUNY, reported on the Speakers' Bureau; and Dr. Harry Ruderman, Hunter College High School, reported on the Greater Metropolitan New York Math Fair, and the MAA High School Contest.

The Section Award to the highest regional scorer in the Putnam Competition was presented to Lorenzo Traldi of Queens College, CUNY.

The following Section officers were elected for 1976–79: Secretary, Lily Christ, John Jay College, CUNY; Treasurer, Howard Kleiman, Queensborough Community College, CUNY.

The proposed amendment to the Section By-Laws affecting the nomination and election of officers was discussed and unanimously approved.

The morning session continued with the following lectures:

1. *The dynamic days of dynamic programming*, by H. N. Shapiro, Courant Institute, NYU.
2. *Set theory and American mathematics at the turn of the century*, by Joseph Dauben, Lehman College, CUNY.

Professor Maurice Nadler of Pace University, Vice-Chairman for Four-Year Colleges, presided at the afternoon session which began with a symposium on *Applications of mathematics in the real world.* Clifford Marshall of the Polytechnic Institute of New York, Farmingdale, spoke on *The role of mathematics in operations research*; Marcel Tennenbaum of Hofstra University spoke on *The role of mathematics in economics*; and Francis O'Grady of the Metropolitan Life Insurance Company spoke on *The Role of Mathematics in Actuarial Science.*

The following student and faculty papers were then given in parallel sessions:

1. *Unitary arrays and square matrices*, by L. Ausubel, student, Benjamin Cardoza H. S.
2. *Quasi-practical numbers*, by H. J. Hindin, Empire State College.
3. *A numerical measure of complexity*, by J. Propp, student, Great Neck North Senior H. S.
4. *The traveling salesman problem*, by Samuel Kohn, New York Institute of Technology.
5. *A trip through topology*, by Roy Pomerantz, student, Shelter Rock Junior H. S.
6. *Triangle fitting*, by M. Berger, student, Benjamin Cardoza H. S.
7. *Finite circle geometries of three dimensions*, by W. F. Orr, Hofstra University.
8. *The smallest regular tetrahedron inscribed in a given tetrahedron*, by L. Gerber, St. John's University.
9. *Infinite transforms on Euclidean space*, by J. Peters, graduate student, Stevens Institute of Technology.
10. *Shape theory and the Lakta-Volterra equations*, by H. Hastings, Hofstra University.
11. *SPEAK: A computer synthesis speech program*, by C. Hansen, student, Cold Spring Harbor H. S.
12. *When is a solution correct to the nearest tenth?*, by G. Zirkel, Nassau Community College.
13. *The theory and application of Bayesian decision analysis*, by A. Cesar, U. S. Merchant Marine Academy.
14. *Stochastic models in neurophysiology*, by J. Augenbaum, student, Touro College.
15. *An exponential alternative to logarithmic differentiation*, by M. Feigenbaum, U. S. Merchant Marine Academy.
16. *An alternative to Gaussian elimination*, by M. Machover, St. John's University.
17. *A computer oriented calculus program*, by S. Gordon, Suffolk County Community College.
18. *Mathematical aspects of CUNY restructuring proposals*, by H. E. Bohigian, John Jay College, CUNY.
19. *International college prep math*, by M. Griffin, United Nations International School.
20. *A class of subgroups of $S_\infty$ that change asymptotically few natural numbers*, by G. Stoller, Brooklyn, New York.
21. *Analyzing as a secondary school activity*, by F. Wilson, United Nations International School.

RORA IACOBACCI, *Secretary*

## APRIL MEETING OF THE MISSOURI SECTION

The annual Spring meeting of the Missouri Section of the MAA was held at Southwest Missouri State University, Springfield, Missouri, on April 9 and 10, 1976. A total of 93 registered for the meeting.

The officers for 1976–77 are as follows: Chairman, F. W. Wilke, University of Missouri-St. Louis; Vice-Chairman, G. C. Schrag, Central Missouri State University; Secretary-Treasurer, H. K. Stumpff, Central Missouri State University; Past Chairman, J. R. Downing, Southwest Missouri State University.

Dean Robert Gilmore of Southwest Missouri State University welcomed the group. Except for invited

addresses, multiple sessions were conducted throughout the meeting. The following presentations were made at the Friday afternoon session:

1. *The role of a mathematician in biophysical and biomedical research*, by J. W. Northrip, Southwest Missouri State University.
2. *A method for computing the radius of convergence of a power series*, by John Lott, K. C. High School Student.
3. *When does a sequence converge to a fixed point of a given function*?, by T. L. Hicks, University of Missouri-Rolla.
4. *The golden ratio*, by Mike Larson, University of Missouri-St. Louis.
5. *On the Mann iteration process in a Hilbert space*, by John Kubicek, Southwest Missouri State University.
6. *Extremal structure of convex sets*, by J. C. Hankins, University of Missouri-Rolla.
7. *Patterns of problem solving as applied to medicine* (*R Mathematics*), by Sherralyn Craven, Central Missouri State University.
8. *SMSU Math Relays and SMSU Junior High Math Tournament*, by J. L. Wise, Southwest Missouri State University.
9. *An application of mathematics to the production of energy*, by Dale Woods, Northeast Missouri State University.
10. *Partial Boolean rings*, by Larry Cammack, Central Missouri State University.
11. *The mathematics of computation: A critical history* (invited address), by Y. L. Luke, University of Missouri-Kansas City.
12. *Distribution-free statistical methods* (invited banquet speaker), by R. V. Hogg, The University of Iowa.

Films were shown before and after the Friday afternoon session and a Mini Computer Demonstration was held before papers were presented. The Saturday morning session consisted of a Sectional Business Meeting and the following presentations:

13. *History of computers*, by George Luffel, University of Missouri-Rolla.
14. *A "$\frac{1}{2}$" theorem for a class of univalent functions*, by J. R. Hatcher, Southwest Missouri State University.
15. *Divisors and complete integral closure in rings with zero divisors*, by R. E. Kennedy, Central Missouri State University.
16. *Representation of integers by special Diophantine equations*, by Shirley Kolmer, St. Louis University.
17. *Generalized permutation matrices*, by G. H. Bernet, Jr., Evangel College.
18. *Combinatorial problems with surprising solutions* (invited address), by D. P. Roselle, MAA Secretary, Virginia Polytechnic Institute.

H. K. STUMPFF, *Secretary-Treasurer*

## APRIL MEETING OF THE NEBRASKA SECTION

The fifty-second annual meeting of the Nebraska Section of the MAA was held on April 23 and 24, 1976, at Kearney State College in Kearney with Professor L. M. Larsen presiding. There were 50 persons present of whom 30 were members of the Association.

Officers for 1976 – 77 were elected as follows: Chairman, Professor S. D. Luke, Nebraska Wesleyan University; Chairman Elect, Professor P. A. Haeder, University of Nebraska at Omaha; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska-Lincoln; Contest Chairman, Professor L. J. Stephens, University of Nebraska at Omaha. Dr. H. O. Pollak represented the MAA and brought greetings to the group.

The following papers were presented:

1. *Integrals of the Henstock type*, by R. Koslaski, University of Nebraska at Omaha.
2. *Finite fields and planar difference sets*, by J. Konvalina, University of Nebraska at Omaha.
3. *Coding theory I: Discrete coding theory I*, by C. P. Downey, University of Nebraska at Omaha.
4. *Coding theory II: Group codes for the Gaussian channel*, by J. Karlof, University of Nebraska at Omaha.
5. *Unequal concurrent Cevians*, by D. M. Mesner, University of Nebraska-Lincoln.
6. *Invited address: How to embed an arbitrary graph in a cube—with engineering applications*, by H. O. Pollak, Bell Laboratories, Murray Hill, New Jersey.
7. *Controllability of non-linear systems*, by R. D. Gahl, Concordia College.
8. *A subspace property of an irreducibly connected space*, by E. Halfar, University of Nebraska-Lincoln.
9. *Panel: Mathematics in the social sciences*, by R. Heckman, Kearney State College, Chairman; M. P. Gessaman, University of Nebraska at Omaha; W. W. Gutzman, University of South Dakota; D. L. Skoug, University of Nebraska-Lincoln; and K. Nickels, Kearney State College.
10. *Adapt: Successes and frustrations*, by M. C. Thornton, University of Nebraska-Lincoln.

11. *Honor roll students, 1974 and 1975 Annual High School Mathematics Examination*, by H. M. Cox, University of Nebraska-Lincoln.

12. *Report on Nebraska-South Dakota Mathematics Contest for High School Students*, by L. J. Stephens, University of Nebraska at Omaha.

13. *Invited address: Relationship between the applications of mathematics and teaching of mathematics*, by H. O. Pollak, Bell Laboratories, Murray Hill, New Jersey.

H. M. Cox, *Secretary-Treasurer*

## APRIL-MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The Fifty-ninth Annual Meeting of the Rocky Mountain Section of the MAA was held on the campus of Ft. Lewis College, Durango, Colorado, April 30 and May 1, 1976. There were eighty-six registrants, including Forest Fisch, Governor of the Section, and A. D. Porter, Chairman of the Section. Dean Larry Johnson, Director of School of Arts and Sciences, welcomed the Section to Ft. Lewis College. The banquet address "Whither Mathematics", was delivered by Professor R. D. Anderson of Louisiana State University. The keynote address, "Yes, 'Virginia', There is such a Thing as a Liberal Arts Course in Mathematics", was given by Professor John Hodges, University of Colorado, and two invited addresses were given: "When Is a Sequence Random", by Dr. Gus Simmons, Sandia Corp., and "Algorithmically Defined Functions", by Professor R. D. Anderson, Louisiana State University.

There were thirty-one contributed papers read at the meeting:

1. *An actuarial option in the traditional mathematics program*, by D. C. Benson, South Dakota School of Mines and Technology.

2. *What mathematics skills do prospective elementary teachers command?*, by Milfried Olson, The University of Wyoming.

3. *Minicourses in mathematics education — a year later*, by A. D. Porter, The University of Wyoming.

4. *A variation on a problem of Davenport and Diophantus*, by B. W. Jones, University of Colorado.

5. *On the Oxtoby-Ulam theorem*, by C. G. Mendez, Metropolitan State College.

6. *Reversible multiples*, by David Ballew and C. A. Grimm, South Dakota School of Mines and Technology.

7. *On internal gravity wave generation by a stationary oscillating source*, by Ed Adams, University of Southern Colorado.

8. *Ill posed linear-operator equations*, by O. N. Strand, National Oceanic and Atmospheric Administration.

9. *Families of fourth order Sturm-Liouville systems*, by F. M. Stein, Colorado State University.

10. *A note on the matrix equation* $\Phi = P\Sigma P^T$, *where P is a projection*, by J. O. Kork, Colorado School of Mines.

11. *On imitating the Euclidean metric*, by Ira Rosenholtz, The University of Wyoming.

12. *Several conjectures on commutativity in algebraic structures*, by K. S. Joseph, Metropolitan State College.

13. *Solving the Gauss equation via exterior algebra*, by Jack Vilms, Colorado State University.

14. *The integral closure of a ring from a topological point of view*, by C. H. Brase, Littleton, Colorado.

15. *Categorical properties of algebraic structures*, by D. D. Cox and R. E. Prather, University of Denver.

16. *Computer use in teaching mathematics at Utah State University*, by J. D. Watson, Utah State University.

17. *Irreducible self-adjoint unbounded representations of* $^*$*-algebras*, by W. M. Scruggs, University of Denver.

18. *An inverse diffraction technique*, by R. D. Mager, University of Denver.

19. *Estimation of parameters in acceleration models by use of ranks*, by Raymond Williams, Fort Lewis College.

20. *Five problems for freshmen and sophomores*, by F. M. Stein, Colorado State University.

21. *Some uses of the programmable calculator in the teaching of probability and statistics*, by J. H. Foster, Weber State College.

22. *Training for prospective and inservice teachers on the role of the hand-held calculator in the mathematics curriculum*, by Steven Kerr, Weber State College.

23. *Undergraduate mathematics education — a user's view*, by William Orth, USAF Academy.

24. *The Jacobian of a certain transformation*, by Hung C. Li, University of Southern Colorado.

25. *Evaluation of the zeta function by Pascal's triangle*, by C. A. Grimm, South Dakota School of Mines and Technology.

26. *A generalized Euclidean algorithm and n-ary continued fractions*, by J. A. Raab, Metropolitan State College.

27. *Converging factors for sequences of linear fractional transformations*, by John Gill, University of Southern Colorado.

28. *The Fibonacci pseudo-group and tri-diagonal matrices*, by H. R. P. Ferguson, Brigham Young University.

29. *Mathematics on the air: making a video tape*, by Corrinne Brass, Littleton, Colorado.

30. *Conversion to metric — a Canadian approach*, by Len Orman, University of Southern Colorado.

31. *Combinations via functional equations*, by Don Snow, Brigham Young University.

The business meeting was convened Saturday morning, May 1, at 8:00 A. M. by Professor Porter who presided. Thirty-eight members attended.

New officers for 1976–77 are: Chairman, Donald Bushnell, Ft. Lewis College; Chairman-Elect, Vern Nelson, Metropolitan State College; Second Vice Chairman, Robert Bitts, Arapahoe Community College; Co-Program Chairmen, Nancy Warren and Freida Holley, Metropolitan State College.

The officers of the new Intermountain Section are: Chairman, E. A. Davis, University of Utah; First Vice Chairman, Stephen Parker, Idaho State University; Second Vice Chairman, Leslie Glaser, University of Utah; and Secretary-Treasurer, Donald Snow, Brigham Young University. The splitting of the Rocky Mountain Section into the Intermountain Section and the Rocky Mountain Section is now official.

DAVID BALLEW, *Secretary-Treasurer*

## APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section was held at Texas A & M University in College Station on April 2–3, 1976. There were 231 registered persons in attendance. This meeting was one of the Centennial Academic Assemblies of Texas A & M University.

Presenting invited addresses were: Professor H. L. Alder, President-Elect of the MAA, who spoke on "Recent Developments in the Theory of Partition Identities"; Professor P. R. Halmos who spoke on "Bounded Matrices"; and Professor Paul Erdös, Centennial Professor of Mathematics at Texas A & M, who spoke on "Combinatorial Problems in Elementary Geometry." Professor Charles Chui of Texas A & M arranged a special session on Approximation theory. Professor Larry Guseman of Texas A & M arranged a special session on Mathematical Pattern Recognition. Professor Howard Rolf, Baylor, organized a panel of representatives of business, industry, and government to discuss the needs of undergraduate programs. Professor Norman Fletcher of Mountain View College organized a special pedagogical session relating to Mathematics in the first two years.

Officers for 1976–77 are: Chairman: G. R. Blakley, Texas A & M University; First Vice Chairman: H. L. Rolf, Baylor University; Second Vice Chairman: R. G. Dean, Stephen F. Austin State University; Past Chairman: J. E. Hodge, Angelo State University; Level I Director: B. D. Langston, Tarrant County Jr. College, Northeast Campus; Level II Director: Archie Brock, East Texas State University; Director at Large: R. H. Cranford, Texas Eastern University; Secretary-Treasurer: J. C. Bradford, Abilene Christian University; MAA High School Contest: J. R. Boone, Texas A & M University.

Contributed papers were:

*Two theorems characterizing increasing k-set contraction mappings*, by K. L. Singh, Texas A & M University.

*Banach lattices whose duals are $l_1$*, by H. E. Lacey, University of Texas at Austin.

*Closed subsets of compact groups*, by A. Y. W. Lau, North Texas State University.

*Orthogonal complements in $C(K)$ spaces*, by R. G. Bilyeu, North Texas State University.

*Further results on expansive mappings*, by R. K. Williams, Southern Methodist University.

*Absolute summability and stretchings of sequences*, by T. A. Keagy, Wayland Baptist College.

*On the summation of infinite series and the gamma function*, by Russell Cowan, Lamar University.

*Summability of sequences determined by certain difference properties*, by D. F. Dawson, North Texas State University.

*On the space $BSV^m[a, b]$*, by F. N. Huggins, University of Texas at Arlington.

*Endomorphism rings of totally projective Abelian groups*, by R. K. O'Callaghan, University of Texas of the Permian Basin.

*Critical maximal ideals*, by R. W. Yeagy, Stephen F. Austin State University; H. S. Butts, Louisiana State University; N. H. Vaughn, North Texas State University.

*Automata that recognize free monoids*, by Tom Head, University of Alaska and Department of Mathematical Sciences, Rice University.

*On consecutive primitive roots*, by M. G. Monzingo, Southern Methodist University.

*Monogenic near-rings*, by Henry Heatherly, University of Southwestern Louisiana.

*The coloring of graphs*, by R. R. Korfhage, Serban Constantin, Judith Feld, and Fred Reagor, Southern Methodist University.

*Characterizations of normed algebras with multiplicative norms*, by R. S. Doran, Texas Christian University.

*A unified method of solving second order linear partial differential equations with constant coefficients*, by W. S. McCulley, Texas A & M University.

*A new computer oriented development of linear algebra*, by Robert Ducharme, University of Texas at San Antonio.

*The rug cutting paradox*, by John Lamb, Jr., East Texas State University.

*Graffiti on teacher evaluations*, by J. W. Strain, Midwestern State University.

*Proofs by coloration*, by George Berzsenyi, Lamar University.

*A survey of models in energy related research*, by C. R. Deeter and A. A. J. Hoffman, Texas Christian University.

*Estimating parameters in mixtures of normal distributions by maximum likelihood*, by H. F. Walker, University of Houston; and B. C. Peters, NRS Postdoc Fellow, NASA/Johnson Space Center.

*Estimating mixture proportions by maximum likelihood*, by B. C. Peters, NRC Postdoc Fellow, NASA/Johnson Space Center.

*Householder transformations and optimal linear combinations*, by H. P. Decell, Jr., University of Houston.

*A two-stage classifier for remotely sensed data*, by Demetrios Kazakos, State University of New York at Buffalo.

*An iterative scheme for estimating an affine transformation*, by T. F. McCabe, Pan American University; and J. L. Solomon, Mississippi State University.

*On canonical complete Tchebycheff systems*, by L. L. Shumaker, University of Texas at Austin.

*On compact perturbations of operators*, by J. D. Ward, Texas A & M University.

*On L-ideals*, by R. R. Smith, Texas A & M University.

*Padé approximation of a class of operator-valued analytic functions*, by F. J. Narcowich, Texas A & M University.

*The use of fixed-point theorems in approximation theory*, by E. W. Cheney, University of Texas at Austin.

*Some questions related with the projection constants*, by C. Franchetti, University of Texas at Austin.

*The Eberlein integral and optimal interpolation projections*, by D. L. Barrow, Texas A & M University.

*Norm inequalities for the radial projection*, by G. Votruba, University of Montana and University of Texas at Austin.

*The dimension of $F^{\alpha}$ over $F$ is $2^{\alpha}$*, by Liam O'Callaghan, Odessa College.

*Remarks on fuzzy sets and fuzzy topology*, by Dennis Kulvici, University of Texas at Austin.

*Project map phase B: A proposed model for the utilization of micro-fiche as a primary medium for mathematical science general and research communication*, by Dennis Kulvici, University of Texas at Austin.

*Eccentricity for assorted configurations*, by J. M. Stark, Lamar University.

*A theorem on difference equations*, by D. K. Cohoon, School of Aerospace Medicine, Biometrics Div., Brooks AFB.

*Spaces of solvability for linear transformations on vector spaces*, by D. K. Cohoon, School of Aerospace Medicine, Biometrics Div., Brooks AFB.

*Effective teaching of mathematics for business and economics*, by Dorothy Lynn Chesnut and Marian Paysinger, The University of Texas at Austin.

*Pitfalls in illustrating limits and derivatives on the handheld calculator*, by R. V. Morgan and T. T. Warnock, Sul Ross State University.

*An interpolative method of computing implicit Runge-Kutta constants*, by S. A. Sim and Jane Morris Able, The University of Texas at Arlington.

*Some result of fixed points through abstract cones*, by B. B. Williams, The University of Texas at Arlington.

J. C. BRADFORD, *Secretary-Treasurer*

## MAY MEETING OF THE ILLINOIS SECTION

The Fifty-fifth annual meeting of the Illinois Section of the MAA was held on the campus of Chicago State University, Chicago, on Friday and Saturday, May 14–15, 1976, with approximately 100 members and guests in attendance. The program featured Professor Leonard Gillman, Treasurer of the MAA, who spoke following the Friday banquet on "How Students Think".

Invited addresses included:

*Computers and number theory*, by J. L. Selfridge, Northern Illinois University.

*Cycles and units from the periods of periodic continued fractions*, by Leon Bernstein, Illinois Institute of Technology.

*Dual optimization problems*, by Lynn McLinden, University of Illinois.

Panel discussions on "Applicable Mathematics in the College Curriculum" and "Problems Arising in Articulation and Placement in the First Two Years of College Mathematics" were held. In addition, concurrent sessions on "An Algebra Course for Prospective Secondary Mathematics Teachers," sponsored by the Illinois Council on Mathematics Education, by Katherine Pedersen of Southern Illinois University, Carbondale, and "Utilization of the Computer and the Hand Held Calculator in Calculus and Pre-calculus Instruction," by Norman Ladd, of Maine Township High School-West, took place.

At the annual business meeting with Chairman Jon Laible of Eastern Illinois University presiding, committee reports were presented, the by-laws amended, and Professor John Christiano of Northern Illinois University elected Chairman for 1976–1977. Professor John Bradburn of Elgin Community College was named Chairman-

elect, Professor Therese Butzen of William Rainy Harper College was elected First Vice-Chairman, and Professor H. C. Saar was re-elected Secretary-Treasurer.

H. C. SAAR, *Secretary-Treasurer*

## MAY MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held at Franklin College, Franklin, on Saturday, May 1, 1976, with approximately 40 persons in attendance. The Chairman of the Section, Rodney Hood of Franklin College, presided.

The program consisted of the following:

1. *Some simple computations*, by P. T. Joshi, Ball State University.
2. *Are we off the track?*, by G. N. Wollan, Purdue University.
3. *Islamic art and mathematics*, by H. W. Alexander, Earlham College.
4. Film: *John von Neumann.*
5. *Some bridges to and from mathematics*, by A. B. Willcox, Executive Director of the MAA.

L. J. Cote, Purdue University, awarded prizes for solving problems appearing in the Indiana School Mathematics Journal to Richard Dukes, Crispus Attucks High School; Rex Dwyer, Yorktown High School; Mark Lasky, Valparaiso High School; Steve Neumann, Highland High School; and Randy Strommer, Arlington High School.

At the business meeting, memberships in the MAA were awarded to M. C. Ng, Purdue University, and A. N. Srivastava, Indiana University, in recognition of their performances on the Putnam examination.

Maynard Thompson, Indiana University, as chairman of the Nominating Committee, presented the following slate of officers for 1976–77 (which was unanimously approved): Chairman, M. C. Gemignani, Indiana University-Purdue University at Indianapolis; Vice-Chairman, G. J. Sherman, Rose-Hulman Institute of Technology; Secretary-Treasurer, D. E. Wilson, Wabash College.

D. E. WILSON, *Secretary*

## MAY MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the MAA was held at Calvin College, Grand Rapids, on May 7–8, 1976. Attendance continued to be high. Despite a conflict with final examinations or other events at certain colleges and the closeness of the forthcoming graph theory conference at Western Michigan University, approximately 150–160 persons, representing most of the state's four-year colleges and several two-year ones, attended the sessions.

In formulating the plans for the meeting, the Program Committee took note of the strong interest in applied mathematics and the role of mathematics in business and industry demonstrated by member colleges following last year's annual meeting in Flint which featured seminars with mathematicians from industry. Some more speakers from industry were invited for that express purpose. The Program Committee also planned some talks that reflected the Centennial (Calvin College is celebrating its Centennial this year) and Bicentennial theme as well as sessions that would be of special interest to the two-year colleges. Dr. Louis DiBello of Computer-Based Education Research Laboratory of the University of Illinois made a presentation of the PLATO Math Project which is used to assist in mathematics instruction via computer and television in nine community colleges in the Chicago area. Dr. Phillip Jones of the University of Michigan reminisced over the events that happened to mathematics curricula and mathematicians in the state of Michigan in his banquet address, at the end of which he donated to the Section a box of coins which he had collected over the years as treasurer of various mathematical associations.

At the business meeting, Professor C. B. Stortz, chairperson of the Michigan Section, made the following announcements: (1) The Michigan Section has been in contact with the Michigan Academy of Science, Arts and Letters over the possibility of holding a meeting at the same time if the opportune moment presents itself. (2) The next Annual Meeting of the Section will be held at Eastern Michigan University on May 6 and 7, 1977. (3) There have been two invitations from member institutions to host the 1978 and 1979 meetings, but the Section hopes to hold its meetings at different parts of the state and welcomes invitations from colleges located outside of the southeastern part of the state. (4) The *Mathematics Magazine* has a new format and copies of the magazine were prominently displayed. (5) The biannual summer seminar to be held at Marquette will focus on operations research.

The secretary-treasurer reported on the financial status of the Section and read part of the letter from Dr.

Willcox informing the Section that the Section's (1975) revised by-laws were under review by the Board of Governors and that it was anticipated that they would be approved in substantially their present form. This was followed by reports by the director of the Michigan Mathematics Prize Competition and the chairman of the Nominating Committee.

The main item of business was the approval by the members of a further revision of the Sections's by-laws which, in their revised form, named the following as members of the Executive Committee of the Section: The officers of the Section (chairperson, the two vice-chairpersons, secretary-treasurer), the Sectional Governor and the Section officers of the past year.

Copies of the (1976) revision were being sent to the MAA Committee on Sections for its recommendation to the Board of Governors of the MAA for approval.

The newly elected officers of the Section are: Chairperson, E. A. Tanis, Hope College; Vice-chairperson, J. E. Adney, Michigan State University; Vice-chairperson (subject to approval of the revised by-laws of the Michigan Section by the Board of Governors of MAA), Newell Remington, Delta College. Each of the above will serve for one year (1976 – 77).

Delia Koo, Eastern Michigan University, will continue as secretary-treasurer. (The three year term of secretary-treasurer does not expire till May, 1977.)

As a result of an informal straw vote, the members present at the business meeting indicated an overwhelming desire to raise the Section dues from the present $2.00 and to establish closer contact with mathematicians in business and industry by soliciting corporate donors from corporations in Michigan.

There was a welcome address by Dr. John Vanden Berg, Vice President for Academic Affairs and Dean, Calvin College, after which the following program was presented:

Invited addresses:

1. *Pegboard solitaire*, by Hugh Montgomery, University of Michigan.
2. *Application of statistics and mathematics in the discovery and development of new drugs for human life*, by John Schultz, The Upjohn Company.
3. *Gödel sentences and the mind-brain problem*, by Paul Zwier, Calvin College.
4. *From hard times to good times — and back!*, by Phillip Jones, University of Michigan.
5. *The PLATO Community College Math Project*, by Louis DiBello, University of Illinois.
6. *Selected applications of mathematics in the aerospace industry*, by Rod Wierenga, Lear Siegler Corporation.
7. *Combinatorial scheduling theory*, by R. L. Graham, Bell Laboratories.

Contributed papers:

1. *Problems that walked in the door*, by Judith Longyear, Wayne State University.
2. *Sampling petition signatures*, by James Stapleton, Michigan State University.
3. *Geometry and algebra*, by P. C. Hammer, Grand Valley State Colleges.
4. *Using numerical techniques in ordinary differential equations*, by John Van Iwaarden, Hope College.
5. *Perfect numbers and powers of two*, by Maurice Eggen, Central Michigan University.
6. *External separation of subspaces*, by Thomas Elsner, General Motors Institute.
7. *Distributive law for classes in non-associative algebraic structures*, by Thomas Storer, University of Michigan.
8. *On the pencil of two antisymmetric matrices and its Pfaffian*, by Brian Murphy and Paul Weiss, Wayne State University.

Student papers:

1. *Maximum entropy spectral analysis*, by Christine Seber, Kalamazoo College.
2. *Self complementary graphs*, by James Diephuis, Hope College.
3. *An application of conformal mapping to measuring specific resistivity of arbitrarily shaped flat samples*, by Yek Soan Cheng, Eastern Michigan University.
4. *Asymptotic solutions of* $x^2y'' + (x^2 + 2)y = 0$, by Toshitake Okada, Western Michigan University.

DELIA KOO, *Secretary-Treasurer*

## MAY MEETING OF THE SEAWAY SECTION

The Spring Meeting of the Seaway Section of the MAA was held at College of St. Rose, Albany, N.Y., on May 1, 1976, with a registered attendance of 85 people, including 67 members of the Association. Professor Mabel D. Montgomery, Chairman of the Section, presided.

At the morning session, E. T. Frankel, Schenectady, formerly of the Department of H. E. W., presented a program of "Mathematical Magic." Jean Smith, Middlesex Community College, Middlesex, Connecticut, talked

on "Math Anxiety." Kenneth Magill, SUNY at Buffalo, presented the Harry M. Gehman Lecture, entitled "Some Topics in Semigroups of Continuous Selfmaps."

At the business meeting, the following officers were elected: Chairman, F. D. Parker, St. Lawrence University; First Vice-Chairman, P. T. Schaefer, State University College at Geneseo; Second Vice-Chairman, R. A. Meili, Mohawk Valley Community College.

It was announced that Erik Hemmingsen, Syracuse University, had been elected to a three year term as Section Governor.

J. E. Quinsey, student at the University of Waterloo, was recognized as receiving the highest score of anyone in the Section in the William Lowell Putnam Mathematical Contest. He will receive a congratulatory check of $10 from the Section.

Frank Hacker, Mohawk Valley Community College, was recognized for his work during the past year as Section Chairman of the MAA High School Mathematics Contest.

During the afternoon the following contributed papers were presented:

*Four-dimensional division algebras*, by C. W. Kohls, Syracuse University.

*Are algebraic expressions just infix polish?*, by Margaret Brown, Skidmore College.

*Fuzzy Algebra*, by Jay Delkin, University of Western Ontario.

*Characterization of a certain binary operation*, by Donald Palmer, student, Rochester Institute of Technology.

*A pre-categorical look at sets, relations, and functions*, by Joseph Bodenrader and William Hartnett, State University College at Plattsburgh.

*A unifying proof in plane synthetic projective geometry*, by David Farnsworth, Eisenhower College.

*The pearl; A metaphor in educational mathematics*, by Larry Copes, Ithaca College.

*Linear algebraic groups acting with Zariski open orbits on a Hypersurface $Z(P)$*, by F. J. Servedio, McMaster University.

EMMET STOPHER, *Secretary-Treasurer*


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## CALENDAR OF FUTURE MEETINGS

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

Fifty-seventh Summer Meeting, University of Washington, August 14–16, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, St. Francis College, Loretto, Pennsylvania, April 22–23, 1977.

FLORIDA, University of South Florida, Tampa, March 4–5, 1977.

ILLINOIS, Chicago Loop College, Chicago, May 6–7, 1977.

INDIANA, Manchester College, North Manchester, November 6, 1976.

INTERMOUNTAIN

IOWA, Drake University, Des Moines, April 22–23, 1977.

KANSAS, Tabor College, Hillsboro, March or April 1977.

KENTUCKY, early April. Deadline for papers 6 wks. bef. mtg.

LOUISIANA-MISSISSIPPI, University of New Orleans, Louisiana, February 25–26, 1977.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Virginia Commonwealth University, Richmond, Virginia, November 20, 1976.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Eastern Michigan University, Ypsilanti, May 6–7, 1977.

MISSOURI, University of Missouri, St. Louis, April 29–30, 1977.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 15–16, 1977.

NEW JERSEY, early November and early May.

NORTH CENTRAL, North Hennepin Community College, Minneapolis, Minnesota, Spring 1977.

NORTHEASTERN, Rhode Island College, Providence, November 27, 1976.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO

OKLAHOMA-ARKANSAS, Oral Roberts University, Tulsa, Oklahoma, April 1–2, 1977.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 wks. bef. mtg.

PHILADELPHIA, Montgomery County Community College, Blue Bell, Pennsylvania, November 20, 1976.

ROCKY MOUNTAIN, Metropolitan State College, Denver, Colorado, April 29–30, 1977.

SEAWAY, Broome County Community College, Binghamton, New York, November 5–6, 1976.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, Loyola Marymount University, Los Angeles, March 12, 1977.

SOUTHWESTERN, Phoenix College, Phoenix, Arizona, Spring 1977.

TEXAS, Baylor University, Waco, April 1–2, 1977.

WISCONSIN, University of Wisconsin, Oshkosh, Spring 1977.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 20–26, 1977.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, St. Louis, Missouri, January 27–30, 1977.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of North Dakota, Grand Forks, June 13–16, 1977.

ASSOCIATION FOR COMPUTING MACHINERY, Olympic Hotel, Seattle, Washington, October 17–19, 1977.

ASSOCIATION FOR SYMBOLIC LOGIC, Chase Park Plaza Hotel, St. Louis, Missouri, January 27–28, 1977.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, Hamilton, Ontario, June 2, 1977 (tentative date).

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Seattle, Washington, August 14–18, 1977.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cincinnati, Ohio, April 20–23, 1977.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Americana Hotel, Miami Beach, November 3–5, 1976.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Commodore Perry Hotel, Toledo, Ohio, November 11–13, 1976.

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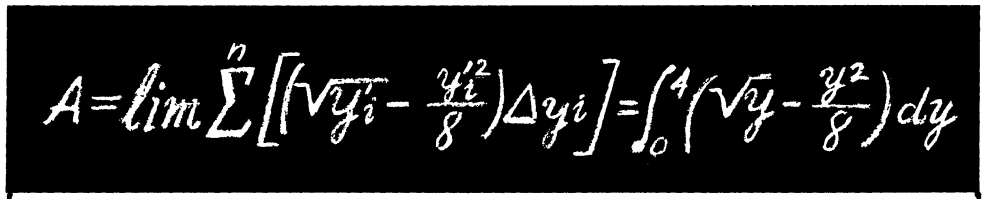
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# THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

AN OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 83 NUMBER 10

CODEN: AMMYAE

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Backlog: Main Articles 19 months, Math. Notes 20 months, Research Problems 20 months, Classroom Notes 18 months, Math. Education 14 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to R. P. BOAS, Department of Mathematics, Northwestern University, Evanston, IL 60201; NOTES, etc.: to the corresponding Associate Editor; REPRINT PERMISSION: to LEONARD GILLMAN, Mathematical Association of America, University of Texas, Austin, Texas 78712 (see also the copyright notice below); ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N.Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036; BACK ISSUES: Contact P. and H. Bliss Co., Middletown, CT 06457.

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PUBLISHED BY THE ASSOCIATION at Washington, D.C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D.C., and additional mailing offices.

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PRINTED IN THE UNITED STATES OF AMERICA

# THE MATHEMATICS CLINIC: AN INNOVATIVE APPROACH TO REALISM WITHIN AN ACADEMIC ENVIRONMENT

JEROME SPANIER

**Introduction.** Recently much has been said and written about the mismatch between traditional academic training in mathematics (both at the undergraduate and graduate levels) and the current and predicted job market for mathematicians. Evidence of this mismatch is widespread. In [1] a steady decline in graduate mathematics enrollments, accompanied by a general shift of student interest toward applied mathematics, is reported. Several new MS and PhD programs are being initiated ([2], [3]) with emphasis on training for non-academic employment. Thought is being given to the special preparation needed to train mathematicians for industrial careers [4] and to the role of the industrial mathematician [5], and new courses in which industrial mathematics is simulated are being designed [6].

In this article we describe experience with a new course developed at the Claremont Colleges, in which industrial and governmental problems are addressed, studied, and solved by teams of students working under faculty supervision. This approach was pioneered within the Engineering Department at Harvey Mudd College, beginning with a pilot activity in 1964 and led to the formation of the Engineering Clinic at that College. During the current academic year, the Engineering Clinic will operate a wide variety of Clinic projects involving about 70 students each semester. The Mathematics Clinic is patterned after this highly successful model.

In response to the changing national funding patterns and job markets, the Claremont Graduate School developed, in 1973, four new Master's programs aimed at training students for immediate employment in industry, government, and two-year college teaching. At the same time, joint BS and MA programs with the same objectives were developed in concert with several of the undergraduate Claremont Colleges. Three of these four concentrations involve various aspects of applied mathematics, and the notion of a Mathematics Clinic was deemed of central importance in these concentrations. A first project was operated at Harvey Mudd College during 1973–74. In 1974–75, the Mathematics Clinic was developed jointly by Claremont Graduate School and Harvey Mudd College, and during the current academic year, Clinic projects are being operated which involve an average of 20 students each semester.

Reaction on the part of students, faculty, and Clinic clients is overwhelmingly positive so far. Student participation in real-world problems provides stimulation simply not achievable in a traditional classroom environment. Clients, too, feel excitement through participation in an educational venture which they judge to be more meaningful than conventional ones. In addition, relationships are formed which may very well lead to jobs for the students and to a reduction in expensive on-the-job training for the employer. Furthermore, Clinic projects produce income which not only covers average project expenses but also provides stipends for graduate students and helps to offset some overhead expenses. Income this year from the Engineering Clinic will total approximately $200,000, while income from the Mathematics Clinic will total about $50,000.

**The Organization of the Clinic.** Claremont Graduate School (CGS) and Harvey Mudd College (HMC), two of the six Claremont Colleges, jointly conduct the Mathematics Clinic for students in their graduate and undergraduate applied mathematics programs. A major purpose of the Clinic is to help students gain the experience required to obtain positions in industry and government without further graduate training. This is accomplished by providing students an opportunity to apply their mathematical skills to one, or if they reregister for the Clinic course, several, of a wide variety of problems of current interest. Students who are working towards a PhD in applied mathematics also obtain enormous benefits from the Clinic experience; they may even be led to a dissertation problem through Clinic involvement. Working together in teams under careful faculty supervision, students in

the Clinic address themselves to problems of sufficient magnitude or complexity that their formulation, analysis, solution, and exposition requires substantial involvement for an academic year, or full-time involvement over a summer. Some problems may require expertise from other disciplines (engineering, physics, economics, etc.); in such cases, advanced undergraduates or graduate students from appropriate departments join the Clinic teams.

The Mathematics Clinic is administered by two co-directors, one from HMC and one from CGS. While the HMC Development Office employs a person to help identify and develop potential Clinic projects, mathematicians are involved in most negotiations with prospective clients regarding a Clinic project. When the co-directors have decided which of the two institutions would supply the faculty for project supervision, then that institution sets up whatever procedures are required to produce a written proposal, including an initial problem statement, which is acceptable both to the institution and the client.

Two standing committees, the Clinic Goals Committee and the Clinic Advisory Committee, help to guide and establish policy for both the Mathematics and the Engineering Clinics. These committees involve college administration, trustees and faculty, as well as Clinic directors, in the operation of the two Clinics.

Problems accepted for the Clinic originate from a variety of sources: industry, business, government, or even other educational institutions. Proposed projects are reviewed by a panel of faculty specialists prior to final acceptance. Projects selected are those:

(a) whose solution is of real interest to the client;

(b) whose contents promise educational value to students of mathematics;

(c) for which a reasonable prediction of successful resolution can be made;

(d) for which the client is willing to allocate a few hours a week of liaison time;

(e) which reasonably can fit the time and other resource constraints of an academic year or a summer schedule;

(f) for which the client can meet the Clinic fee.

The charge for a Mathematics Clinic project is established as a fixed fee, based on estimated average project expenses, faculty released time, and an average share of overhead. This formula has proved to be much simpler to implement than one based on an attempt to meet expenses plus overhead for each individual project.

Students participate in Mathematics Clinic activity by registering for a course with that name, either at the undergraduate or the graduate level. Regular course credit is given for this participation, and the faculty supervisor is provided some released time for each Clinic project he directs.

The student team for each project normally consists of 2–5 undergraduates and 1 or 2 graduate students, the latter assigned as team leader(s). Team leaders receive stipends for their participation, in addition to course credit. While the assignment of grades is the responsibility of the faculty supervisor, team leaders are expected to direct the day-to-day activities of the team, assign specific student responsibilities, and, in general, oversee the team operation on as many levels as is practical. In this way team leaders obtain realistic management experience, in addition to the experience of working on a mathematical or computational problem of obvious value. Participating undergraduate students are expected to spend an average of 135 hours per semester in project activity, while graduate students average closer to 200 hours per semester during the regular academic year. For summer projects these estimates are doubled so that a single summer project is equivalent to two semester projects during the regular academic year. In addition, all participating students receive salaries for summer project work, and faculty supervisors also receive summer salary for their involvement.

A variety of documents, ranging from purchase orders and simple letters of agreement to full scale research contracts, are used to finalize an agreement between each client and the Clinic. Each such document sets forth an agreed-upon statement of work, schedules for reporting Clinic progress to the client, and a schedule for payments to the Clinic. Each Clinic sponsor will normally receive at least one progress report, roughly midway in the project's life, and a bound final report. In addition,

throughout the academic year project teams give oral presentations describing the progress of their work at regular weekly meetings. All faculty and students are invited to these presentations, and attendance by Clinic sponsor representatives is also welcomed. At the end of each academic year, a "Projects Day" open house permits each student team to present its results to a group of invited client representatives and guests. Besides these oral presentations and the written reports which are part of each project activity, faculty and students have published novel results obtained while working on a client's project. Such project-related work may be submitted for publication under conditions negotiated with each client separately. Normally, publication approval will need to be obtained only within the year following delivery of the final project report. The Clinic team and the two institutions agree to assign patent rights to sponsors. It is Clinic policy to decline any "classified" project.

**Operation of the Clinic.** It is clear from the description above that the Clinic must operate on a timetable which has little margin for error. Students may indicate their intent to participate in Clinic projects before the start of each semester, but their total number and range of skills is not determined until after the start of each semester. Availability of faculty supervisors is also subject to some fluctuation, especially until final negotiations with Clinic project sponsors are concluded. Despite these staffing uncertainties, dialogues with potential sponsors must and do take place in advance of each semester. Prior to the beginning of each semester, the Clinic co-directors circulate a list of possible Clinic projects and their descriptions to all interested students. Students express their preferences for specific project work based on this list. Despite the obvious "cart and horse" problem, a reasonably good match between client needs and student-faculty capabilities and interests has been maintained during each semester of operation.

After the start of each semester each Clinic project meets much as would a seminar. While there is quite a bit of variation in this regard, a typical schedule would call for two meetings per week of the full Clinic team plus separate working meetings of team subsets. Additionally, team leaders report progress to the faculty supervisor each week, which permits planning discussions to be held and problem areas to be explored. Invariably, "mini-seminars" are held during sessions which involve the full Clinic team. (Other sessions, involving "working" subgroups, are also held to meet specific project objectives.) These mini-seminars consist of abbreviated and intensive courses which permit the team to study very current methods of direct relevance to the project problem. These sessions nearly always cover ground at the research frontiers of the problem area and give students invaluable training not obtainable in traditional courses. Such "crash courses" in new techniques are, of course, frequently undertaken in an industrial environment because they are obviously necessary to create and maintain competitive market positions. Since the time constraints of each project are also established by agreement with the client, the Clinic experience is completely faithful to the environment of the sponsoring organization.

Because of the need to protect company confidentiality in many cases, it is not possible to discuss current projects in any detail. However, I shall attempt to outline a recent, fairly typical project activity in an effort to communicate some of the technical flavor of the Mathematics Clinic.

A project sponsor was interested in developing a mathematical modeling capability to support instrumentation it designs and builds for air pollution monitoring. Study showed that there are two, rather different, types of problems which must be treated, depending on whether the polluted air is "well-mixed" or not. The former problem, typified by air masses over large urban areas, is rather well modeled by turbulent diffusion theory, whereas near isolated large sources of pollution, diffusion theory fails to predict observed pollutant concentrations. A Mathematics Clinic team consisting of two graduate and two undergraduate students developed two completely different models for the sponsor. The first, based on a so-called "Gaussian puff" formula, is useful downwind from an isolated, elevated point source. The basic assumption of this model (developed in 1973) is that puffs of effluent are released at regular intervals and move downstream under piecewise-constant conditions of wind speed and atmospheric stability, maintaining a three-dimensional Gaussian distribution of concentration

within the puff volume. Linear superposition is then used to express the pollutant concentration at any point and time as the sum of concentrations arising from each individual puff. The resulting semi-empirical model predicts downwind concentrations in good agreement with observed values.

A very different approach to the problem of modeling the well-mixed air over large urban areas was taken. The usual starting point is a coupled system of partial differential equations whose computer solution is quite costly. It was decided to develop a diffusion model capable of accurate prediction of *relative* concentrations at ground level, deliberately sacrificing the ability to predict accurate absolute levels of concentration and ignoring dependence of concentration on altitude. The resulting novel model, which made use of recent work [7] involving the fractional calculus, then provided the potential for accurately assessing the impact on ground level concentrations of a perturbation—for example, addition of a new roadway, or a new power generating station—at a fraction of the usual cost. By focusing on ground level concentrations and ignoring dependence on altitude, a vast simplification in the formulation was achieved, enabling large savings in cost per problem to be realized.

Both models required computer implementation; the student team performed considerable analysis and all of the programming and debugging. The project sponsor was well-pleased with the work, distributing copies of the final Clinic report widely within its organization.

The breadth of disciplines and problem areas treated within Clinic projects is truly impressive. Problems studied have involved chemical coatings for metals, business forecasting models, nuclear-reactor theory, underground-fluid-pressure modeling, and juror utilization, to name only a few. Only rarely have entirely classical techniques been completely satisfactory for dealing with the problem at hand. Techniques used have ranged from experimental design and response-surface statistical methods, to sophisticated applications of finite element and finite difference methods for solving complex nonlinear initial/boundary-value problems. Problems with substantial computer science components have also been tackled and solved. Despite the hard work implicit in such team-project experiences, the satisfaction gained in contributing to a problem of immediate interest to a paying sponsor keeps morale high.

**Summary.** We have described what we believe to be an exciting and unconventional new device for training students of applied mathematics. The benefits of the Mathematics Clinic for students seem quite clear; relevant training, eminently practical experience, and enhanced job prospects, as well as financial reward. Subtler benefits accrue from the student's exposure to open-ended and sometimes loosely-defined problems, from experience gained working as a team member under time and resource constraints, and from enhanced communication skills, both written and oral, acquired through the Clinic experience. Participating sponsors also benefit in many ways: They invariably feel good about supporting a novel and meaningful educational experience, and they may occasionally obtain solutions to problems of great importance to them. Thus far, in fact, Mathematics Clinic sponsors have unanimously expressed the opinion that they have benefited from the project activity. While it is clear that a perfect record cannot be maintained, a success rate of 80% or more seems well within reach. Sponsors may substantially strengthen their capability in a given problem area at a fraction of the cost of hiring the equivalent permanent staff. And they are automatically involved in a recruitment program (whose dollar value would be difficult to assess) which provides the possibility of filling key positions in their organization very efficiently.

It is the educational value of the Mathematics Clinic, however, which is of foremost importance, we think. Perhaps it represents a partial solution to the vexing problem of PhD oversupply in mathematics, as well as a tool to enrich the training of those destined to obtain a PhD in mathematics.

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## WHITEHEAD'S PROBLEM IS UNDECIDABLE

PAUL C. EKLOF

**1. Introduction.** The working mathematician, unless he is studying the foundations of mathematics, usually does not find it necessary to make explicit references to axioms of set theory — except perhaps to invoke the Axiom of Choice or the Continuum Hypothesis. As long as his arguments can be carried out within the framework of a commonly accepted system such as Zermelo–Frankel set theory (*ZF*), his set-theoretic assumptions can remain unexpressed. However, mathematicians have known since the work of Gödel that there are mathematical statements that are undecidable (i.e., neither provable nor refutable) on the basis of Zermelo–Frankel set theory. (In fact, *any* consistent axiomatization of set theory which can be effectively written down will have such undecidable statements: see, for example, Monk [**13**; Thm 1].) In recent years, a number of concrete examples of undecidable statements have been discovered. Probably the most famous is the Continuum Hypothesis ($2^{\aleph_0} = \aleph_1$) proved undecidable in *ZFC* ( = *ZF* + Axiom of Choice) by Gödel and Cohen. Other examples belong to topology and analysis. (See Rudin [**16**] and Shoenfield [**18**].) In this paper, we are going to discuss an algebraic example: a famous problem which had resisted the best efforts of mathematicians for many years before it was recently proved by Shelah to be undecidable in *ZFC*. His method of proof is to show that two axioms, each consistent with *ZFC*, yield contradictory answers to the problem.

In order to state the problem, we need some definitions. Let $1_A$ denote the identity function on $A$ and let $\boldsymbol{Z}$ denote the group of integers. A surjective homomorphism of abelian groups $\pi: B \to A$ is said to **split** if there is a homomorphism $\rho: A \to B$ such that $\pi\rho = 1_A$. An abelian group $A$ is called a ***W*-group** if it satisfies the property: for all surjective homomorphisms $\pi : B \to A$, if the kernel of $\pi$ is isomorphic to $\boldsymbol{Z}$, then $\pi$ splits. It is not hard to see that a free abelian group is a $W$-group. (See Corollary 2.4.) Whitehead's Problem asks whether the converse is true. In homological terms, Whitehead's Problem asks whether $\mathrm{Ext}(A, \boldsymbol{Z}) = 0$ implies $A$ is free (see section 3).

The problem also has an equivalent formulation in terms of topological groups: is every compact arcwise-connected abelian group a product of copies of the circle group, $\boldsymbol{R}/\boldsymbol{Z}$? The equivalence follows from the Pontryagin Duality Theorem.

It was proved by Stein in 1951 that every countable $W$-group is free. (See Theorem 4.1.) But for uncountable groups only partial results have been obtained in *ZFC*.

Set-theorists have studied various hypotheses which may consistently be added to *ZF*. (We shall assume that *ZF* is consistent.) One of these is Gödel's Axiom of Constructibility, denoted $V = L$. (Gödel defined a subclass $L$ of the class $V$ of all sets and proved that the axioms of *ZF* as well as the Axiom of Choice and the Continuum Hypothesis are true in $L$. The sets in $L$ are called constructible sets. The Axiom of Constructibility asserts that every set is constructible.) Gödel proved the following:

1.1 THEOREM [5]. (i) *ZF + V = L is consistent*;
(ii) *ZF + V = L implies the Axiom of Choice and the Continuum Hypothesis.*

It follows from the theorem that *ZFC* is consistent (assuming, of course, that *ZF* is consistent).

Another axiom that is important to us is Martin's Axiom (MA), which was introduced originally in connection with Souslin's Problem (see Shoenfield [18; §5]). We shall state (a weak form of) MA in section 7. Here we state only the important consistency theorem proved by Solovay and Tennenbaum using the forcing technique introduced by Cohen.

1.2 THEOREM [19]. *ZFC + MA* $+ 2^{\aleph_0} > \aleph_1$ *is consistent.*

Now we can state Shelah's theorem, which shows that at least for cardinality $\aleph_1$ Whitehead's Problem is not decidable on the basis of *ZFC*. (See also Section 8.)

1.3 THEOREM [17]. (i) *ZFC + V = L implies that every W-group of cardinality* $\aleph_1$ *is free*;
(ii) *ZFC + MA* $+ 2^{\aleph_0} > \aleph_1$ *implies that there is a W-group of cardinality* $\aleph_1$ *which is not free.*

It follows that either the affirmative or the negative answer to Whitehead's Problem is consistent with *ZFC*. The reader may still ask: which answer is "true"? The question presumes a "platonist" philosophy, i.e., that sets really exist in some sense so that it is meaningful to ask about the "truth" of a given assertion about sets (see Monk [13]). Neither of the hypotheses $V = L$ or $MA + 2^{\aleph_0} > \aleph_1$ is one which mathematicians would agree is intuitively "true" about sets. Thus Shelah's theorem does not settle the "truth" about Whitehead's Problem. It is conceivable that in the future some new axiom about sets will come to be as accepted as are the axioms of *ZFC* and that it will yield a definitive solution to Whitehead's Problem (as well as to other undecidable problems, such as the Continuum Hypothesis).

In the remainder of this paper, we shall give the algebraic portion of the proof of the undecidability of Whitehead's Problem. (The set-theoretic pre-requisites, i.e., Theorems 1.1 (i), 1.2, 6.1 and 7.1 will not be proved here.) The two parts of Theorem 1.3 are proved, respectively, in sections 6 and 7.

We begin in the next section with a study of properties of free groups.

**2. Free groups.** Throughout the rest of the paper "group" will mean abelian group. Functions which are group homomorphisms will be denoted by lower case Greek letters, and functions which are simply set mappings will be denoted by lower case Roman letters.

Recall that a group is **free** if it has a basis, that is, a linearly independent generating set. One of the most important results about free groups is the following, which is a special case of a theorem about modules over a principal ideal domain. (See, for example, Lang [11; Thm. 4, p. 45].)

2.1 THEOREM. *A subgroup of a free group is free.*

Another useful result is the so-called Fundamental Theorem of Abelian Groups which in the case of torsion-free groups may be stated as follows. (See, for example, Lang [11; Thm. 7, p. 49].)

2.2 THEOREM. *A finitely-generated torsion-free group is free.*

We shall prove a characterization of free groups in terms of the notion of splitting which will immediately imply that every free group is a *W*-group. First, let us call a homomorphism $\rho : A \to B$ a **splitting homomorphism for** $\pi : B \to A$ if $\pi\rho = 1_A$. Note that $\rho$ is necessarily injective since $\rho(a) = 0$ implies $a = \pi\rho(a) = 0$.

2.3 THEOREM. *A group A is free if and only if every homomorphism onto A splits.*

*Proof.* Suppose that $A$ is free and that $\pi : B \to A$ is surjective. If $X = \{x_i : i \in I\}$ is a basis of $A$, choose $b_i$ in $B$ for each $i$ in $I$ such that $\pi(b_i) = x_i$. Since $X$ is a basis of $A$ there is one and only one

homomorphism $\rho : A \to B$ such that $\rho(x_i) = b_i$ for each $i \in I$. Clearly $\rho$ is a splitting homomorphism for $\pi$.

To prove the converse, consider a free group $F$ with basis $X = \{x_a : a \in A\}$. Let $\pi : F \to A$ be the unique homomorphism such that $\pi(x_a) = a$ for all $a \in A$. By hypothesis, there is a splitting homomorphism $\rho : A \to F$ for $\pi$. Since $\rho$ is injective, $A$ is isomorphic to a subgroup of $F$; therefore, by Theorem 2.1, $A$ is free.

2.4 COROLLARY. *Every free group is a W-group.*

2.5 COROLLARY. *Suppose $B$ is a subgroup of $A$ such that $B$ and $A/B$ are both free. Then $A$ is free; moreover, any basis of $B$ extends to a basis of $A$.*

*Proof.* Let $\pi : A \to A/B$ be the canonical surjection (i.e., $\pi(a) = a + B$ for all $a \in A$). Since $A/B$ is free, the theorem implies that there is a splitting homomorphism $\rho$ for $\pi$. Then $A = \rho(A/B) \oplus B$. (Indeed, for any $a \in A$, $a = \rho\pi(a) + (a - \rho\pi(a))$, which is the unique representation of $a$ as a sum of elements of $\rho(A/B)$ and $B$ respectively.) If $Y$ is a basis of $A/B$, then $\rho(Y)$ is a basis of $\rho(A/B)$ since $\rho$ is injective. Therefore, if $X$ is any basis of $B$, then $\rho(Y) \cup X$ is a basis of $A$. This proves the corollary.

An important element in the analysis of Whitehead's Problem is the study of ascending chains of groups. Consider an ascending chain of sets

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_\nu \subseteq \cdots, \qquad \nu < \alpha$$

indexed by an ordinal $\alpha$. This chain is called a **smooth chain** if for every limit ordinal $\lambda < \alpha$, $A_\lambda = \bigcup_{\nu<\lambda} A_\nu$. The chain is called **strictly increasing** if for every $\nu < \alpha$, $A_\nu \neq A_{\nu+1}$. Finally, the chain is a **chain of groups** if for every $\nu < \alpha$, $A_\nu$ is a group which is a subgroup of $A_{\nu+1}$.

2.6 THEOREM. *Let $A$ be the union of a smooth chain of groups $\{A_\nu \mid \nu < \alpha\}$ such that $A_0$ is free and for every $\nu < \alpha$, $A_{\nu+1}/A_\nu$ is free. Then $A$ is free; moreover, for every $\nu < \alpha$, $A/A_\nu$ is free.*

*Proof.* Let $X_0$ be a basis of $A_0$. We shall construct by transfinite induction on $\nu < \alpha$ a smooth chain of sets

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_\nu \subseteq \cdots, \qquad \nu < \alpha$$

such that $X_\nu$ is a basis of $A_\nu$. This will suffice since $X = \bigcup_{\nu<\alpha} X_\nu$ will be a basis of $A$ and $\{x + A_\nu \mid x \in X - X_\nu\}$ will be a basis of $A/A_\nu$. Suppose we have already exhibited a chain

$$X_0 \subseteq X_1 \cdots \subseteq X_\nu \cdots, \qquad \nu < \mu$$

with the desired properties, for some $\mu < \alpha$. We must produce $X_\mu$. If $\mu$ is a limit ordinal, let $X_\mu = \bigcup_{\nu<\mu} X_\nu$; then $X_\mu$ is a basis of $\bigcup_{\nu<\mu} A_\nu$ which equals $A_\mu$ by the definition of a smooth chain. If $\mu$ is a successor ordinal, say $\mu = \delta + 1$, then $A_{\delta+1}/A_\delta$ is free by hypothesis, so by Corollary 2.5, $X_\delta$ extends to a basis $X_{\delta+1}$ of $A_{\delta+1}$. This completes the proof.

**3. Properties of *W*-groups.** In this section, and this section only, we shall assume familiarity with the rudiments of homological algebra (e.g., Chapters 1 and 3 of Jans [**8**]) in order to prove three properties of $W$-groups. Only the *statements* of the following three theorems are used in the rest of the paper, so the reader who is willing to accept them on faith — or is able to supply his own proofs — can skip the remainder of this section.

3.1 THEOREM. *A subgroup of a W-group is a W-group.*

3.2 THEOREM. *Every W-group is torsion-free.*

3.3 THEOREM. *If $B_0$ is a subgroup of $B_1$, such that $B_1$ is a W-group but $B_1/B_0$ is not a W-group, then there exists a homomorphism $\psi : B_0 \to Z$ which does not extend to a homomorphism: $B_1 \to Z$.*

In abelian group theory, Ext is usually defined as a group of equivalence classes of short exact sequences (see Fuchs [4; Chapter IX]) but here we shall define it in terms of free resolutions for the benefit of those familiar with an introductory text in homological algebra such as Jans [8] or Northcott [14].

Let us recall some definitions. If $A$ and $C$ are groups, $\mathrm{Hom}(A, C)$ is the group of all homomorphisms $\varphi : A \to C$, where addition is defined by $(\varphi_1 + \varphi_2)(a) = \varphi_1(a) + \varphi_2(a)$. For every group $C$, a homomorphism $\sigma : A_1 \to A_2$ induces a homomorphism

$$\sigma' : \mathrm{Hom}(A_2, C) \to \mathrm{Hom}(A_1, C)$$

defined by $\sigma'(\varphi) = \varphi\sigma$ for $\varphi \in \mathrm{Hom}(A_2, C)$.

Recall that a sequence

$$\cdots \to A_{i+1} \xrightarrow{\sigma_{i+1}} A_i \xrightarrow{\sigma_i} A_{i-1} \to \cdots$$

is **exact** if $\mathrm{Ker}\,\sigma_i = \mathrm{Im}\,\sigma_{i+1}$ for every $i$.

A **free resolution** of a group $A$ is a short exact sequence

$$0 \to F_1 \xrightarrow{\delta} F_0 \xrightarrow{\varepsilon} A \to 0, \tag{*}$$

where $F_0$ (and hence $F_1$) is free. Given a free resolution (*) of $A$, we define

$$\mathrm{Ext}(A, C) = \mathrm{Hom}(F_1, C)/\mathrm{Im}\,\delta'.$$

(This is called $\mathrm{Ext}^1(A, C)$ in [8], $\mathrm{Ext}^1_{\mathbf{Z}}(A, C)$ in [14].) It may be proved that $\mathrm{Ext}(A, C)$ does not depend on the choice of the free resolution (see [8; p. 35]). The following theorem is crucial (see [8; p. 41f] for a proof).

3.4 THEOREM. *For any exact sequence*

$$0 \to A_1 \xrightarrow{\sigma_1} A_2 \xrightarrow{\sigma_2} A_3 \to 0$$

*and any group $C$ there is an exact sequence*

$$0 \to \mathrm{Hom}(A_3, C) \xrightarrow{\sigma_2'} \mathrm{Hom}(A_2, C) \xrightarrow{\sigma_1'} \mathrm{Hom}(A_1, C) \to \mathrm{Ext}(A_3, C) \to \mathrm{Ext}(A_2, C) \to \mathrm{Ext}(A_1, C) \to 0.$$

In order to make use of Theorem 3.4, we need to express in terms of Ext the property of being a $W$-group.

3.5 THEOREM. *A group $A$ is a $W$-group if and only if* $\mathrm{Ext}(A, \mathbf{Z}) = 0$.

*Proof.* Suppose $A$ is a $W$-group and consider a free resolution (*). We must prove that $\mathrm{Im}\,\delta' = \mathrm{Hom}(F_1, \mathbf{Z})$. Given $\varphi_1 : F_1 \to \mathbf{Z}$, define $B = (\mathbf{Z} \oplus F_0)/I$ where $I = \{(\varphi_1(y), -\delta(y)) \mid y \in F_1\}$. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \xrightarrow{\delta} & F_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle 1_A} & & \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{\iota} & B & \xrightarrow{\pi} & A & \longrightarrow & 0 \end{array} \tag{3.5.1}$$

where the homomorphisms $\pi$, $\theta$ and $\iota$ are defined, respectively, by: $\pi((n, x) + I) = \varepsilon(x)$; $\theta(x) = (0, x) + I$; $\iota(n) = (n, 0) + I$. The bottom line of (3.5.1) is exact.

Since $A$ is a $W$-group, there is a $\tau : B \to \boldsymbol{Z}$ such that $\tau\iota = 1_{\boldsymbol{Z}}$. If we let $\varphi_0 = \tau\theta$, it is easily verified that $\delta'(\varphi_0) = \varphi_1$. Hence $\delta'$ is onto and $\operatorname{Ext}(A, \boldsymbol{Z}) = 0$.

Conversely suppose $\operatorname{Ext}(A, \boldsymbol{Z}) = 0$ and consider a short exact sequence

$$0 \to \boldsymbol{Z} \xrightarrow{\iota} B \xrightarrow{\pi} A \to 0. \tag{3.5.2}$$

Let $F_0$ be as in (*) and let $\theta : F_0 \to B$ be a surjective homomorphism such that $\pi\theta = \varepsilon$. (Note that $\theta$ exists since $F_0$ is projective; see [**8**; pp. 7, 8].)

Then there is a homomorphism $\varphi_1 : F_1 \to \boldsymbol{Z}$ such that we have a commutative diagram (3.5.1). By hypothesis there exists $\varphi_0 : F_0 \to B$ such that $\varphi_0\delta = \varphi_1$. We claim that $\varphi_0(x) = 0$ if $x \in \operatorname{Ker}\theta$. Indeed if $x \in \operatorname{Ker}\theta$, then $\varepsilon(x) = \pi\theta(x) = 0$, so $x = \delta(y)$ for some $y \in F_1$; therefore $\varphi_0(x) = \varphi_0\delta(y) = \varphi_1(y) = 0$, the last since $\iota$ is injective and $\iota\varphi_1(y) = \theta\delta(y) = \theta(x) = 0$. Thus $\varphi_0$ induces a map $\tau : B \to \boldsymbol{Z}$. We claim that $\tau\iota = 1_{\boldsymbol{Z}}$. Indeed, if $n \in \boldsymbol{Z}$, $\tau\iota(n) = \varphi_0(x)$ where $\theta(x) = \iota(n)$. Now $\varepsilon(x) = \pi\theta(x) = \pi\iota(n) = 0$, so by exactness, $x = \delta(y)$ for some $y \in F_1$. Hence $\varphi_0(x) = \varphi_0\delta(y) = \varphi_1(y)$. Moreover, $\iota\varphi_1(y) = \theta\delta(y) = \theta(x) = \iota(n)$. Since $\iota$ is injective, $n = \varphi_1(y) = \tau\iota(n)$. Therefore (3.5.2) splits and $A$ is a $W$-group. The theorem is proved.

We are now ready to prove the three theorems stated at the beginning of this section.

*Proof of* 3.1. Suppose $A_1$ is a subgroup of a $W$-group $A_2$. There is an exact sequence

$$0 \to A_1 \xrightarrow{\sigma_1} A_2 \xrightarrow{\sigma_2} A_2/A_1 \longrightarrow 0,$$

where $\sigma_1$ is the inclusion map. By Theorem 3.4 there is an exact sequence

$$\operatorname{Ext}(A_2, \boldsymbol{Z}) \to \operatorname{Ext}(A_1, \boldsymbol{Z}) \to 0.$$

Since $A_2$ is a $W$-group, Theorem 3.5 implies $\operatorname{Ext}(A_2, \boldsymbol{Z}) = 0$, and therefore $\operatorname{Ext}(A_1, \boldsymbol{Z}) = 0$. Again by Theorem 3.5, $A_1$ is a $W$-group.

*Proof of* 3.2. If $A$ is not torsion-free, then there exists $a \in A$ such that $\langle a \rangle$ is a non-zero finite cyclic group.

By Theorem 3.1 it suffices to prove that $\boldsymbol{Z}/n\boldsymbol{Z}$ is not a $W$-group for any $n > 0$. Consider the canonical projection

$$\pi : \boldsymbol{Z} \to \boldsymbol{Z}/n\boldsymbol{Z}.$$

The kernel of $\pi$ is isomorphic to $\boldsymbol{Z}$, but clearly $\pi$ does not split (since $\boldsymbol{Z}$ is torsion-free). Hence $\boldsymbol{Z}/n\boldsymbol{Z}$ is not a $W$-group.

*Proof of* 3.3. Consider the exact sequence

$$0 \to B_0 \xrightarrow{\sigma_1} B_1 \to B_1/B_0 \to 0,$$

where $\sigma_1$ is the inclusion map. By Theorem 3.4 there is an exact sequence

$$\operatorname{Hom}(B_1, \boldsymbol{Z}) \xrightarrow{\sigma_1'} \operatorname{Hom}(B_0, \boldsymbol{Z}) \to \operatorname{Ext}(B_1/B_0, \boldsymbol{Z}) \to \operatorname{Ext}(B_1, \boldsymbol{Z}).$$

By hypothesis and Theorem 3.5, $\operatorname{Ext}(B_1, \boldsymbol{Z}) = 0$ and $\operatorname{Ext}(B_1/B_0, \boldsymbol{Z}) \neq 0$. Therefore $\sigma_1'$ is not surjective, which is precisely the conclusion of Theorem 3.3.

**4. Countable $W$-groups.** The principal goal of this section is to prove the following result, which is a converse of Corollary 2.4 for countable groups. Shorter proofs than ours may be given, but our proof has the advantage that it is a paradigm for the proof of Theorem 1.3 (i).

4.1 THEOREM [20]. *Every countable W-group is free.*

Before embarking on the proof of 4.1 we shall prove an important characterization of countable free groups. A subgroup $B$ of a *torsion-free* group $A$ is called a **pure** subgroup if $A/B$ is torsion-free. (A warning: this definition is not the proper definition of pure if $A$ is not torsion-free; see Kaplansky [10; p. 14].) If $B$ is a subgroup of $A$ the **pure closure** of $B$ in $A$ is the subgroup $B' = \{a \in A \mid na \in B \text{ for some } n \neq 0\}$. It is clearly a pure subgroup of $A$. If $A$ is free and $B$ is finitely-generated, then $B'$ is free by Theorem 2.1. Clearly $B'$ has the same dimension as $B$ so if $B$ is finitely-generated so is $B'$. We have therefore proved that if $A$ is free then every finitely-generated subgroup of $A$ is contained in a finitely-generated pure subgroup of $A$. The following result known as *Pontryagin's Criterion*, says that the converse is true for countable torsion-free groups.

4.2 THEOREM [15]. *Let $A$ be a countable torsion-free group such that every finitely-generated subgroup of $A$ is contained in a finitely-generated pure subgroup of $A$. Then $A$ is free.*

*Proof.* Since $A$ is countable we can enumerate the elements of $A$ in a sequence:

$$A = \{a_n \mid n < \omega\}.$$

We define by induction on $n < \omega$ a (smooth) chain $\{B_n \mid n < \omega\}$ of finitely-generated pure subgroups of $A$. Let $B_0 = 0$. If $B_n$ has been defined, let $B_{n+1}$ be a finitely-generated pure subgroup of $A$ containing $B_n \cup \{a_n\}$. The union of this chain is clearly $A$. Consider the quotient $B_{n+1}/B_n$: it is torsion-free because $B_n$ is pure in $A$; it is finitely-generated because $B_{n+1}$ is finitely-generated. Therefore, by Theorem 2.2, $B_{n+1}/B_n$ is free. The proof of the theorem is completed by an application of Theorem 2.6.

Theorem 4.2 is *not* true if we remove the hypothesis of countability. In section 7, we shall exhibit a torsion-free group of cardinality $\aleph_1$ which satisfies a condition even stronger than the hypothesis of Theorem 4.2 and yet is not free.

Let us adopt the convention that whenever $C$ is a set (or group) of the form $B \times \mathbf{Z}$, $\pi$ will denote the projection map onto the first factor, i.e., $\pi(b, n) = b$ for all $(b, n) \in B \times \mathbf{Z}$. In context there will be no ambiguity as to the domain of $\pi$. If $B$ is a group we define a **$(B, \mathbf{Z})$-group** to be a group $C$ whose underlying set is $B \times \mathbf{Z}$ such that $\pi : C \to B$ is a homomorphism and $(0, n) + (0, m) = (0, n + m)$ for all $n$, $m$ in $\mathbf{Z}$. The simplest example is $B \oplus \mathbf{Z}$, i.e., the set $B \times \mathbf{Z}$ equipped with the group operation given by $(b_1, n_1) + (b_2, n_2) = (b_1 + b_2, n_1 + n_2)$. Notice that for any $(B, \mathbf{Z})$-group $C$, the kernel of $\pi$ is $\mathbf{Z}$, so if $\pi : C \to B$ does not split then $B$ is not a $W$-group.

4.3 LEMMA. *Let $B_0$ be a subgroup of $B_1$ such that $B_1$ is a $W$-group but $B_1/B_0$ is not a $W$-group. Let $C_0$ be a $(B_0, \mathbf{Z})$-group and $\rho$ a splitting homomorphism for $\pi : C_0 \to B_0$. Then there is a $(B_1, \mathbf{Z})$-group $C_1$ which is an extension of $C_0$ such that $\rho$ does not extend to a splitting homomorphism for $\pi : C_1 \to B_1$.*

*Proof.* Since $\pi : C_0 \to B_0$ splits, there is an isomorphism $\tau : B_0 \oplus \mathbf{Z} \to C_0$ given by $\tau(b, n) = \rho(b) + (0, n)$. Notice that $\tau^{-1}\rho(b) = (b, 0)$ for all $b \in B_0$. Hence we may assume that $C_0 = B_0 \oplus \mathbf{Z}$ and that $\rho(b) = (b, 0)$ for all $b \in B_0$. Let $\tilde{C}_1 = B_1 \oplus \mathbf{Z}$ and let $\psi : B_0 \to \mathbf{Z}$ be the homomorphism given by Theorem 3.3. Define $\gamma : C_0 \to \tilde{C}_1$ by $\gamma(b, n) = (b, n + \psi(b))$. Suppose that there is a splitting homomorphism $\tilde{\rho}_1 : B_1 \to \tilde{C}_1$ for $\pi : \tilde{C}_1 \to B_1$ such that $\tilde{\rho}_1 \mid B_0 = \gamma \circ \rho$. Let $\varphi = \pi \circ \tilde{\rho}_1 : B_1 \to \mathbf{Z}$. Then for any $b \in B_0$, $\varphi(b) = \pi\tilde{\rho}_1(b) = \pi\gamma\rho(b) = \psi(b)$. Thus $\varphi$ is an extension of $\psi$, which contradicts Theorem 3.3. We conclude that no such $\tilde{\rho}_1$ exists. If $\gamma$ were inclusion we would be done; since it is not, we must employ a little trick. Define a set map $f : \tilde{C}_1 \to B_1 \times \mathbf{Z}$ by

$$f(b, n) = \begin{cases} (b, n) & \text{if } b \notin B_0 \\ (b, n - \psi(b)) & \text{if } b \in B_0. \end{cases}$$

Clearly $f$ is a bijection and $f \circ \gamma$ is the inclusion of $B_0 \oplus \mathbf{Z}$ into $B_1 \times \mathbf{Z}$. Let $C_1$ be $B_1 \times \mathbf{Z}$ with the

group structure which makes $f$ a group isomorphism (i.e., for any $u, v \in B \times \mathbf{Z}$, $u + v = f(f^{-1}(u) + f^{-1}(v))$. Then $C_1$ is an extension of $C_0$ and by the property of $\gamma$ proved above, there is no splitting homomorphism $\rho_1 : B_1 \to C_1$ for $\pi : C_1 \to B_1$ which extends $\rho : B_0 \to C_0$.

*Proof of* 4.1. By Theorem 3.2, we know that $A$ is torsion-free, so to prove Theorem 4.1 it will suffice to prove that $A$ satisfies the hypothesis of Pontryagin's Criterion. Assume it does not; that is, there is a finitely-generated subgroup $B_0$ of $A$ which is not contained in a finitely-generated pure subgroup of $A$. Let $B$ be the pure closure of $B_0$ in $A$. By our assumption $B$ is not finitely-generated. Hence $B$ is the union of a strictly-increasing chain of finitely-generated groups

$$B_0 \subsetneqq B_1 \subsetneqq \cdots \subsetneqq B_n \subsetneqq \cdots, \quad n < \omega.$$

Note that $B/B_0$ is a torsion group by the definition of the pure closure. We shall construct by induction on $n$ a chain of groups

$$C_0 \subsetneqq C_1 \subsetneqq \cdots \subsetneqq C_n \subsetneqq \cdots, \quad n < \omega,$$

such that $C_n$ is a torsion-free $(B_n, \mathbf{Z})$-group. The union $C = \bigcup_n C_n$ will then be a torsion-free $(B, \mathbf{Z})$-group. Our goal is to define the $C_n$'s so that $\pi : C \to B$ does not split, thereby contradicting Theorem 3.1.

Let $S$ be a finite set of generators for $B_0$. We claim that if $C$ is torsion-free, any homomorphism $\rho : B \to C$ is completely determined by its values on $S$. Indeed for any $b \in B$, there is an $n \neq 0$ such that $nb \in B_0$. Now $\rho(nb)$ is clearly determined by the values of $\rho$ on $S$ and since $C$ is torsion-free, the equation $nx = \rho(nb)$ has only one solution in $C$, viz $x = \rho(b)$.

Let $\{g_n \mid n < \omega\}$ be a list of all the (set) maps $g_n : S \to S \times \mathbf{Z}$ such that $\pi g_n = 1_S$. (There are only a countable number of such maps since $S$ is finite and $\mathbf{Z}$ is countable.) Now we proceed to define the $C_n$'s. Let $C_0$ be $B_0 \oplus \mathbf{Z}$. If $C_n$ has been defined, we consider two cases. In the first case, if $g_n$ extends to a splitting homomorphism $\rho$ for $\pi : C_n \to B_n$, let $C_{n+1}$ be an extension of $C_n$ such that $\rho$ does not extend to a splitting homomorphism for $\pi : C_{n+1} \to B_{n+1}$. (Note that $C_{n+1}$ exists by Lemma 4.3 since $B_{n+1}/B_n$ is torsion and therefore not a $W$-group by Theorem 3.2.) If the first case does not hold, let $\rho$ be any splitting homomorphism for $\pi : C_n \to B_n$ and define $C_{n+1}$ as above. (At least one such $\rho$ exists by Theorem 2.2 since $B_n$ is finitely generated, torsion-free and therefore free.)

To complete the proof of 4.1 we must show that $\pi : C \to B$ does not split. If there is a splitting homomorphism $\rho : B \to C$ for $\pi$ then $\rho \mid S = g_n$ for some $n$. But then $\rho \mid B_n$ is a splitting homomorphism for $\pi : C_n \to B_n$ which is an extension of $g_n$ and extends to a splitting homomorphism for $\pi : C_{n+1} \to B_{n+1}$. This contradicts the construction of $C_{n+1}$ and hence the proof is finished.

Theorem 4.1 yields some information about arbitrary $W$-groups. Let us call a group $A$ **$\aleph_1$-free** if every subgroup of cardinality less than $\aleph_1$ (i.e. countable) is free. Putting together Theorems 3.1 and 4.1 we obtain the following result.

4.4 COROLLARY. *Every $W$-group is $\aleph_1$-free.*

**5. Chase's condition.** We shall consider a generalization of the hypothesis of Pontryagin's Criterion (Theorem 4.2). First of all, notice that "$\aleph_1$-free" is a natural generalization of "torsion-free" since by Theorem 2.2 "torsion-free" is equivalent to "every finitely-generated subgroup is free." Taking this as our cue, we define a generalization of "pure." If $A$ is $\aleph_1$-free, call a subgroup $B$ of $A$ an **$\aleph_1$-pure** subgroup if $A/B$ is $\aleph_1$-free. We shall refer to the following as **Chase's condition.**

(5.1) *$A$ is an $\aleph_1$-free group such that every countable subgroup of $A$ is contained in a countable $\aleph_1$-pure subgroup of $A$.*

(This terminology is not standard. We have chosen it since Chase [1] proved, under the assumption of

the Continuum Hypothesis, that every $W$-group satisfies (5.1).) As we shall see later (Theorem 7.3), Chase's condition does *not* imply that $A$ is free (unless $A$ is countable). But it does play a central role in our proof of Theorem 1.3. In this section we shall give a necessary and sufficient condition for a group of cardinality $\aleph_1$ satisfying Chase's condition to be free. We begin by expressing the condition in terms of ascending chains. (Here $\omega_1$ is the first uncountable ordinal, i.e. the first ordinal of cardinality $\aleph_1$.)

5.2 LEMMA. *If $A$ is a group of cardinality $\aleph_1$, $A$ satisfies Chase's condition* (5.1) *if and only if $A$ is the union of a smooth chain of countable free groups*

$$A_0 \subseteq \cdots \subseteq A_\upsilon \subseteq \cdots, \qquad \upsilon < \omega_1$$

*such that $A_0 = 0$ and for each ordinal $\upsilon < \omega_1$, $A_{\upsilon+1}$ is $\aleph_1$-pure in $A$.*

*Proof.* Suppose $A$ satisfies (5.1). Since $A$ has cardinality $\aleph_1$ we can list all the elements of $A$ in a sequence of length $\omega_1$:

$$A = \{a_\upsilon \mid \upsilon < \omega_1\}.$$

Define $A_\upsilon$ by induction on $\upsilon < \omega_1$. Let $A_0$ be 0. If $A_\mu$ has been defined for all $\mu < \upsilon$ consider two cases. First, if $\upsilon$ is a limit ordinal, let $A_\upsilon = \bigcup_{\mu<\upsilon} A_\mu$. (A countable union of countable sets is countable.) Second, if $\upsilon$ is a successor ordinal, say $\upsilon = \mu + 1$, let $A_\upsilon$ be a countable $\aleph_1$-pure subgroup of $A$ which contains $A_\mu \cup \{a_\mu\}$. The chain constructed in this fashion clearly satisfies the properties stated in the lemma. Conversely, if $A$ is the union of a smooth chain of the type described, then $A$ satisfies Chase's condition because any countable subgroup $B$ of $A$ is contained in $A_{\upsilon+1}$ for some $\upsilon < \omega_1$. This completes the proof of the lemma.

Referring to the notation of the lemma, let $E$ be the set of all limit ordinals $\lambda < \omega_1$ such that $A_\lambda$ is *not* $\aleph_1$-pure in $A$. We shall prove that the "size" of $E$ determines whether or not $A$ is free. First we need some definitions.

A function $f: \omega_1 \to \omega_1$ is called **normal** if it is strictly increasing (i.e., $\mu < \upsilon$ implies $f(\mu) < f(\upsilon)$) and continuous (i.e., for any limit ordinal $\lambda$, $f(\lambda) = \sup\{f(\upsilon) \mid \upsilon < \lambda\}$). Notice that if $f$ is normal then the image of $f$ is unbounded in $\omega_1$ since the image is uncountable. A subset $S$ of $\omega_1$ is called **stationary** if the image of every normal function has non-empty intersection with $S$. Examples of stationary subsets of $\omega_1$ are: $S = \omega_1$; $S$ = the set of all limit ordinals $< \omega_1$; $S$ = the set of all limit ordinals $< \omega_1$ which are not of the form $\alpha + \omega$; $S$ = any set which contains the image of a normal function. (We use only the first two examples.)

Making use of the same notation as in Lemma 5.2 and the comments following the lemma, we can state the main theorem of this section.

5.3 THEOREM [2]. *The group $A$ is free if and only if $E$ is not a stationary subset of $\omega_1$.*

*Proof.* Suppose $E$ is not stationary in $\omega_1$. Let $f: \omega_1 \to \omega_1$ be a normal function whose image does not intersect $E$, and let $\tilde{A}_\upsilon = A_{f(\upsilon)}$. Then since $f$ is an unbounded continuous function, $\{\tilde{A}_\upsilon \mid \upsilon < \omega_1\}$ is a smooth chain whose union is $A$. Since the range of $f$ does not intersect $E$, $\tilde{A}_\upsilon$ is $\aleph_1$-pure in $A$ for every $\upsilon < \omega_1$. Hence $\tilde{A}_{\upsilon+1}/\tilde{A}_\upsilon$ is free for every $\upsilon < \omega_1$, by the definition of $\aleph_1$-pure. Therefore Theorem 2.6 implies that $A$ is free.

Conversely, suppose that $A$ is free and let $X$ be a basis of $A$. We assert that there is a smooth chain $\{X_\upsilon \mid \upsilon < \omega_1\}$ of subsets of $X$, and a normal function $f: \omega_1 \to \omega_1$ such that for every $\upsilon < \omega_1$, $X_\upsilon$ is a basis of $A_{f(\upsilon)}$. If this is the case, then $E$ is not stationary in $\omega_1$. Indeed, for every $\upsilon < \omega_1$, $f(\upsilon)$ is not in $E$ since $A/A_{f(\upsilon)}$ is isomorphic to the free group generated by $X - X_\upsilon$ and hence $A_{f(\upsilon)}$ is $\aleph_1$-pure in $A$.

We define $X_\upsilon$ and $f(\upsilon)$ by induction on $\upsilon$. Let $X_0 = \varnothing$ and $f(0) = 0$. Suppose $X_\mu$ and $f(\mu)$ have been defined for all $\mu < \upsilon$. If $\upsilon$ is a limit ordinal, let $X_\upsilon = \bigcup_{\mu<\upsilon} X_\mu$ and $f(\upsilon) = \sup\{f(\mu) \mid \mu < \upsilon\}$. Then $X_\upsilon$ is a basis of $A_{f(\upsilon)}$ since $A_{f(\upsilon)} = \bigcup_{\mu<\upsilon} A_{f(\mu)}$. If $\upsilon = \mu + 1$, let $Y_0$ be a countable subset of $X$

properly containing $X_\mu$ and let $\sigma_0$ be an ordinal such that $Y_0 \subseteq A_{\sigma_0}$. Let $Y_1$ be a countable subset of $X$ such that $A_{\sigma_0}$ is contained in $\langle Y_1 \rangle$. By induction on $n$ we obtain a chain of countable subsets of $X$

$$X_\mu \subsetneqq Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n \subseteq \cdots, \qquad n < \omega$$

and a sequence of ordinals

$$f(\mu) < \sigma_0 \leqq \cdots \leqq \sigma_n \leqq \cdots, \qquad n < \omega$$

such that for every $n < \omega$,

$$Y_n \subseteq A_{\sigma_n} \subseteq \langle Y_{n+1} \rangle.$$

If we let $X_\upsilon = \bigcup_{n<\omega} Y_n$ and $f(\upsilon) = \sup\{\sigma_n : n < \omega\}$, then $X_\upsilon$ is a basis of $A_{f(\upsilon)}$. This completes the proof of the theorem.

**6. The axiom of constructibility.** We shall make use of the following rather amazing consequence of Godel's axiom $V = L$ which was discovered by Jensen [9]. (Although we cannot give the proof here, this result is a relatively easy consequence of the formal definition and some elementary properties of $L$. We could have simply taken this consequence of $V = L$ as our axiom — replacing $V = L$ in 1.3 (i) — but we preferred to indicate its relation to the well-known Axiom of Constructibility.)

6.1 THEOREM. *Assume $V = L$. Let $C$ be a set which is the union of a strictly increasing smooth chain of countable sets $\{C_\upsilon \mid \upsilon < \omega_1\}$, and let $E$ be a stationary subset of $\omega_1$. Then there is a sequence $\{S_\upsilon \mid \upsilon \in E\}$ such that $S_\upsilon \subseteq C_\upsilon$ for all $\upsilon \in E$ and such that for any subset $X$ of $C$, the set of $\upsilon$ in $E$ with $X \cap C_\upsilon = S_\upsilon$ is stationary in $\omega_1$.*

We actually make use of the following corollary of the theorem:

6.2 COROLLARY. *Assume $V = L$. Let $B$ be a set which is the union of a strictly increasing smooth chain of countable sets $\{B_\upsilon \mid \upsilon < \omega_1\}$ and let $Y$ be any countable set. Let $E$ be a stationary subset of $\omega_1$. Then there is a sequence of functions $\{g_\upsilon : B_\upsilon \to B_\upsilon \times Y \mid \upsilon \in E\}$ such that for any function $h : B \to B \times Y$ satisfying $h(B_\upsilon) \subseteq B_\upsilon \times Y$ for all $\upsilon$, there is an $\upsilon$ in $E$ such that $h$ restricted to $B_\upsilon$ is $g_\upsilon$.*

*Proof.* To prove the corollary from the theorem let $C_\upsilon = B_\upsilon \times (B_\upsilon \times Y)$ and $C = B \times (B \times Y)$. Let $\{S_\upsilon \mid \upsilon \in E\}$ be the sequence given by the conclusion of Theorem 6.1. Note that $S_\upsilon$ is a subset of $B_\upsilon \times (B_\upsilon \times Y)$. If $S_\upsilon$ is a function from $B_\upsilon$ to $B_\upsilon \times Y$, let $g_\upsilon = S_\upsilon$. Otherwise let $g_\upsilon$ be any function from $B_\upsilon$ to $B_\upsilon \times Y$. If $h$ is any function from $B$ to $B \times Y$ we may regard $h$ as a subset of $B \times (B \times Y)$. By Theorem 6.1 there is an $\upsilon$ (in fact a stationary set of them) such that $h \cap C_\upsilon = S_\upsilon$. Because of the hypothesis that $h(B_\upsilon) \subseteq B_\upsilon \times Y$, we see that $h \cap C_\upsilon$ equals $h$ restricted to $B_\upsilon$, which, therefore, equals $g_\upsilon$. This completes the proof of the corollary.

The following theorem is the key to the solution of Whitehead's Problem under the assumption $V = L$. It is proved by techniques like those used in the proof of Theorem 4.1. In the proof of 4.1 we did not need any special assumptions because every potential splitting homomorphism for $\pi$ was already determined by its values on $B_0$. In the situation of the following theorem where $B = \bigcup_{\upsilon<\omega_1} B_\upsilon$ is uncountable, we need Corollary 6.2 in order to obtain a list of homomorphisms on the $B_\upsilon$'s which includes a restriction of every potential splitting homomorphism for $\pi$.

6.3 THEOREM. *Assume $V = L$. Let $B$ be the union of a strictly increasing smooth chain $\{B_\upsilon \mid \upsilon < \omega_1\}$ of countable free groups such that $E = \{\upsilon < \omega_1 \mid B_{\upsilon+1}/B_\upsilon$ is not free$\}$ is stationary in $\omega_1$. Then $B$ is not a $W$-group.*

*Proof.* Just as in the proof of Theorem 4.1 we define by induction a smooth chain of groups $\{C_\upsilon \mid \upsilon < \omega_1\}$ such that $C_\upsilon$ is a $(B_\upsilon, \boldsymbol{Z})$-group and the union $C$ is a $(B, \boldsymbol{Z})$-group such that $\pi : C \to B$

does not split. By Corollary 6.2 there is a set of functions

$$\{g_\upsilon : B_\upsilon \to B_\upsilon \times Z \mid \upsilon \in E\}$$

such that for every $h : B \to B \times Z$ with $\pi h = 1_B$, there is a $\mu$ in $E$ such that $h \mid B_\mu = g_\mu$.

Let $C_0$ be any $(B_0, Z)$-group. Suppose that $C_\mu$ has been defined for all $\mu < \upsilon$. If $\upsilon$ is a limit ordinal, let $C_\upsilon = \bigcup_{\mu<\upsilon} C_\mu$. If $\upsilon$ is a successor ordinal, say $\upsilon = \mu + 1$, consider two cases. First, if $\mu \in E$ and $g_\mu : B_\mu \to B_\mu \times Z$ is a splitting homomorphism for $\pi : C_\mu \to B_\mu$, let $C_\upsilon$ be an extension of $C_\mu$ such that $g_\mu$ does not extend to a splitting homomorphism for $\pi : C_\upsilon \to B_\upsilon$. (Note that $C_\upsilon$ exists by Lemma 4.3, since $B_{\mu+1}/B_\mu$ is not free and therefore, by Theorem 4.1, is not a $W$-group.) In the second case, if $\mu \notin E$ or $g_\mu$ is not a splitting homomorphism for $\pi : C_\mu \to B_\mu$, let $C_\upsilon$ be any $(B_\upsilon, Z)$-group extending $C_\mu$ (cf. proof of 4.1).

Let $C$ be the union of the chain $\{C_\upsilon \mid \upsilon < \omega_1\}$. If there is a splitting homomorphism $\rho : B \to C$ for $\pi : C \to B$ then there is a $\mu \in E$ such that $\rho \mid B_\mu = g_\mu$. We are clearly in the first case of the definition. But then, just as in the proof of 4.1, we obtain a contradiction of our construction of $C_{\mu+1}$. Hence $\pi$ does not split and the theorem is proved.

*Proof of Theorem* 1.3(i). Let $A$ be a $W$-group of cardinality $\aleph_1$. We shall prove that $A$ is free in two steps. First we shall prove that $A$ satisfies Chase's condition (5.1) and then we shall use Theorem 5.3 to prove that $A$ is free.

Suppose, first that $A$ does not satisfy Chase's condition. Since Corollary 4.4 says $A$ is $\aleph_1$-free, this means: (*) there is a countable subgroup $B_0$ of $A$ such that for any countable subgroup $C$ of $A$ which contains $B_0$ there is a countable subgroup $C'$ containing $C$ such that $C'/C$ is not free. It follows that there is a strictly increasing smooth chain $\{B_\upsilon \mid \upsilon < \omega_1\}$ of countable subgroups of $A$ such that for each $\upsilon < \omega_1$, $B_{\upsilon+1}/B_\upsilon$ is not free. (The chain is constructed by transfinite induction using (*).) If $B$ is the union of this chain then, by Theorem 6.3, $B$ is not a $W$-group (since in this case $E = \omega_1$). We have thus obtained a contradiction of Theorem 3.1. Hence $A$ must satisfy Chase's condition.

By Lemma 5.2, $A$ is the union of a smooth chain $\{A_\upsilon \mid \upsilon < \omega_1\}$ such that for each $\upsilon < \omega_1$, $A_{\upsilon+1}$ is $\aleph_1$-pure in $A$. Let $E = \{\lambda < \omega_1 \mid A_\lambda \text{ is not } \aleph_1\text{-pure in } A\}$ and let $E' = \{\lambda < \omega_1 \mid A_{\lambda+1}/A_\lambda \text{ is not free}\}$. We assert that $E = E'$. Certainly $E' \subseteq E$. Conversely, if $\lambda$ is not in $E'$, then for every $\upsilon > \lambda$, $A_\upsilon/A_\lambda$ is free by Corollary 2.5 since

$$(A_\upsilon/A_\lambda)/(A_{\lambda+1}/A_\lambda) \cong A_\upsilon/A_{\lambda+1}.$$

It follows that $A/A_\lambda$ is $\aleph_1$-free since every countable subgroup of $A/A_\lambda$ is contained in $A_\upsilon/A_\lambda$ for some $\upsilon > \lambda$. Thus $\lambda$ is not in $E$ and the assertion is proved.

By Theorem 6.3, $E'$ is not a stationary subset of $\omega_1$ because $A$ is a $W$-group. Since $E$ equals $E'$, Theorem 5.3 implies that $A$ is free. The proof of Theorem 1.3(i) is complete.

**7. Martin's axiom.** We shall state a special case of Martin's axiom in the form of a theorem. The reader, if he is familiar with the general form of MA (see Shoenfield [**18**]; also Martin–Solovay [**12**]) should find it easy to supply a proof of the theorem. Otherwise, he may take it as an axiom.

7.1 THEOREM. *Assume MA. Let $A$ and $B$ be sets of cardinality $< 2^{\aleph_0}$ and let $P$ be a family of functions with the following properties*:

(7.1.1) *for every $f$ in $P$, $f$ is a function from a subset of $A$ into $B$*;

(7.1.2) *for every $a$ in $A$ and every $f$ in $P$, there exists $g$ in $P$ such that $f \subseteq g$ and $a$ is in the domain of $g$*; *and*

(7.1.3) *for every uncountable subset $P'$ of $P$, there exist $f_1$, $f_2$ in $P'$ and $f_3$ in $P$ such that $f_1 \neq f_2$ and $f_3$ extends both $f_1$ and $f_2$.*

*Then there exists a function $g : A \to B$ such that for every finite subset $F$ of $A$ there exists $f$ in $P$ with $F$ contained in the domain of $f$ and $g \mid F = f \mid F$.*

(An example which shows that condition (7.1.3) — called the **countable antichain condition** — is necessary is: $P$ = the set of all one-one functions from a finite subset of an uncountable set $A$ into a countable set $B$.)

We shall use Theorem 7.1 to prove the following:

7.2 THEOREM. *Assume MA* $+ 2^{\aleph_0} > \aleph_1$. *Let A be any group of cardinality* $\aleph_1$ *which satisfies Chase's condition. Then A is a W-group.*

Before proving 7.2 let us see that it implies Theorem 1.3 (ii). The following theorem is proved in *ZFC*.

7.3 THEOREM [6]. *There is a group A of cardinality* $\aleph_1$ *which satisfies Chase's condition but is not free.*

*Proof.* We shall define by induction a smooth chain $\{A_\upsilon \mid \upsilon < \omega_1\}$ of countable groups satisfying the following three properties:

(i) for every $\upsilon < \omega_1$, $A_\upsilon$ is free;

(ii) for every $\mu < \upsilon < \omega_1$, $A_\upsilon/A_{\mu+1}$ is free; and

(iii) for every limit ordinal $\lambda < \omega_1$, $A_{\lambda+1}/A_\lambda$ is *not* free.

If $A$ is the union of such a chain, $A$ satisfies the requirements of the theorem. Indeed, $A$ satisfies Chase's condition by Lemma 5.2 and properties (i) and (ii). (Notice that for every $\mu < \omega_1, A_{\mu+1}$ is $\aleph_1$-pure in $A$ because every countable subgroup of $A/A_{\mu+1}$ is contained in $A_\upsilon/A_{\mu+1}$ for some $\upsilon < \omega_1$.) By (iii) the set of limit ordinals $\lambda$ such that $A_\lambda$ is not $\aleph_1$-pure in $A$ equals the set of all limit ordinals, which is clearly a stationary set. Therefore by Theorem 5.3, $A$ is not free.

Now let us define the chain. Let $A_0 = 0$ and suppose that for some $\delta < \omega_1$, a smooth chain $\{A_\upsilon \mid \upsilon < \delta\}$ has been defined which satisfies (i), (ii) and (iii). We must define $A_\delta$ and verify the three properties for the chain $\{A_\upsilon : \upsilon < \delta + 1\}$. There are three cases.

*Case* 1: $\delta = \upsilon + 1$, where $\upsilon$ is *not* a limit ordinal. Let $A_\delta = A_\upsilon \oplus Z$. The three properties are easily verified. (Property (ii) uses Corollary 2.5.)

*Case* 2: $\delta = \lambda$, a limit ordinal. Let $A_\lambda = \bigcup_{\upsilon<\lambda} A_\upsilon$. Choose a strictly increasing sequence $\{\sigma_n \mid n < \omega\}$ whose limit is $\lambda$ such that $\sigma_n$ is a successor ordinal for every $n$. Then $A_\lambda = \bigcup_{n<\omega} A_{\sigma_n}$, and by (ii), $A_{\sigma_{n+1}}/A_{\sigma_n}$ is free for every $n < \omega$. By Theorem 2.6, $A_\lambda$ is free, and $A/A_{\sigma_n}$ is free for every $n < \omega$. Hence (i) holds; and (ii) follows from Corollary 2.5 since if $\mu < \lambda$, $A_{\mu+1}$ is contained in $A_{\sigma_n}$ for some $n < \omega$. Property (iii) is obvious in this case.

*Case* 3: $\delta = \lambda + 1$, where $\lambda$ is a limit ordinal. This is the most difficult case. Let $\{\sigma_n \mid n < \omega\}$ be as in Case 2, except that for convenience we require $\sigma_0 = 0$. By the proof of Theorem 2.6 we know that there exists a smooth chain of sets $\{X_n \mid n < \omega\}$ such that $X_n$ is a basis of $A_{\sigma_n}$. For each $n > 1$, choose $x_n \in X_n - X_{n-1}$. Let $Y_n = X_n - \{x_n\}$ for each $n > 1$ and let $B$ be the subgroup of $A_\lambda$ generated by $\bigcup_n Y_n$. Let $P = \prod_{n=1}^{\infty} \langle x_n \rangle$. Define $A_{\lambda+1}$ to be the subgroup of $B \oplus P$ generated by $A_\lambda$ and $\{z_m : 1 \leqq m < \omega\}$, where $z_m$ is the element of $P$ represented (in the obvious sense) by the formal sum

$$z_m = \sum_{n \geqq m} (n!/m!)x_n.$$

It is an easy exercise to verify that $\bigcup_n Y_n \cup \{z_m : 1 \leqq m < \omega\}$ is a basis of $A_{\lambda+1}$. Thus (i) holds. For each $k < \omega$, $A_{\lambda+1}/A_{\sigma_k}$ is isomorphic to the subgroup of $A_{\lambda+1}$ generated by

$$\bigcup_{n>k} (Y_n - Y_k) \cup \{z_m : k + 1 \leqq m < \omega\}.$$

Thus $A_{\lambda+1}/A_{\sigma_k}$ is free; property (ii) follows from Corollary 2.5 (cf. Case 2). As for property (iii), notice that $m!z_m - z_1 \in A_\lambda$ for every $m \geqq 1$. Thus $z_1 + A_\lambda$ is a non-zero element of $A_{\lambda+1}/A_\lambda$ which is divisible by $n$ for every $n > 0$. Since a free group does not have any elements of this type, property (iii) and the theorem are proved.

*Proof of* 7.2. Let $A$ be a group of cardinality $\aleph_1$ satisfying Chase's condition and let $\pi : B \to A$ be a surjective homomorphism with kernel $Z$. (All we really need is that the kernel is countable.) We shall prove that $\pi$ splits by applying Theorem 7.1 to the set $P$ of all homomorphisms $\varphi : S \to B$ satisfying:

(7.2.1) $\pi\varphi = 1_S$; and $S$ is a finitely-generated pure subgroup of $A$.

If we prove that $P$ satisfies (7.1.1), (7.1.2) and (7.1.3), then by Theorem 7.1 there is a function $g : A \to B$ which agrees with a member of $P$ on any given finite subset of $A$. It follows that $g$ is a homomorphism and that $\pi g = 1_A$. Hence $\pi$ splits.

Since (7.1.1) is obvious for $P$, it remains to verify (7.1.2) and (7.1.3). Property (7.1.2) is a consequence of the following lemma.

7.4 LEMMA. *If $\varphi$ is in $P$ and $F$ is a finite subset of $A$, then there is a function $\varphi'$ in $P$ such that $\varphi'$ extends $\varphi$ and $F$ is contained in the domain of $\varphi'$.*

*Proof.* Let $S$ be the domain of $\varphi$. Since $A$ is $\aleph_1$-free, there is a finitely-generated pure subgroup $S'$ of $A$ which contains $S \cup F$ (cf. the remarks preceding Theorem 4.2). Now $S'/S$ is finitely-generated and torsion-free (since $S$ is pure in $A$) and therefore, by Theorem 2.2, $S'/S$ is free. By Corollary 2.5 there is a basis of $S'$ of the form $X \cup Y$ where $X$ is a basis of $S$. If $x \in X$, define $\varphi'(x) = \varphi(x)$. If $y \in Y$ define $\varphi'(y) = b_y$, where $b_y$ is some element of $B$ such that $\pi(b_y) = y$. This defines a homomorphism $\varphi' : S' \to B$ which has the desired properties; thus the lemma is proved.

The proof of property (7.1.3) requires a little more work. Let us first prove it under the assumption that $P'$ is an uncountable subset of $P$ such that there is a *pure* subgroup $A'$ of $A$ which is free and which contains the domains of all the elements of $P'$. Choose a basis $X = \{x_\nu \mid \nu < \omega_1\}$ of $A'$. Because of Lemma 7.4 we may assume that the domain of each $\varphi$ in $P'$ is generated by a finite subset of $X$. Moreover, since a countable union of countable sets is countable, we may assume (replacing $P'$ by an uncountable subset if necessary) there is an $m$ such that for every $\varphi$ in $P'$, the domain of $\varphi$ is generated by exactly $m$ elements of $X$. Let $P' = \{\varphi_\nu \mid \nu < \omega_1\}$ and let $Y_\nu \subseteq X$ be a basis of the domain of $\varphi_\nu$. Since $Y_\nu$ has cardinality $m$ for $\nu$, there is a subset $T$ of $X$ which is maximal with respect to the property that $T$ is contained in $Y_\nu$ for uncountably many $\nu$. (Possibly $T = \varnothing$.) Notice that since the kernel of $\pi$ is countable there are only countably many functions on $T$ which belong to $P$. Hence we may assume without loss of generality that $\varphi_\mu$ and $\varphi_\nu$ agree on $T$ whenever $T \subseteq Y_\mu$ and $T \subseteq Y_\nu$. Renumbering, we may assume that $T \subseteq Y_0$. For each $y \in Y_0 - T$, there are only countably many $\nu$ such that $y \in Y_\nu$ (by the maximality of $T$). Hence there exists $\nu \neq 0$ such that $Y_\nu \cap Y_0 = T$. Since $\varphi_0$ and $\varphi_\nu$ agree on $T$ they have a common extension $\psi : \langle Y_0 \cup Y_\nu \rangle \to B$. Now $\langle Y_0 \cup Y_\nu \rangle$ is a pure subgroup of $A'$ since it is generated by a subset of a basis of $A'$. Therefore $\langle Y_0 \cup Y_\nu \rangle$ is a pure subgroup of $A$ since

$$(A/\langle Y_0 \cup Y_\nu \rangle)/(A'/\langle Y_0 \cup Y_\nu \rangle) \cong A/A'.$$

(We make use of the easily proved fact that if $A/B$ and $B$ are torsion-free, then $A$ is torsion-free.) Hence $\psi$ is an element of $P$ and property (7.1.3) is proved under the special assumption. The general case of (7.1.3) follows from the next lemma. (Recall that we are assuming that $A$ satisfies Chase's condition. Here we finally use that hypothesis!)

7.5 LEMMA. *For any uncountable subset $P'$ of $P$ there is a free subgroup $A'$ which is pure in $A$ and an uncountable subset $P''$ of $P'$ such that $\operatorname{dom}(\varphi) \subseteq A'$ for every $\varphi$ in $P''$.*

*Proof.* Suppose $P' = \{\varphi_\nu \mid \nu < \omega_1\}$ where $\varphi_\nu : S_\nu \to B$. Replacing $P'$ by an uncountable subset if necessary, we may assume there is an $m$ such that $S_\nu$ has a basis of cardinality $m$ for all $\nu < \omega_1$. There is a pure subgroup $T$ of $A$ maximal with respect to the property that $T$ is contained in uncountably many $S_\nu$. We may assume that $T$ is contained in $S_\nu$ for all $\nu$. By Theorem 2.2 and Corollary 2.5 we may assume there is a (finite) basis of $S_\nu$ of the form $X \cup Y_\nu$ where $X$ is a basis of $T$.

We shall construct $A'$ as the union of a smooth chain $\{A_\nu \mid \nu < \omega_1\}$ such that for each $\nu < \omega_1$, $A_\nu$ is a pure subgroup of $A$ and $A_{\nu+1}/A_\nu$ is free. Then $A'$ will be free by Theorem 2.6; moreover, $A'$ will be pure in $A$ since it is a union of pure subgroups of $A$.

Let $A_0 = T$. Suppose we have defined $\{A_\mu \mid \mu < \nu\}$ and a strictly increasing sequence of ordinals $\{\sigma_{\mu+1} \mid \mu < \nu\}$ such that $Y_{\sigma_{\mu+1}} \subseteq A_{\mu+1}$. If $\nu$ is a limit ordinal let $A_\nu = \bigcup_{\mu<\nu} A_\mu$. If $\nu$ is a successor ordinal, $\nu = \delta + 1$, let $C_\delta$ be a countable $\aleph_1$-pure subgroup of $A$ which contains $A_\delta$. (Here we use the hypothesis that $A$ satisfies Chase's condition.) There exists $\sigma_\nu > \sigma_{\mu+1}$ for all $\mu < \nu$ such that $\langle Y_{\sigma_\nu}\rangle \cap C_\delta = 0$. (Otherwise, since $C_\delta$ is countable there would be an element $c \in C_\delta$ and uncountably many $\tau < \omega_1$ such that $c \in \langle Y_\tau\rangle$; but then the pure closure of $T + \langle c\rangle$ would contradict the maximality of $T$.) Let $A_\nu$ be the pure closure of $A_\delta + \langle Y_{\sigma_\nu}\rangle$. Since $(Y_{\sigma_\nu}) \cap C_\delta = 0$, it follows that $A_\nu \cap C_\delta = A_\delta$. Hence $A_\nu/A_\delta$ is isomorphic to a countable subgroup of $A/C_\delta$ and is therefore free — since $C_\delta$ is $\aleph_1$-pure in $A$. If we let $P'' = \{\varphi_{\sigma_{\mu+1}} : \mu < \omega_1\}$ we see that the conclusions of the lemma are satisfied and hence Theorem 7.2 is proved.

**8. Generalizations.** Theorem 1.3 (ii) may be generalized as follows. Assuming $MA + 2^{\aleph_0} > \aleph_1$, for every uncountable cardinal $\kappa$, there is a non-free $W$-group of cardinality $\kappa$. This follows because if $A$ is a non-free $W$-group of cardinality $\aleph_1$ then the direct sum of $\kappa$ copies of $A$ is a $W$-group, because in general

$$\operatorname{Ext}\left(\bigoplus_{i\in I} A_i, C\right) = \prod_{i\in I} \operatorname{Ext}(A_i, C).$$

Shelah (Is. J. Math., 21 (1975) 319–349) has shown that Theorem 1.3 (i) generalizes also, i.e., assuming $V = L$, every $W$-group is free. The proof is by induction on the cardinality of the $W$-group. At singular cardinalities the proof requires the following result of Shelah's: if $A$ is a group of singular cardinality $\kappa$ such that every subgroup of $A$ of cardinality $< \kappa$ is free, then $A$ is free.

Also, recently Shelah has proved that a negative answer to Whitehead's problem for groups of cardinality $\aleph_1$ is consistent with $ZFC + 2^{\aleph_0} = \aleph_1$. This together with Theorems 1.3 (i) and 1.1 (ii) implies that Whitehead's problem is undecidable in $ZFC + 2^{\aleph_0} = \aleph_1$. (An assertion to the contrary in Math. Reviews, vol. 50 # 2362 is in error.)

The proof of 7.3 can be generalized to prove that there are $2^{\aleph_1}$ pairwise non-isomorphic groups of cardinality $\aleph_1$ which satisfy Chase's condition.

The proof of Theorem 1.3 which we have given is somewhat different in detail from that in [17], although the key ideas are the same. The advantage of the proof given here is that it may easily be generalized to obtain other undecidability results. See [3] for these as well as related results.

Work on this paper was partially supported by NSF grant GP-43910.

I wish to thank Professors S. Feferman, U. Felgner and R. Solovay for their helpful comments.

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## MUCH ADO ABOUT NOTHING

MORRIS MARDEN

Let me begin by explaining my use of the Shakespearean title for this talk. Ever since I have been studying the *zeros* of functions, some of my non-mathematical friends have poked fun at me as being the world's expert on *nothing*. As I have been the innocent victim of this pun for almost fifty years during which I have tried my best to *do much about zeros*, the appropriateness of the title is obvious. Of course, Shakespeare applied his title to a comedy, so I hope that I have not misled anyone to expect anything but a very serious mathematical talk today.

By request of the committee on arrangements*, this talk is to be a retrospect on some of my past research. In order to give the talk some unity, I shall limit it to a single topic: the extension of Rolle's theorem to the complex plane. This is, of course, the theorem which states that between any pair of real zeros of a real differentiable function $f$ lies at least one zero of its derivative; that is, at least one critical point of $f$. It is a theorem which one meets in any introductory course in the calculus. Yet its extension to the complex plane is by no means trivial. In fact Rolle's theorem does not hold for arbitrary analytic functions of a complex variable as is shown by the example $f(z) = e^{zi} - 1$. This function has zeros at $z = 0$ and $z = 2\pi$ but its derivative $f'(z) = ie^{zi}$ has no zeros whatsoever.

However, for certain classes of analytic functions of a complex variable some analogues to Rolle's theorem have been found; namely, for polynomial, rational, entire and meromorphic functions. I shall describe these analogues including my own contributions. In general I shall omit proofs but give physical interpretations where possible. Now, these analogues of Rolle's theorem fall into two types. The first type pertains to functions for which the location of *all* the zeros and poles is prescribed. The second type pertains to functions for which the location of only *some* zeros and poles is prescribed, the remaining ones being unspecified.

The theorems of the first type aim to generalize the corollary to Rolle's theorem that any interval of the real axis containing all the zeros of a given polynomial $P$ also contains all the critical points of $P$. The initial complex variable discovery along these lines goes back to the earliest days of complex variables and in fact to the man who was most influential in getting the geometric representation of complex variables accepted. I refer to Gauss who in 1836 stated that the critical points of a polynomial $P$ are the multiple zeros of $P$ and the equilibrium points in a field of force due to particles at the zeros of $P$, attracting according to the inverse distance law. Gauss' mechanical interpretation may be easily

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* Lecture given on May 9, 1975 as part of the Colloquium at the University of Wisconsin-Milwaukee honoring the author's retirement from the UWM.

verified if we write $P(z) = \Pi_{j=1}^{p}(z - z_j)^{m_j}$, take its logarithmic derivative

$$\frac{P'(z)}{P(z)} = \sum_{k=1}^{p} \frac{m_k}{z - z_k}, \qquad n = \sum_{k=1}^{p} m_k \tag{1}$$

and then interpret vectorially each term in this sum. An immediate consequence of Gauss' theorem is that all the critical points of $P$ lie in the convex hull of the zeros of $P$. However, this conclusion was not published by Gauss, but was first announced and proved in 1874 by a French engineer named Lucas. Since a convex region may be regarded as the intersection of all its covering disks, an equivalent to Lucas' theorem is the statement that all the critical points of $P$ lie on any closed disk containing all the zeros of $P$.

Now, an $n$th degree polynomial may be regarded as a rational function which has an $n$th order pole at infinity. A natural generalization of Lucas' theorem is therefore the one given by Laguerre in 1880 that, if all the zeros of an $n$th degree polynomial $P$ lie in a circular region $C$, not containing the point $\zeta_0$, then all the critical points of the function $f(z) = P(z)/(z - \zeta_0)^n$ also lie in $C$. (By a circular region we mean the closure of not merely the interior of a circle but also the exterior of a circle or a half-plane.) Laguerre's theorem was in turn generalized in 1904 by Bôcher to read that, if $P(z)$ is an $n$th degree polynomial all of whose zeros lie in a circular region $C_1$, and if $Q(z)$ is an $n$th degree polynomial all of whose zeros lie in circular region $C_2$, and if $C_1 \cap C_2 = \varnothing$, then no critical point of the function $F(z) = P(z)/Q(z)$ lies in the complement of $C_1 \cup C_2$.

If we set $P(z) = \Pi_{k=1}^{p}(z - z_k)^{m_k}$, and $Q(z) = \Pi_{k=1}^{q}(z - \zeta_k)^{\mu_k}$, $\Sigma_{k=1}^{q}\mu_k = n = \Sigma_{k=1}^{p} m_k$, we see that

$$\frac{F'(z)}{F(z)} = \sum_{k=1}^{p} \frac{m_k}{z - z_k} + \sum_{k=1}^{q} \frac{(-\mu_k)}{z - \zeta_k}. \tag{2}$$

Thus, the zeros of $F'(z)/F(z)$ are the equilibrium points in a field of force due to positive masses $m_k$ at the points $z_k$ and negative masses $(-\mu_k)$ at the points $\zeta_k$. Two alternative physical interpretations of the zeros of $F'(z)/F(z)$ are possible; namely,

(i) the equilibrium points in a field due to two dimensional positive and negative electric charges at the zeros and poles of $F$, respectively; or

(ii) the stagnation points in an incompressible flow generated by two dimensional sources and sinks at the zeros and poles of $F$, respectively.

So far I have described the work of Gauss, Lucas, Laguerre and Bôcher. Their work was generalized by Walsh in a series of papers published between 1918 and 1922, which included the following three beautiful results.

THEOREM W1. *If an $n$-th degree polynomial $P$ has $m_1$ zeros in a circle $C_1$, center at $c_1$ and radius $r_1$, and the remaining $m_2 = n - m_1$ zeros lie in a circle $C_2$, center at $c_2$ and radius $r_2$, then any critical point of $P$ not in $C_1$ or $C_2$ lies in a third circle $C_3$ with center at $c_3 = (m_2c_1 + m_1c_2)/n$ and radius $r_3 = (m_2r_1 + m_1r_2)/n$.* [See Fig. 1.]

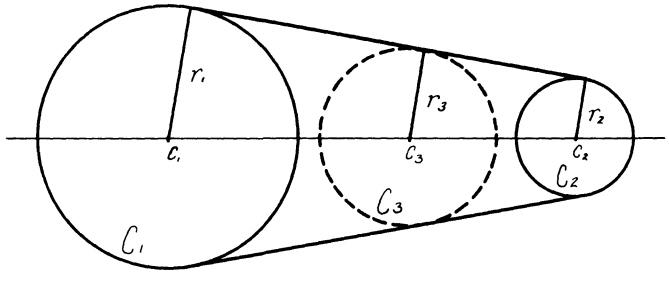



FIG. 1

THEOREM W2. *If the zeros of a polynomial $P$ lie in a set of $p+1$ circles $C_0, C_1, \ldots, C_p$ having an external center 0 of similitude, the critical points of $P$ not in any circle $C_k$ lie in the set of $p$ circles $C_1', C_2', \ldots, C_p'$ also having 0 as an external center of similitude and having the same common tangents as the circles $C_k$.* [See Fig. 2.]

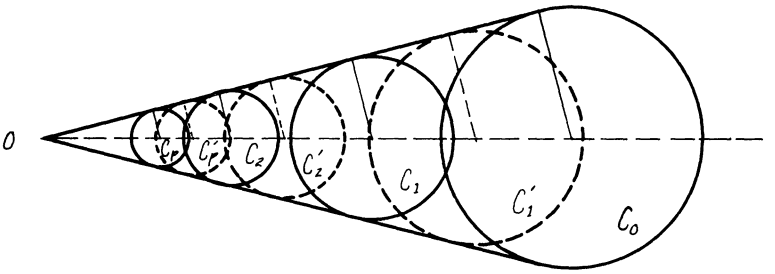



FIG. 2

THEOREM W3. *If all the zeros of an $n$-th degree polynomial $P(z)$ lie in the union of two circular regions $C_1$ and $C_2$ and if all the zeros of an $n$-th degree polynomial $Q(z)$ lie in a circular region $C_3$, then any critical point of $F(z) = P(z)/Q(z)$ which does not lie in $C_1 \cup C_2 \cup C_3$, lies in a fourth circular region $C_4$. The boundary of $C_4$ is the locus of the point $z_4$ defined by a constant cross-ratio $(z_1, z_2, z_3, z_4) = \alpha$ when $z_1, z_2, z_3$ describe the boundaries of $C_1$, $C_2$, $C_3$ respectively.*

Now, Walsh's proofs were geometrical and mechanical and in the case of his cross-ratio theorem long and complicated. Attempts to prove the latter result analytically were made by the American mathematician Coble and later by the Japanese mathematician Nakahara but both attempts were unsuccessful. The task of finding a correct analytical proof was assigned to me by Walsh as a Ph.D. thesis subject. Ultimately I found that I could as well solve the general problem of locating the critical points of any rational function whose zeros and poles are prescribed to lie in a given set of $p+1$ arbitrary circular regions. That is, the problem of locating the equilibrium positions in a field due to positive masses placed in a subset of those circular regions and negative masses in the remaining circular regions.

The problem implies basically a mapping from a space of $(p+1)$ complex variables into that of one complex variable. If $Z(z_0, z_1, \ldots, z_p)$ denotes the algebraic function defined by the equation

$$\sum_{k=0}^{p} \frac{m_k}{Z - z_k} = 0, \tag{3}$$

we need to determine the set of regions in the $Z$-plane onto which this function maps the polycircular region $\Delta = C_0 \times C_1 \times \cdots \times C_p$.

My thesis was published in 1930, but I was able to return in 1936 with a much simpler and broader treatment. The new approach enabled me to replace the original nonlinear problem by a linear one, as follows. Choosing $Z$ as an arbitrary but fixed point exterior to all the given regions $C_k$, we then consider the locus of the point $W$ defined by the equation

$$W = \sum_{k=0}^{p} \frac{m_k}{Z - z_k} \tag{4}$$

as each $z_k$ varies independently over its circular region $C_k$. Since the corresponding point $w_k = 1/(Z - z_k)$ varies over the interior of some disk $\Gamma_k$, we are therefore seeking the locus of the point $W = \Sigma_{k=0}^{p} m_k w_k$ when the $w_k$ vary independently over the disks $\Gamma_k$. The point $W$ is essentially the

centroid of the points $w_k$. Its locus is a disk $\Gamma$ whose center and radius can be easily found. Finally for $Z$ to be a critical point, $W = 0$ must, according to equation (3), lie in $\Gamma$. This then leads to the principal theorem of my thesis which may be stated as follows.

THEOREM. *For $j = 0, 1, \ldots, p$ let $f_j(z)$ denote a polynomial of degree $n_j$ having all of its zeros in the circular region $\sigma_j C_j(z) \leqq 0$ where $\sigma_j = \pm 1$ and*

$$C_j(z) \equiv |z - c_j|^2 - r_j^2 \equiv (x - c_j')^2 + (y - c_j'')^2 - r_j^2.$$

*Then every finite critical point $Z$ of the rational function*

$$f(z) = \frac{f_0(z) f_1(z) \cdots f_q(z)}{f_{q+1}(z) f_{q+2}(z) \cdots f_p(z)} \qquad 0 \leqq q \leqq p, \tag{5}$$

*satisfies at least one of the $p+2$ inequalities $\sigma_j C_j(Z) \leqq 0$, $j = 0, 1, \ldots, p$,*

$$\frac{E(Z)}{C_0(Z) C_1(Z) \cdots C_p(Z)} = \sum_{j=0}^{p} \frac{n m_j}{C_j(Z)} - \sum_{\substack{j=0 \\ k=j+1}}^{p} \frac{m_j m_k \tau_{jk}}{C_j(Z) C_k(Z)} \leqq 0,$$

*where $m_j = n_j$ or $-n_j$ according as $j \leqq q$ or $j > q$, $n = \Sigma_0^p m_j$,*

$$\tau_{jk} = |c_j - c_k|^2 - (\mu_j r_j - \mu_k r_k)^2, \qquad \mu_j = \sigma_j \operatorname{sg} m_j.$$

In applying this theorem to the case $q = p$, we assume that rational function (5) reduces to the polynomial $f(z) = f_0(z) f_1(z) \cdots f_p(z)$.

Let us interpret the theorem geometrically. According as $\sigma_k = +1$ or $-1$, the given region $\sigma_k C_k(z) \leqq 0$ is the closed interior or exterior of the circle with center $c_k$ and radius $r_k$. The quantity $\tau_{jk}$ when positive is the square of the common external or internal tangent of the circles $C_j$ and $C_k$ according as $\mu_j \mu_k = 1$ or $-1$. The number $n$ is the degree of the numerator minus the degree of the denominator in (5). It is not too difficult to verify that my theorem includes as special cases those of Lucas, Laguerre, Bôcher and Walsh.

In particular, let us examine the case of 3 circles ($p = 2$), when

$$E(z) = n m_0 C_1(z) C_2(z) + n m_1 C_2(z) C_0(z) + n m_2 C_0(z) C_1(z)$$
$$- m_0 m_1 \tau_{01} C_2(z) - m_0 m_2 \tau_{02} C_1(z) - m_1 m_2 \tau_{12} C_0(z).$$

If $n = 0$, the equation $E = 0$ reduces to a linear combination of $C_0$, $C_1$, $C_2$ and hence represents a fourth circle, thus proving Walsh's Cross-Ratio Theorem (W3). If $n \neq 0$, $E(z) = n^2(x^2 + y^2)^2 + \cdots$ and so $E = 0$ represents a bicircular quartic. If $E(z)$ were factorable, it would have the form

$$E(z) = n^2(x^2 + y^2 + \alpha_1 x + \beta_1 y + \gamma_1)(x^2 + y^2 + \alpha_2 x + \beta_2 y + \gamma_2)$$

and so the bicircular quartic would degenerate into a pair of circles. But this does not usually happen. As an illustration, let us take circles $C_0$, $C_1$, $C_2$ of equal radius $r$ and centers $c_0 = 2i$, $c_1 = -3 - i$, $c_2 = 3 - i$ (vertices of an isosceles right triangle, see Fig. 3A) and a polynomial $P(z)$ of degree $n = 3m$ having $m$ zeros in each circle $C_0$, $C_1$, $C_2$. Then according to my theorem any critical point of $P$ which is outside all the given $C_k$, lies in a region bounded by the ovals of the bicircular quartic $E(z) = 0$ where except for a constant factor

$$E(z) = [(x - \sqrt{2})^2 + y^2 - r^2][(x + \sqrt{2})^2 + y^2 - r^2] - 4r^2.$$

This bicircular quartic has always the points $s_1 = -\sqrt{2}$ and $s_2 = \sqrt{2}$ as singular foci. It consists of [see Fig. 3B]

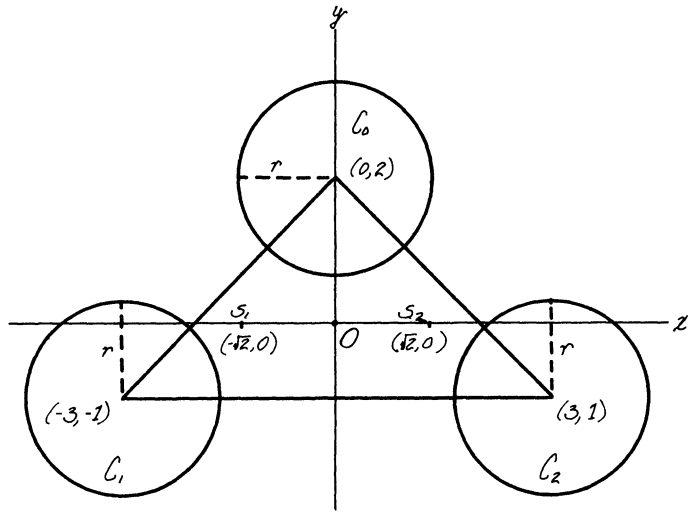



FIG. 3A

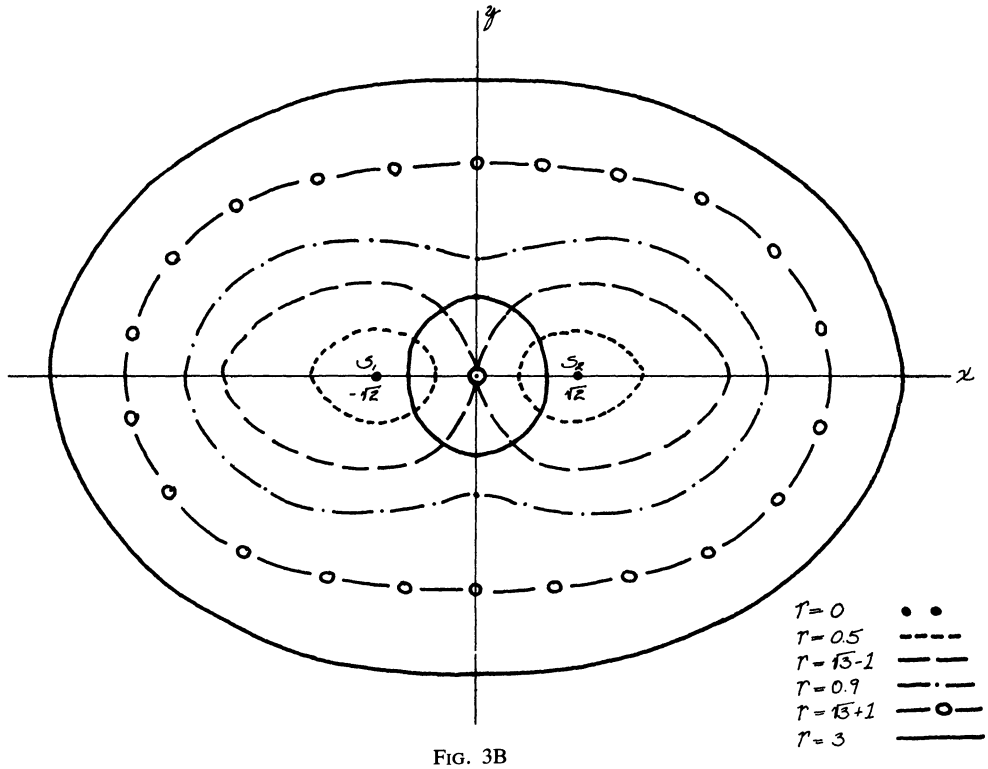



FIG. 3B

(a) just the points $s_1$, $s_2$ when $r = 0$;
(b) two disjoint ovals, one enclosing $s_1$ and the other enclosing $s_2$, when $0 < r < \sqrt{3} - 1$;
(c) two ovals touching at the origin when $r = \sqrt{3} - 1$;
(d) a single oval when $\sqrt{3} - 1 < r < \sqrt{3} + 1$;
(e) an oval and an isolated double point at the origin when $r = \sqrt{3} + 1$;
(f) two ovals, one enclosed in the other, when $r > \sqrt{3} + 1$.

Since the locus of the critical points for any $r_0 > 0$ contains the locus of the critical points for any $r < r_0$, the locus in cases (e) and (f) is the simply connected region bounded by the outer oval.

In the general case of $p+1$ circular regions $C_k$ with $n \neq 0$, the equation $E(z) = 0$ has the leading term $n^2(x^2+y^2)^p$ and so represents a $p$ circular $2p - ic$ curve, which consists only of ovals, at most $p$ in number. Each oval is a circle if $E(z)$ has $p$ factors in the domain of reals. This is so in the case of Walsh's multi-circle Theorem W2. Also, if the regions $C_j$ are the interiors of circles of equal radius $r$ and with centers at the vertices of a regular polygon with $p+1$ sides, and if all $m_j$ are equal, I have shown [see Fig. 4] that the $p$-circular $2p - ic$ curve degenerates into a set of circles concentric at the center of the polygon. I was disappointed not to be able to find more general conditions on the choice of the circular regions $C_j$ and other parameters in order for the curve $E(z) = 0$ to degenerate into a set of circles. It would have been nice to have found such conditions since they would have indicated when the function $Z(z_0, z_1, \ldots, z_p)$ maps the polycircular region $\Delta = C_0 \times C_1 \times \cdots \times C_p$ of the space $C^{p+1}$ onto a set of circular regions in the $Z$ plane. However, I was able to prove that for all choices of radii $r_k$ of the circles $C_k$ the $p$-circular $2p - ic$ curve $E(z) = 0$ has the same singular foci. Hence the singular foci are the isolated points into which this curve degenerates when $r_k = 0$ for all $k$. That is, they are the equilibrium positions when all the masses originally in each disk $C_k$ are concentrated at the center of the disk $C_k$.

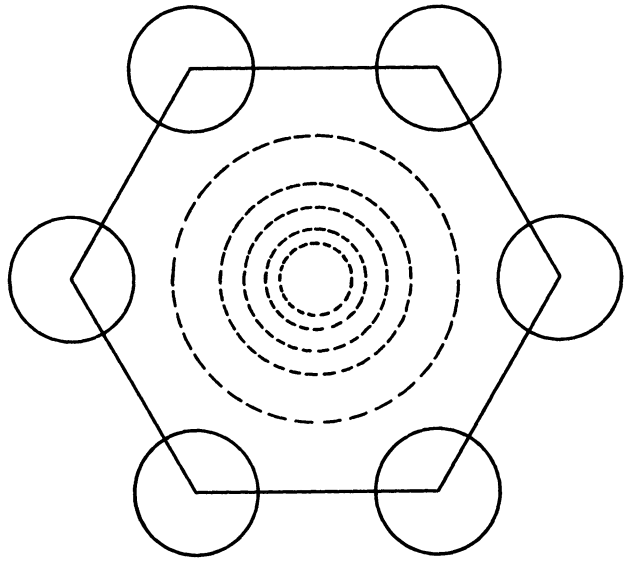


FIG. 4

So far I have reported on results pertaining to the regional location of *all* the critical points of a rational function when the location of *all* the zeros and poles of the function is prescribed. Now let us consider a second type of result which specifies the regional location of *some* of the critical points of a polynomial $P$ when the location of only *some* zeros of $P$ is prescribed. A first such result was discovered by J. W. Alexander in 1915 in connection with his study of univalent functions. He proved that, if a disk of radius $R$ contains two zeros of a polynomial $P$, the concentric disk of radius $R' = R \csc(\pi/n)$ contains at least one critical point. In 1917, Kakeya proposed the more general problem of finding the smallest number $\varphi(n, p)$ such that, if a disk $C$ of radius $R$ contained $p$ zeros of a polynomial of degree $n \geqq p \geqq 2$, then the concentric disk $C'$ of radius $R' = R\varphi(n, p)$ contains at least $p-1$ critical points. From a mechanical standpoint, we are assuming that $p$ particles of a system of $n$ particles are known to lie in a disk $|z| \leqq R$ and we are asking for the smallest $\varphi(n, p)$ such that at least $p-1$ equilibrium points lie in the concentric disk $|z| \leqq R\varphi(n, p)$, regardless of the position of the

remaining $n-p$ particles of the system. Now, $\varphi(n,n)=1$ by Lucas' theorem and $\varphi(n,2)=\csc(\pi/n)$ by Alexander's theorem. Kakeya proved the existence of $\varphi(n,p)$ for arbitrary $p$, but was able to calculate $\varphi(n,p)$ explicitly only for $p=2$ verifying thereby Alexander's result. In 1922 Szegö showed that Alexander's result could also be inferred from the theorems discovered in 1902 by Grace and in 1907 by Heawood.

While I was in Paris during 1930, Montel called my attention to the Kakeya problem and to the additional result $\varphi(n,n-1)=(1+1/n)^{1/2}$ derived by Biernacki in his doctoral thesis written in 1927 under Montel. However, it was not until six or seven years later that there occurred to me a new approach to Kakeya's problem. At that time I developed the following identity connecting any $p$ zeros of a polynomial $P$ with any $n-p+1$ critical points of $P$. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$, $2 \leqq p \leqq n$, be any $p$ zeros of an $n$th degree polynomial $P$ and $\beta_1, \beta_2, \ldots, \beta_q$ be any $q$ distinct critical points of $P$ with $p+q=n+1$. Then

$$\sum \frac{D_{j_1 j_2 \cdots j_q}}{(\beta_1-\alpha_{j_1})(\beta_2-\alpha_{j_2})\cdots(\beta_q-\alpha_{j_q})}=0, \tag{6}$$

where $j_1, j_2, \ldots, j_q$ run independently through the values $1, 2, \ldots, p$ and $D_{j_1 j_2 \cdots j_q}=k_1!\,k_2!\cdots k_q!$ with $k_\nu$ equal to the number of times $\alpha_\nu$ occurs in the product $(\beta_1-\alpha_{j_1})(\beta_2-\alpha_{j_2})\cdots(\beta_q-\alpha_{j_q})$. This identity is a generalization of the familiar one

$$\sum_{j=1}^{n} \frac{1}{\beta_k-\alpha_j}=0$$

which connects any one zero $\beta_k$ of $f'(z)/f(z)$ with the $n$ zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $f$.

From identity (6) I was able to infer by quite elementary methods that

$$\varphi(n,p) \leqq \csc(\pi/2q), \qquad q=n-p+1. \tag{7}$$

It is clear from my derivation, my upper bound to Kakeya's function $\varphi(n,p)$ is not the least upper bound. Perhaps the least upper bound could be deduced from identity (6) by applying more sophisticated extremal methods.

More recently [2] I studied polynomials with just three distinct zeros

$$f(z)=(z-z_1)^\lambda (z-z_2)^\mu (z-z_3)^\nu,$$

where $\lambda+\mu=p$, $\nu=n-p$, on the assumption that $z_1$ and $z_2$ lie on the closed unit disk. For this class of polynomials I have proved that at least $p-1$ critical points lie in the disk $|z| \leqq r$ where $r \leqq [2-(p/n)]^{1/2}$, with equality holding for even $p$. This implies a lower bound on $\varphi(n,p)$:

$$\varphi(n,p) \geqq [2-p/n]^{1/2}.$$

My conjecture is that $\varphi(n,p)=[2-p/n]^{1/2}$ for $p \geqq 3$. This conjecture is based upon my belief that the optimal polynomial is one for which the $p$ given zeros on the closed disk $|z| \leqq R$ coincide at two distinct points of the circle $|z|=R$.

The estimates on $\varphi(n,p)$ have at least two applications. The first is to obtaining the following sufficient condition for an $n$th degree polynomial $P$ to be at most $p$-valent ($p \leqq n$): if $P'(z)$ has at most $p-1$ zeros on the unit disk, then $P(z)$ is at most $p$-valent on the disk $|z| \leqq \sin[\pi/2(n-p)]$. For, otherwise the polynomial $f(z)=P(z)-\gamma$ for some constant $\gamma$ would have at least $(p+1)$ zeros on this disk and hence at least $p$ critical points would be on the disk

$$|z| \leqq \sin[\pi/2(n-p)]\csc[\pi/2(n-p)]=1$$

in contradiction to the hypothesis.

A second application is to the problem on which Landau, Fejér, Montel, Van Vleck and others had worked in hopes of finding a polynomial approach to the Picard theorem. The problem is to find the

smallest number $R = R(a_0, a_1, \ldots, a_p; n_1, n_2, \ldots, n_k)$ such that on the disk $|z| \leqq R$ lie at least $p$ zeros of the lacunary polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k},$$

where $1 \leqq p < n_1 < n_2 < \cdots < n_k$. For example, using (7), one may show that every polynomial

$$f(z) = a_0 + a_p z^p + a_{n_1} z^{n_1} + a_{n_2} z^{n_2} + \cdots + a_{n_k} z^{n_k}$$

has at least $p$ zeros on the disk

$$|z| \leqq \left[\prod_{j=1}^{k} \left(\frac{n_j}{n_j - p}\right) \left|\frac{a_0}{a_p}\right|\right]^{1/p} \csc^k(\pi/2p).$$

Obviously, a closer determination of the quantity $\varphi(n, p)$ would lead to an improvement of these bounds. In particular, if a bound on $\varphi(n, p)$ could be found which remains finite as $n \to \infty$, the theorems would be applicable to entire functions.

Before leaving the subject of critical points of polynomials, let me call your attention to a recent conjecture erroneously attributed to L. Ilieff. The conjecture appears to have been made by B. Sendov in 1959 to Gloreschkov, in 1962 to M. Marden and also to other mathematicians, including L. Ilieff. The latter presented the conjecture at a conference in 1966 attended by W. K. Hayman who then listed it among his Research Problems [**3,** (4)] as originating from Ilieff. It may be stated as follows. Given that $P(z)$ is an $n$th degree polynomial all of whose zeros are on the unit disk; to prove that the circle of radius one with center at any of the zeros of $P$ contains at least one critical point of $P$. There have recently been quite a few papers published on this problem, but for the most part they treat only special cases [**3**]. In other words, the Sendov conjecture has not yet been proved or disproved in the general case.

Let us now move from polynomials and rational functions to entire and meromorphic functions and discuss the progress made so far in extending Rolle's theorem to these functions. The investigation of the critical points of entire functions is complicated by the fact that in contrast with polynomials, an entire function is not completely determined by its zeros. If $P(z)$ is an $n$th degree polynomial with zeros $a_1, a_2, \ldots, a_n$, then up to a multiplicative constant it has the form $P(z) = (z - a_1)(z - a_2)\cdots(z - a_n)$. However, according to the theorem of Weierstrass and Hadamard, if an entire function $f$ of finite order $\rho$ has the zeros $a_1, a_2, a_3, \ldots$, is factorization has the form

$$f(z) = e^{Q(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) \exp\left[\left(\frac{z}{a_j}\right) + \frac{1}{2}\left(\frac{z}{a_j}\right)^2 + \cdots + \frac{1}{p}\left(\frac{z}{a_j}\right)^p\right],$$

where $p \leqq \rho$ and $Q(z) = c_0 + c_1 z + \cdots + c_q z^q$, $q = [\rho]$ and where $[\rho]$ means the largest integer not exceeding $\rho$. The coefficients $c_j$, in general, have no relation to the given zeros of $f$. The logarithmic derivative of $f$ may be written in the form

$$\frac{f'(z)}{f(z)} = Q'(z) + \sum_{j=1}^{\infty} \frac{z^p}{a_j^p (z - a_j)} \tag{8}$$

and this also involves the arbitrary coefficients $c_j$. For a good reason, these coefficients were called "plaguy factors" by the late E. B. Van Vleck in 1929 when he addressed the American Mathematical Society on entire functions [**4**]. Essentially because of these coefficients, the earlier extensions of Rolle's theorem attempted by Laguerre, Cesàro, Pólya and others were limited to cases where $p = 0$ or 1 and $q = 2$, or where $q = 0$.

In the course of several papers from 1949 to 1971, I succeeded in establishing for the logarithmic derivative of entire functions a new representation [**5**] which seems to be better suited for the study of the critical points than is the Weierstrass-Hadamard representation. The new representation may be

stated as follows. Let $f(z)$, an entire function of finite order $\rho$, have the zeros $a_1, a_2, \ldots, a_m, a_{m+1}, \ldots$, and critical points $\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n, \ldots$. Then

$$\frac{f'(z)}{f(z)} = \frac{\psi(z)}{\varphi(z)} \sum_{j=m+1}^{\infty} \frac{\varphi(a_j)}{\psi(a_j)(z-a_j)}, \tag{9}$$

where $\varphi(z) = 1$ if $m = 0$, and $\varphi(z) = \prod_{j=1}^{m}(z-a_j)$, $m \neq 0$; $\psi(z) = \prod_{k=1}^{n}(z-\zeta_k)$ and $n = m + [\rho]$. The series is uniformly convergent on every compact set in $\boldsymbol{C}$ for every choice of $m$, $0 \leqq m < \infty$. In (9) $\zeta_1, \zeta_2, \ldots, \zeta_n$ play the rôle of arbitrary parameters instead of the $c_k$ in (8), but at least they have relevant geometric meaning.

One of the applications of (9) is to the theorem of Laguerre and Borel which reads essentially as follows. Let a real entire function $f$ of finite order $\rho$ have at most $m$ non-real zeros. Let $M$ be the set consisting of one critical point between each pair of adjacent zeros of $f$. Then at most $n = [\rho] + m$ critical points, real and non-real, lie in the complement of $M$. Whereas Borel [6] apologetically took eight pages of his book on entire functions to prove this theorem, a half page of print suffices if representation (9) is used.

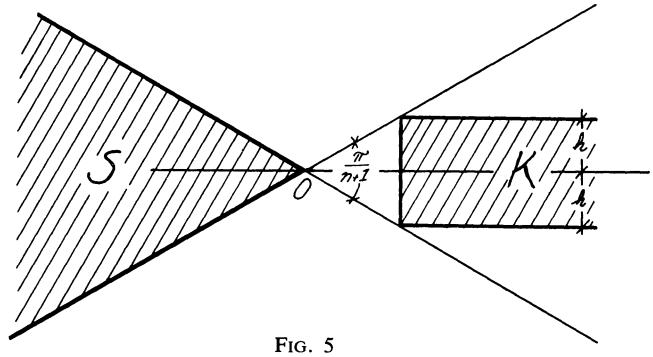



FIG. 5

Among the new theorems that (9) leads to is the following: Let $f$ be an entire function of order $\rho$ having all its zeros in a convex region $K$. Let $S$ consist of the points from which $K$ subtends an angle not exceeding $\pi/(n+1)$, $n = [\rho]$. Then at most $n$ critical points of $f$ lie in $S$. Thus, if $K$ is [see Fig. 5] the semi-infinite strip $\mathcal{R}(z) \geqq h \cot[\pi/2(n+1)]$, $|\mathcal{I}(z)| \leqq h$, $S$ is the sector $|\arg(-z)| \leqq \pi/2(n+1)$. In particular, for entire functions $F$ of order less than one we have $n = 0$, $S \equiv K$, and thus the critical points $F$ lie in the convex hull of the zeros of $F$. That is, Lucas' theorem holds for entire functions of order less than 1. An example is the function

$$f(z) = \prod_{k=-p}^{p} \cos(z - ki)^{1/2}$$

which is of order $\rho = 1/2$, has all its zeros in the semi-infinite strip $\mathcal{R}(z) > 0$, $|\mathcal{I}(z)| \leqq p$, and therefore has all its critical points in the same strip.

A representation [7] similar to (9) is also valid for meromorphic functions of finite order $\rho$. A consequence of that representation is the striking identity

$$\sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - a_j)(\zeta_1 - a_j)\cdots(\zeta_n - a_j)} = \sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - b_j)(\zeta_1 - b_j)\cdots(\zeta_n - b_j)}$$

which connects the zeros $a_j$ and poles $b_k$ of $f$ with any $n+1$ critical points $\zeta_j$ of $f$, with $n = [\rho]$. This identity leads to a number of theorems such as the following. Let $f(z)$ be a meromorphic function of finite order $\rho$ and let $n = [\rho]$. Let $S_1$ be the sector $|\arg z| < \pi/2(n+1)$ and $S_2$ the sector

$|\arg(-z)| < \pi/2(n+1)$. Let $T_1$, $T_2$, and $T_3$ be respectively the strips

$$\mathcal{I}(z) > h, \quad \mathcal{I}(z) < -h, \quad \text{and} \quad |\mathcal{I}(z)| \leqq h.$$

Then, if (see Fig. 6) $f$ has all its zeros in $H_1 = S_1 \cap T_1$ and all its poles in $H_2 = S_1 \cap T_2$, then it has at most $n$ critical points in $H_3 = S_2 \cap T_3$. Furthermore, the new representation for the logarithmic derivative of a meromorphic function may be useful in other respects, as for instance in simplifying the derivations of some Nevalinna relations as given in Hayman's book **[8]** on meromorphic function.

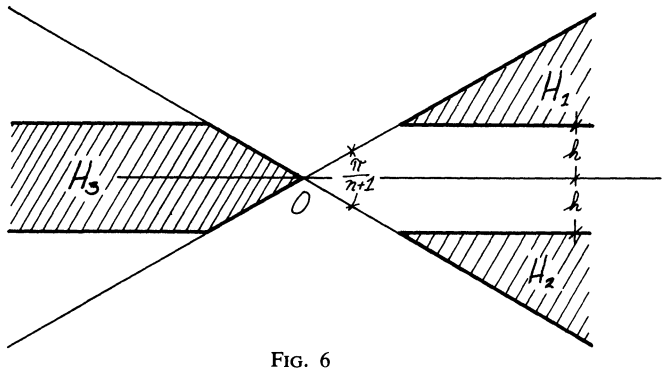



FIG. 6

To summarize, I have tried in my talk today to give an indication of the progress made by myself and others towards extending Rolle's theorem to certain classes of analytic functions of a complex variable. Though our methods have not carried this extension to arbitrary analytic functions, they have incidentally enabled my students and me to obtain some results on a number of other topics which turn out to be structurally related to the problem of locating the critical points of polynomials. These include the following:

(1) Mean Value Theorems in the Complex Plane.

(2) Zeros of infrapolynomials (extremal polynomials).

(3) Critical points of Green's function and linear combinations of Green's functions.

(4) Zeros of polynomial solutions of the Lamé differential equation

$$A(z)w'' + B(z)w' + C(z)w = 0,$$

where $A$, $B$, and $C$ are polynomials of degrees $p$, $p-1$, and $p-2$ respectively. (This equation includes the hypergeometric equation and thus the Legendre and some other well-known linear differential equations of second order.)

(5) Critical points of distance polynomials in $R^3$

$$F(x, y, z) = \prod_{k=1}^{n} [(x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2].$$

(6) Null sets of abstract homogeneous polynomials.

(7) Value distribution of harmonic polynomials in $R^N$.

Of course, these topics belong to the most classical parts of classical analysis. Yet our activity and that of many others in recent times give further evidence that even this aspect of classical analysis is not obsolete or even obsolescent.

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## MATHEMATICAL NOTES

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### COMMENTS AND COMPLEMENTS

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The purpose of this biannual article is to provide an opportunity for our readers to comment on articles that have been published in the Mathematical and Classroom Notes sections of the MONTHLY. Because of the nature of the material we seek to publish, it is not surprising that some duplication of results and ideas already in the literature occurs. Such duplication may not be undesirable, depending upon availability, presentation, and so on. In this regard, we are pleased to present the following information received from our readers during the past two years.

**Calculus.** R. P. Boas writes concerning his article *L'Hospital's rule without mean-value theorems* (this MONTHLY, 76 (1969) 1051–3) that a similar proof is given by F. Lettenmeyer, *Über die sogenannte Hospitalsche Regel* (J. Reine Angew. Math., 174 (1936) 246–7) and that the abstract *Simplified proof of l'Hospital's theorem on indeterminate forms* by E. V. Huntington (Bull. Amer. Math. Soc., 29 (1923) 207) is concerned with the same problem. Back to back articles concerning $e$ appeared in volume 81, 1974 of the MONTHLY — R. F. Johnsonbaugh, *Another proof of an estimate for e*, 1011–2, and H. Samelson, *To e via convexity*, 1012–4. Hugh Thurston has informed us that Johnsonbaugh's proof is very much like an old one from Chrystal's *Algebra*, vol. 2, p. 77 (Edinburgh, 1899). Aaron Galuten tells about a short proof that $\lim_{n\to\infty}(1+x/n)^n = e^x$ (defined by its power series) that appears in *Eléments de la Théorie des Fonctions Elliptiques*, vol. I, by Tannery and Molk (Gauthier-Villars et Fils, Paris 1893, 101–2). Samelson has discovered that J. Aczél's article *A természetes logaritmus és az*

*exponenciális függvény bevezetéséröl* (Matematikai Lapok, 7 (1956) 101–5 — in Hungarian with Russian and German summary) contains an approach to $e$ very much like his.

**Analysis.** Concerning J. A. Eidswick's article *Alternatives to Taylor's Theorem in Proving Analyticity* (this MONTHLY, 82 (1975) 929–31), R. P. Boas has observed that Theorem 1 goes back at least to the article *A condition for the validity of Taylor's expansion* by T. W. Chaundy (Messenger of Math., 45 (1915) 115–9) where a similar proof appears. Jerry E. Vaughan has written that Theorem 2 of R. Hansell and E. Wolk's article *On continuous functions and convergence of nets* (this MONTHLY, 83 (1976) 365–6) is contained in Theorem 5.1 of *Relations among continuous and various non-continuous functions* by R. V. Fuller (Pac. J. Math., 25 (1968) 495–509). Concerning Theorem 1 on the negativity of the real parts of the zeros of the Bessel polynomials in his article *Remarks on the Bessel polynomials* (this MONTHLY, 80 (1973) 1034–40) and the subsequent article *Remarks on the zeros of the Bessel polynomials* by A. Wragg and C. Underhill (this MONTHLY, 83 (1976) 122–6), C. W. Barnes has informed us that at the February 1961 meeting of the Louisiana-Mississippi section of the M. A. A. he read a paper in which he reported his Theorem 1 (recorded in this MONTHLY, 68 (1961) 589–91).

**Algebra.** The publication of A. J. Insel's *Nilpotent transformations and the decomposition of a vector space* (this MONTHLY, 81 (1974) 160–2) has prompted Keith A. Rowe to bring to our attention the article *Eine Bemerkung zur Jordanschen Normalform von Matrizen* by V. Ptak (Acta. Sci. Math., 17 (1956) 190–4). Rowe writes that while the usual proofs of the decomposition of a finite dimensional vector space $V$ into a direct sum of cyclic subspaces with respect to a nilpotent linear transformation $T$ use induction on the index of nilpotency $q$ (as does Insel's proof), Ptak gives a direct proof. A very rough outline of Ptak's proof is as follows. Let $x_0 \in V$ be such that $T^{q-1}x_0 \neq 0$. Then there is a linear functional $f_0 \in V^*$ such that $f_0(T^{q-1}x_0) \neq 0$. Let $V' = \{x \in V: f_0(T^k x) = 0 \ (k \geq 0)\}$. Then $V = V_0 \oplus V'$ where $V_0$ is the subspace spanned by all vectors of the form $T^k x_0$ $(k \geq 0)$.

The publication of C. M. Bang's *A condition for two matrices to be inverses of each other* (this MONTHLY, 81 (1974) 764–7) has caused three proofs of "$AB = I$ implies $BA = I$", $A$ and $B$ $n \times n$ matrices, to come to our attention:

1. (Desmond Fearnley-Sander) Let $\mathbf{A}$, $\mathbf{B}$ be linear transformations on an $n$-dimensional vector space $V$ such that $\mathbf{AB} = \mathbf{I}$. Let $X = (x_1, \ldots, x_n)$ be a basis for $V$ and write $\mathbf{B}X = (\mathbf{B}x_1, \ldots, \mathbf{B}x_n)$. Suppose $\mathbf{AB} = \mathbf{I}$. Then $\Sigma_i \alpha_i \mathbf{B}x_i = 0 \Rightarrow \mathbf{B}(\Sigma_i \alpha_i x_i) = 0 \Rightarrow \Sigma_i \alpha_i x_i = 0 \Rightarrow \alpha_i = 0$ $(i = 1, \ldots, n)$. Hence $\mathbf{B}X$ is a basis for $V$. Let $y \in V$. Then $y = \Sigma_i \alpha_i \mathbf{B}x_i$ and $\mathbf{BA}y = \Sigma_i \alpha_i \mathbf{BAB}x_i = \Sigma_i \alpha_i \mathbf{B}x_i = y$. Thus $\mathbf{BA} = \mathbf{I}$. It is now straightforward to transform to matrices.

2. (A. Wilansky)

LEMMA 1. $\mathbf{AB} = \mathbf{I} \Rightarrow \mathbf{BA}$ *idempotent.*

*Proof.* $\mathbf{BABA} = BIA = BA$.

LEMMA 2. *The only idempotent with inverse is I.*

*Proof.* $A^2 = A \Rightarrow A^{-1}A^2 = I \Rightarrow A = I$.

Now suppose $AB = I$. Then it follows from the determinant product rule that $\det BA \neq 0$; thus $BA$ is invertible and $BA = I$.

3. (Harley Flanders and K. Pearson (independently).)

The $n \times n$ matrices form a vector space of dimension $n^2$ (suffices to know dimension is finite). Thus $I, A, \ldots, A^m$ $(m = n^2)$ are linearly dependent so that there is a relation $c_r A^r + \cdots + c_m A^m = 0$, where $0 \leq r \leq m$ and $c_r \neq 0$. Assuming $AB = I$, multiply by $B^{r+1}$ to obtain $c_r B + c_{r+1} I + \cdots + c_m A^{m-r-1} = 0$. Hence $B$ is a polynomial in $A$ so that $BA = I$.

H. Niederreiter has informed us that the main theorem in Craig M. Cordes' article *Permutations*

*mod m in the form* $x^n$ (this MONTHLY, 83 (1976) 32–3) can be obtained by combining Theorems 1.9 and 1.10 on pages 311–12 of the book *Uniform distribution of sequences* by L. Kuipers and H. Niederreiter (Wiley-Interscience, New York, 1974). An additional reference is *Uniform distribution modulo m of monomials* by B. Zane (this MONTHLY, 71 (1964) 162–4).

**General.** Concerning the article *On Fermat's equation in* $C(t)$ by Newcomb Greenleaf (this MONTHLY, 76 (1969) 808–9), Victor Albis has written that an elementary algebraic proof that the equation $X^n + Y^n + Z^n = 0$ ($n \geqq 3$) has no non-trivial solutions in the field $C(t)$ of rational functions over $C$ is given in an old article by Korkine (C. R. Acad. Sci. Paris, 90 (1880) 303–4). Joseph Zaks has informed us that there is an elementary computational mistake in the proof given by B. Fisher in *On a problem of Besicovitch* (this MONTHLY, 80 (1973) 785–7) which makes the proof invalid. Zaks has corrected the mistake as follows (refer to page 786).

(1) replace line 11 by: $\frac{S}{2^n}\left(1+(1-x)^2+\frac{x^2}{2}\right)$.

(2) replace line 13 by: $\frac{S}{2^{n-1}}\left(1-x-\frac{x^2}{4}\right)<\frac{S}{2^{n-1}}(1-\frac{1}{2}x)$.

(3) replace $2^{n-2}$ in line 14 by $2^n$.

(4) through the rest of page 786 replace $(1-x)$ by $(1-\frac{1}{2}x)$.

(5) replace last expression in line 18 by $\frac{1}{2}Sx^2$.

(6) replace line $-10$ by: $\frac{1}{2}Sx^2+\frac{1}{2}S_1x^2<\frac{1}{2}Sx^2+\frac{1}{2}Sx^2(1-\frac{1}{2}x)$.

(7) replace line $-4$ by: $\frac{1}{2}Sx^2[1+(1-\frac{1}{2}x)+\cdots+(1-\frac{1}{2}x)^{r-1}]<Sx$.

(8) in line $-3$, let $x=\varepsilon/S$.

The estimate of the area of triangle $A_rB_rC_r$ on the last line of page 786 is still correct.

Phyrne Youens Bacon has supplied us with the following counterexample to the uniqueness claim in Lemma I of A. D. Keedwell's *Self-collineations of desarguesian projective planes* (this MONTHLY, 82 (1975) 59–63). Let $K$ be a non-commutative field (e.g., the quaternions) and consider the projective plane with coordinates over $K$. Let $j$, $k$ be elements of $K$ such that $kj \neq jk$ and let $I$ be the identity matrix. Both $I$ and $B = kI$ leave the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$ fixed but $(1,j,0)I \neq (1,j,0)B$. The existence statement of Lemma I is, however, sufficient for the proof of the theorem. D. R. Hughes has written that Lemma II in Keedwell's article is folklore.

Both D. K. Chakerian and John Oman have informed us that the conjecture proved by A. Sudberry in his article *The quadrilateral inequality in two dimensions* (this MONTHLY, 82 (1975) 629–32) had previously been established by L. M. Kelly, D. M. Smiley, and M. F. Smiley in *Two dimensional spaces are quadrilateral spaces* (this MONTHLY, 72 (1965) 753–4). Richard Askey has discovered that inequality (3) in M. S. Klamkin's *Extension of the Weierstrass Product Inequalities II* (this MONTHLY, 82 (1975) 741–2) follows immediately from inequality (4) by letting $A_i = 1 - 2a_i$. Concerning the conclusion of the article *An algebraic characterization of limits* by J. D. Gray (this MONTHLY, 82 (1975) 825–7), A. Wilansky writes that if we define $Sx = \lim x_{2n}$, we have a regular method which sums the divergent series $\{(-1)^n\}$, hence is not ordinary convergence, but satisfies $Sxy = SxSy$.

In regard to his article *Directional differentiation in the plane and tangent vectors on* $C^r$ *manifolds* (this MONTHLY, 82 (1975) 641–5), H. G. Ellis has written that E. Heil has brought to his attention the fact that a definition of tangent vector, not essentially different from the one he proposed, appears on page 77 of the book *Differential- und Integralrechnung II* (Springer-Verlag, Hiedelberg, 1968) by H. Grauert and W. Fisher. Grauert and Fisher's definition uses along with the linearity condition (1) in

Ellis' article the alternative conditions (2') and (3') (misprinted as (3)) that are stated in the last paragraph. Concerning A. D. Wadha's *An interesting subseries of the harmonic series* (this MONTHLY, 82 (1975) 931–3) Robert Baillie writes that he has computed the sums $S_i = \Sigma 1/n$ ($n$ has no digit equal to $i$) for $i = 0, 1, \ldots, 9$ to twenty decimal places. Further information can be obtained by writing to him at Computer-Based Educ. Res. Lab., University of Illinois, Urbana, IL 61801.

The article *Probabilities in Proofreading* by G. Polya (this MONTHLY, 83 (1976) 42) has stimulated a lot of reader response. V. N. Murty has informed us that the estimate obtained by Polya for the number of unnoticed misprints was obtained by Edward Deming and Chandra Sekhar (J. Amer. Stat. Assoc., 44 (1949) 101–15) and that demographers use it to estimate vital events. Ralph Winter writes that if the number $C$ of misprints noticed by both proofreaders is 0, then Polya's estimate is undefined. He then adds that if most probable numbers (rather than expected numbers) are used, the estimate becomes $(A - C)(B - C)/(C + 1)$. L. Glickman has informed us that the problem Polya solves appears as Exercise 23 on page 170 of W. Feller's book *An Introduction to Probability Theory and its Applications*, vol. I, 3rd edition (J. Wiley, New York, 1968).

Richard I. Loebl has written that Chandler Davis has informed him that an example similar to that in his article *The non-commutative triangle inequality fails* (this MONTHLY, 83 (1976) 259–60) appears in Davis' article *Notions generalizing convexity for functions defined as spaces of matrices* (Amer. Math. Soc. Proc. Symp. Pure Math. VII, 1963).

**Research problems.** Due to the unusual amount of activity concerning the research problem posed by I. Cahit in *Are all complete binary trees graceful* (this MONTHLY, 83 (1976) 35–7) and the fact that the next updating article for the Research Problems section is scheduled for December, 1977, we are reporting here that the question posed by Cahit has been answered in the affirmative by several people. It has been discovered also that the solution is immediately implicit in the article *Labelling of balanced trees* by R. G. Stanton and C. R. Zarnke (Congressus Numerantium VIII, Proc. 4th S. E. Conf. on Combinatorics, Graph Theory and Computing, Boca Raton, 1973, 479–95). More details will be given in the December 1977 article by Richard K. Guy.

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## RESEARCH PROBLEMS

EDITED BY RICHARD GUY

*In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)*

### THE POWERS THAT BE

R. B. EGGLETON, P. ERDŐS AND J. L. SELFRIDGE

If $n$ is a fixed positive integer, $a$ any integer greater than unity and $m$ the unique integer such that

$$a^m \leqq n < a^{m+1},$$

then we shall call $a^m$ a **maximal power** for $n$, $a^{m+1}$ a **minimal power** above $n$, and $m$ the **exponent** of $a$ for $n$. (The exponent is just the characteristic of the base $a$ logarithm of $n$.) In particular, if the prime $p$

has exponent $m$ for $n$ we call $p^m$ a **maximal prime-power** for $n$ and $p^{m+1}$ a **minimal prime-power** above $n$, and $m$ will be referred to as a $P$-**exponent** for $n$.

It is easily seen that the least common multiple $L_n$ of the positive integers up to $n$ is equal to the product of the maximal prime-powers for $n$. More generally, let $A_k$ be the largest possible least common multiple of any set of $k$ positive integers not exceeding $n$. Then the $\pi(n)$ maximal prime-powers for $n$ are a suitable set for showing that $A_{\pi(n)} = L_n$, where $\pi(n)$ denotes the number of primes up to $n$. As no positive integer up to $n$ is divisible by two different maximal prime-powers for $n$ it follows that the sequence $(A_k)_{1 \leqq k \leqq \pi(n)}$ is strictly monotonic increasing. We have investigated [**1, 2**] a number of properties of this sequence. The questions we shall raise here have their origins in these investigations.

Let any two integers greater than unity be called **exponentially equivalent** for $n$ if they have the same exponent for $n$. This is an equivalence relation on the set of integers greater than unity and, by restriction, also on the set $P$ of primes. Let $E(n)$ denote the set of exponents for $n$, that is, the exponents corresponding to the various exponential equivalence classes for $n$; similarly let $E_P(n)$ denote the set of $P$-exponents for $n$. If $m$ is an exponent for $n$, let $a_m$ and $b_m$ denote respectively the smallest and largest integer in the corresponding exponential equivalence class; if $m$ is also a $P$-exponent for $n$, let $p_m$ and $q_m$ denote respectively the smallest and largest prime in the corresponding exponential equivalence class. (Note that $b_0$ and $q_0$ do not exist.) We illustrate these ideas in the following tabulation.

For $n = 90$:

| $m$ | $a_m$ | $b_m$ | $p_m$ | $q_m$ |
|---|---|---|---|---|
| 0 | 91 | – | 97 | – |
| 1 | 10 | 90 | 11 | 89 |
| 2 | 5 | 9 | 5 | 7 |
| 3 | 4 | 4 | – | – |
| 4 | 3 | 3 | 3 | 3 |
| 5 | – | – | – | – |
| 6 | 2 | 2 | 2 | 2 |

Thus 90 has 6 exponential equivalence classes, $E(90) = \{0, 1, 2, 3, 4, 6\}$. Since 4 is alone in its equivalence class, this class does not contain a prime and $E_P(90) = \{0, 1, 2, 4, 6\}$.

There are several questions which now arise. We shall mention some of these, and follow up with such comments or partial solutions as we can offer.

Q1. When is $a_m = b_m$?

Q2. If $m, m' \in E(n)$ and $m < m'$, for which integers $n$ do we have $a_{m'} = b_{m'}$ whenever $a_m = b_m$?

Q3. When is $p_m = q_m$?

Q4. When does $a_m$ exist while $p_m$ fails to exist?

Q5. Are there infinitely many $n$ for which $p_m = a_m$ or for which $q_m = b_m$, whenever $m \in E_P(n)$?

Q6. How many exponential equivalence classes does $n$ have, that is, what is the cardinality of $E(n)$?

Q7. What is the cardinality of $E_P(n)$?

An asymptotic answer to Question 1 is not difficult. It is clear that $a_m = b_m$ must hold if $n^{1/m} - n^{1/(m+1)} < 1$ and cannot hold if $n^{1/m} - n^{1/(m+1)} \geqq 2$. The reader may wish to verify that if $n$ is sufficiently large, $a_m = b_m$ holds if $m \geqq \log n/(\log\log n - 2\log\log\log n)$ and fails if $m < \log n/(\log\log n - (2-\varepsilon)\log\log\log n)$ for any given $\varepsilon > 0$. The phenomenon in Question 2 occurs for small $n$, but for $15625 = 5^6 \leqq n < 16384 = 4^7$ we have $a_5 = b_5 = 6$ and $a_6 = 4$, $b_6 = 5$. For sufficiently large $n$, the number of integers $m$ satisfying $1 < n^{1/m} - n^{1/(m+1)} < 2$ is large enough to ensure that there is an exponential equivalence class containing two integers with larger exponent than that of some

other class which contains only one integer. Thus the set of integers $n$ described in Question 2 is finite; what is its largest member?

For Question 3, it is probable that the asymptotic answer is the same as for Question 1, though in this case the distribution of the primes complicates the matter, and the same difficulty arises in Question 4. For Question 5, we note for example that $p_m = a_m$ whenever $m \in E_P(100)$, and $q_m = b_m$ whenever $m \in E_P(127)$. However, one should be able to prove that only finitely many integers $n$ have either of these properties, but finding the largest is probably very difficult. The calculations for Question 1 are adequate to obtain an asymptotic answer to Question 6, since $m \in E(n)$ certainly holds as long as $n^{1/m} - n^{1/(m+1)} \geqq 1$. If $m_0$ is the largest exponent satisfying this inequality, the number of exponents not counted in this way is at most $n^{1/m_0}$, which is of smaller order than $m_0$. Hence $|E(n)| \sim \log n/\log\log n$. We conjecture the same asymptotic answer for Question 7, though we cannot prove this. It has long been known (cf. [6]) that for all sufficiently large $x$ there is a prime between $x$ and $x + x^\theta$, where $\theta < 1$ is a suitable constant; this implies that $|E_P(n)| > c \log n/\log\log n$ for some positive constant $c < 1$. We can show, however, that $E_P(n) = E(n)$ holds for only finitely many integers $n$. Indeed, $E_P(4095) = E(4095) = \{0, 1, 2, 3, 4, 5, 7, 11\}$ is the largest such example, because 4 is alone in its exponential equivalence class for $4096 = 4^6 \leqq n < 15625 = 5^6$ and for $16384 = 4^7 \leqq n$, and 6 is alone in its class for $15625 = 5^6 \leqq n < 16807 = 7^5$.

We shall now consider the numbers $\alpha_m = p_m^m$, $\beta_m = q_m^m$, $\gamma_m = p_m^{m+1}$ and $\delta_m = q_m^{m+1}$. Denote the sequences of these numbers by $\alpha(n) = \{\alpha_m\}$, $\beta(n) = \{\beta_m\}$, $\gamma(n) = \{\gamma_m\}$ and $\delta(n) = \{\delta_m\}$, where the subscripts run through $E_P(n)$ by increasing magnitude for $\alpha$ and $\gamma$, and likewise through $E_P(n)\backslash\{0\}$ for $\beta$ and $\delta$. (The corresponding sequences of powers of $a_m$ and $b_m$ are also interesting, but do not necessarily have all terms distinct, e.g., $a_3^3 = a_6^6 = 64 = b_3^3 = b_6^6$ when $n = 90$.) We now illustrate.

For $n = 90$:

| $m$ | $\alpha_m$ | $\beta_m$ | $\gamma_m$ | $\delta_m$ |
|---|---|---|---|---|
| 0 | 1 | — | 97 | — |
| 1 | 11 | 89 | 121 | 7921 |
| 2 | 25 | 49 | 125 | 343 |
| 4 | 81 | 81 | 243 | 243 |
| 6 | 64 | 64 | 128 | 128 |

Since $p_m$ is approximately $n^{1/(m+1)}$, the sequence $\alpha(n)$ increases monotonically, at least initially. Similarly $q_m$ is approximately $n^{1/m}$, so $\delta(n)$ decreases monotonically, at least initially. There are no such compelling reasons for $\beta(n)$ or $\gamma(n)$ to be monotonic, even initially, though if it happens that $n^{1/m} - q_m$ does not decrease too rapidly as $m$ increases then $\beta(n)$ will decrease initially; likewise if $p_m - n^{1/(m+1)}$ does not decrease too rapidly as $m$ increases then $\gamma(n)$ will increase initially.

We are now in a position to raise several more questions, also to be followed by comments and answers as far as we know them.

Q8$\alpha$. Are there infinitely many integers $n$ for which $\alpha(n)$ increases monotonically throughout?

Q8$\beta$, Q8$\gamma$ are the corresponding questions about $\beta(n)$ and $\gamma(n)$.

Q9$\delta$. Are there infinitely many integers $n$ for which $\delta(n)$ decreases monotonically throughout?

Q9$\beta$, Q9$\gamma$ are the corresponding questions about $\beta(n)$ and $\gamma(n)$.

Q10. Are there infinitely many integers $n$ for which $\alpha(n)$ increases throughout and simultaneously $\delta(n)$ decreases throughout?

Q11. To what point can we be sure that $\alpha(n)$ will increase monotonically?

Q12. To what point can we be sure that $\delta(n)$ will decrease monotonically?

For Question 8$\alpha$ it is easy to find integers $n$ for which $\alpha(n)$ increases throughout. This is the case

for $1 \leqq n < 9$, $16 \leqq n < 27$, $32 \leqq n < 81$, $256 \leqq n < 625$, $1024 \leqq n < 2187$, $8192 \leqq n < 15625$ and $16777216 = 2^{24} \leqq n < 19487171 = 11^7$.

For $n = 2^{24}$:

| $m$ | $p_m$ | $\alpha_m$ | $m$ | $p_m$ | $\alpha_m$ |
|---|---|---|---|---|---|
| 0 | 16777259 | 1 | 6 | 11 | 1771561 |
| 1 | 4099 | 4099 | 8 | 7 | 5764801 |
| 2 | 257 | 66049 | 10 | 5 | 9765625 |
| 3 | 67 | 300763 | 15 | 3 | 14348907 |
| 4 | 29 | 707281 | 24 | 2 | 16777216 |
| 5 | 17 | 1419857 | | | |

However, the question is whether there are infinitely many integers $n$ for which $\alpha(n)$ increases throughout. We conjecture a negative answer, and possibly one can prove this. By computer we verified that outside the seven intervals listed, there are no cases with $n \leqq 10^{250000}$ in which $\alpha(n)$ increases throughout. (The corresponding question can be asked for such sequences as $\{a_m^m\}$, where $m$ runs through $E(n)$. Perhaps $n = 15624$ is the last case in which this sequence increases throughout.) It would be interesting to decide if there are infinitely many $n$ for which those terms of $\alpha(n)$ corresponding to exponents $m < (\log n)^{\frac{1}{2}}$ are increasing. For Question $8\gamma$ we note that $\gamma(n)$ can increase throughout only when $\alpha(n)$ does so. Thus $1 \leqq n < 3$ and $16 \leqq n < 23$ are the only intervals with $n \leqq 10^{250000}$ in which $\gamma(n)$ increases throughout; we conjecture that there are no later examples.

For Question $8\beta$ we have the examples $2 \leqq n < 5$ and $n = 8$. Moreover, in this case we can prove that there are only finitely many integers $n$ for which $\beta(n)$ increases monotonically. A classical result of Ingham [6] states that for any $\varepsilon > 0$ and all sufficiently large $x$ there is a prime between $x$ and $x + x^{(5/8)+\varepsilon}$. Hence we deduce that between any two sufficiently large cubes there exists a prime. Ingham's result also ensures that 3 and 6 are $P$-exponents for any sufficiently large $n$. Thus if $n$ is sufficiently large $\beta(n)$ contains the cubes $\beta_3$ and $\beta_6$, and between them there is a prime, so $\min\{\beta_3, \beta_6\} < \beta_1$, showing that $\beta(n)$ does not increase monotonically. Incidentally, it has never been proved that between every two consecutive cubes (or every two consecutive tetrahedral numbers) there is a prime: it would be nice to have such results available.

The parts of Question 9 have a similar status to the corresponding parts of Question 8. It may be easier to resolve Question 10 negatively than to resolve either Question $8\alpha$ or Question $9\delta$ alone. The interval $1331 \leqq n < 2048$ is the last interval with $n \leqq 10^{250000}$ for which $\alpha(n)$ and $\delta(n)$ are both monotonic throughout.

For Question 11 we note that a result of Huxley [5] improves Ingham's result used above by replacing the exponent $\frac{5}{8}$ by $\frac{7}{12}$. Thus for any given $\varepsilon > 0$, if $n^{1/m}$ is sufficiently large, then $p_{m-1} < n^{1/m} + n^{(7+\varepsilon)/12m}$. Also $n^{1/(m+1)} < p_m$, so a short calculation shows that if $n$ is sufficiently large, $\alpha_{m-1} < \alpha_m$ holds so long as $m \leqq (\frac{5}{24} - \varepsilon)\log n/\log\log n$. How good is this estimate? (Probably the result is still true if the coefficient is replaced by $\frac{1}{4} - \varepsilon$, and perhaps even by $\frac{1}{2} - \varepsilon$.) Are there infinitely many $n$ for which this monotonicity fails before the $P$-exponent reaches $\log n/\log\log n$? This discussion essentially carries over for Question 12 as well.

We shall now introduce two functions (from $P$ to the integers) related to the monotonicity problems for $\alpha$ and $\delta$. In the present context, let $p_i$ denote the $i$th prime and let $m_i$ denote the exponent of $p_i$ for $n$. For a given $k$, suppose $n$ is the smallest integer for which $2^{m_1} > 3^{m_2} > \cdots > p_k^{m_k}$. (Recall that $p_i^{m_i+1} > n$ for each $i \leqq k$.) This sequence is conveniently recorded by the function $f(p_k) = m_1$, since $n = 2^{m_1}$. Similarly suppose $n'$ is the smallest integer for which

$$2^{m_1'+1} < 3^{m_2'+1} < \cdots < p_k^{m_k'+1},$$

where $m_i'$ is the exponent of $p_i$ relative to $n'$. The function $g(p_k) = m_1'$ records this sequence.

As part of our computations related to the monotonic behavior of $\alpha$, we obtained the following data:

| $p_k$ | $f(p_k)$ | $p_k$ | $f(p_k)$ | $p_k$ | $f(p_k)$ |
|---|---|---|---|---|---|
| 2 | 0 | 11 | 10 | 23 | 5627 |
| 3 | 1 | 13 | 40 | 29 | 14501 |
| 5 | 2 | 17 | 40 | 31 | 330861 |
| 7 | 5 | 19 | 106 | 37 | 658110 |

Here are three further questions.

Q13. What are the corresponding values of $g$, and how do both sequences of values continue?

Q14. What is the asymptotic behavior of $f$ and $g$?

Q15. Does $f$ increase strictly for $p_k \geqq 17$? (Probably not.)

In closing, we note the following result related to the problems we have raised. Let $p, q$ be primes such that $1 < p < q \leqq n$, and let $p^r$ and $q^s$ be the highest powers of $p$ and $q$ dividing $n!$; then $p^r > q^s$ if $r > s$. This result is the content of a problem originally proposed by Erdös [3], with published solution due to Harrington [4].

The research of the first author has been supported by the Foundation for Number Theory Computing.

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## CLASSROOM NOTES

EDITED BY RICHARD A. BRUALDI

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### MOTIVATING EXISTENCE—UNIQUENESS THEORY FOR APPLICATIONS ORIENTED STUDENTS

ARTHUR DAVID SNIDER

When one is teaching a course in partial differential equations to an applications-oriented audience, a dilemma arises about halfway through the semester: namely, the available techniques for constructing solutions (method of characteristics, transforms, separation of variables, Green's

functions) have been exhausted and the professor wishes to expose the students to the rich, beautiful existence and uniqueness theory that has been developed over the years, but the spectre of Relevance rears its ugly head and, usually taking the earthly form of a bright engineering student, asks if General Motors actually pays people merely to verify that a given problem has one and only solution. Therefore it would certainly be advantageous to have an example wherein one of these non-constructive theorems is used in an essential way in determining the solution to a physical problem. This note attempts to fill this need by focusing attention on one specific aspect of an initial-boundary value problem for the wave equation.

One considers then the physical situation of a semiinfinite vibrating string with one end fixed. The equation of motion is the one-dimensional wave equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}, \quad x>0, t>0, \tag{1}$$

with initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{2}$$

and the boundary condition

$$u(0,t) = 0 \tag{3}$$

(see ref. [1] for details). One then shows by standard arguments that the solution to this problem is unique, and that travelling waves are possible.

Now one is in a position to ask the class what happens to a travelling wave moving to the left as in Fig. 1, when it hits the boundary.

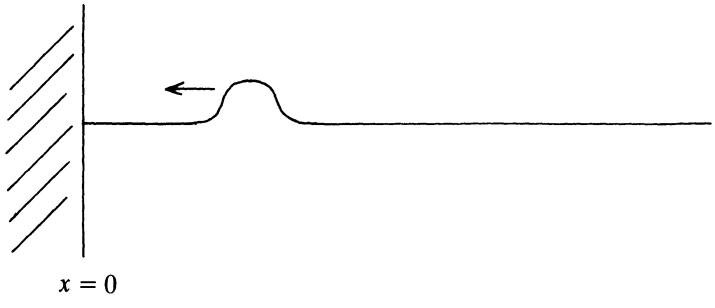



FIG. 1

Nearly everyone will acknowledge that it is reflected, but probably the class will be divided as to whether the reflected wave rides above or below the equilibrium position (Fig. 2).

Indeed, it is extremely difficult to analyze the mechanics of what is going on at $x = 0$, and the formulation of a convincing argument on this basis would probably warrant a note in the *American Journal of Physics*. Assuming that such an argument is not forthcoming, the situation is ripe for a definitive analysis which we outline below (see ref. [1] for a more complete treatment).

We shift our attention momentarily to the doubly infinite string with two antisymmetric waves travelling in opposite directions as in Fig. 3.

By linearity we know that these waves will pass through each other, canceling at $x = 0$, and then continue on unimpaired. But restricting our observations to the right of $x = 0$, we observe that this motion satisfies equations (1), (2), and (3). Furthermore, it is the only solution satisfying (1), (2), and (3), *by the uniqueness theorem*! Consequently, it coincides with the solution of the original problem, with the fixed endpoint, and we conclude that the reflected wave in that situation must ride below the

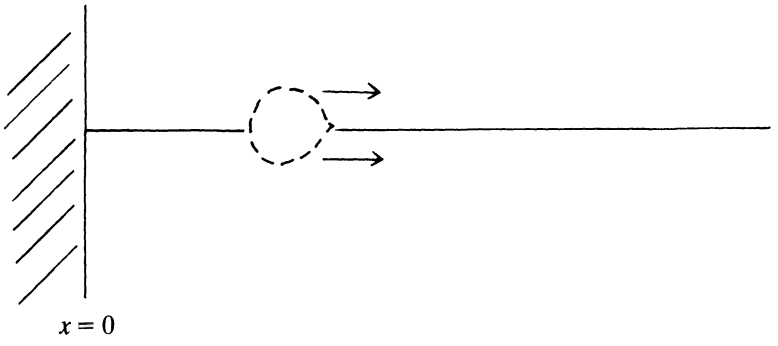



FIG. 2

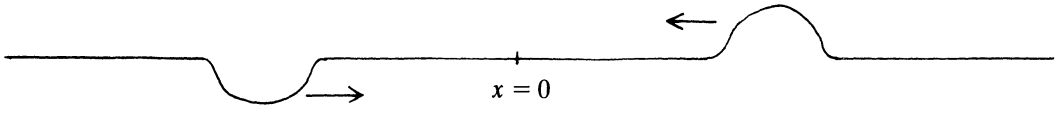



FIG. 3

equilibrium position, without question. Thus we have a triumph, at least for uniqueness theory, in the realm of physics.

A subsequent demonstration of the phenomenon with a Slinky borrowed from the physics department ices the cake, and the demon Relevance will have been exorcised (temporarily) from the souls of the practical-minded students.

**Reference**

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## A SHORT PROOF OF RUNGE'S THEOREM

SANDY GRABINER

Suppose that $K$ is a compact subset of the complex plane, and that $S$ is a set containing one point in each component of the complement of $K$ in the extended complex plane. Let $B(S)$ be the collection of all functions which are uniform limits, on $K$, of sequences of rational functions all of whose poles lie in $S$. Runge's theorem for compact sets states that $B(S)$ contains every function which is analytic on a neighborhood of $K$. There are other versions of Runge's theorem for approximations on open sets, but these are simple topological consequences of Runge's theorem for compact sets [**6**, Th. 13.9, p. 290 and prob. 7, p. 295].

Despite the importance of Runge's theorem in both classical and functional analysis, the usual proofs have drawbacks which often cause it to be omitted from elementary courses. The classical "translation of poles" argument is somewhat tedious [**2**, pp. 194–200] [**7**, 172–176]. The elegant abstract proofs that are based on the observation that $B(S)$ is a Banach space require at least the

Hahn–Banach theorem [**3**, pp. 60–61] and sometimes the Riesz representation theorem as well [**6**, pp. 288–289], [**5**].

The proof we give is elementary, even though it is motivated by the fact that $B(S)$ is a Banach algebra, not just a Banach space. The explicit connections with Banach algebra methods will be pointed out after the proof.

We base the proof of Runge's theorem on three lemmas.

LEMMA 1. *Every function analytic on a neighborhood of $K$ is a uniform limit on $K$ of rational functions, all of whose poles lie in the complement of $K$ with respect to the finite complex plane.*

For a proof see [2, Lemma 1.5, p. 193] or [7, Lemma (1.1), p. 171]. This lemma, or the version of Cauchy's integral theorem of which it is an easy consequence, is the starting point for both the classical and abstract proofs of Runge's theorem.

LEMMA 2. *If $\rho$ and $U$ are open subsets of the complex plane with $\rho \subseteq U$ and with the boundary of $\rho$ disjoint from $U$, then $\rho$ contains every component of $U$ which it meets.*

*Proof.* Suppose that $H$ is a component of $U$ which contains a point $s$ in $\rho$. Let $G$ be the component of $\rho$ which contains $s$. Since $H$ is connected, it must either equal $G$ or contain a boundary point of $G$. But each boundary point of $G$ is a boundary point of $\rho$, and cannot belong to $U$.

LEMMA 3. *If $\lambda$ does not belong to $K$, then $(z-\lambda)^{-1}$ belongs to $B(S)$.*

*Proof.* For large enough $|\lambda|$, the Taylor series for $(z-\lambda)^{-1}$ converges uniformly on $K$; so we can assume that $S$ is a subset of the finite complex plane. Let $U$ be the complement of $K$ in the finite plane and let $\rho = \{\lambda : (z-\lambda)^{-1} \in B(S)\}$. If $\lambda$ belongs to $\rho$ and if $|\mu - \lambda| < \operatorname{dist}(\lambda, K)$, then $\mu$ also belongs to $\rho$ since $(z-\mu)^{-1} = (z-\lambda)^{-1}(1-(\mu-\lambda)/(z-\lambda))^{-1}$ is the uniformly converging sum of a series in powers of $(z-\lambda)^{-1}$. Thus $\rho$ is an open subset of $U$. Suppose that $\mu$ is a boundary point of $\rho$, and choose a sequence $\{\lambda_n\}$ in $\rho$ with limit $\mu$. Since $\mu$ does not belong to $\rho$ we must have $|\mu - \lambda_n| \geqq \operatorname{dist}(\lambda_n, K)$ for all $n$. Taking limits we see that $\operatorname{dist}(\mu, K) = 0$, so that $\mu$ does not belong to $U$. Finally, notice that each component of $U$ contains a point of $S$, and therefore meets $\rho$. Hence, by Lemma 2, $\rho = U$.

Since $B(S)$ is an algebra closed under uniform limits (i.e., a Banach algebra), Runge's theorem is an immediate consequence of Lemmas 1 and 3.

Observe that the norm of $(z-\lambda)^{-1}$ in the Banach algebra $B(S)$ is just $1/\operatorname{dist}(\lambda, K)$; and thus the above proof is really an adaptation of Rickart's results on invertible elements in a Banach algebra [**4**, ths. (1.4.6) and (1.4.7), p. 12]. Compare also Bonsall and Duncan [**1**, th. 11, p. 24] who go in the other direction, obtaining Banach algebra results by adapting a standard proof of Runge's theorem.

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## MATHEMATICAL EDUCATION

EDITED BY PAUL T. MIELKE AND SHIRLEY HILL

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### A COURSE "BACKGROUNDS OF SCHOOL MATHEMATICS" AT THE UNIVERSITY OF UTRECHT, THE NETHERLANDS

A. F. MONNA

Among the students for the so-called "*doctoraal examen*" (after which most students leave university) in mathematics, we find in the Netherlands future teachers of the last years of the classical and modern *gymnasium* and higher vocational (technical or commercial) schools as well as future workers in scientific institutes or in industry. Some students of the latter category continue their study at the university after this *doctoraal examen* to obtain their Ph.D.

In Utrecht we split up the students for the *doctoraal examen* in three main streams: 1°. research in pure mathematics, 2°. applied mathematics and 3°. teaching. There are different curricula for each of these streams. For the first there is already a reasonable specialization, for instance in algebra, analysis or the foundations of mathematics. For the last category the curriculum is rather broad; we think that a broad program is more suitable for these students in view of their future career.

In the teachers' curriculum we have courses in foundations, logic, numerical mathematics, statistics, and history of mathematics, apart from the usual subjects from algebra, geometry and analysis. It is about one special course, called "Background of school mathematics," which we tried out in Utrecht several times, that we shall give some information in this note.

Those who become teachers will have to teach pupils of the gymnasium at the ages of 12–18, or pupils of the higher vocational schools, aged 16–19. They will have to deal with the ordinary subjects of school mathematics for pupils in the age range of 12–16 and some more advanced subjects for some students in the age range of 16–19, such as differential and integral calculus, differential equations with some applications to physics, complex numbers and some linear algebra. Evidently there is a big gap between these subjects and university mathematics. This is an old problem. Already Felix Klein occupied himself with it, and his activities resulted in his three books on *Elementarmathematik vom höheren Standpunkte aus* still worthwhile reading but evidently not up to date anymore. We were thinking of a course in the spirit of these books. Such a course should not be a course in didactics or mathematical education. The subjects should have some relation with school mathematics and should be chosen in such a way as to show future teachers that elementary mathematics are connected with interesting and sometimes difficult mathematical problems. So we came to plan a course "Backgrounds of school mathematics." We still don't have a fixed program; each time the course is given we choose somewhat different topics. Some examples may illustrate our ideas.

1. There is a tendency in modern geometrical education towards a more phenomenological approach. Thus many schools use a foundation of geometry starting from **translations, rotations** and **reflexions**, somewhat in the sense of F. Bachmann's book, *Aufbau der Geometrie aus dem Spiegelungsbegriff*. Although, evidently, this theory cannot be brought into the school in an exact way, it is particularly useful for the teacher to know the backgrounds of this approach. We gave a short introduction to this theory.

2. The theory of real numbers furnishes interesting topics. In university courses real numbers are

introduced as equivalence classes of Cauchy sequences, in school (possibly) as decimal fractions. What is the relation?

The theory and exercises on inequalities are a starting point for dealing with order and absolute value on groups, rings and fields. We treated **ordered groups**, **rings** and **fields**, **valuations** on a field, questions on unicity of orderings. Examples other than the classical real and complex numbers, e.g., the $p$-adic numbers, the ring of polynomials with non-archimedean ordering and a valuation derived from the degree. The relation of these subjects with the introduction of segments in distance geometry, for instance $d(P, Q) = \max(|x_P - x_Q|, |y_P - y_Q|)$ were also covered.

3. In school mathematics **quadratic equations** play an important role. In a background course some information on cubic equations and biquadratic equations (for which there is no time in the ordinary algebra program) may be useful. This provides a worthwhile complement to the courses in Galois theory (and the unsolvability of the equation of the fifth degree). But on the quadratic equation also a lot can be said: equations in fields different from the real or complex numbers, in fields with characteristic 2; quadratic equations in non-commutative rings (e.g., matrix rings) or fields (quaternions). A quite obvious remark should be that it is better to say: a quadratic equation has solutions if the discriminant is a square instead of the classical statement that it should be positive.

4. There are several interesting topics in analysis. For instance the subject **area, volume and integral**. How shall we introduce the concept of area? Here we can use quite a number of different approaches. One can follow the geometrical way of introducing the concept of area for polygons in the plane. This approach is connected with the concept of *Zerlegungsgleichheit*; see Hilbert's classical book *Grundlagen der Geometrie*. How to dissect a square into a finite number of parts so as to form an equilateral triangle? There are nice examples in Hadwiger's book [4]. Why is the subject so much more difficult in three dimensions? Is it possible to dissect a cube to build a regular simplex? We also can follow the way of introducing area as a kind of homomorphism of the class of nice geometrical figures into the set of non-negative real numbers, where disjoint unions correspond to addition. This is what is done in analysis. How can one define a homomorphism of the set of segments (with invariance with respect to translations) into the set of real numbers? This may lead as far as the definition of Haar measure on the additive group of the reals and, more generally, on the additive group of the $n$-dimensional vector space over the reals. In this context we discussed Cauchy's famous functional equation $f(x + y) = f(x) + f(y)$, and considered the problem of continuous and discontinuous solutions, connected with the existence of a Hamel-basis for the real numbers. Haar measure on the multiplicative group of positive real numbers can be used to define the logarithmic function. One can treat the Stieltjes integral as a possibility of a discrete measure.

In this topic there is room to discuss something about unconstructive methods (Hamel-basis for the real numbers and Zorn's lemma) and constructive analysis; see Bishop [2]. It is worthwhile mentioning here intuitionism and the work of L. E. J. Brouwer and Heyting.

5. Still another subject is **derivatives and differentials**. On the one hand one can discuss the relations between the concepts of integral and derivatives and one can speak about the Perron- and Denjoy integral (see [3]). On the other hand one can start from differential equations, a subject that is in the school curriculum, and special physical applications, where one is accustomed to work with $dx$ and $dy$. What are differentials? Are they infinitesimals in the sense of Leibniz or are they to be introduced as linear maps? In this context we spoke about non-standard analysis (using the theory of filters). One can also try to give an algebraic definition of the derivative in connection with the concept of a derivation in algebra.

6. As a last example of an instructive topic we mention **maxima and minima**. We started with the isoperimetrical property of the circle and the sphere and gave the very instructive (but incorrect) proof of Steiner. We dealt with problems on geodesics on polyhedra and on curved surfaces (cones and cylinders). One then comes to the calculus of variations and the equation of Euler–Lagrange; this can

be illustrated by Fermat's principle. The existence and uniqueness of geodesics on the sphere can be compared with the same question about geodesics in the plane when the distance is defined by

$$d(P, Q) = \max(|x_P - x_Q|, |y_P - y_Q|), \text{ or } d(P, Q) = |x_P - x_Q| + |y_P - y_Q|.$$

The specialization in the three streams takes place after 3 or 4 semesters; the curriculum for the *doctoraal examen* lasts 8–10 semesters. Many students take the course "Backgrounds of school mathematics" in the 5th or 6th semester and it seems that they are interested in these subjects. The final examination for this course has posed some problems to us. We feel that it does not make much sense to let the students memorize parts of the course. So we decided on a form of examination by assigning groups of two or three students the task of writing together a paper on a subject, connected with school mathematics and if possible with the subjects treated in the course. This proved to be difficult. The majority of the students have at that time no teaching experience. Their subjects—which they could choose themselves—tended primarily toward pure mathematics, with only a faint connection with the daily practice of secondary school teaching. Those who had already some school experience got far better results than those who had not. Perhaps we shall have to try other methods of examination.

As a curiosity we mention some subjects that were chosen by our students:
Polytopes in 4-dimensional euclidean space;
Homomorphisms in school mathematics;
Prime numbers and irreducible polynomials;
Ordering, valuations and the definition of real numbers;
Decimal fractions, continued fractions and real numbers;
The infinite (ordering and set theory);
The cubic equation, historical development;
The role of negation in language and mathematics;
Quadratic equations in rings and fields;
Analytic versus synthetic geometry;
The theorem of Pythagoras.

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# PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

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*All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

## ELEMENTARY PROBLEMS

*Solutions of Elementary Problems should be sent to Problems Group, Faculty of Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before March 31, 1977.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

E 2623. *Proposed by Ivan Niven, University of Oregon*

It is well known that if $m$ and $n$ are positive integers and $(m, n) = 1$, then $(m + n - 1)!/m!n!$ is an integer. For which positive integers $k$ is it true that there are infinitely many pairs of positive integers $m, n$ such that $(m + n - k)!/m!n!$ is an integer?

E 2624. *Proposed by Robert M. Hashway, West Warwick, Rhode Island*

James T. Rogers, *The Calculating Book*, Random House, Inc., (1974) states that the number 3025 has a "remarkable quirk." If it is "split in two parts," 30 and 25, the sum of those parts is 55, and its square $55^2 = 3025$, the original number. Such numbers can be obtained by solving the Diophantine equation

$$a + b \cdot 10^k = (a + b)^2$$

where $0 < a, b < 10^k$. Find all solutions when $k \leqq 5$.

E 2625. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

Let $A_i$ $(i = 0, 1, 2, 3 \pmod 4)$ be four points on a circle $\Gamma$. Let $t_i$ be the tangent to $\Gamma$ at $A_i$ and let $p_i$ and $q_i$ be the lines parallel to $t_i$ passing through the points $A_{i-1}$ and $A_{i+1}$, respectively. If $B_i = t_i \cap t_{i+1}$, $C_i = p_i \cap q_{i+1}$, show that the four lines $B_iC_i$ have a common point.

E 2626*. *Proposed by Richard Johnsonbaugh, Chicago State University*

Is there a positive continuous function $f$ on $[1, \infty)$ such that $\Sigma_{n=1}^{\infty} f(n) = \infty$ but $\Sigma_{n=1}^{\infty} a^n f(a^n) < \infty$ for all $a > 1$?

E 2627. *Proposed by Ron Evans, University of Wisconsin, Madison*

Let $m$ and $n$ be fixed integers greater than 1, $n$ odd. Suppose $n$ is a quadratic residue modulo $p$ for all sufficiently large prime numbers $p \equiv -1 \pmod{2^m}$. Show that $n$ is a square.

E 2628. *Proposed by Richard J. Hall, London, England*

Let $a, b, c$ be distinct positive integers, at least two of which are prime. Show that $a^{1/n}$, $b^{1/n}$ and $c^{1/n}$ cannot be terms of an arithmetic progression.

## SOLUTIONS OF ELEMENTARY PROBLEMS

### A Necessary but not Sufficient Condition for Primeness

E 435 [1940, 488; 1941, 269]. *Proposed by David Segal*

Show that the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

is a necessary and sufficient condition for $p$ to be an odd prime.

II. *Comment and completion of the solution by Wells Johnson, Bowdoin College.* In his beautiful partial solution, Professor Brinkmann proved the more general congruences

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}B_{p-3}p^3 \equiv \binom{2p^2-1}{p^2-1} \pmod{p^4},$$

where $B_n$ denotes the $n$th Bernoulli number. The last congruence shows that, if there is a prime $p$ which divides $B_{p-3}$, then the sufficiency of the original congruence for the primeness of $p$ would be false. Such a prime would have to be irregular by definition. Brinkmann argues, "... although in the first few cases (of irregular primes) $B_{p-3}$ is not the Bernoulli number that $p$ divides, there does not seem to be any reason why this should not happen eventually."

Indeed, there is such a prime: $p = 16843$. This was first found (though not explicitly reported) by Selfridge and Pollak (*Notices AMS* **11** (1964), 97), and later confirmed by me (*Math. Comp.* **29** (1975), 113–120) and S. S. Wagstaff, Jr. (*Notices AMS* **23** (1976), A-53). Thus Segal's original congruence is satisfied by $q = p^2$ which is not prime. I have searched for other examples up to 150,000 but with no luck. Interest in this arises from the fact that in 1857, Kummer proved that the first case of the Fermat conjecture is true for all prime exponents $p$ such that $p \nmid B_{p-3}$. We still don't know if there are infinitely many such primes.

### Hilbert Function and Counting

E 2546 [1975, 756]. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Let $n$ be a positive integer and let $S$ be a set of $n$-tuples of nonnegative integers with the property that if $(a_1, \ldots, a_n) \in S$ and if $0 \leqq b_i \leqq a_i$ for $i = 1, 2, \ldots, n$, then $(b_1, \ldots, b_n) \in S$. Let $H(m)$ be the number of elements of $S$ whose coordinates sum to $m$. Prove that $H(m)$ is a polynomial in $m$ for $m$ sufficiently large. (E.g., if $S$ is finite, then $H(m) = 0$ for $m$ sufficiently large.)

*Solution I by the proposer.* Let $I$ be the ideal of the polynomial ring $A = \mathbf{R}[X_1, \ldots, X_n]$ which is generated by all monomials $X_1^{a_1} \ldots X_n^{a_n}$ such that $(a_1, \ldots, a_n) \notin S$. Then

$$H(m) = \dim(A_m / I_m),$$

where $A_m$ is the subspace of $A$ consisting of homogeneous polynomials of degree $m$ and $I_m = I \cap A_m$. Thus $H(m)$ is the Hilbert function of the $\mathbf{Z}$-graded algebra $A/I$ and hence is a polynomial in $m$ for sufficiently large $m$. (See Atiyah and Macdonald, *Introduction to Commutative Algebra*, Chapter 11.)

*Solution II by the proposer.* (This solution is more elementary but longer.) Let $\mathbf{N}$ be the set of

nonnegative integers. For $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbf{N}^n$ we set $\mathbf{a} \leqq \mathbf{b}$ if $a_i \leqq b_i$ for $1 \leqq i \leqq n$. Then $(\mathbf{N}^n, \leqq)$ is a partially ordered set. Let $M$ be the set of minimal elements of the subset $\mathbf{N}^n \setminus S$. Then $M$ is an anti-chain (which means that if $\mathbf{a}, \mathbf{b} \in M$ and $\mathbf{a} \neq \mathbf{b}$ then neither $\mathbf{a} \leqq \mathbf{b}$ nor $\mathbf{b} \leqq \mathbf{a}$). The following lemma implies that $M$ is finite.

LEMMA. *If $S \subset \mathbf{N}^n$ is an anti-chain then $S$ is finite.*

*Proof.* We use induction on $n$ and note that the assertion is valid for $n = 1$. Now let $n > 1$ and assume that $S$ is an infinite anti-chain in $\mathbf{N}^n$. Fix $\mathbf{a} = (a_1, \ldots, a_n) \in S$ and put $S' = S \setminus \{\mathbf{a}\}$, $T = \{(i, j) \mid 1 \leqq i \leqq n, 0 \leqq j < a_i\}$. Define a map $f: S' \to T$ by $f(b) = (i, j)$ if $i$ is the smallest index such that $b_i < a_i$ and $j = b_i$. At least one of the sets $S'_{ij} = f^{-1}(\{(i, j)\})$ is infinite, say $S'_{rk}$. Then by suppressing the $r$th coordinate from each member of $S'_{rk}$ we obtain an infinite anti-chain in $\mathbf{N}^{n-1}$, contradicting the induction hypothesis. The lemma is proved.

For any finite subset $S \subset \mathbf{N}^n$ let $\sup(S)$ be the least upper bound of $S$ in $\mathbf{N}^n$. Explicitly, we have

$$\sup(S) = (b_1, \ldots, b_n),$$

where $b_i = \max_{\mathbf{a} \in S} a_i$, $\mathbf{a} = (a_1, \ldots, a_n)$.

For $\mathbf{a} \in \mathbf{N}^n$, $m \in \mathbf{N}$ let $J(\mathbf{a}, m)$ be the number of $\mathbf{b} \in \mathbf{N}^n$ such that $\mathbf{a} \leqq \mathbf{b}$ and $b_1 + \cdots + b_n = m$. By the Inclusion-Exclusion Principle we have

$$H(m) = \sum_{k=0}^{r} \sum_{S \in M_k} (-1)^k J(\sup(S), m),$$

where $M_k$ is the set of $k$-subsets of $M$ and $r = |M|$. It remains to note that $J(\mathbf{a}, m)$ is a polynomial in $m$ for $m$ sufficiently large. In fact, we have

$$J(a, m) = \begin{cases} \binom{m + n - s - 1}{n - 1} & \text{if } m \geqq s \\ 0 & \text{if } 0 \leqq m < s, \end{cases}$$

where $s = a_1 + \cdots + a_n$.

Correct solutions were also received from Carl Hurd, Robert Patenaude, M. J. Pelling (Nigeria), and the Temple University Problem Solving Group. Most of the incorrect solutions assumed that $S$ is a cartesian product of $n$ sets of integers.

**An Inequality**

E 2547 [1975, 756]. *Proposed by T. S. Bolis, State University College of New York at Oneonta*

Let $p$ and $q$ be positive numbers with $p + q = 1$. Show that for all $x$,

$$pe^{x/p} + qe^{-x/q} \leqq e^{x^2/8p^2q^2}.$$

*Solution by L. Kuipers, Mollens, Switzerland, and H. Niederreiter, University of California at Los Angeles (jointly).* If $f(y) = \exp(y^2/2 + 2py) - pe^{2y}$ then $f'(y) = e^{2y}g(y)$, where $g(y) = (2p + y)\exp(y^2/2 - 2qy) - 2p$ and hence $g'(y) = (y + p - q)^2\exp(y^2/2 - 2qy)$. It follows that $g'(y) \geqq 0$ for all $y$, $g(y) \geqq 0$ for $y \geqq 0$ and $g(y) \leqq 0$ for $y \leqq 0$. Hence $f(y) \geqq f(0) = q$ for all $y$. This last inequality is equivalent to the given one (put $x = 2pqy$).

Also solved by Aage Bondesen (Denmark), Thad Dankel, Jr., Gerd Fricke, Samuel Glidewell & Joseph Grimland, Jr., William Habakkuk, Harry Lass, Benjamin Lepson, O. P. Lossers (Netherlands), L. E. Mattics, W. W. Meyer, Peter Montgomery, Ram Murty & Kumar Murty (Canada), William Nuesslein, M. J. Pelling (Nigeria), Otto Ruehr, Rollin Sandberg, St. Olaf Problem Group, R. J. Serfling, Temple University Problem Solving Group, Paul Vojta, David Voss, and the proposer.

*Editor's comment.* The solution above shows that in fact the proposed inequality holds for any pair of nonzero real numbers $p$ and $q$ with $p + q = 1$.

**Simplices of Equal Volumes**

E 2548 [1975, 756]. *Proposed by Murray S. Klamkin, University of Waterloo*

Let $A_0, A_1, \ldots, A_n$ be distinct points of $n$-space which lie in a hyperplane. Suppose that these points are parallel projected into another hyperplane and that their images are $B_0, B_1, \ldots, B_n$ respectively. Prove that for any $r = 0, 1, \ldots, n$, the volumes of the simplexes spanned by $A_0, A_1, \ldots, A_r, B_{r+1}, B_{r+2}, \ldots, B_n$ and by $B_0, B_1, \ldots, B_r, A_{r+1}, A_{r+2}, \ldots, A_n$ are equal.

*Solution by Aage Bondesen, Espergaerde, Denmark.* Call the two given hyperplanes $\alpha$ and $\beta$, and let the given parallel projection $p$ be from $\alpha$ to $\beta$. The set of midpoints of the segments $Ap(A)$, for $A$ in $\alpha$, is a hyperplane $\gamma$. Let $r$ be the affine reflection in $\gamma$ such that $r(A) = p(A)$ for $A$ in $\alpha$. Since $r$ is volume-preserving, the desired result is immediate.

Also solved by Leon Gerber, Clark Givens, Michael Goldberg, M. G. Greening (Australia), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), Stephen Maurer, Temple University Problem Solving Group, and the proposer. Partially solved by Eric Rosenthal, G. T. Sallee, and Paul Vojta.

**Adding Edges to Get an Euler Path**

E 2549 [1975, 756]. *Proposed by David Singmaster, Polytechnic of the South Bank, London, England*

Let $G$ be a connected graph with $2k$ vertices of odd degree. It is well known that $G$ can be covered by a *$k$-part Euler path*, i.e., a union of $k$ edge-disjoint paths having no repeated edges. (When $k = 0$ or $k = 1$, we get an *Euler circuit.*) When can $G$ be covered by a single path with at most $k - 1$ repeated edges?

*Solution by Aage Bondesen, Espergaerde, Denmark.* The question is equivalent to the following: When can we add to $G$ at most $k - 1$ new edges, each joining two adjacent vertices of $G$, so that the resulting graph $G'$ has at most two odd vertices?

This is clearly possible if and only if the odd vertices of $G$ can be partitioned into pairs $\{a_i, b_i\}$, $1 \leqq i \leqq k$ so that $a_i$ and $b_i$ are adjacent for $1 \leqq i \leqq k - 1$.

Also solved by Anthony Barkauskas & David Bange, Lowell Beineke, Edward Dixon, Michael Ecker, Landy Godbold, Temple University Problem Solving Group, and the proposer.

**Signs of Successive Derivatives**

E 2550 [1975, 756]. *Proposed by I. J. Schoenberg, University of Wisconsin*

Let $q > 1$ and let $n$ be a natural number. Show that the polynomial

$$P(x) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{x^k - 1}{q^k - 1}$$

has the property that for $k = 1, 2, \ldots, n$

$$(-1)^{k+1} P^{(k)}(x) > 0 \qquad \text{if} \quad x \leqq q.$$

*Solution by Mourad Ismail, University of Toronto.* By routine induction we have

$$f_k(x) = (-1)^{k+1} P^{(k)}(x) = \frac{n!}{(n-k)!} \sum_{j=k}^{n} (-1)^{j-k} \binom{n-k}{j-k} \frac{x^{j-k}}{q^j - 1},$$

for $k = 1, 2, \ldots, n$. It follows that

$$f_k(x) - q^{-k} f_k\left(\frac{x}{q}\right) = \frac{n!\, q^{-k}}{(n-k)!} \left(1 - \frac{x}{q}\right)^{n-k}$$

Thus $f_k(x) > q^{-k} f_k(xq^{-1})$ for $x \leqq q$. The assertion of the problem follows by repeated application of this inequality because $x \leqq q$ implies $xq^{-1} \leqq q$, $f_k(x)$ is continuous and $f_k(0) > 0$.

More generally, the same conclusion holds if

$$P(x) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} \frac{x^k - 1}{q^{k+\alpha} - 1},$$

where $q > 1$ and $\alpha \geqq 0$,

Also solved by Aage Bondesen (Denmark), David Bressoud, Philip Castevens, James Hickman, A. A. Jagers (Netherlands), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), Ram Murty & Kumar Murty (Canada), St. Olaf Problems Group, Otto Ruehr, Temple University Problem Solving Group, University of South Alabama Problem Group, and the proposer.

**Oscillation of Partial Sums**

E 2551 [1975, 756]. *Proposed by Hugh L. Montgomery, University of Michigan*

Let $r_1, \ldots, r_n$ be real numbers such that $-1 \leqq r_i \leqq 1$ for $i = 1, 2, \ldots, n$ and such that $r_1 + \cdots + r_n = 0$. It is easy to see that there is a permutation $\pi$ of $\{1, \ldots, n\}$ with the property that all of the partial sums $S_k(\pi) = \Sigma_{i=1}^{k} r_{\pi(i)}$ $(k = 1, 2, \ldots, n)$ lie in the interval $[-1, 1]$. Strengthen this as follows: Show that there exists a permutation $\pi$ such that

$$\max_k S_k(\pi) - \min_k S_k(\pi) < 2 - n^{-1}.$$

Show also that if the right-hand side of the above is replaced by $2 - 4n^{-1}$, then the assertion is false for certain arbitrarily large $n$.

I. *Solution revised from those by the University of South Alabama Problem Group and by Aage Bondesen, Espergaerde, Denmark.* We use induction on $n$ and note that the cases $n = 1$ and $n = 2$ are true. Let $r_1, \ldots, r_{n+1}$ $(n \geqq 2)$ be an admissible sequence and consider two cases.

CASE 1. $|r_i| \leqq n/(n+1)$ for some $i$. Then we can assume that $0 \leqq r_{n+1} \leqq n/(n+1)$ and $r_n \leqq 0$. The sequence $r_1', \ldots, r_n'$, where $r_i' = r_i$ for $i < n$ and $r_n' = r_n + r_{n+1}$ is admissible and by induction hypothesis there is a permutation $\sigma$ such that if

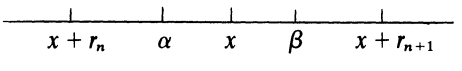
$$\alpha = \min_{1 \leqq k \leqq n} \sum_{i=1}^{k} r'_{\sigma(i)}, \quad \beta = \max_{1 \leqq k \leqq n} \sum_{i=1}^{k} r'_{\sigma(i)},$$

then $\beta - \alpha \leqq 2 - 1/n$. Let $\sigma(s) = n$ and put

$$x = \sum_{i=1}^{s-1} r'_{\sigma(i)}.$$

We will be done (see the figure) if either

$$\beta - (x + r_n) \leqq 2 - 1/(n+1) \quad \text{or} \quad (x + r_{n+1}) - \alpha \leqq 2 - 1/(n+1).$$




If none of these holds, then $\beta - \alpha + r_{n+1} - r_n > 4 - 2/(n+1)$ which gives a contradiction

$$\beta - \alpha > \left(4 - \frac{2}{n+1}\right) - \left(2 - \frac{1}{n+1}\right) = 2 - \frac{1}{n+1}.$$

CASE 2. $|r_i| > n/(n+1)$ for $1 \leqq i \leqq n+1$. We can assume that $r_i > 0$ for $1 \leqq i \leqq k$ and $r_i < 0$ for $k+1 \leqq i \leqq n+1$. Put $s_i = -r_{k+i}$ for $1 \leqq i \leqq n+1-k$. Then

$$k \geqq r_1 + \cdots + r_k > k - 1,$$

$$n + 1 - k \geqq s_1 + \cdots + s_{n+1-k} > n - k.$$

Since $r_1 + \cdots + r_k = s_1 + \cdots + s_{n+1-k}$ it follows that $n+1 = 2k$. We may further assume that $r_i - s_i \leqq 0$ for $1 \leqq i \leqq m$ and $r_i - s_i > 0$ for $m+1 \leqq i \leqq k$. When $m = k$, we have $r_i = s_i$ for $1 \leqq i \leqq k$ and we can use the sequence $(r_1, -s_1, \ldots, r_k, -s_k)$. Otherwise $m \leqq k-1$ and each partial sum $S$ of this sequence satisfies

$$-(k-1)\cdot\frac{2}{n+1} \leqq S \leqq 1.$$

Since $2(k-1)/(n+1) = 1 - 1/k = 1 - 2/(n+1)$ we are done.

II. *Counterexample for the second part.* Let $n$ be odd, say $n = 2k+1$, and set $r_i = k/(k+1)$ for $1 \leqq i \leqq k+1$, $r_i = -1$ for $k+1 < i \leqq n$. If two consecutive terms in a sequence $(r_{\pi(1)}, \ldots, r_{\pi(n)})$ are both $-1$ or both $k/(k+1)$ then

$$\max S_k(\pi) - \min S_k(\pi) \geqq 2k/(k+1) > 2 - 4/n.$$

Otherwise $S_1(\pi) = k/(k+1)$, $S_{2k}(\pi) = -k/(k+1)$ and $S_1(\pi) - S_{2k}(\pi) > 2 - 4/n$.

Also solved by Carl Hurd, Temple University Problem Group, and the proposer. (All solvers gave the counterexample II.)

## ADVANCED PROBLEMS

*All solutions of Advanced Problems should be sent to J. Barlaz, Hill Center, Rutgers University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before March 31, 1977.*

*An asterisk (*) means neither the proposer nor the editors supplied a solution.*

6120. *Proposed by Jack Fishburn, University of Wisconsin at Madison*

Let $f$ be a continuous function from the closed unit disk into the reals. If the line integral of $f$ over every chord is zero, must $f$ be identically zero?

What if the continuity of $f$ is replaced by measurability? Must $f = 0$ almost everywhere?

6121*. *Proposed by Harry D. Ruderman, Hunter College Campus School*

For all positive integers $a_1, a_2, \ldots, a_n$ the following is always an integer:

$$\prod_{i=1}^{n} (na_i)! \Big/ \left[ n \prod_{i=1}^{n} (a_i!) \right]^{n-1} \left( \sum_{i=1}^{n} a_i \right)!.$$

Prove the conjecture for $n = 3$. Is it true in general?

6122*. *Proposed by Albert A. Mullin, Fort Hood, Texas*

Does there exist a compact set $S \subset E^2$ such that for each $x \in E^2 \backslash S$ there exist precisely two nearest points of $S$? Clearly, $S$ cannot be convex.

6123. *Proposed by E. G. Kundert, University of Massachusetts*

Let $s$ be any integer $\geqq 2$ and let $\varepsilon_i$ be the following function defined on the integers:

$$\varepsilon_i = \begin{cases} 0 & \text{if } i \equiv 0, 6 \\ 1 & \text{if } i \equiv 2, 4, 7, 11 \\ -1 & \text{if } i \equiv 1, 5, 8, 10 \\ 2 & \text{if } i \equiv 9 \\ -2 & \text{if } i \equiv 3 \end{cases} \mod 12$$

Show that the following identity holds:

$$\sum_{1\leq i,j\leq s} \varepsilon_i\varepsilon_j \binom{j+1}{s-i}\binom{s+1}{j+1}3^{[i/2]+[j/2]-[(s-2)/2]} = -3\varepsilon_s.$$

6124*. *Proposed by Thomas E. Elsner, General Motors Institute, Flint, Michigan*

Let $Y$ be a compactification of completely regular space $X$. Is there a base $B$ for $Y$ such that the smallest algebra of sets containing $B$ has no element in $Y - X$?

6125. *Proposed by Simeon Reich, Tel Aviv University, Israel*

For a given $n \times n$ matrix $A$ of rank $r$, and an integer $k$, $1 \leq k \leq r$, a best rank-$k$ approximation of $A$ is a matrix $A_{(k)}$ satisfying $\|A - A_{(k)}\| = \inf\{\|A - X\|: X \text{ is an } n \times n \text{ matrix of rank } k\}$ where $\|A\| = (\text{trace } A^*A)^{1/2}$.

Show that if $A$ is normal, then $A^j_{(k)}$ is a best rank-$k$ approximation of $A^j$ for all $j \geq 1$, but that this is no longer true for arbitrary $A$. (Cf. Ben-Israel and Greville, *Generalized Inverses*, Wiley, New York, 1974, p. 250.)

## SOLUTIONS OF ADVANCED PROBLEMS

### Undersequences versus Subsequences

5437 [1966, 1019]. *Proposed by Hewitt Kenyon, George Washington University*

We say that a sequence $x$ is eventually in $A$ if $x_n \in A$ for all sufficiently large $n$; a sequence $y$ in an undersequence to a sequence $x$ if and only if whenever $x$ is eventually in a set $A$, then $y$ is eventually in $A$; we shall also say that $x$ is greater than $y$ in this case. Two sequences will be called equivalent when each is an undersequence of the other.

Suppose that $S$ is a nonempty set of sequences, every finite subset of which has a common undersequence. Which of the following statements are necessarily true?

(a) If $y$ is an undersequence of $x$, then there exists a subsequence $z$ of $x$ which is equivalent to $y$.
(b) If $S$ is finite, then $S$ has a common subsequence.
(c) If $S$ is finite, then $S$ has a greatest common subsequence.
(d) If $S$ is finite, then $S$ has a greatest common undersequence.
(e) If $S$ is countable, then $S$ has a common subsequence.
(f) If $S$ is countable, then $S$ has a common undersequence.
(g) If $S$ is countable, then $S$ has a greatest common undersequence.
(h) $S$ has a common undersequence.

*Solution by Peter Flor.* For any sequence $x$, let $A_x$ be the set of values assumed by $x$ a finite number of times, and $B_x$ the set of values assumed by $x$ an infinite number of times so that $A_x \cup B_x$ is the range of $x$. Obviously, (1) both $A_x$ and $B_x$ are at most denumerable, and (2) if $A_x$ is finite then $B_x$ is not empty; (3) to every pair of disjoint sets $A$ and $B$ satisfying (1) and (2), there exist sequences $x$ such that $A = A_x$ and $B = B_x$. $y$ is an undersequence of $x$ if and only if $B_x \supset B_y$ and the set-theoretic difference $A_y - (A_x \cup B_x)$ is finite. $x$ and $y$ are equivalent if and only if $B_x = B_y$ and the symmetric difference of $A_x$ and $A_y$ is finite.

(a) *is true.* Remove from $x$ every value not in the range of $y$, and all occurrences but the first of every value in $B_x \cap A_y$. The remaining subsequence $z$ of $x$ is equivalent to $y$.

(b) *is true.* Let $S = \{x, y, \ldots, z\}$, and let $u$ be an undersequence of every sequence in $S$. *Given any positive integer $n$, there exist integers $p \geq n$, $q \geq n, \ldots, r \geq n$ such that $x_p = y_q = \cdots = z_r$.* For if $A_u$ is infinite, there are infinitely many values of $u$ which are also values of all sequences in $S$. If, on the

other hand, $A_u$ is finite, then $B_u$ is not empty. Every element of $B_u$ occurs infinitely often in every sequence $x, y, \ldots, z$. From the italicized statement above, the existence of a common subsequence of $x, y, \ldots, z$ follows immediately.

(c) *is false.* Consider the set $S$ consisting of the equivalent sequences $(1, 2, 3, 4, 5, 6, \ldots)$ and $(2, 1, 4, 3, 6, 5, \ldots)$. Both $u = (1, 3, 5, \ldots)$ and $v = (2, 4, 6, \ldots)$ are subsequences of the given sequences, but no common subsequence can be greater than both $u$ and $v$ since it can assume at most one of the values $2n-1$ and $2n$, for every $n$.

(d) *is true.* Let $u$ be an undersequence of every $x \in S$. So $B_u \subset \cap\{B_x \mid x \in S\}$, and $A_u - \cap\{A_x \cup B_x \mid x \in S\}$ is finite. Let $B = \cap\{B_x \mid x \in S\}$, $A = \cap\{A_x \cup B_x \mid x \in S\} - B$. Construct a sequence $v$ satisfying $A = A_v$, $B = B_v$. Every such $v$ is a maximal $u$.

(e) *is false.* Let $S$ consist of all sequences $(k, k+1, k+2, k+3, \ldots)$ where $k$ runs through the positive integers.

(f) *is true.* Let $S$ consist of the sequences $x^{(1)}, x^{(2)}, \ldots$. The hypothesis allows us to construct, by recursion on $k$, a sequence $u$ such that for every $k$ and for every $i$, $1 \leqq i \leqq k$, the values $u_i, u_{i+1}, \ldots, u_k$ occur in $x^{(i)}$, *in that order* (not necessarily in immediate succession). That this is possible may be seen easily with the help of the italicized statement in the proof of (b). $u$ is an undersequence of every $x^{(i)}$: indeed $(u_i, u_{i+1}, \ldots)$ is a subsequence of $x^{(i)}$, by construction.

(g) *is false.* Let $S = \{x^{(0)}, x^{(1)}, \ldots\}$ where $x^{(k)}$ is the sequence of multiples of $2^k$. For a sequence $u$ to be an undersequence of every sequence in $S$, it is necessary and sufficient that given any $k$, all but finitely many values of $u$ are divisible by $2^k$. So, in particular, given any sequence $p$ of positive integers, the sequence $(p_1, 2p_2, 4p_3, 8p_4, \ldots)$ is an undersequence of every sequence in $S$. Suppose that $u$ is a maximal common undersequence. Then $B_u = \varnothing$, so $A_u$ is infinite, and for every sequence $p$, the set $\{p_1, 2p_2, 4p_3, \ldots\} - A_u$ is finite, i.e., there exists a finite set $C_p$ such that $\{p_1, 2p_2, 4p_3, \ldots\} \subset A_u \cup C_p$. From now on, consider only those sequences $p$ which assume odd values only. Since $u$ is an undersequence of every sequence of $S$, $A_u$ contains only a finite number of odd multiples of $2^k$, for every $k$. The same is true of every set $A_u \cup C$ if $C$ is finite. So, for every finite $C$, there exists a function $f_C$ such that $\{p_1, 2p_2, 4p_3, \ldots\} \subset A_u \cup C$, all $p_i$ odd, entails $p_i \leqq f_C(i)$, for every $i$. There are only denumerably many $C$; so the set of functions $f_C$ is denumerable and can be arranged in a sequence $f_1, f_2, \ldots$. Choose $p_i$ odd, $p_i > \max\{f_k(i) \mid k = 1, 2, \ldots, i\}$. Then $\{p_1, 2p_2, 4p_3, \ldots\} \not\subset A_u \cup C$, for every finite set $C$ of positive integers. This is a contradiction.

Finally. (h) *is false.* Let $A$ be denumerable, and let $\mathcal{F}$ be a filter in $A$ such that no sequence is eventually in every set of $\mathcal{F}$. (Such a filter has been constructed by R. Arens: see Kelley, *General Topology*, p. 77, Example E.) Every set in $\mathcal{F}$ is denumerable, and thus can be arranged in a sequence. Since $\mathcal{F}$ is a filter, there is a common undersequence for every finite set of sequences constructed in this way. But there is no undersequence common to the whole system since such a sequence would have to be eventually in every set of $\mathcal{F}$.

Also solved by Edmund Butler, Charles Riley, and the proposer.

## $1/n^3 \sin(n\pi\theta)$

5952 [1974; 175, 1033; 1975, 679]. *Proposed by J. Gilles, University of Charleroi, Belgium.*

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin(n\pi\sqrt{2})} = -\frac{13\pi^3}{360\sqrt{2}}.$$

III. *Comment by J. Barkley Rosser and Hugh L. Montgomery (independently).* It is not correct to say (as in Solution I) that $\Sigma 1/n^3 \sin(n\pi\theta)$ converges when $\theta$ is irrational. Bruce C. Berndt (author of Solution I) has noted the same situation and informs us that $\theta$ is to be a quadratic irrational.

Rosser's counterexample follows: Define

$$\phi(0)=1, \qquad \phi(N+1)=2^{N\phi(N)}, \qquad \theta=\sum_{N=0}^{\infty}\frac{1}{\phi(N)}.$$

Let $N$ be large, and take $n=\phi(N)$. Then

$$\frac{1}{n^3\sin(n\pi\theta)}\cong\frac{2^{N\phi(N)}}{\pi\{\phi(N)\}^4}.$$

This goes to infinity with increasing $N$, whereas terms of a convergent series have to approach zero.

**Prime Decomposition of Integers**

6015 [1975, 183]. *Proposed by the late C. W. Anderson*

Let $n=q_1^{a_1}q_2^{a_2}\cdots q_k^{a_k}$, $k>1$, be the prime decomposition of the integer $n$, and define $\mathrm{Ind}(n)=\max\{a_i\,|\,1\leqq i\leqq k\}$. Show that

$$\overline{\mathrm{Ind}(N)}=\lim_{m\to\infty}\frac{1}{m}\sum_{k=2}^{m}\mathrm{Ind}(k)=1+\sum_{n=2}^{\infty}\frac{\mu(n)}{n(n-1)}=1.705211\ldots,$$

where $\mu$ is the Moebius function.

*Solution by T. Salat, Bratislava, Czechoslovakia.* The result is known. See I. Niven, *Averages of exponents in factoring integers*, Proc. Amer. Math. Soc., 22 (1969), 356–363, where it is proved that

$$\mathrm{Ind}(N)=1+\sum_{k=2}^{\infty}\{1-1/\zeta(k)\}.$$

$\zeta$ is the Riemann $\zeta$-function.

Also solved by Ram Murty & Kumar Murty (Canada), and the proposer.

**A Large Modified Factorial**

6016* [1975, 184]. *Proposed by C. J. Moreno, University of Illinois*

Let $D(n)$ be the function defined by $D(n)=\Pi p$, where the product runs over those primes $p$ such that $p-1$ divides $2n$. Find an asymptotic formula for the summatory function

$$\sum_{n\leqq x}D(n).$$

By a well-known result, such estimate gives the average size of the denominators of the Bernoulli numbers.

*Comments by Paul Erdös, Budapest, Hungary.* I do not believe that there is an asymptotic formula for $\Sigma_1^x D(n)$. Pracher (*Monatshefte der Math.*, 1954) proved that if $f(n)$ denotes the number of primes $p$ for which $(p-1)\,|\,n$, then for infinitely many $n$ and a fixed $c$

$$f(n)>n^{c/(\log\log n)^2}.$$

This implies that $D(n)>\exp(n^{c/(\log\log n)^2})$ for infinitely many values of $n$; thus $\Sigma_{n=1}^x D(n)$ grows very fast and no asymptotic expression should be expected. I do believe that

$$\overline{\lim}\, D(n) \Big/ \sum_{k=1}^{n-1} D(k) = \infty,$$

but this has never been proved.

I have proved that the set of integers $n$ for which 2 is the only divisor of the form $p-1$ has positive density. Instead of $D(n)$ we could consider an alternate problem concerning $D_1(n)$ where $D_1(n)$ is the least common multiple of all $(p-1)|2n$. Here I expect that one could obtain the formula

$$\sum_{n=1}^{x} D_1(n) \approx cx^2/\log x.$$

**Constructing 'Smaller' Norms**

6017 [1975, 184]. *Proposed by Albert Wilansky, Lehigh University*

In a popular text it is proposed to find a strictly smaller norm for any normed space $E$ by first constructing a strictly larger norm on $E'$. Show that this construction must fail. The new norm must be equivalent to the old. Give a correct construction.

*Solution by the proposer.* Let $D$ be the unit disc corresponding to the new larger norm on $E'$, $p$ the dual norm on $E$ (whose unit disc is $D^0$), $n$ the original norm on $E$. Since $D$ is absorbent, $D^0$ is weakly bounded, hence $n$-bounded. Thus $p(x) \leqq 1$ implies $n(x) \leqq M$, hence $n \leqq Mp$. Conversely (assuming, as we may, $D$ is included in the original unit disc of $E'$) we have that $D^0$ includes the $n$ unit disc. Thus $n$, $p$ are equivalent.

To give an accurate construction, let $F$ be the completion of $(E, n)$. Bessaga-Klee (Israel J. Math. 2(1964) 211–220) give $F$ a smaller non-complete norm $p$. Then $p$ is strictly smaller than $n$ on $E$ for if not, $(E, n)$ and $(E, p)$ would have the same completion.

Also solved by George Crofts.

**Differentiable functions in $\mathbb{R}^2$**

6018 [1975, 307]. *Proposed by Antonio Marquina, Valencia, Spain*

Does there exist a real-valued function $f(x, y)$ defined in every point of $\mathbb{R}^2$, satisfying the following properties:

(i) For every point $(x, y)$, $f(x, y)$ is continuous;

(ii) For every point $(x, y)$, there exist the two partial derivatives $D_x f$ and $D_y f$;

(iii) The function $f(x, y)$ is not differentiable in $(x, y)$ for every point of $\mathbb{R}^2$?

*Solution by M. J. Pelling, University of Benin, Nigeria.* The answer is no. For $D_x f$ and $D_y f$ are functions of Baire class 1 and hence their sets of continuity points, $A$ and $B$ say, are dense $G_\delta$ sets in $\mathbb{R}^2$. By Baire's theorem $A \cap B$ is also a dense $G_\delta$ set and $f(x, y)$ has a differential at every point of it since continuity of both partials at a point suffices to establish the existence of a differential.

The general question of continuity of derivatives is discussed in *Derivatives*, by A. M. Bruckner and J. L. Leonard in Part II (*Papers in Analysis*) of this MONTHLY for April 1966.

Also solved by Thad Dankel, Jr., Michael Evans, Rev. William Habakkuk (pseudonym), and C. J. Neugebauer.

# REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

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*Mathematics Applied to Deterministic Problems in the Natural Sciences.* By C. C. Lin and L. A. Segel. Macmillan, New York, 1974. xix + 604 pp. $15.95. (Telegraphic Review, February 1975.)

The words "applied mathematics" decorate the covers of many books these days, but precious little of the practice of applied mathematics seeps through into the pages bound between those covers. Of the select few that do indeed introduce the reader to applied mathematics, this book may well be the best. I know of none better.

It is a lucid, carefully written account of the "construction, analysis, and interpretation of mathematical models that shed light on significant problems in the natural sciences." Since it is *not* merely a collection of methods, *Mathematics Applied* can initiate the novice, whether student or teacher, into the full circle of thought that carries an applied mathematician from the formulation of a model of a physical problem through its analysis and back to the interpretation of results in the light of reality.

For example, the simple pendulum provides a prototype problem for the introduction of two concepts, scaling, which precisely identifies the small terms in an equation, and regular perturbation methods, which can successively improve linearizing approximations, such as the replacement of the sine of a small angle by the angle itself.

The next chapter then applies these techniques to a model of a process proposed to explain secretions observed in kidney and similar tissues. After a description of the physiological setting and the motivation for considering the particular process, the differential equations comprising this model are carefully derived by arguments that can be applied to other phenomena as well.

Scaling and a regular perturbation analysis of the model follow. The pitfalls in the latter clearly demonstrate the fallacy of attempting to follow blindly a fixed prescription for "the method." Finally, the results of the analysis are compared with numerical calculations made by the physiologists who first proposed the process, and the parameters of the model are interpreted physiologically.

This discussion is contained in the text's middle section (six chapters), "Some Fundamental Procedures Illustrated on Ordinary Differential Equations," which also treats dimensional analysis, singular perturbation methods (where the case study is a model of enzyme kinetics), linear stability analysis, and the phase plane. The first section (five chapters) presents "An Overview of the Interaction of Mathematics and Natural Science" in the context of such diverse phenomena as whirlpool galaxies and slime mold amoebae. The final five chapters form an "Introduction to Theories of Continuous Fields," the fertile ground of continuum mechanics from which much of pure and applied mathematics has grown. Entry into this last section is eased by first developing many of the basic concepts of mechanics in a single dimension.

The overall effect of the text is one of challenge, enthusiasm, and growing confidence. The satisfaction the authors find in facing the problems open to applied mathematicians is communicated clearly, and their success as teachers builds the reader's confidence in facing those same challenges himself.

Eloquent discussions of major issues and careful accounts of derivations and calculations seldom leave the reader dangling between lines. Well-chosen exercises, neither trivial nor impossible, complement the text. Hints or answers are provided for a good number of them.

*Mathematics Applied* is a perfect text for the core course in applied mathematics that (I believe) belongs in every mathematics curriculum. Its contents and point of view are in the spirit of many of the recommendations of the CUPM for "Applied Mathematics in the Undergraduate Curriculum" (CUPM January, 1972). Its style and effectiveness are products of years of classroom testing by the authors, who are themselves distinguished teachers and applied mathematicians.

I have twice used pre-publication versions of the text in courses populated by seniors majoring in mathematics and physics, master's candidates in mathematics, and a handful of engineers and other science majors. This first exposure to the combined use of physical *and* mathematical reasoning challenged most of the students and demanded increased scientific maturity. (The magnitude of the challenge was much like that encountered in the transition from solving routine calculus problems to the first course in analysis or modern algebra.) In spite of the pressure, the text worked well because it is designed to develop this kind of dual reasoning.

Beyond introductory courses in calculus, differential equations, and college physics, the book is essentially self-contained. This feature may appeal to the non-expert who has volunteered or been elected to "bell the cat" by teaching his department's core course in applied mathematics; *Mathematics Applied* provides more than enough fact, wit, and wisdom for a full year course in the practice and philosophy of applied mathematics.

In short, classical applied mathematics belongs in the undergraduate curriculum, and *Mathematics Applied* is the most effective vehicle I have seen for getting it there.

PAUL DAVIS, Worcester Polytechnic Institute

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T = textbook
S = supplementary reading
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
P = professional reading
L = undergraduate library purchase

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(13-15), L*. *Mathematical Gems II*. Ross Honsberger. Dolciani Math. Expos., No. 2. MAA, 1976, ix + 182 pp, $5. 14 brief essays (on e.g., checker-jumping, box packing, art gallery theorem) illustrating surprising yet sophisticated insights from elementary (high school!) mathematics. Problems and references make this volume ideal for math clubs or problems seminars. LAS

GENERAL, T(13: 1). *Elements of Mathematics, Second Edition*. James W. Armstrong. Macmillan, 1976, x + 356 pp, $10.95. Mathematics without asking students to do computations. A rather miscellaneous assemblage of topics for liberal arts majors. LH

GENERAL. *The DDT System of Place-Value Notation*. Leo D. Holloway, Jr. Vantage Pr, 1976, iv + 83 pp, $4.95. A slightly eccentric presentation of the "digitally-directed ternary" notation system for numbers. This system deals implicitly, without the introduction of extra symbols, with negatives and complex numbers. JAS

GENERAL, P. *Lecture Notes in Mathematics-514: Séminaire Bourbaki vol. 1974/75, Exposés 453-470*. Springer-Verlag, 1976, iv + 276 pp, $10.30 (P).

GENERAL, L*. *The High School Mathematics Library, Sixth Edition*. William L. Schaaf. NCTM, 1976, vi + 74 pp, $2.50 (P). A list of recommended titles. Major changes from the fifth (1973) edition: deletion of sections on Survey Mathematics, Slide Rule, Mathematics for Parents, and new stress on computers, expository and recreational mathematics, and metric measures. LAS

GENERAL, L?. *Basic Mathematics*. Joseph P. Lerro, Jr. Cahners Books, 1976, 202 pp, $12.95. A handbook which organizes frequently used equations, facts and formulas from algebra, geometry and calculus into graphs, charts, diagrams, figures, tables and nomograms. This book might be useful for engineers, physicists, chemists, draftsmen, technicians and students. CEC

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1974, V. 30*. AMS, 1976, iii + 260 pp, $28.40.

BASIC, T(13: 1). *Elementary Algebra, Second Edition*. Robert G. Moon, Robert D. Davis. Merrill, 1975, xii + 468 pp, $9.95 (P). Consists of 26 units organized into six modules: Whole numbers; Integers; Real numbers; Operations with algebraic expressions; Exponents and radicals. Detailed solutions of examples show step by step algebraic manipulation. Designed for use with an audio cassette but would be useful by itself. RBK

BASIC, T. *Basic Mathematics for Technical Occupations*. Ronald L. Bohuslov. P-H, 1976, xii + 512 pp, $12.95. The student is told: "You do not need to have had previous experience in mathematics to use this text." Also: "This book will help you learn the mathematical skills that will put you into positions of high responsibility and top income." Mainly treats the arithmetic of whole numbers and fractions, with other basic topics (e.g., rounding off, percentage, areas and volumes) thrown in.DFA

BASIC, T(13). *Algebra and Geometry: An Introduction with Applications*. Seymour Hayden, John Mineka. Williams & Wilkins, 1976, xiii + 399 pp, $10.95 (P). Review of high school algebra, geometry. "Advanced topics" such as exponents, factoring are included as options. LH

PRECALCULUS, T(13: 2). *Elementary Algebra for College Students, Fourth Edition*. Irving Drooyan, William Wooton. Wiley, 1976, viii + 326 pp, $10.95; *Student's Supplement, Second Edition*, 1976, iii + 236 pp, $2.95 (P). A straightforward and traditional treatment of algebra for college students. The approach is intuitive with exercises well-chosen to engage students' interest. JW

PRECALCULUS, T*(13: 1). *Precalculus Mathematics*. David G. Crowdis, Brandon W. Wheeler. Glencoe Pr, 1976, vii + 426 pp, $12.95. A solid, well-written traditional precalculus course which includes an optional introduction to BASIC programming. An adequate number of problems are included and amongst them are computer applications whenever appropriate. CEC

PRECALCULUS, T(13: 1). *College Algebra*. William G. Ambrose. Macmillan, 1976, x + 374 pp, $11.95. Standard college algebra. Solid presentation stressing computations. Extensive problem sets. LH

EDUCATION, S(16-18), P*, L. *The Psychology of Mathematical Abilities in Schoolchildren*. V.A. Krutetskii. Trans: Joan Teller. U of Chicago Pr, 1976, xvi + 417 pp, $22. Report of a 1955-66 Soviet research program to explore the nature and structure of mathematical abilities. Using interview techniques Krutetskii focuses on the process of mathematical thinking while strongly disagreeing with the American reliance on test scores as ability indicators. Includes consideration of problem solving, gifted students, and the notion of a "mathematical cast of mind." Editors predict an impact on mathematics education similar to that of Piaget. PSJ

EDUCATION, T*(13-14: 1), L. *The Mathematics of the Elementary School.* Edward G. Begle. McGraw, 1975, x + 453 pp, $12. Includes geometry, numbers and numeration systems, arithmetic operations on whole numbers and rationals, probability, flow charts, and statistics, among other topics. Clear expositions of the mathematics ("modern") of the elementary school with excellent thought questions throughout. Comprehensive problem sets. PSJ

EDUCATION, T(15-17: 1). *Mathematics and the Elementary Teacher, Third Edition.* Richard W. Copeland. Saunders, 1976, xi + 405 pp, $11.50. Methods text for pre- and in-service elementary teachers. Revision includes new chapters on psychology of learning, ordering and seriation, exceptional child, and understanding time. Also more consideration of metric system. (Second edition, TR, August 1974.) PSJ

EDUCATION, S(15-17), P, L. *Measurement in School Mathematics, 1976 Yearbook.* Doyal Nelson, Robert E. Reys. NCTM, 1976, xi + 244 pp, $7.50. The first of a new breed of *annual* NCTM yearbooks. Independent essays (15) on the measurement theme in three groupings: Learning and Teaching Measurement; Activities; and Resources. One nonthematic essay on Low-Stress Algorithms. For K-12 teachers. PSJ

EDUCATION, S, P*, L. *Classroom Ideas from Research on Computational Skills.* Marilyn N. Suydam, Donald J. Dessart. NCTM, 1976, vi + 58 pp, $2.50 (P). Seeks to help teachers gain a better understanding of how to help children learn computational skills with meaning by summarizing selected research results on computational skills with whole and rational numbers. Considers introducing, reinforcing, maintaining, transferring and applying computational skills. Omits technical details of research methodology such as types of statistics used while focusing on the information of more immediate use to the classroom teacher. Bibliography. PSJ

HISTORY, P. *Diocles on Burning Mirrors: The Arabic Translation of the Lost Greek Original.* Ed: G.J. Toomer. Springer-Verlag, 1976, ix + 249 pp, $27.90. Definitive translation (English and Arabic on facing pages) of work by Diocles (fl. 190-180 BC) dealing with burning mirrors (parabolic and spherical), solution of a problem from Archimedes' *Sphere and Cylinder*, and doubling of the cube (via intersection of two parabolas, and via the cissoid). PJC

HISTORY, P. *Opera matematica, V. I.* Grigore C. Moisil. Editura Academiei Romania, 1976, 376 pp, Lei 31. Papers, mostly in differential equations, from 1928 to 1957. JAS

FOUNDATIONS, T(14-16: 2). *Number and Structure.* W.G. Malcolm. Reed Educ, 1975, ix + 176 pp, $5.95 (P). An axiomatic approach to the real number system which develops its properties as complete ordered field. Probably more useful for its emphasis on axiomatics than for its fairly standard content. JW

FOUNDATIONS, P. *Proceedings of the Sixth International Symposium on Multiple-Valued Logic.* Stephen Y.H. Su. IEEE Computer Society, 1976, vii + 272 pp, (P). Held at Utah State University, May 1976. Includes "tutorials" introducing general areas. More computer-related papers than previously. JAS

FOUNDATIONS, P. *Problems in the Constructive Trend in Mathematics. VI.* Ed: V.P. Orevkov, N.A. Sanin. Proc. of Steklov Inst. of Math., No. 129. AMS, 1975, iii + 271 pp, $55.60 (P). Four papers on aspects of constructive mathematics. LAS

FOUNDATIONS, P. *Proceedings of the 1975 International Symposium on Multiple-Valued Logic.* George Epstein, *et al.* IEEE Computer Society, 1975, 475 pp, (P). Held at Indiana University, May 1975. JAS

COMBINATORICS, P. *Research Bibliography of Two Special Number Sequences, Revised Edition.* H.W. Gould. Henry W. Gould, 1239 College Ave., Morgantown, W. VA 26505, 1976, x + 38 pp, $2 (P). 169 references on the Bell sequence $(B(n) = e^{-1}\Sigma_1^\infty k^n/k!)$ and 431 on the Catalin sequence $(C(n) = (2n)!/n!(n+1)!)$, preceded by a brief historical introduction. LAS

ALGEBRA, P. *Abelian Groups and Modules.* A.P. Mišina, L.A. Skornjakov. Amer. Math. Soc. Transl. Ser. 2, V. 107. AMS, 1976, 160 pp, $20. A translated and updated version of the six-year-old Russian original which treats the theory of purity, torsion, and completeness (divisibility). Contains a section on open problems. JAS

ALGEBRA, S(17-18), P. *Lecture Notes in Mathematics-513: Nilpotent Groups.* Robert B. Warfield, Jr. Springer-Verlag, 1976, viii + 115 pp, $7.40 (P). A self-contained volume on the theory of nilpotent groups containing and inspired in part by recent developments in algebraic topology concerning localizations and completions of nilpotent groups. This volume includes earlier notes by the author on the same subject. JAS

ALGEBRA, P. *Indecomposable Representations of Graphs and Algebras.* Vlastimil Dlab, Claus Michael Ringel. Memoirs No. 173. AMS, 1976, iii + 57 pp, $6.40 (P).

ALGEBRA, T*(15-17: 1-3). *Lectures in Abstract Algebra I, III.* Nathan Jacobson. Grad. Texts in Math., V. 30, 32. Springer-Verlag, 1975. *I: Basic Concepts*, xii + 217 pp, $10.80; *III: Theory of Fields and Galois Theory*, xi + 323 pp, $14.80. Reprints of the remaining two volumes in the classic set of three. The reprint of the second volume was noted in a TR, December 1975. JAS

FINITE MATHEMATICS, T(13-14: 3). *Fundamentals of Finite Mathematics.* Robert L. Childress. P-H, 1976, xvi + 544 pp, $12.95. This book develops a well-chosen sequence of topics from finite mathematics in a rigorous but clear and exciting way. It enjoins the student to follow rigorous developments by the careful choice of problems and topics which emphasize applicability. JW

CALCULUS, S(13). *Programmed Guide to accompany Calculus with Analytic Geometry*, by Earl W. Swokowski. Roy A. Dobyns. Prindle, 1976, vi + 297 pp, $4.95 (P).

CALCULUS, T(13-14: 2, 3). *Mathematical Analysis, Business and Economic Applications, Third Edition.* Jean E. Weber. Har-Row, 1976, xii + 691 pp, $17.95. A revision of the second edition (authors: Draper and Klingman, TR, November 1974). Additional examples, exercises, and rewording of several sections make this even more attractive. Applications of calculus outside economics and business go essentially unnoticed. TAV

CALCULUS, T(13-14: 3). *Introductory Mathematical Analysis: For Students of Business and Economics, Second Edition.* Ernest F. Haeussler, Jr., Richard S. Paul. Reston, 1976, xiii + 769 pp, $15. Improved from the earlier edition (TR, May 1974) by the addition of a chapter on algebra review, an expansion of the linear algebra section to include linear programming, a discussion of the adjoint matrix, and the Leontief input-output methods. TAV

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-524: Séminaire Pierre Lelong (Analyse) Année 1974-75.* Ed: Pierre Lelong. Springer-Verlag, 1976, 222 pp, $9.50 (P). Complex analysis, especially extensions to infinite-dimensions. Contains those presentations from the seminar which have not already appeared elsewhere. JAS

DIFFERENTIAL EQUATIONS, P. *Nonoscillation, Disconjugacy and Integral Inequalities.* Shmuel Friedland. Memoirs No. 176. AMS, 1976, v + 78 pp, $6.80 (P). Shows that the infima of appropriate integral functionals are constants for nonoscillation and disconjugacy criteria; computes certain of these infima; and obtains known and new criteria. JAS

DIFFERENTIAL EQUATIONS, P. *New Developments in Differential Equations.* Ed: Wiktor Eckhaus. Math. Stud., V. 21. North-Holland, 1976, viii + 248 pp, $23.25 (P). Proceedings of the second Scheveningen Conference, held in August 1975. Emphasis is on non-linear analysis. JAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-515: Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications.* Ed: R.M. Miura. Springer-Verlag, 1976, viii + 295 pp, $11.50 (P). Report on a research workshop on contact transformations, Vanderbilt University, September, 1974. Includes papers on theory and on applications to physical problems.JAS

DIFFERENTIAL EQUATIONS, T(18: 1), P. *The Hopf Bifurcation and Its Applications.* J.E. Marsden, M. McCracken. Appl. Math. Sci., V. 19. Springer-Verlag, 1976, xiii + 408 pp, $14.80 (P). The Hopf bifurcation denotes the formation of periodic orbits emanating from a stable fixed point as a parameter crosses a critical value. Discusses the method of invariant manifolds. Some chapters contain ideas and approaches of different authors. Exercises. References. Index. RJA

DIFFERENTIAL EQUATIONS, P. *Transactions of the Moscow Mathematical Society for the Year 1974, V. 31.* AMS, 1976, v + 305 pp, $38.80 (P). Dedicated to I.G. Petrouskiĭ; includes a vigorous obituary by P.S. Aleksandrov, A.N. Kolmogorov and O.A. Oleĭnik. Mathematical contents devoted to differential equations. LAS

NUMERICAL ANALYSIS, P. *Numerische Methoden der Approximationstheorie, Band 3.* L. Collatz, H. Werner, G. Meinardus. Int. Ser. Num. Math., V. 30. Birkhauser, 1976, 333 pp, Sfr. 44 (P). Proceedings of the conference held at Oberwolfach in May 1975. JAS

NUMERICAL ANALYSIS, P. *Approximation Theory II.* Ed: G.G. Lorentz, C.K. Chui, L.L. Schumaker. Acad Pr, 1976, xi + 588 pp, $23.50. Proceedings of a symposium held in Austin, Texas in January, 1976; a sequel to the similar January 1973 symposium published as *Approximation Theory* in 1973 (TR, May 1974). JAS

NUMERICAL ANALYSIS, T(16: 1), S*, L*. *Numerical Solution of Partial Differential Equations.* G.D. Smith. Oxford U Pr, 1969, viii + 179 pp, $2.50 (P). Reprint of an excellent survey of standard techniques for parabolic, hyperbolic and elliptic equations. Also discusses convergence and stability. Includes exercises and worked solutions. RWN

NUMERICAL ANALYSIS, P. *Approximation Theory.* Ed: Zbigniew Ciesielski, Julian Musielak. Reidel, 1975, xii + 289 pp, $35. Proceedings of a conference held in Poznán, August 1972. JAS

FUNCTIONAL ANSLYSIS, S(18), P. *Lecture Notes in Mathematics-485: Geometry of Banach Spaces--Selected Topics.* Joseph Diestel. Springer-Verlag, 1975, xi + 282 pp, $11.50 (P). Aimed at providing geometric perspective on the Radon-Nikodym theorem for vector-valued measures. Incorporates many recent results. Assumes a background in functional analysis. Bishop-Phelps property, renorming theorems, Lindenstrause result re the Krein-Milman property. LH

FUNCTIONAL ANALYSIS, T(17-18: 1, 2), L. *Linear Topological Spaces.* John L. Kelley, Isaac Namioka. Grad. Texts in Math., V. 36. Springer-Verlag, 1976, xv + 256 pp, $14.80. A reprint of the 1963 original Van Nostrand edition. JAS

FUNCTIONAL ANALYSIS, P. *On Integral Functionals with a Variable Domain of Integration.* I.I. Daniljuk. Proc. of Steklov Inst. of Math., No. 118. AMS, 1976, iv + 144 pp, $21.60 (P). Problems studied in this exposition of the calculus of variations, where variations are considered not only in the integrand but in the domain of integration, arose in connection with boundary value problems in two dimensional fluid flow. Primary object of study is a "second term" in the first variation equation which is due to the variation of the domain. RBK

ANALYSIS, P, L*. *Studies in Harmonic Analysis.* Ed: J.M. Ash. Stud. in Math., V. 13. MAA, 1976, xv + 319 pp, $15. Eleven papers from an "expository conference" held at DePaul University, June 29-July 2, 1974. Authors include A. Zygmund (on history), C. Fefferman, E.M. Stein, G.L. Weiss. LAS

COMPUTER SCIENCE, T(15: 1), L. *Computational Structures, An Introduction to Non-numerical Computing.* Patrick A.V. Hall. Am Elsev, 1975, xv + 193 pp, $10.50. For a second level course presuming programming. Data structures, storage allocation, string manipulation, tables, files, sorting. Includes some analysis of algorithms. Exercises. RWN

COMPUTER SCIENCE, T(13-14: 1), S. *Structured Programming in APL.* Dennis P. Geller, Daniel P. Freedman. Winthrop Pub, 1976, xix + 324 pp, $8.95 (P). Moderately paced introduction to APL. Algorithm design employing structure charts. Data structures and operators. Programming structures. Examples. Exercises. System dependent. RWN

COMPUTER SCIENCE, T(14). *Learning COBOL Fast, A Structured Approach.* Claude J. DeRossi. Reston, 1976, xi + 212 pp, $8.95 (P). How to handle business-oriented data files. Teaches by example. No explicit exercises. Structured programming here means modularization only and is not emphasized. RWN

COMPUTER SCIENCE, T(13-14: 1). *Assembler Language with ASSIST.* Ross A. Overbeek, W.E. Singletary. SRA, 1976, x + 372 pp, $14.95. Teaches assembly language programming in ASSIST, a subset of the assembly language for 370's which includes some macros for I/O. Moderately paced. Examples, exercises. Source tapes are said to be available without charge. RWN

COMPUTER SCIENCE, P. *Analytic Computational Complexity.* Ed: J.F. Traub. Acad Pr, 1986, ix + 239 pp, $11. Invited papers from the April 1975 conference at Carnegie-Mellon University. LAS

COMPUTER SCIENCE, S(16-17), L. *Computer Design Development, Principal Papers.* Ed: Earl E. Swartzlander, Jr. Hayden, 1976, 310 pp, $13.95. 17 important papers. Authors: Shannon, Karnaugh, MacCluskey, Booth, Wilkes, et al.. Groupings: Logic Design, Arithmetic, Algorithms, Computer Architecture. RWN

COMPUTER SCIENCE, P. *Advances in Computers, V. 14.* Ed: Morris Rubinoff, Marshall C. Yovits. Acad Pr, 1976, xii + 287 pp, $24.50. Five expositions of topics of current interest: feasible computations, programming systems, context-free languages, statistical processes, secure systems. LAS

COMPUTER SCIENCE, T(14-16: 1), S, L. *Let's Talk LISP.* Laurent Siklóssy. P-H, 1976, xv + 237 pp, $11.95. LISP is a non-numeric programming language used extensively in artificial intelligence work. The reader cannot fail to learn recursion. Presents complete description of all the language features. Includes a chapter on LISP solutions of several sophisticated problems. Exercises. Four appendices. Bibliography. Index. RJA

COMPUTER SCIENCE, T(13: 1), S. *Introduction to Computer Applications for Non-Science Students (BASIC).* William Ralph Bennett, Jr.. P-H, 1976, xv + 207 pp, $8.95 (P). Contains initial chapters on the programming language BASIC and applies it to problems in elementary calculus: derivatives, series, integrals. Later chapters on graphic display and language. Material for both science and humanities people deals with realistic problems and contains many comments of historical relevance. Chapter references. Problems. Appendix. Index. RJA

COMPUTER SCIENCE, T(13-14: 1), S, P, L. *Techniques of Program Structure and Design.* Edward Yourdon. P-H, 1975, xv + 364 pp, $16.95. Attempts to define what a good program is and proceeds to describe how to write one. Contains chapters on top-down design, modular programming, structured programming, program testing and others. The appendices present four major programming problems whose solutions should illustrate the design principles set forth earlier. Problems. References. Index. RJA

COMPUTER SCIENCE, T(13-14: 1), S, L. *Systematic Programming, An Introduction.* Niklaus Wirth. P-H, 1973, xiv + 169 pp, $13.95. Equates programming and formulation of algorithms. Emphasizes the design of algorithms by examining problems and techniques from various areas of application. Discusses program verification. Uses an ALGOL-like notation for writing algorithms. An exciting, ambitious, out-of-the-ordinary introduction to programming. Exercises. Appendices (one on PASCAL). Index. RJA

COMPUTER SCIENCE, T(14-16: 1), S, L. *Applying Data Structures.* T.G. Lewis, M.Z. Smith. HM, 1976, xii + 290 pp, $15.95. Initial chapters provide motivation for study of data structures and are more intuitive. Later chapters take a more formal approach. Emphasizes that how data is structured depends on how it is to be manipulated. The algorithms are described in a combination of English and algebraic notation. Presents an analytical model for memory management. Exercises with selected answers. Three appendices. Glossary. References. Index. RJA

COMPUTER SCIENCE, S(13), P. *Pascal User Manual and Report, Second Edition.* Kathleen Jensen, Niklaus Wirth. Springer-Verlag, 1975, 167 pp, $5.90 (P). The manual section is a tutorial on the programming language PASCAL. Contains many examples, several appendices, and references. The report contains the definition of the language. Both parts contain indices. RJA

COMPUTER SCIENCE, T(14-16: 1), S, L. *A Discipline of Programming.* Edsger W. Dijkstra. P-H, 1976, xvii + 217 pp, $14.95. Previous programming knowledge is required to fully appreciate this text. Each chapter presents a problem and an algorithmic solution given in a mini-language developed for use in this text. Underlying the presentation of each topic is the author's belief that it is understanding the essential concepts (rather than the code) that must be stressed. Exercises. RJA

COMPUTER SCIENCE, P. *Proceedings of the 1976 Army Numerical and Computers Analysis Conference.* US Army Research Office, Durham, N.C., 1976, xi + 524 pp, $13 (P). The theme of this thirteenth conference, held in Durham, N.C., in February, 1976 was "Impact of Mini-computers and Micro-processors on Scientific Computation in Army Research and Development." JAS

COMPUTER SCIENCE, P*. *Finite Orthogonal Series in the Design of Digital Devices (Analysis, Synthesis and Optimization).* M.G. Karpovsky. Wiley, 1976, xi + 251 pp, $35. This book is concerned with methods for the analysis, synthesis and optimization of digital control devices, based on the use of finite orthogonal series. Almost all the results and methods described are original researches of the author and E.S. Moskalev. CEC

COMPUTER SCIENCE, T(16-17: 1), P. *Digital Picture Processing.* Azriel Rosenfeld, Avinash C. Kak. Comp. Sci. and Appl. Math.. Acad Pr, 1976, xii + 457 pp, $28.50. Supersedes *Picture Processing by Computer* by the first author. Mathematical and computational techniques for the digitizing, rotation and description of images. Applies transforms, probability, and geometry. Compares natural and artificial perception. RWN

COMPUTER SCIENCE, T*(14-16: 1), S, L*. *Discrete Mathematical Structures for Computer Science.* Ronald E. Prather. HM, 1976, xv + 549 pp, $14.50. An excellent, well-motivated introduction to discrete (especially algebraic and combinatorial) structures and their application to topics in computer science. The structures include monoids, boolean algebras, groups, lattices and graphs. Applications include finite state machines, Turing machines, circuits, and codes. Numerous exercises. References. RWN

COMPUTER SCIENCE, T(14-15: 1), S*, L. *Scientific and Engineering Problem-Solving with the Computer.* William Ralph Bennett, Jr. P-H, 1976, xix + 457 pp, $17.95. A well-motivated introduction to programming and problem-solving. Numerous interesting examples. Well illustrated. Although some examples are non-scientific, e.g., cryptography, many require a year of calculus and physics. Uses BASIC. Exercises. References. RWN

SYSTEMS THEORY, P. *Applications of Fuzzy Sets to Systems Analysis.* C.V. Negoiță, D.A. Ralescu. Wiley, 1975, 187 pp, $19.75. A mathematically precise framework is developed in which to treat systems of phenomena which cannot be characterized precisely. This theory is then applied to logic, systems theory, automata, decision making, and clustering problems. Includes thorough bibliography. CEC

APPLICATIONS, P. *Proceedings of the Eighth International Conference on Numerical Taxonomy.* Ed: George F. Estabrook. Freeman, 1975, xvii + 429 pp, $12.50. Papers plus discussions thereon from the conference held at Oeiras, Portugal, 1974. JAS

APPLICATIONS, P. *Proceedings of the Twenty-First Conference on the Design of Experiments.* US Army Research Office, Durham, N.C., 1976, xv + 496 pp, $12.75 (P). Some of the papers and invited addresses from an October 1975 conference at the Walter Reed Medical Complex in Washington, D.C. JAS

APPLICATIONS, P. *Transactions of the Twenty-First Conference of Army Mathematicians.* US Army Research Office, Durham, N.C., 1975, xvii + 712 pp, $18.75 (P). Transactions of the May 14-16, 1975 conference at El Paso, Texas, JAS

APPLICATIONS, P. *Sur les Méthodes Numériques en Sciences Physiques et Économiques.* J.L. Lions, G.I. Marchouk. Dunod (US Distr: SMPF), 1974, xii + 298 pp, 190 F. Two papers on applications to physics and economics. The first deals with iterative methods of minimizing functionals. The second deals with numerical methods (decomposition and decentralization) in optimization problems with constraints. PJM

APPLICATIONS (AERODYNAMICS), P. *Collected Works of Robert T. Jones.* William R. Sears. Ames Research Center, 1976, xvi + 1025 pp, $15.25 (P). Reprints of some sixty papers in aerodynamics in honor of Jones' 65th birthday. JAS

APPLICATIONS (BIOLOGY), P. *Pharmacokinetics.* Inst Math Appl, 1975, viii + 182 pp, £3.00 (P). Pharmacokinetics is the study of the interaction of drugs in the animal system in a dynamic framework. These notes, the report of a conference on the subject, describe the variety of mathematical models used; e.g., Markov chains, ordinary and stochastic differential equations. Each paper contains references. TAV

APPLICATIONS (BUSINESS), T(13: 2). *Mathematics for Business and Social Sciences, An Applied Approach.* Abe Mizrahi, Michael Sullivan. Wiley, 1976, xiii + 699 pp, $13.95. For social science and business majors. Linear algebra, probability, mathematics of finance, calculus; limits of sums in an appendix. Clear examples. LH

APPLICATIONS (BUSINESS), T(13-14: 2). *Mathematics: Fundamentals for Managerial Decision-Making, Second Edition.* Michael L. Kovacic. Prindle, 1975, xiv + 626 pp, $13.95. Carries business students from review of algebra to double integrals with side trips to probability and linear programming. "How" rather than "why." Large number of examples. LH

APPLICATIONS (ECOLOGY), P. *Systems Analysis and Simulation in Ecology, V. III.* Ed: Bernard C. Patten. Acad Pr, 1975, xv + 601 pp, $39.50. Twelve lengthy papers on ecosystem modelling growing out of a symposium held in March 1973 at the University of Georgia; V. IV, forthcoming, completes the report from this symposium. LAS

APPLICATIONS (ECOLOGY), P. *Lecture Notes in Biomathematics-5: Mathematical Analysis of Decision Problems in Ecology.* Ed: A. Charnes, W.R. Lynn. Springer-Verlag, 1975, viii + 421 pp, $15.10 (P). Papers from a NATO conference in Istanbul, July 1973, covering land use management, pollution control, and disease control. LAS

APPLICATIONS (ECONOMICS), T(14-15: 1, 2), S, L. *An Introduction to Mathematical Economics: Methods and Applications.* G.C. Archibald, Richard G. Lipsey. Har-Row, 1976, xv + 576 pp, $15.95. Designed for the student of economics who suffers from mathophobia, this text surveys mathematical

methods in calculus and linear algebra and relates each to realistic economics examples. In addition to fulfilling its primary purpose, it is an excellent source of applications for mathematics teachers. LAS

APPLICATIONS (ECONOMICS), T(13: 1-3). *Mathematics with Applications in Management and Economics, Fourth Edition.* Earl K. Bowen. Irwin, 1976, xxi + 721 pp, $14.50. Adds numerous applications to the previous edition (TR, May 1973), and contains a less theoretical treatment of calculus. Other topics: linear equations and inequalities (with linear programming), matrices, mathematics of finance, probability. DFA

APPLICATIONS (ECONOMICS), S(16-17), P. *Differential Games and Other Game-Theoretic Topics in Soviet Literature, A Survey.* Alfred Zauberman. New York U Pr, 1975, xi + 227 pp, $12.50. The first of a planned series, this volume represents an annotated bibliography which grew up into an extensive introductory survey of the Soviet contributions to game theory with applications in operations research. JAS

APPLICATIONS (ENERGY), P. *Energy: Mathematics and Models.* Ed: Fred S. Roberts. SIAM, 1975, xxiv + 279 pp, $16 (P). Proceedings of a SIMS (SIAM Institute for Mathematics and Society) conference held at Alta, Utah, July 1975. Most papers deal directly or indirectly with forecasting; few papers use mathematics more advanced than linear programming. LAS

APPLICATIONS (ENGINEERING), S(16), P. *Sparsity: The Practical Application to Systems Analysis.* A. Brameller, R.N. Allan, Y.M. Hamam. Pitman, 1976, ix + 177 pp, $15.90. Relationship between graphs and linear networks. A review of elimination techniques for linear systems. Extensions for sparse matrices. Applications to linear programming and non-linear problems. RWN

APPLICATIONS (ENGINEERING), T(15: 1). *Matrix Analysis of Discrete Elastic Systems.* Hayrettin Kardestuncer. Springer-Verlag, 1974, 45 pp, $4.30 (P). Uses Castigliano theorems (not stated) to obtain matrix formulas for the analysis of elastic systems. Considers improvement of design and analysis of large systems by decomposition. No examples or index. References. RBK

APPLICATIONS (ENGINEERING), S(17), P. *Iterative Structural Design.* William R. Spillers. North-Holland, 1975, xii + 187 pp, $18. On the practice of using optimization techniques for structural design. The main method employs the Newton-Raphson-Kantoravich method to iteratively solve the Kuhn-Tucker conditions. Numerous examples. Bibliography. RWN

APPLICATIONS (LAW), T. *Practical Logic.* Zygmunt Ziembiński. Reidel, 1976, xv + 437 pp, $29. Logic for law students, with emphasis on semantic and philosophical considerations rather than on formalized deduction. Part III is entitled "The Intellectual Work of Lawyers", and an appendix by Zdzisław Ziemba treats deontic logic (logic of permission, prohibition and obligation). The book is in its eighth edition in Poland, where it has been used for 20 years. PJC

APPLICATIONS (OPTIMAL CONTROL), P. *Banach Center Publications, V. 1: Mathematical Control Theory.* Ed: Szymon Dolecki, Czesław Olech, Jerzy Zabczyk. PWN, 1976, 166 pp. Twenty-two research papers (one written in French, one in Russian, the others in English) appear in this proceedings of a working conference on the mathematical theory of optimal control, held at Zakopane, Poland, January 7-14, 1974. I-CH

APPLICATIONS (PHYSICS), S. *Science Brain-Twisters, Paradoxes, and Fallacies.* Christopher P. Jargocki. Scribner, 1976, viii + 183 pp, $9.95. Which is heavier: humid air or dry air? Most of the intriguing puzzles of this book are based on qualitative application of broad physical principles. Few are specifically mathematical. (Answer: dry air, since water vapor is lighter than the air it displaces.) PJC

APPLICATIONS (PHYSICS), S(16-17), L. *Conceptual Foundations of Quantum Mechanics, Second Edition.* Bernard d'Espagnat. Benjamin, 1976, xxxvii + 301 pp, $16.50 (P); $26.50. Assuming some familiarity with the experiments leading to the formulation of quantum mechanics and the basic ideas of the subject, the author reviews the elements of quantum mechanics and then discusses problems of interpretation for non-separability, measurement, and "knowledge and the physical world." References at the end of each chapter, and exercises. Good supplement to a mathematically rigorous course in quantum mechanics. PJM

APPLICATIONS (PHYSICS), T(16-17). *Physique Statistique.* Ya. P. Terletskii. Pr U Quebec, 1975, xii + 351 pp, (P). Translation of a Russian text on statistical physics. Includes classical statistical mechanics, theory of equilibria, and quantum statistical theory. PJM

APPLICATIONS (PHYSICS), P. *Quantum Mechanics, Determinism, Causality, and Particles.* Ed: M. Flato, *et al.* Math. Physics & Appl. Math., V. 1. Reidel, 1976, x + 252 pp, $29. A volume of essays in honor of Louis de Broglie. Sixteen papers including philosophical and logical issues raised by quantum mechanics, historical reviews, statistical aspects, particle aspects. JAS

*Reviewers Whose Initials Appear Above*

Richard J. Allen, St. Olaf; David F. Appleyard, Carleton; Paul J. Campbell, St. Olaf; Clifton E. Corzatt, St. Olaf; Loren Haskins, Carleton; Ih-Ching Hsu, St. Olaf; Paul S. Jorgensen, Carleton; Roger B. Kirchner, Carleton; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; James White, Carleton.

# NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

*Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least five months before publication can take place.*

## PERSONAL ITEMS

*Colorado State University*: Associate Professors H. H. Frisinger and R. P. Osborne have been promoted to Professors.

*Ohio University, Athens*: Associate Professor D. O. Norris has been promoted to Professor and appointed Chairman of the Department of Mathematics; Assistant Professor P. S. Malcom has been promoted to Associate Professor.

Professor Max Kramer, San José State University, retired on May 26, 1976, with the title of Professor Emeritus.

Dr. Maurice Nadler, Chairman of the Department of Mathematics at Pace University, retired on September 1, 1976, with the title of Professor Emeritus.

## 1977 AUSTRALIAN APPLIED MATHEMATICS CONFERENCE

This Conference will be held in the Southern Hemisphere's Summer at Terrigal, N.S.W., Australia, from February 6–February 9, 1977. Terrigal is a seaside town approximately 80 km north of Sydney, popular for its beaches and as a tourist resort.

The Conference is the annual meeting place of applied mathematicians in Australia. Papers to be presented will be on research work on applied or applicable mathematics, and on the teaching of applied mathematics. Overseas visitors will be particularly welcomed although, unfortunately, financial support will not be available.

Enquiries for further details should be directed to the Secretary, Dr. N. G. Barton, School of Mathematics, University of New South Wales, P.O. Box 1, Kensington, N.S.W. 2033, Australia.

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# MATHEMATICAL ASSOCIATION OF AMERICA

*Official Reports and Communications*

## THE FIFTY-SIXTH SUMMER MEETING OF THE ASSOCIATION

The Fifty-Sixth Summer Meeting of the Mathematical Association of America was held at the University of Toronto, Toronto, Ontario, Canada, from Thursday, August 26, to Saturday, August 28, 1976, in conjunction with the meetings of the American Mathematical Society, the Association for Women in Mathematics, Pi Mu Epsilon, and the Mathematicians Action Group. There were registered 1088 persons, including 661 members of the Association.

Sessions of the Association were held on Thursday and Friday mornings and on Saturday morning and afternoon. All sessions were held in Convocation Hall. Presiding officers at the three Earle Raymond Hedrick Lectures by Professor M. D. Davis were President H. O. Pollak, Professor J. R. Shoenfield, and Second Vice-President Betty J. Hinman; at the lecture by Professor N. G. Gunderson, Professor P. A. Lindstrom; at the lecture by Professor Cathleen S. Morawetz, Professor Deborah T. Haimo; at the lecture by Dr. L. A. Shepp, Professor R. A. Ross; at the lecture by Professor Samuel Karlin, Professor D. Z. Djokovic; at the lecture by Professor H. S. M. Coxeter, Professor F. A. Sherk; at the lecture by Professor H. S. Tropp, Professor K. O. May; and at the lecture by Professor N. S. Mendelsohn, Professor K. D. Fryer.

Also, Professor R. P. Boas presented his Retiring Presidential Address and President H. O. Pollak presided.

## FIRST SESSION OF THE ASSOCIATION

Welcome on behalf of the University by Professor G. de B. Robinson of the University of Toronto.

The Earle Raymond Hedrick Lectures: *Some Mathematical Applications of Logic*, Lecture I: *Unsolvable Problems* by Professor M. D. Davis, Courant Institute of Mathematical Sciences.

Studying effective computability leads to many unsolvable problems. That every recursive (effectively computable set) of positive integers is precisely the set of positive values assumed by some polynomial for positive integer arguments, yields the unsolvability of Hilbert's tenth problem and other interesting applications to number theory.

Nonstandard analysis, another application of logic, enlarges the "standard universe" of mathematical discourse to a nonstandard universe, containing infinitesimals and infinite quantities, (with the same properties as the standard universe insofar as these are appropriately expressible) vindicates Leibniz ideas about calculus, and furnishes mathematics with powerful new tools.

*Stamp Out Math Boredom*, by Professor N. G. Gunderson, University of Rochester.

Thematic philately is a vehicle for arousing interest in mathematics. Aspects of the history and applications of mathematics can be related to portraits of mathematicians, formulas, notation, uses, and even theorems which have appeared on stamps and in postal markings. Slides of items in such a collection were shown.

*Geometrical Optics and the Singing of Whales*, by Professor Cathleen S. Morawetz, Courant Institute of Mathematical Sciences, New York University.

Nature provided phenomena to prod the applied mathematician to understanding, the analyst to rigorizing and the teacher to explaining. An example is the miraculous, melodious, long distance singing of whales. It can be understood by the asymptotics of geometrical optics. Special cases can be rigorized; much can be explained at a simple level; open problems abound, and some of these were presented.

## SECOND SESSION OF THE ASSOCIATION

*Hedrick Lecture II*: *Diophantine Sets*, by Professor M. D. Davis.

*Business Meeting of the Association*: presentation of Lester R. Ford Awards, tribute to a distinguished Association member, and announcement of a gift to the Association.

*Retiring Presidential Address*: *Convergence Divergence and the Computer*, by Professor R. P. Boas, Northwestern University.

Looking at infinite series with an eye to what actual computation can say about the behavior of their partial sums, one sees tests for convergence and divergence in a different light from the usual textbook discussions.

## THIRD SESSION OF THE ASSOCIATION

*Hedrick Lecture III*: *Nonstandard Analysis*, by Professor M. D. Davis.

*Mathematics of Genetics and Evolution*, by Professor Samuel Karlin, Stanford University.

Evolution theory and mathematical genetics is the study of temporal and spatial changes of frequencies of types (e.g., genes, gametes, etc.) in populations subject to various ecological and genetic influences. Two main problem areas of mathematical genetics were dealt with in the lecture. The first class of models concerns multi gene interactions and measurement of gene frequency associations. The analysis of these models involves a mixture of finite dimensional non-linear analysis superimposed on certain algebraic structures (sums of Kronecker and tensor products, Schur products of matrices, some facets of group theory). The second class of models concerns selection migration interaction. A population is distributed over a range acted on by temporally and spatially varying selection forces. The objective is then to correlate the dynamic and equilibrium behavior of the observed frequency configurations and the underlying nature of the population structure and the selection regime.

*Geometrical Models*, by Professor H. S. M. Coxeter, University of Toronto.

Various models were exhibited and described: the five Platonic solids, the four star polyhedra of Kepler and Poinsot, most of the thirteen Archimedean solids and a few of the analogous star polyhedra, some solids with rhombic faces, Petrie's two regular "sponges" or skew polyhedra (all faces congruent and regular, all angles equal), sets of three mirrors hinged together to make polyhedral kaleidoscopes like those used in the film *Symmetries of the cube*. A set of four congruent spheres, all touching one another, motivated a plea for a model that has never yet been made: a sequence of six or seven balls whose radii are successive powers of $\frac{1}{2}(\sqrt{2}+1+\sqrt{2\sqrt{2}-1})+1.8832\ldots$, stuck together so that every *five* consecutive balls all touch one another.

## FOURTH SESSION OF THE ASSOCIATION

*The Role of Mathematics in the Design of an X-Ray Tomographic Human Body Scanner*, by Dr. L. A. Shepp, Bell Telephone Laboratories.

Passing a thin beam of X-rays along a line $L$ in one fixed plane section of an object, say a head, allows one to measure the line integral $P(L)$ of the density $f = f(x, y)$ of the object along $L$. Randon's theorem says that if $F$ is continuous with compact support then knowing $P(L)$ for all $L$ determines $f$ uniquely. Many algorithms for reconstructing $f$ from a finite number of line integrals $P(L)$ have been proposed. These are compared using simulated body sections formed from ellipses. The role of rigor and of mathematics in both the algorithms and the scanner design was discussed.

*J. C. Fields, The Fields Medal, and Mathematical Research*, by Professor H. S. Tropp, Humboldt State University.

In the President's Address to the Royal Canadian Institute, entitled "Universities, Research and Brain Waste" (Toronto, November 8, 1919), John Charles Fields made some strong statements regarding the appalling state of mathematical instruction and the lack of emphasis on research in universities in Canada and the U. S. in the early decades of the 20th Century. This talk examined the environment of university level mathematics in North America. It also dealt with the establishment of the Fields Medal by the International Mathematical Congress in 1932 in the context of Field's philosophy, personal experiences and the intellectual and political forces of his time.

*Modern Algebraic Approach to Combinatorial Designs*, by Professor N. S. Mendelsohn, University of Manitoba.

The last thirty years have seen spectacular advances in the theory of combinatorial design. This has been due almost entirely to the application of sophisticated algebraic tools. These include cyclotomy theory, rational equivalence of quadratic forms, universal algebra, especially the theory of varieties of algebra. Conversely, combinatorial constructs have proved useful in algebra and logic. In this talk a survey was made of some of the more important combinatorial designs, such as projective planes, orthogonal latin squares, room designs, pairwise balanced designs, block designs and $t$-designs and the algebraic connection was made.

## SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Faculty of Education Auditorium on Thursday and Friday at 7:00 P.M. The Film Program was devoted to films featuring mathematicians whose work has had great impact on American mathematics. All of the films shown are from the MATHEMATICS TODAY series produced by the MAA. The following films were shown on Thursday:

7:00–8:01 P.M. LET US TEACH GUESSING, George Polya
8:05–9:00 P.M. CHALLENGE IN THE CLASSROOM, R. L. Moore
9:05–9:48 P.M. GÖTTINGEN AND NEW YORK, Richard Courant

The following films were shown on Friday:

7:00–7:48 P.M. PITS, PEAKS, AND PASSES, Marston Morse
7:55–8:55 P.M. FIXED POINTS, Solomon Lefschetz
9:00–10:03 P.M. JOHN VON NEUMANN, a documentary (b&w)

## MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Wednesday at 9:00 A.M. in the Council Chamber in the Galbraith Building with thirty-nine members present. Among the items of business transacted were the following:

The Board elected Professor R. P. Boas at Northwestern University as Editor of the AMERICAN MATHEMATICAL MONTHLY for the period January 1, 1977 through December 31, 1981. Professor Boas was elected to fill the remainder of the term of Professor Alex Rosenberg who resigned as Editor.

The Board heard a report from Dr. Gordon Raisbeck, Chairman of the Committee on Corporate Members, on the Spring Meeting of the Northeastern Section held at the University of New Hampshire on June 18–19, 1976. The theme of this meeting was *Non-Academic Mathematics* and the program featured several industrially employed mathematicians.

A report was received from the Committee on High School Contests in which it was reported that certain precautions have been taken to ensure the validity of the Examination. It was also reported that the Annual High School Mathematics Examination will appear on the 1976–77 *Advisory List of National Contests and Activities.* The 1977 Examination will be given on March 8.

It was reported that the Fifth U.S.A. Mathematical Olympiad was held on May 4, 1975. Ninety-six students participated and seven of the eight highest scorers and their parents were honored at an Awards Ceremony at the National Academy of Science and a dinner at the State Department in Washington, D.C. on June 15, 1976. Professor A. M. Gleason of Harvard University delivered the address at the Awards Ceremony. A grant of $6500 from the International Business Machines Corporation to defray expenses of the Awards Ceremony was gratefully acknowledged.

A training session for the eight students selected to represent the United States in the International Mathematical Olympiad and sixteen other students was held at the U. S. Naval Academy during the period June 16–July 6, 1976, This session was greatly assisted by the members of the mathematics faculty at the Naval Academy and was funded by grants of $1800 from the Office of Naval Research and $5553 from the National Science Foundation.

It was reported that the United States entry in the International Mathematical Olympiad, held on July 12 and 13 in Lienz, Austria, finished in third place. The U.S. team was awarded one first prize and four second prizes in the competition. The first prize was awarded to Mark Kleiman and the second prizes were awarded to Randall Dougherty, Paul Herdeg, Dan Knierem, and Miller Puckette. The first five teams and their scores were announced to be U.S.S.R. 250, Great Britain 214, U.S.A. 188, Bulgaria 174, Austria 167. A grant of approximately $7500 from the Army Research Office to pay travel expenses to Austria was gratefully acknowledged.

The Board heard a report from Dr. Eileen Poiani, Director of Women and Mathematics: A Secondary School Lectureship Program Designed to Interest More Women in Mathematics, on the operation of WAM during the 1975–76 academic year and the plans for the 1976–77 academic year. The Board voted to accept a grant of $7500 from the International Business Machines Corporation to conduct the Program during 1976–77.

The Board voted to approve revisions of the By-Laws of the Michigan and Texas Sections as recommended by the Committee on Sections.

The Committee on Visiting Lecturers and Consultants reported that it had added a number of outstanding mathematical lecturers who are employed by government or industry. These speakers are located primarily in the Northeast, but efforts are underway to locate additional nonacademic mathematicians who are willing to serve as lecturers in other parts of the country.

The Board endorsed an action voted by the Council of the American Mathematical Society with regard to the periodical EMPLOYMENT INFORMATION FOR MATHEMATICIANS. This action is designed to ensure that prospective employees are informed of the status of the non-discrimination statement requested of foreign employers which list positions in EIM.

The Board approved the following schedule of meetings:

Fifty-ninth Summer Meeting: Virginia Polytechnic Institute and State University, August 21–23, 1979.

Sixty-first Annual Meeting: Atlanta, Georgia, January 6–8, 1978.

Sixty-third Annual Meeting: San Antonio, Texas, January 5–7, 1980.

The meeting in Atlanta had previously been approved, but with different dates. Other meetings already approved are:

Sixtieth Annual Meeting: St. Louis, Missouri, January 29–31, 1977.

Sixty-second Annual Meeting: Milwaukee, Wisconsin, January 13–15, 1979.

Sixty-fourth Annual Meeting: San Francisco, California, January 10–12, 1981.

Fifty-seventh Summer Meeting: University of Washington, August 14–16, 1977.

The *ad hoc* Committee to Review the Guidelines for Evaluation submitted a revision of the guidelines approved by the Board at its meeting on August 27, 1972. The intended use of such guidelines is that they be forwarded to each regional accreditation commission for possible use to help institutions correct weaknesses in their mathematics programs. The revised guidelines were approved by the Board.

The Executive Director reported that the membership of the Association as of July 1, 1976 was 17,174 compared with 17,368 a year earlier. As of the same date, there were 407 academic members of the Association compared with 387 a year earlier.

## BUSINESS MEETING OF THE ASSOCIATION

A business meeting was held at 10:00 A.M. on Friday in Convocation Hall with President Pollak presiding.

The eleventh set of Lester R. Ford Awards was presented by President Pollak to authors of expository articles published in the MONTHLY and MATHEMATICS MAGAZINE during 1975. These Awards, in the amount of $100 each, were presented for six articles (for further details on these Awards, see the August-September issue of the

MONTHLY, page 587). Professors Branko Grünbaum, D. W. McLaughlin and J. J. Price were present to accept the Awards.

President Pollak next announced to the meeting that the Board of Governors of the Association had voted to award an honorary life membership to Mr. Martin Gardner and read the following resolution which had been approved by the Board at its meeting the previous day:

"By this resolution, the Board of Governors of the Mathematical Association of America at its meeting on August 25, 1976, in Toronto expresses its gratitude to Martin Gardner for the substantial contributions he has made to the public appreciation of mathematics by his superb exposition in his texts and his column "Mathematical Games" in the SCIENTIFIC AMERICAN. The enjoyment and humor which he conveys have been an inspiration to many and are a model for all.

The Board knows many members of the Association who have expressed admiration of Martin Gardner and consider him the main force which has brought mathematics to the attention of the public.

In deep appreciation of these contributions, the Board herewith presents Martin Gardner with an honorary life membershp in the Association."

Martin Gardner was unable to be present at the meeting. The plaque and a copy of the resolution were forwarded to him by the Secretary. When notified of the fact that he was to be presented an honorary life membership in the Association, Martin Gardner had written to the Secretary as follows:

"It goes without saying that I consider it an enormous honor to have been given a life membership in the Mathematical Association. At the same time I am somewhat embarrassed. The reason is that I consider myself more a journalist and popularizer of mathematics than a genuine mathematician. My knowledge of math is extremely limited, except perhaps for the field of recreational mathematics in which I specialize, and which fortunately is low-level math.

Let me hasten to add, lest this be taken as false modesty, that I believe it is an asset for a mathematical journalist not to know *too much* about math. If it is a hard struggle to understand something, a journalist knows where the traps are and can better avoid them when he writes for laymen.

Gilbert Chesterton once said that anything worth doing is worth doing badly. It is true that I never took a college course in math — my major was philosophy — but math was my favorite subject in high school, and I have never lost that enthusiasm and love for math that is characteristic of the amateur.

I believe that a basic knowledge of mathematics is essential for a liberal education, and of course everybody knows that mathematics is indispensable for the growth of science and technology. If the Association feels that my scribblings have contributed to a better public awareness of the beauty and usefulness of math, then I am pleased beyond measure. Thank you all very very much."

This message was read at the Business Meeting by the Secretary.

President Pollak announced to the Business Meeting that Carol Ryrie Brink has given a gift of $50,000 to the Association in memory of her late husband, Professor Raymond W. Brink. This gift will be used to establish a new series of publications, the *Raymond W. Brink Selected Mathematical Papers.* President Pollak called Mrs. Brink to the stage and read the following resolution:

"By this resolution, the Board of Governors of the Mathematical Association of America, at its meeting on August 25, 1976, in Toronto, expresses its deepest appreciation to Mrs. Carol Ryrie Brink for her generous gift to the Association in memory of her husband, Professor Raymond W. Brink. This gift has enabled the Association to establish a new regular series of publications, *The Raymond W. Brink Selected Mathematical Papers.* Through this series of publications the Association will establish a lasting reminder to Professor Brink who served the Association in many significant ways including terms as President (1941–42), as Vice-President (1940–41), and as Governor (1934–39) and (1943–48).

The Board is particularly pleased that Mrs. Brink has made this gift to the Association at the present time when our organization has become dependent on support from private sources. The Board is, therefore, not only grateful to Mrs. Brink for her generosity but also for her wish that the gift will stimulate others to contribute to the projects of the Association.

The Board pledges to use Mrs. Brink's gift to make the *Raymond W. Brink Selected Mathematical Papers* an outstanding series of which Professor Brink would have been especially proud."

Mrs. Brink was then presented with a printed copy of this resolution. In accepting the copy of the resolution, Mrs. Brink expressed her thanks to the Association for honoring the memory of Professor Brink. She recalled that the MAA had been a great interest of his and that Professor Brink had been particularly interested in mathematical exposition. Thus, the series, *The Raymond W. Brink Selected Mathematical Papers,* seems an especially appropriate memorial to Professor Brink.

Mrs. Brink was accompanied at the Business Meeting by her son, David Brink, who was introduced by President Pollak.

President Pollak then called upon Professor E. F. Beckenbach to report on the volumes now in preparation for *The Raymond W. Brink Selected Mathematical Papers.* These include selected mathematical papers in algebra, analysis, calculus, geometry, and precalculus. The calculus volume is now in print and will be volume 2 in the series. The precalculus and algebra volumes are now nearing publication and will be volumes 1 and 3 in the series.

The Secretary reported on some of the actions taken by the Board of Governors. In particular, he explained the structure of the Olympiad activities and the generous support for these activities made available by the Army Research Office, International Business Machines Corporation, the National Science Foundation, the Office of Naval Research, and the U. S. Naval Academy. The Secretary announced that the 1977 International Mathematical Olympiad will be held in Yugoslavia.

The Secretary paid tribute to those responsible for the delightful meeting at the University of Toronto. Clearly, many persons must be involved to look after the many details involved in arranging so complex a meeting. The Secretary thanked all of the members of the Committee on Arrangements and the Program Committee. He singled out, in particular, the Chairmen of these Committees, Professors P. G. Rooney and G. de B. Robinson. Ms. Annette Sunter was also thanked for her hard work on behalf of the Association.

## MEETING OF SECTION OFFICERS

The meeting of representatives of the Sections was held on Thursday, August 26, 1976, at 7:00 P.M. in Galbraith Council Chamber. Dean L. H. Lange, Chairman of the Committee on Sections, presided. Fifty-eight persons were present, representing twenty-six of the twenty-nine Sections.

President H. O. Pollak welcomed the Section Officers to the meeting. He said that the MAA officers learn a great deal from the meetings of the Section Officers and expressed his hope that any concerns of the Sections would be made known at this meeting. President Pollak noted that, due to a trip to China the previous spring, he would not be able to visit all of the Sections during his tenure as President. However, he will continue visiting Sections after his term as President. He said that he has enjoyed his visits to the Section meetings and found these visits a good way to keep in touch with the activities of the Sections.

President-Elect H. L. Alder emphasized that the Sections are the lifeblood of the Association. He expressed the hope that, at this meeting, the Section Officers would get many new ideas for increasing the effectiveness and vitality of their Sections and, what is more important, would be able to take the necessary steps to implement them in their Sections wherever appropriate. He invited everyone to let the national officers have any ideas for actions they can take to help the Sections in any way whatsoever.

Dean Lange also called upon the Secretary and Dr. A. B. Willcox, Executive Director, to welcome the representatives of the Sections. He then introduced Professor Leonard Gillman, Treasurer, and Professor Betty J. Hinman, Second Vice-President. There next followed a roll call of the official representatives of the Sections.

Dean L. H. Lange reported that, with the addition of the Intermountain Section, there are now twenty-nine Sections. He said that the chief interest of the Committee on Sections was to find good ideas to disseminate among the Sections. In short, the goal is to keep Sections not only alive but lively.

Dean Lange continued as follows: "This Committee plans these meetings to provide a prime vehicle for the exchange of ideas about the activities and responsibilities of the twenty-nine Sections, and to inform the membership of the MAA, via its Section Officers, of these ideas as well as key actions of the national officers and committees. The conscientious follow-up work by the Section Officers is of very great importance to the welfare of the Association in all that it tries to accomplish, for it is heavily dependent upon the efforts of its individual members working through those local section organizations. This Committee is also principally engaged in the supervision of changes in Section By-Laws presented to the Board of Governors for approval."

Dean Lange called upon President-Elect Alder to introduce Professor C. R. Deeter, Governor of the Texas Section. Professor Alder noted that he had visited the meeting of the Texas Section in the spring of 1976 and had been amazed at the excellent organization of that Section.

Professor Deeter began his remarks by crediting Professor J. C. Bradford, Secretary of the Texas Section, for much of the success of the programs of the Texas Section. He then reported that some of the things which help the Texas Section to successfully work toward carrying out the purposes of the Association are:

1. The people — both individuals and departments
2. Newsletter
3. Cooperation with the Texas Association of Academic Administrators in the Mathematical Sciences
4. Support of compilation and publication of "College Mathematics in Texas 1974–75"
5. Continuing efforts to involve TYC faculties
6. By-Laws which fit the needs of the Section
7. Routine planning and execution of many activities, without forcing routine results
8. Provision of Executive Committee with adequate representation of all constituencies of the Section
9. Responsible fiscal management
10. Miscellaneous smaller factors which, in total, have considerable effect.

The publication, *College Mathematics in Texas 1974–75,* is a compilation of all the mathematics courses offered by colleges in Texas. It lists the course description, number of hours, and textbook in use. This publication is available from Professor G. E. Mattingly, Department of Mathematics, Sam Houston State University, Huntsville, Texas, 77340. The cost is $8.00.

Professor L. A. Guillou, North Central Section and member of the Committee on Sections, distributed the recently completed *Guide for Host Institutions for Preparation of North Central Section Meetings.* It was distributed to Section Officers for the purpose of:

1. Obtaining suggestions for improvement of the document for use by the North Central Section
2. Providing a model for other Sections to develop a "Guide" suited to their needs
3. Determining whether it is desirable for some committee such as the Committee on Sections to prepare a "Guide" for all the Sections.

Professor C. A. Lathan, Editor of THE TWO-YEAR COLLEGE MATHEMATICS JOURNAL, reported that the TYCMJ is now an official journal of the Association. He noted that one Section has already invited a representative of the TYCMJ to attend its meeting to speak on the journal. He said the TYCMJ has articles, a problems section, a column by Professor George Polya, a section to report computer uses, a column on two-year college news, and a book review section. He said the journal should assist Sections in recruiting two-year college faculty.

Professor M. W. Pownall, Chairman of the Committee on Visiting Lecturers and Consultants, asked Section Officers to report at their meetings the desirability of combining visits by speakers. The Committee on Visiting Lecturers and Consultants attempts to promote such arrangements both because of the economies and because this promotes closer relationships between the schools involved. Professor Pownall also reported that a number of persons employed by government or industry have been added to the Program of Visiting Lecturers and Consultants.

Professor Eileen L. Poiani, Director of the lectureship program *Women and Mathematics* (WAM), reported that WAM is a program which sends female scientists to high schools to speak to tenth grade girls, their teachers, counselors and, whenever possible, their parents. The program is designed to encourage more girls to continue academic training in mathematics. During the 1975–76 academic year, thirty visits were scheduled in the New York/New Jersey area and twenty-seven in the San Francisco Bay area. Professor Poiani also reported that IBM has made a grant of $7500 for operation of WAM in the New York/New Jersey, Chicago and San Francisco Bay areas during the 1976–77 academic year. Professor Poiani will serve as Director for WAM during the 1976–77 academic year, and the Regional Coordinators will be:

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|---|---|
| New York/New Jersey | Susan Devlin and Mary J. Cross of Bell Telephone Laboratories |
| Chicago | Mary L. Boas of DePaul University |
| San Francisco | Jean Pedersen of University of Santa Clara |

Professor Jean Pedersen, Coordinator for WAM in the San Francisco Bay area during 1975–76, reported that 2500 girls were visited by WAM speakers in that area. A typical visit consisted of a mathematics lecture followed by an informal conference with students about individual career interests. Boys often were present at the lectures.

The Bay area WAM program has received applications for repeat visits during the coming year as well as numerous requests from lower level schools, which fall outside the program. Honoring these requests, however, would advance the goals of the WAM program. It was suggested that the MAA should expand the WAM program or initiate separate programs to accommodate these requests.

Dr. Gordon Raisbeck, Chairman of the Committee on Corporate Members, said that far more pupils who study mathematics in junior college, college and university undergraduate classes spend their careers in industry and government than in faculties of mathematics. Even though the MAA devotes its attention primarily to what teachers of college mathematics do, it should be very sensitive to what industrial and government users of mathematics do and how they can be effectively educated and trained. Yet government and industrial mathematicians find little evidence of such sensitivity in the everyday affairs, meetings and publications of the MAA and the behavior of many academic mathematicians. The Sections of the MAA can foster mutual understanding, increase the interdependence between academic and non-academic mathematicians, and foster the recruiting of MAA Corporate Members by involving industrial and government mathematicians in MAA activities, especially those in which academic mathematicians also participate: addressing meetings, serving as appointed and elected officers, and others. Only then are efforts to recruit more Corporate Members likely to be successful.

Professor D. B. Coleman asked what the arguments were for an industrial mathematician joining the Association. President Pollak listed a number of reasons including the excellent exposition in the MONTHLY, the industrial mathematician's desire to get the information on the training of students back into the educational community, and enjoyment of mathematics meetings. He said that all mathematicians like to be a mathematician in the fullest sense of the word and *the* organization that has all of the professional activities of a mathematician at its heart is the MAA.

Professor H. M. Cox, Executive Director of the Annual High School Mathematics Examination, addressed those present as follows: "I express appreciation to you the Section Officers of MAA for your selection or confirmation of the selection of Regional Contest chairmen. These are the men and women who, along with the high school mathematics teachers, continue to make the Annual High School Mathematics Examination a success. The 1977 Examination will be given on Tuesday, March 8.

It is also my privilege and my pleasure to present to you my successor as Executive Director of the High School Mathematics Examination, my colleague on the faculty of the University of Nebraska, Professor Walter Mientka."

Professor Ronald M. Davis, Chairman of the Maryland-District of Columbia-Virginia Section, reported on the Summer Seminar sponsored jointly by the Md.-D.C.-Va. Section and Salisbury State College. The seminar was titled *Models — Deterministic and Stochastic* and was held on June 7–11, 1976, at Salisbury State College, Salisbury, Maryland.

Professor Davis noted that 40% of the participants in the Summer Seminar were from Sections other than Md.-D.C.-Va. He said that the Section hopes to continue such seminars on an annual or biannual basis.

Professor Boas, Editor-Elect of the MONTHLY, asked for suggestions to improve the usefulness of that journal. A number of suggestions, including reinstatement of the biographical sketches of the authors, reducing lead time for announcements of Section activities, and making the Personal Items section more complete were made.

The Secretary said that Dr. Raoul Hailpern should be sent information on the meetings of the Sections for inclusion in the MONTHLY. Such information should include the date and place of the meeting and the date, if any, by which contributed papers are to be received.

Professor J. A. Seebach, Co-Editor of MATHEMATICS MAGAZINE, reported on the recent changes in the format of the journal and editorial policy. Professor Seebach asked that he be sent the programs of the meetings of the Sections and that persons be aware of the desire of the Editors of MATHEMATICS MAGAZINE to publish well-written manuscripts on all aspects of mathematics.

Dr. A. B. Willcox, Executive Director, circulated a sheet listing the benefits of Academic Membership in the MAA. An Academic Member is entitled to:

1. A subscription to the AMERICAN MATHEMATICAL MONTHLY
2. A subscription to MATHEMATICS MAGAZINE
3. A subscription to the TWO-YEAR COLLEGE MATHEMATICS JOURNAL
4. A copy of the COMBINED MEMBERSHIP LIST
5. The privilege of nominating one Institutional Nominee
6. The privilege of nominating additional students as Institutional Nominees at $10 per nominee
7. The privilege of purchasing MAA texts for presentation as student awards at members' prices

The Executive Director noted that the value of items 1–5 is $77 and that an Academic Member pays only $60 annually.

In answer to a question of whether there should be kickback to Sections on dues paid by individual members, the Executive Director noted that the following counter arguments have been advanced:

1. Many MAA members who reside in a Section do not participate in the activities of the Section. That is, membership in a Section is involuntary.
2. It would be difficult to get Sections to agree on a uniform dues structure. Some Sections do not collect dues.
3. This would require a raise in the dues.

It was also suggested that the MAA might increase institutional memberships by, say, $5 and pay this amount to the Section where the institutional member is located.

There followed a discussion of the finances of the Sections. It was said by several persons that local collection of Section dues (registration fees) at the local, rather than national, level was the most practical. It was also reported that, in addition to the annual payment to Sections, the MAA conducts a program of officers visits to Sections and a program of special grants to Sections. These grants, in the amount of $300, are made to Sections to finance special projects and are awarded on a competitive basis. The Executive Director encouraged Section Officers to apply for these grants.

Dr. Willcox noted that Sections are eligible for a non-profit, third class mailing permit and that first class postage can be reduced if the mailings are sorted by zip codes. Finally, the Executive Director noted that certain Section Secretaries save on postage by bundling all announcements to members located at the same institution.

Dr. Willcox said that visits to Section meetings by officers of the Association can be arranged by writing to Secretary Roselle.

Professor P. A. Lindstrom, Genesee Community College, Batavia, New York, gave a short report of a survey that he had conducted. Through the help and cooperation of the MAA Washington Office, a questionnaire dealing with the relationship of the two-year colleges and the Sections of the MAA was sent to each Section Officer in April, 1975. Responses were received from at least one officer from twenty-two of the Sections. Some general results of the survey show that some Sections are receiving the support of its two-year college members along with active participation of these people, whereas other Sections are not receiving support of these people or are doing very little to encourage the support and participation of its two-year college members.

There followed a discussion of how the MAA might better serve the two-year community. Suggestions included maintenance of a list of two-year faculty and two-year institutions within each Section. It was also reported by Professor D. J. Albers that the next CBMS survey will report that there has been a considerable recent drop in the percentage of two-year institutions whose programs are coordinated with those of four-year institutions.

Professor Albers suggested that the MAA should encourage individual four-year institutions to invite faculty from two-year colleges to visit. This promotes cooperation and coordination of programs between the schools involved.

It was suggested that MAA Sections might increase articulation among two-year and four-year institutions by holding meetings in conjunction with those attended by two-year faculty. Also mentioned was the possibility of fostering relationships similar to that the Texas Section has with the Texas Association of Academic Administrators.

Professor J. A. Schumaker, Governor of the Illinois Section, said that two-year faculty cooperate with MAA members in revising the curriculum guide developed by that Section and the Illinois Junior College Board. Professor E. A. Tanis, Chairman of the Michigan Section, noted that the Michigan Section elects a two-year faculty member as a Vice-President and that this person serves on the Program Committee. Similar procedures are used in other Sections, including the Kansas Section.

Dean Lange next led a discussion about the problems of communications among Sections. He said that he felt meetings of the Section Officers were an excellent way to promote the exchange of ideas among the Sections.

Professor E. A. Klotz, Philadelphia Section, asked that there be more printed material circulated in advance of meetings of Section Officers. A revision of the Guidelines for Section Officers would be helpful, he said.

Professor D. D. Bushnell, Rocky Mountain Section, suggested sending a letter to each newly elected Section Officer giving a listing of the publications and materials available from the MAA. Dean Lange concurred and also suggested that the Section Officers should meet annually instead of each year and one-half. He noted that the present schedule makes it possible for Section Chairmen to complete their terms of office without being eligible to attend a meeting of Section Officers. Dean Lange said a request for an annual meeting is being forwarded to the officers of the Association.

The final topic discussed at this meeting was Section newsletters. Each Section which has a newsletter was requested to send a copy of its newsletter to the Officers of each of the other Sections. The Executive Director said he would distribute Section newsletters he receives to all of the Section Officers and agreed to send peel-off labels for all Section Officers to each Section Secretary.

The Executive Director asked the Section Officers to inform the Washington Office of all changes in Section Officers.

Professor G. R. Blakley, Texas Section, moved for adjournment.

## DINNER IN HONOR OF THIRTY-YEAR MEMBERS

On Friday, the Association held a dinner in honor of its thirty-year members at 6:30 P.M. in the New College Cafeteria. The dinner was planned as a pleasant occasion to recall the services of the Association's senior members and spouses. It provided the opportunity to renew old friendships and to inform members of the current activities and future plans of the Association. Dr. C. V. Newsom served as toastmaster and President H. O. Pollak brought greetings to those present. The speakers were Professor Burton W. Jones, "Needs of the Future"; Professor G. B. Price, "The MAA in 1976"; and President-Elect H. L. Adler, "Plans for the Future".

Following the dinner, Professor G. B. Price was presented the Mu Alpha Theta Service Award by Professor R. L. Wilson.

Eighty persons attended the dinner.

## MEETINGS OF OTHER ORGANIZATIONS

The Pi Mu Epsilon Fraternity held its sessions for contributed papers on Wednesday at 2:00 P.M. and on Thursday at 3:00 P.M. in 1072 Sidney Smith Hall. The Pi Mu Epsilon Governing Council Luncheon was held at 12:00 noon on Wednesday and the banquet was held the same day at 5:30 P.M. On Thursday, a Dutch treat breakfast was held at 8:00 A.M. All of these functions were held in Wilson Hall Cafeteria in New College. Following the banquet Wednesday, the Pi Mu Epsilon's J. S. Frame Lecture *The Pappus Configuration and Its Groups*, was delivered by Professor H. S. M. Coxeter at 7:00 P.M. in 2135 Sidney Smith Hall.

The Association for Women in Mathematics sponsored a panel discussion, *History of Women in Mathematics*, on Thursday, August 26, at 12:00 noon in 2135 Sidney Smith Hall. AWM President Lenore Blum of Mills College served as moderator. Professor Lida K. Barrett of the University of Tennessee presented an overview. Other speakers were Professor Mary W. Gray of American University who spoke on Sophie Germain; Professsor Linda Keen of the Graduate School and University Center (CUNY) who spoke on Sonya Kovalevski; Professor Emilana Noether of the University of Connecticut who spoke on *Emmy Noether — Twentieth Century Mathematician and Woman*; Professor Martha K. Smith of the University of Texas at Austin who spoke on *Emmy Noether — Her Work and Influence.* The AWM business meeting followed the panel discussion. The same organization held an open Executive Committee meeting at 5:30 P.M. on Wednesday, August 25, in 2129 Sidney Smith Hall.

The Council of the Conference Board of the Mathematical Sciences met at 2:30 P.M. on Friday, August 27 in the Council Chamber in the Galbraith Building.

The Mathematicians Action Group held a panel discussion on *Other Mathematical Models in Economics* at 8:00 P.M. on Thursday, August 26 in Room 2135 Sidney Smith Hall. The MAG Business Meeting was held on Tuesday, August 24 at 7:00 P.M. in 2135 Sidney Smith Hall.

The American Mathematical Society held sessions from Tuesday, August 24 through Friday, August 27. One set of Colloquium Lectures, consisting of four one-hour talks, was presented by Professor J. K. Moser of the Courant Institute of Mathematical Sciences, New York University. The title of these lectures was *Recent Progress in Dynamical Systems.* The first lecture was presented on Tuesday at 10:00 A.M., the second on Wednesday at 9:00 A.M., and the third and fourth on Thursday and Friday at 1:30 P.M. Each of these lectures was delivered in Convocation Hall.

The AMS Committee on Employment and Educational Policy sponsored an open meeting on the job market on Tuesday, August 24, at 8:30 P.M. in Convocation Hall. The panel included Professors C. W. Curtis, J. W. Jewett, and J. A. Nohel; Professor W. H. Fleming acted as moderator. Professor Jewett reported on the current CBMS survey. An open discussion followed the panel discussion.

Invited addresses were given as follows:

*Tuesday, August 24*

1:30–2:30 P.M. Professor Eugenio Calabi, University of Pennsylvania, *Nearly Flat Triangulations of Riemannian Manifolds*

2:45–3:45 P.M. Professor Per Enflo, University of Stockholm, *On the Invariant Subspace Problem for Banach Spaces*

4:00–5:00 P.M. Dr. L. A. Shepp, Bell Telephone Laboratories, *Optimal Reconstruction of a Function from Its Projections.*

*Thursday, August 26*

2:45–3:45 P.M. Professor Michael Aschbacher, California Institute of Technology, *Determining the Finite Simple Groups*

4:00–5:00 P.M. Professor Edward Nelson, Princeton University, *Principles and Practice of Non-Standard Analysis*

5:15–6:15 P.M. Professor Marian B. Pour-El, University of Minnesota, Minneapolis, *New Directions in Computability Theory.*

## ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements consisted of P. G. Rooney, Chairman; D. F. Andrews, E. J. Barbeau, Jr., H. Botta, T. H. Callahan, W. H. Gottschalk, Lee Lorch, S. J. Pierce, D. P. Roselle, R. A. Ross, J. R. Vanstone, and G. L. Walker.

Registration headquarters were located in the Entrance Lobby of Sidney Smith Hall. Dormitory rooms and dining facilities were provided by the University of Toronto. The Mathematical Sciences Employment Register was conducted on a limited basis with facilities provided for applicants and employers to display résumés and job listings. Books and educational media were displayed in Room 2096 of Sidney Smith Hall on Tuesday through Friday.

A beer party was held in Wetmore Hall Cafeteria at 8:00 P.M. on Wednesday, August 25. A variety of entertainment events were available in the Toronto area during the meeting.

DAVID P. ROSELLE, *Secretary*

## ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting at the University of Toronto, Toronto, Canada on August 25, 1976, elected to membership the twenty-eighth set of applicants for academic membership (for election of the other twenty-seven sets, see the April, 1976 issue of the MONTHLY, page 315 and the references cited there). Approval of election was given to the following applicants for academic membership:

De Paul University
Inter American University
Lake Superior State College
Mount Vernon Nazarene College
Southwest Missouri State University
SUNY at Oneonta
University of Northern Colorado
University of Texas-San Antonio

## MAY MEETING OF THE WISCONSIN SECTION

The Wisconsin Section of the MAA held its forty-fourth annual Spring Meeting at Beloit College, Beloit, Wisconsin, on May 7 and at UW Janesville–Rock County Center, Janesville, Wisconsin, on May 8, 1976. Ninety-four attended the meetings.

The following invited addresses were presented:

50 *Years Ago — Mathematical Personalities in the Middle West, Especially in Wisconsin*, by Emeritus Dean M. H. Ingraham, University of Wisconsin.

*What is an Automoton*, by F. E. Hohn, University of Illinois.

*Prime Generating Functions and Congruences*, by H. L. Alder, University of California at Davis.

*Geometry is the Dual of Algebra*, by Daniel Zelinsky, Northwestern University.

The following contributed papers were presented:

*Generalized Pythagorean triplets*, by Larry Davis, UW Whitewater.

*Scheduling triangular and dual meets*, by John Omen and Norbert Kuenzi, UW Whitewater.

*Mathematical models for regulating wild things*, by G. G. Walter, UW Milwaukee.

*A design of a diagnostic test–retest paradigm leading to placement in calculus*, by Nicholas Bystrom, Northland College.

*Iterative methods vs. direct methods for the solution of linear systems of equations*, by Y. J. Kim, UW Whitewater.

*Periodic points of fold functions: the approach to chaos*, by P. D. Straffin, Jr., Beloit College.

*Big problems are getting smaller: some mathematical dichotomies*, by Al Ristow, UW Milwaukee.

*Found: geometric art; may be claimed in any classroom*, by J. L. Teeters, UW Eau Claire.

*MAA contest problems and solutions*, by R. D. Wagner.

*A trigonometric proof of Ptolemy's theorem*, by Cindy Visgar, Sophomore, UW Janesville.

*Connectedness and completeness of the real line*, by Laura McCarten, Sophomore, UW Janesville.

*All topological spaces are nice*, by Douglas Harris, Marquette University.

*That intermediate algebra — precalculus sequence, what are we to do with it?*, by J. D. Wine, UW La Crosse.

*Stalking the odd triperfect*, by Rudolph Najar, UW Whitewater.

*A report on post graduation work environment and its curricular implications*, by Tom Renfrow, Beloit College.

*Multiperiodic functions and some of their applications*, by Nancy Fincke, UW Stevens Point.

*Randomized response: are revised models better*, by D. M. O'Brien, UW La Crosse.

The program included mathematical films presented by R. G. Long, Lawrence University.

Officers for the next year are: F. G. Florey, UW Superior, Chairman; G. B. Klatt, UW Whitewater, Vice-Chairman; and Tom Renfrow, Secretary-Treasurer.

R. D. WAGNER, *Secretary-Treasurer*

## ACKNOWLEDGMENT

The editors wish to thank the following mathematicians who have refereed manuscripts for Volume 83: H. L. Abbott, Shreeram Abhyankar, Janos D. Aczel, Roy Adler, Patrick Ahern, R. D.

Anderson, George E. Andrews, Tom M. Apostol, Richard Askey, Krishna Athreya, R. G. Ayoub, George Bachman, W. Baker, T. F. Banchoff, David Barnette, R. G. Bartle, K. Jon Barwise, Anatole Beck, Edwin F. Beckenbach, Barry Belkin, Grahame Bennett, Sterling Berberian, Elwyn R. Berlekamp, Bruce Berndt, Richard Bieberich, R. H. Bing, B. J. Birch, Errett A. Bishop, Michael N. Bleicher, R. Blumenthal, Ralph P. Boas, William M. Boothby, C. B. Boyer, Fred G. Brauer, Leon Breiman, Alfred Brousseau, Ezra A. Brown, W. Dale Brownawell, Thomas H. Brylawski, W. Buchholz, R. C. Buck, Peter Bullen, David Burton, James Cannon, D. G. Cantor, Leonard Carlitz, B. C. Carlson, Jack G. Ceder, Gulbank Chakerian, Joshua Chover, Kai Lai Chung, Earl Coddington, Charles Conley, R. L. Colley, H. S. M. Coxeter, Michael Crandall, Donald W. Crowe, M. Davis, Louis J. DeLuca, John de Pillis, Allen Devinatz, Harold Diamond, Joaquin B. Diaz, Raymond F. Dickman, R. C. Di Prima, Joseph L. Doob, Lester Dubins, Ed L. Dubinsky, Peter L. Duren, Verena Dyson, Michael Edelstein, R. Eggleston, R. B. Eggleton, David Eisenbud, P. C. Eklof, Klaus Eldridge, Ronald J. Evans, Howard W. Eves, Edward R. Fadell, Herbert Federer, Walter Feit, Tom Foregger, Frank Forelli, J. B. Fraleigh, J. Sutherland Frame, Stanley P. Franklin, Gerald Freilich, A. Friedli, David Gale, Paul R. Garabedian, A. M. Garsia, George Gasper, Jr., M. A. Geraghty, M. Gerstell, R. K. Getoor, Peter M. Gibson, Robert P. Gilbert, Irving L. Glicksberg, Casper Goffman, S. Goldberg, J. Goldman, L. J. Goldstein, Michael Golomb, Henry W. Gould, Judith V. Grabiner, J. L. Gross, Emil Grosswald, Branko Grünbaum, Heinrich W. Guggenheimer, Horishi Gunji, Franklin Haimo, Paul R. Halmos, David Hayes, Simon Hellerstein, Melvin Henriksen, Israel N. Herstein, Edwin Hewitt, D. Higgs, Donald Higman, R. V. Hogg, W. Charles Holland, John M. Horvath, Anne Hudson, J. E. Humphreys, H. J. Iglarsh, I. Martin Isaacs, John R. Isbell, Robert C. James, James P. Jans, H. Jensen, G. K. Kalish, William M. Kantor, Irving Kaplansky, Robert H. Kasriel, Nicholas D. Kazarinoff, J. B. Keller, J. L. Kelly, Murray Klamkin, Victor L. Klee, Jr., Erwin Kleinfeld, Donald E. Knuth, Bernard Kolman, Elaine H. Koppelman, Frank Kosier, James D. Keulbs, H. W. Kuhn, Kenneth Kunen, R. A. Kunze, Thomas G. Kurtz, Jeffrey D. Lagarias, J. Lamperti, Serge Lang, J. D. Lawson, Solomon Leader, Walter Leighton, William LeVeque, Jacob Levin, Larry Levy, D. J. Lewis, S. P. Lipshitz, C. J. Long, L. H. Loomis, W. F. Lucas, D. Luckham, Norman Y. Luther, David Lutzer, Wilhelmus Luxemburg, W. MacIntyre, Saunders Mac Lane, Bernie Madison, A. K. Magid, A. B. Manaster, Marvin Marcus, Michael Marcus, Morris Marden, Joseph M. Martin, S. B. Maurer, Kenneth May, L. F. McAuley, E. A. Michael, John W. Milnor, George Minty, Cleve Moler, R. W. Moller, J. W. Moon, Charles B. Morrey, Jr., Francis Murray, Robert H. McDowell, Daniel R. McMillan, Edward H. McShane, Alex Nagel, Morris Newman, Peter Ney, Keith Nicholson, Harald Niederreiter, J. C. C. Nitsche, Ivan Niven, Stanley Ogilvy, Michael Olinick, Ingram Olkin, B. O'Neill, G. Owen, J. C. Oxtoby, B. N. Parlett, Seymour V. Parter, Donald Passman, Martin Pettet, D. T. Piele, Mark Pinsky, Vera S. Pless, William F. Pohn, Barth Pollak, Carl Pomerance, Murray H. Protter, Hilary Putnam, R. M. Redheffer, Edgar Reich, S. Reiter, Daniel G. Rider, Joel Robbin, F. S. Roberts, Hartley Rogers, Steven Robinson, Alex Rosenberg, Murray Rosenblatt, Azriel Rosenfeld, Barkley Rosser, G. L. Rothschild, Joseph Rotman, Halsey L. Royden, Mary Ellen Rudin, Walter Rudin, H. J. Ryser, Edward B. Saff, G. T. Salee, Leo Sario, Richard Savage, Helmut H. Schaefer, Hans Schneider, I. J. Schoenberg, Dan R. Scholz, Seymour Schuster, Abraham Schwartz, Richard Scoville, J. A. Seebach, M. E. Shanks, V. L. Shapiro, Daniel Shea, R. Sibson, N. J. A. Sloane, John R. Smart, David A. Smith, Joel Smoller, Ernst Snapper, Laurie J. Snell, R. I. Soare, Louis Solomon, E. H. Spanier, Joel Spencer, R. S. Spira, Frank L. Spitzer, Ivar Stakgold, Harold Stark, J. Stasheff, L. A. Steen, A. Sterrette, B. M. Stewart, Steve Stigler, Karl R. Stromberg, W. C. Swift, E. Swinnerton-Dyer, Angus E. Taylor, Olga Taussky Todd, E. W. Trost, A. W. Tucker, A. G. Tucker, W. T. Tutte, Dietrich Uhlenbrock, James K. Washenberger, Wolfgang Wasow, W. C. Waterhouse, Daniel Waterman, James H. Wells, James G. Wendel, Arthur T. White, Roger Wiegand, Stephen Willard, W. T. Williams, J. J. Willmore, D. E. Wilson, Robert L. Wilson, Jr., Theophil Worosz, H. Wuszing, N. D. Ylvisaker, G. S. Young, Lawrence Zalcman, Daniel Zelinsky, J. J. Zwislocki.

## CALENDAR OF FUTURE MEETINGS

Sixtieth Annual Meeting, St. Louis, Missouri, January 29–31, 1977.

Fifty-seventh Summer Meeting, University of Washington, August 14–16, 1977.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, St. Francis College, Loretto, Pennsylvania, April 22–23, 1977.

FLORIDA, University of South Florida, Tampa, March 4–5, 1977.

ILLINOIS, Chicago Loop College, Chicago, May 6–7, 1977.

INDIANA

INTERMOUNTAIN

IOWA, Drake University, Des Moines, April 22–23, 1977.

KANSAS, Tabor College, Hillsboro, April 2, 1977.

KENTUCKY, early April. Deadline for papers 6 wks. bef. mtg.

LOUISIANA-MISSISSIPPI, University of New Orleans, Louisiana, February 25–26, 1977.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Spring. Deadline for papers 2 wks. bef. mtg.

MICHIGAN, Eastern Michigan University, Ypsilanti, May 6–7, 1977.

MISSOURI, University of Missouri, St. Louis, April 29–30, 1977.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 15–16, 1977.

NEW JERSEY, early November and early May.

NORTH CENTRAL, North Hennepin Community College, Minneapolis, Minnesota, Spring 1977.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, San Francisco State University, February 26, 1977.

OHIO

OKLAHOMA-ARKANSAS, Oral Roberts University, Tulsa, Oklahoma, April 1–2, 1977.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 wks. bef. mtg.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, Metropolitan State College, Denver, Colorado, April 29–30, 1977.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 wks. bef. mtg.

SOUTHEASTERN, University of Alabama, Huntsville, April 1–2, 1977.

SOUTHERN CALIFORNIA, Loyola Marymount University, Los Angeles, March 12, 1977.

SOUTHWESTERN, Phoenix College, Phoenix, Arizona, April 22–23, 1977.

TEXAS, Baylor University, Waco, April 1–2, 1977.

WISCONSIN, University of Wisconsin, Oshkosh, Spring 1977.

## FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Denver, February 20–26, 1977.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, St. Louis, Missouri, January 27–30, 1977.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, University of North Dakota, Grand Forks, June 13–16, 1977.

ASSOCIATION FOR COMPUTING MACHINERY, Olympic Hotel, Seattle, Washington, October 17–19, 1977.

ASSOCIATION FOR SYMBOLIC LOGIC, Chase Park Plaza Hotel, St. Louis, Missouri, January 27–28, 1977.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, Hamilton, Ontario, June 2, 1977 (tentative date).

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Seattle, Washington, August 14–18, 1977.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Cincinnati, Ohio, April 20–23, 1977.

OPERATIONS RESEARCH SOCIETY OF AMERICA, San Francisco Hilton, May 9–11, 1977.

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**GEOMETRIC INEQUALITIES, by N. D. Kazarinoff, NML-04**

**THE CONTEST PROBLEM BOOK. Problems from the Annual High School Mathematics Contests sponsored by the MAA, NCTM, Mu Alpha Theta, The Society of Actuaries, and the Casualty Actuarial Society. Covers the period 1950-1960. Compiled and with solutions by C. T. Salkind. NML-05.**

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**GEOMETRIC TRANSFORMATIONS, by I. M. Yaglom, translated by Allen Shields, NML-08**

**CONTINUED FRACTIONS, by C. D. Olds, NML-09**

**GRAPHS AND THEIR USES, by Oystein Ore, NML-10**

**HUNGARIAN PROBLEM BOOKS I and II, based on the Eötvös Competitions 1894-1905 and 1906-1928. Translated by E. Rapaport, NML-11 and NML-12.**

**EPISODES FROM THE EARLY HISTORY OF MATHEMATICS, by A. Aaboe, NML-13**

**GROUPS AND THEIR GRAPHS, by I. Grossman and W. Magnus, NML-14**

**THE MATHEMATICS OF CHOICE, by Ivan Niven, NML-15**

**FROM PYTHAGORAS TO EINSTEIN, by K. O. Friedrichs, NML-16**

**THE MAA PROBLEM BOOK II. A continuation of NML-05 containing problems and solutions from the Annual High-School Mathematics Contests for the period 1961-1965.**

**FIRST CONCEPTS OF TOPOLOGY, by W. G. Chinn and N.E. Steenrood, NML-18**

**GEOMETRY REVISITED, by H.S.M. Coxeter, and S. L. Greitzer, NML-19**

**INVITATION TO NUMBER THEORY, by Oystein Ore, NML-20**

**GEOMETRIC TRANSFORMATIONS II, by I. M. Yaglom, translated by Allen Shields, NML-21**

**ELEMENTARY CRYPTANALYSIS—A Mathematical Approach, by Abraham Sinkov, NML-22**

**INGENUITY IN MATHEMATICS, by Ross Honsberger, NML-23**

**GEOMETRIC TRANSFORMATIONS III, by I. M. Yaglom, translated by Abe Shenitzer, NML-24**

**THE MAA PROBLEM BOOK III. A continuation of NML-05 and NML-17, containing problems and solutions from the Annual High School Mathematics Contests for the period 1966-1972.**



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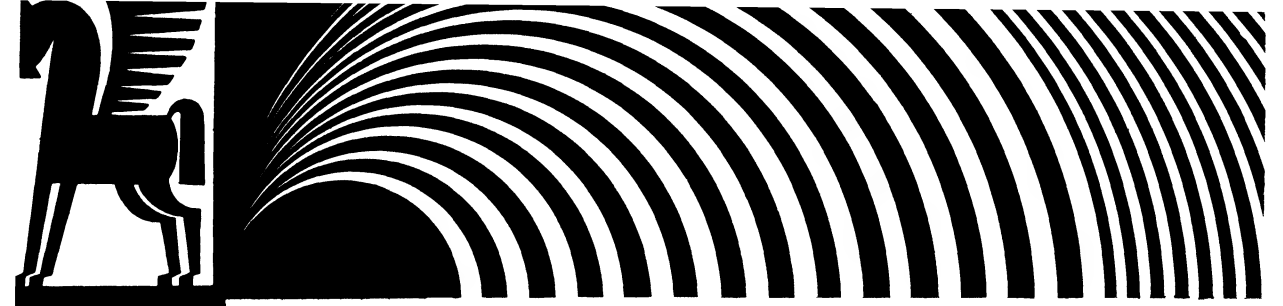
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